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ON REGULAR IMPLICIT OPERATIONS

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1 – Introduction

This paper is concerned with the structure of semigroups of implicit operations. Implicit operations were shown by Reiterman [15] to be an important theoretical tool in the study of pseudovarieties of finite algebras. Specifically, every pseudovariety may be defined by a set of formal equalities of implicit operations on any larger pseudovariety. This result has been used to solve specific problems in the theory of finite semigroups [1], [6], [8], [10].

The main difficulty that one faces when trying to use Reiterman's Theorem lies in the non-explicit character of implicit operations on a given pseudovariety $\mathbf{V}$. Indeed, implicit operations are defined as new operations on the members of $\mathbf{V}$ that commute with all old homomorphisms. Among these operations there are, of course, all those that may be constructed from the old operations and the component projections by composition, which are called "explicit operations". But, in general, there are many other implicit operations.

Thus, it is worthwhile to endow sets of implicit operations with some structure. As a first step, one collects implicit operations on $\mathbf{V}$ according to their arity, denoting by $\overline{\Omega}_n\mathbf{V}$ the set of all n-ary such operations. Then we may operate on this set just as in the members of $\mathbf{V}$ by defining the operations pointwise. Moreover, we get for $\overline{\Omega}_n\mathbf{V}$ a structure of a compact algebra by taking the initial topology for all homomorphisms from $\overline{\Omega}_n\mathbf{V}$ into members of $\mathbf{V}$, which are viewed as discrete spaces [15], [11], [2].

In the case of semigroups, most results that have been obtained so far depend on the availability of a certain kind of factorization for every implicit operation on a given pseudovariety $\mathbf{V}$ of semigroups. Since $\overline{\Omega}_n\mathbf{V}$ is a semigroup, its regular elements will, in general, play an important role. In section 3, we give two characterizations of regular implicit operations: regularity is a pointwise property and it may also be described in terms of $\mathbf{V}$-recognizable languages. Now, for

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some pseudovarieties $\mathbf{V}$, it turns out that every element of $\overline{\Omega}_n\mathbf{V}$ is a product of regular and explicit elements. The best understood case where this happens is the pseudovariety of $\mathcal{J}$-trivial semigroups [5]. This paper presents an extension of this result to the pseudovariety $\mathbf{DS}$ of semigroups in which each $\mathcal{J}$-class is a subsemigroup (section 4) and the study of semidirect products of the form $\mathbf{V} * \mathbf{D}_k$ (section 5) where $\mathbf{D}_k$ is the pseudovariety of semigroups in which every product of k factors is a right zero. In the second case, we obtain a representation of $\overline{\Omega}_n(\mathbf{V} * \mathbf{D}_k)$ which allows us to show that this semigroup has the desired factorization property if we take for $\mathbf{V}$ the pseudovariety $\mathbf{Com}$ of commutative semigroups. Further structural results are obtained for $\overline{\Omega}_n(\mathbf{Com} * \mathbf{D}_k)$ in section 6, including a description of the minimal ideal. We also give an example showing that $\overline{\Omega}_n\mathbf{V}$ does not always have the mentioned factorization property.

A preliminary version of this paper was presented at the Colloquium on Semigroups, Szeged, August, 1987.

2 – Preliminaries

For general background on semigroups see, e.g., Howie [13]. A pseudovariety $\mathbf{V}$ is a class of finite algebras of a given type, closed under formation of homomorphic images, subalgebras and finitary direct products.

Definition 2.1. Let X be a set and $\mathbf{V}$ a pseudovariety. An implicit operation is a family $\pi = (\pi_A)_{A\in\mathbf{V}}$ such that:

i) for each $A \in \mathbf{V}$, $\pi_A\colon A^X \to A$ is a function;

ii) for each homomorphism $\varphi\colon A \to B$ with $A, B \in \mathbf{V}$, the following diagram commutes:

$$\begin{array}{ccc} A^X & \xrightarrow{\pi_A} & A \\ \big\downarrow{\scriptstyle \varphi^X} & & \big\downarrow{\scriptstyle \varphi} \\ B^X & \xrightarrow{\pi_B} & B \end{array}$$

The set of all implicit operations on $\mathbf{V}$, denoted by $\overline{\Omega}_X\mathbf{V}$, is an algebra of the same type as the elements of $\mathbf{V}$. We endow $\overline{\Omega}_X\mathbf{V}$ with the initial topology for all homomorphisms into elements of $\mathbf{V}$. The subalgebra of $\overline{\Omega}_X\mathbf{V}$ generated by the projections is denoted by $\Omega_X\mathbf{V}$.

It is possible to show that $\overline{\Omega}_X\mathbf{V}$ is a compact algebra admitting $\Omega_X\mathbf{V}$ as a dense subalgebra [15], and $\overline{\Omega}_X\mathbf{V}$ is the free object over X in the variety of topological algebras generated by $\mathbf{V}$ (all topologies considered here are supposed to be Hausdorff) [2]. The algebra $\Omega_X\mathbf{V}$ may be naturally identified with the free object on $|X|$ generators in the variety generated by $\mathbf{V}$. We also denote by $\overline{\Omega}_n\mathbf{V}$ and $\Omega_n\mathbf{V}$ the algebras $\overline{\Omega}_X\mathbf{V}$ and $\Omega_X\mathbf{V}$, respectively, in case X is a set with n elements.

For a variety $\mathcal{V}$ and a set A we denote by $\Omega_A\mathcal{V}$ the free object in $\mathcal{V}$ on A.

The next theorem is the analog for pseudovarieties of Birkhoff's Theorem for varieties. To state it, it is convenient to use the following definition.

Definition 2.2. A *pseudoidentity for* $\mathbf{V}$ is a formal equality $\pi = \rho$ with π and ρ in some $\overline{\Omega}_n\mathbf{V}$. An algebra $A \in \mathbf{V}$ is said to *satisfy* the pseudoidentity $\pi = \rho$ if $\pi_A = \rho_A$. For a subclass $\mathcal{C}$ of $\mathbf{V}$ and a set Σ of pseudoidentities for $\mathbf{V}$, we write $\mathcal{C} \models \Sigma$ in case every member of $\mathcal{C}$ satisfies every pseudoidentity in Σ.

It is easy to prove [7] that, if $(\pi_n)_{n\in\mathbb{N}}$ is a sequence in $\overline{\Omega}_X\mathbf{V}$ and $\pi \in \overline{\Omega}_X\mathbf{V}$, then $\lim\limits_{n\in\mathbb{N}} \pi_n = \pi$ if and only if

$$\text{(1)} \qquad \forall p \in \mathbb{N} \ \ \exists k \in \mathbb{N}\colon \Big(A \in \mathbf{V},\ |A| \le p,\ m \ge k\Big) \ \Rightarrow\ A \models \pi_m = \pi\ .$$

For a set Σ of pseudoidentities for $\mathbf{V}$, we also let the *subclass of* $\mathbf{V}$ *defined by* Σ be

$$[\![\Sigma]\!]_{\mathbf{V}} = \Big\{A \in \mathbf{V}\colon\ A \models \Sigma\Big\}\ .$$

Theorem 2.3 (Reiterman [15]). *Let* $\mathbf{W}$ *be any subclass of a pseudovariety* $\mathbf{V}$. *Then* $\mathbf{W}$ *is itself a pseudovariety if and only if there is some set* Σ *of pseudoidentities for* $\mathbf{V}$ *such that* $\mathbf{W} = [\![\Sigma]\!]_{\mathbf{V}}$.

In general we write $[\![\Sigma]\!]$ instead of $[\![\Sigma]\!]_{\mathbf{S}}$, where $\mathbf{S}$ denotes the class of all finite semigroups.

The following is a list of some pseudovarieties of semigroups which will appear in the sequel. We represent by x^ω the implicit operation defined so that, for a semigroup S and an element s of S, $(x^\omega)_S(s)$ is the idempotent, denoted by s^ω, of the subsemigroup of S generated by s. The semigroups considered in the following classes are all finite.

$$\begin{array}{lcl}
\mathbf{Sl} & = & \{\text{semilattices}\} = [\![x^2 = x, xy = yx]\!]\ , \\
\mathbf{Com} & = & \{\text{commutative semigroups}\} = [\![xy = yx]\!]\ , \\
\mathbf{LG} & = & \{\text{semigroups such that all idempotents are } \mathcal{J}\text{-equivalent}\} \\
 & = & [\![(x^\omega y^\omega x^\omega)^\omega = x^\omega]\!]\ , \\
\mathbf{G} & = & \{\text{groups}\} = [\![x^\omega y = yx^\omega = y]\!]\ , \\
\mathbf{J} & = & \{\mathcal{J}\text{-trivial semigroups}\} = [\![x^{\omega+1} = x^\omega, (xy)^\omega = (yx)^\omega]\!]\ , \\
\mathbf{K} & = & \{\text{semigroups such that idempotents are left zeros}\} = [\![x^\omega y = x^\omega]\!]\ , \\
\mathbf{D} & = & \{\text{semigroups such that idempotents are right zeros}\} = [\![yx^\omega = x^\omega]\!]\ .
\end{array}$$

For a nonvoid set A we denote by A^+ (A^*) the free semigroup (monoid) generated by A. As the class $\mathbf{S}$ satisfies no nontrivial identity, $\Omega_A\mathbf{S}$ may be identified with A^+.

Recall that an element a of a semigroup S is *regular* if there exists $x \in S$ such that $axa = a$. In this work we look for some characterization of regular implicit operations on certain pseudovarieties of semigroups. The case of unary implicit operations is easy and is given by the following proposition.

Proposition 2.4 [7]. *If* $\mathbf{V}$ *is a pseudovariety of semigroups and* $\pi \in \overline{\Omega}_1\mathbf{V}$, *then* π *is either regular or explicit.*

For implicit operations of arity greater than 1, the situation is much more complicated as is shown below. On the other hand, even in the case of unary implicit operations, not every one is a composite of explicit operations and the operation x^ω [7]. The following is an example of a regular binary implicit operation on $\mathbf{S}$ which can not be constructed using only explicit and unary operations [7].

Example 2.5. Let μ be the endomorphism of $\{x,y\}^+$ (i.e., $\Omega_2\mathbf{S}$), defined by $\mu(x) = xy$ and $\mu(y) = yx$. Then the sequence given by $w_n = \mu^n(x)$ satisfies $w_{n+2} = w_n[\mu^n(y)]^2 w_n$, and so any accumulation point of $(w_n)_{n\in\mathbb{N}}$ in $\overline{\Omega}_2\mathbf{S}$ is a regular operation. The sequence $(w_n)_{n\in\mathbb{N}}$ is known as the Thue-Morse sequence [14].

3 – Systems of equations in $\overline{\Omega}_n\mathbf{V}$

In the following, $\mathbf{V}$ denotes a pseudovariety of an arbitrary type, n, r, s nonnegative integers, α_i, β_i $(i \in I)$ $(r+s)$-ary implicit operations, and $\pi_1, ..., \pi_r$ n-ary implicit operations.

Definition 3.1. The system of equations

$$\alpha_i(\pi_1, ..., \pi_r, x_1, ..., x_s) = \beta_i(\pi_1, ..., \pi_r, x_1, ..., x_s) \quad (i \in I) \tag{2}$$

is said *to have a solution in* $\overline{\Omega}_n\mathbf{V}$ if there are $\rho_1, ..., \rho_s \in \overline{\Omega}_n\mathbf{V}$ such that (2) holds for $x_j = \rho_j$ $(j = 1, ..., s)$.

Definition 3.2. The system (2) is said to be *pointwise solvable* in $\mathbf{V}$ if, for each $A \in \mathbf{V}$ and all $a_1, ..., a_n \in A$, there are $b_1, ..., b_s \in A$ such that

$$\begin{aligned} \alpha_{iA}\Big(\pi_{1A}(a_1, ..., a_n), ..., \pi_{rA}(a_1, ..., a_n), b_1, ..., b_s\Big) &= \\ = \beta_{iA}\Big(\pi_{1A}(a_1, ..., a_n), ..., \pi_{rA}(a_1, ..., a_n), b_1, ..., b_s\Big) & \quad (i \in I)\,. \end{aligned} \tag{3}$$

These two definitions are related by the following compactness theorem.

Theorem 3.3. *For a pseudovariety* $\mathbf{V}$ *of an arbitrary type, the system of equations (2) admits a solution in* $\overline{\Omega}_n\mathbf{V}$ *if and only if it is pointwise solvable in* $\mathbf{V}$.

Proof: If $x_j = \rho_j$ $(j = 1, \dots, s)$ gives a solution of (2) in $\overline{\Omega}_n\mathbf{V}$, then, for $A \in \mathbf{V}$ and $a_1, \dots, a_n \in A$, the elements of A given by $b_j = \rho_j(a_1, \dots, a_n)$ $(j = 1, \dots, s)$ satisfy (3), so that (2) is pointwise solvable.

Conversely, suppose (2) is pointwise solvable in $\mathbf{V}$. By [2], the topological algebra $\overline{\Omega}_n\mathbf{V}$ may be viewed as a projective limit. Specifically, consider any directed set $(D, \leq)$, $A_d \in \mathbf{V}$, $a_{1d}, \dots, a_{nd} \in A_d$ $(d \in D)$, and any homomorphisms $\varphi_{cd}\colon A_d \to A_c$ $(c \leq d)$ such that:

i) A_d is generated by $\{a_{1d}, \dots, a_{nd}\}$;

ii) for every $A \in \mathbf{V}$ and $a_1, \dots, a_n \in A$, there is $d \in D$ and a homomorphism $\psi\colon A_d \to A$ such that $\psi(a_{kd}) = a_k$ $(k = 1, \dots, n)$;

iii) $\varphi_{dd} = id_{A_d}$;

iv) $c \leq d \leq e \implies \varphi_{ce} = \varphi_{cd} \circ \varphi_{de}$;

v) $\varphi_{cd}(a_{kd}) = a_{kc}$ for $k = 1, \dots, n$.

Then, by [2, Thm. 2.2], we have the following projective limit description of $\overline{\Omega}_n\mathbf{V}$ given by an isomorphism

$$\left(\overline{\Omega}_n\mathbf{V}; pr_1, \dots, pr_n\right) \simeq \varprojlim_{d \in D} \left(A_d; a_{1d}, \dots, a_{nd}\right)$$

under $\pi \mapsto (\pi_{A_d}(a_{1d}, \dots, a_{nd}))_{d \in D}$ where pr_k denotes the n-ary projection into component k.

Since (2) is pointwise solvable in $\mathbf{V}$, for each $d \in D$, there are $b_{1d}, \dots, b_{sd} \in A_d$ such that

$$(4_d) \qquad \alpha_{iA_d}\left(\pi_{1A_d}(a_{1d}, \dots, a_{nd}), \dots, \pi_{rA_d}(a_{1d}, \dots, a_{nd}), b_{1d}, \dots, b_{sd}\right) = \\ = \beta_{iA_d}\left(\pi_{1A_d}(a_{1d}, \dots, a_{nd}), \dots, \pi_{rA_d}(a_{1d}, \dots, a_{nd}), b_{1d}, \dots, b_{sd}\right) \quad (i \in I)\ .$$

By i), for each $d \in D$, there is $\rho_{jd} \in \Omega_n\mathbf{V}$ such that

$$(5_d) \qquad b_{jd} = \rho_{jd_{A_d}}(a_{1d}, \dots, a_{nd})\ .$$

Since $\overline{\Omega}_n\mathbf{V}$ is compact and, for every cofinal subset of D, properties i)–iv) still hold, we may assume that each net $(\rho_{jd})_{d \in D}$ converges to some ρ_j in $\overline{\Omega}_n\mathbf{V}$

$(j = 1, \dots, s)$. Denote $(a_{1d}, \dots, a_{nd})$ by $\vec{a}_d$. To show that (2) holds with $x_j = \rho_j$ $(j = 1, \dots, s)$, by ii) it suffices to verify that, for each $d \in D$,

$$\alpha_{iA_d}\Big(\pi_{1A_d}(\vec{a}_d), \dots, \pi_{rA_d}(\vec{a}_d), \rho_{1A_d}(\vec{a}_d), \dots, \rho_{sA_d}(\vec{a}_d)\Big) = \tag{6}$$
$$= \beta_{iA_d}\Big(\pi_{1A_d}(\vec{a}_d), \dots, \pi_{rA_d}(\vec{a}_d), \rho_{1A_d}(\vec{a}_d), \dots, \rho_{sA_d}(\vec{a}_d)\Big) \quad (i \in I)\ .$$

But, for $d \in D$, there is $c_d \in D$ such that, for $j = 1, \dots, s$ and $c \geq c_d$,

$$\rho_{jcA_d}(a_{1d}, \dots, a_{nd}) = \rho_{jA_d}(a_{1d}, \dots, a_{nd})\ . \tag{7}$$

Now, let $c \geq d, c_d$. Then, applying φ_{dc} to both members of each equation in (4_c) taking into account (5_c), (7) and v), and the characteristic property of implicit operations — they commute with homomorphisms — we obtain (6). Hence (2) is solvable in $\overline{\Omega}_n\mathbf{V}$. ∎

The following result contains some applications of Theorem 3.3 for pseudovarieties of semigroups.

Corollary 3.4. *Let* $\mathbf{V}$ *be a pseudovariety of semigroups and* $\pi \in \overline{\Omega}_n\mathbf{V}$. *Then*

i) π *is regular if and only if it is pointwise regular;*

ii) *if* π *is not explicit then there are* ρ, π' *and* π'' *in* $\overline{\Omega}_n\mathbf{V}$ *such that* $\pi = \pi'\rho^\omega\,\pi''$.

Proof: For i), apply Theorem 3.3 to the equation $\pi = \pi\, x\, \pi$.

For ii), assuming π is not explicit, consider the equation $\pi = x'\, y^\omega\, x''$. By the preceding theorem, it suffices to prove that this equation is pointwise solvable. Let $(w_k)_{k\in\mathbb{N}}$ be a sequence in $\Omega_n\mathbf{V}$ converging to π, which implies that $\lim_{k\to\infty} |w_k| = \infty$.

For any semigroup S in $\mathbf{V}$, let k be such that $S \models \pi = w_k$ and $|w_k| \geq |S|$ according to (1). Then, given $s_1, \dots, s_n \in S$, $w_k(s_1, \dots, s_n)$ admits a factorization $w_k(s_1, \dots, s_n) = s\, e\, t$ for some $s, e, t \in S$ with $e = e^2$ [12], proving local solvability of our equation. ∎

To formulate another characterization of regular implicit operations using Corollary 3.4, we first state a definition.

Definition 3.5. Let $\mathbf{V}$ be a pseudovariety of an arbitrary type. We say that a subset L of $\Omega_n\mathbf{V}$ is $\mathbf{V}$-*recognizable* if there are $F \in \mathbf{V}$ and a homomorphism $\varphi\colon \Omega_n\mathbf{V} \longrightarrow F$ such that $L = \varphi^{-1}(\varphi(L))$.

There is a topological characterization of $\mathbf{V}$-recognizable subsets of $\Omega_n\mathbf{V}$: $L \subseteq \Omega_n\mathbf{V}$ is $\mathbf{V}$-recognizable if and only if its closure $\overline{L}$ in $\overline{\Omega}_n\mathbf{V}$ is open and $L = \overline{L} \cap \Omega_n\mathbf{V}$ [4].

We may now express regularity of implicit operations in terms of combinatorial properties of sequences of "words". We say that a sequence of elements of a set X *lies eventually* in a subset A of X if all but a finite number of terms of the sequence lie in A.

Theorem 3.6. *Let* $\mathbf{V}$ *be a pseudovariety of semigroups. Then, if* $(w_k)_{k\in\mathbb{N}}$ *is a sequence in* $\Omega_n\mathbf{V}$ *with limit* π *in* $\overline{\Omega}_n\mathbf{V}$, *the following are equivalent:*

i) π *is regular;*

ii) *for every* $\mathbf{V}$*-recognizable language* $L \subseteq \Omega_n\mathbf{V}$,

$$\pi \in \overline{L} \implies \pi \in \overline{L}\,\overline{\Omega}_n\mathbf{V}\,\overline{L}\,;$$

iii) *for every* $\mathbf{V}$*-recognizable language* $L \subseteq \Omega_n\mathbf{V}$,

$$(w_k)_{k\in\mathbb{N}} \text{ is eventually in } L \implies (w_k)_{k\in\mathbb{N}} \text{ is eventually in } L\,\Omega_n\mathbf{V}L\,.$$

Proof: Of course i) implies ii). To prove that ii) is equivalent to iii) just note that, since the product operation is continuous in $\overline{\Omega}_n\mathbf{V}$ and $\overline{\Omega}_n\mathbf{V}$ is compact, $\overline{L\Omega_n\mathbf{V}L} = \overline{L}\,\overline{\Omega}_n\mathbf{V}\,\overline{L}$ and apply the above characterization of $\mathbf{V}$-recognizability.

Finally, suppose that ii) holds. By Corollary 3.4 i), to prove i), it suffices to verify that π assumes only regular values. Let $S \in \mathbf{V}$ and let $\varphi\colon \overline{\Omega}_n\mathbf{V} \to S$ be a continuous homomorphism. Then the set $K = \varphi^{-1}(\varphi(\pi))$ is the closure of a $\mathbf{V}$-recognizable subset of $\Omega_n\mathbf{V}$, namely $K = \overline{L}$ with $L = K \cap \Omega_n\mathbf{V}$. Indeed, since φ is continuous, K is a clopen set and so, as $\Omega_n\mathbf{V}$ is dense in $\overline{\Omega}_n\mathbf{V}$, it follows that $K = \overline{L}$. By ii), we have $\pi \in K\overline{\Omega}_n\mathbf{V}K$. Hence $\pi = \pi_1\,\rho\,\pi_2$ for some $\pi_1, \pi_2 \in \overline{\Omega}_n\mathbf{V}$ such that $\varphi(\pi_1) = \varphi(\pi_2) = \varphi(\pi)$, whence $\varphi(\pi) = \varphi(\pi)\,\varphi(\rho)\,\varphi(\pi)$ and π is regular in S. ∎

4 – Regular implicit operations on DS

The rest of this paper is concerned with semigroups.

In this section, which is based on the second author's doctoral thesis [9], we give combinatorial characterizations of regular implicit operations on some pseudovarieties. More precisely, for certain pseudovarieties $\mathbf{V}$, given a sequence in $\Omega_n\mathbf{V}$ with limit π in $\overline{\Omega}_n\mathbf{V}$, we obtain conditions determining when π is regular.

For a pseudovariety $\mathbf{V}$ and $\pi = \pi(x_1, ..., x_n) \in \overline{\Omega}_n\mathbf{V}$ we say that π depends on x_i if there exists $S \in \mathbf{V}$ such that π_S depends on the i^{th}-component, and define $c(\pi)$ as the set of all such x_i. For $\pi \in \overline{\Omega}_n\mathbf{V}$, $c(\pi)$ is said to be the *content of* π.

Lemma 4.1 [9]. *If* $\mathbf{V}$ *is a pseudovariety, then the content function*

$$c\colon \overline{\Omega}_n\mathbf{V} \longrightarrow \mathcal{P}\{x_1,\dots,x_n\}$$

is a continuous homomorphism (where $\mathcal{P}\{x_1,\dots,x_n\}$ *is seen as a discrete topological semigroup under union, and, therefore, as an element of* $\mathbf{Sl}$*) if and only if* $\mathbf{Sl}\subseteq\mathbf{V}$.

Example 4.2. Let $\pi=(xy)^\omega\in\overline{\Omega}_n\mathbf{V}$ with $\mathbf{Sl}\subseteq\mathbf{V}$. Then

$$c((xy)^\omega)=\lim_n c((xy)^{n!})=\lim_n\{x,y\}=\{x,y\}\ .$$

Definition 4.3. The pseudovariety $\mathbf{DS}$ is the class of all finite semigroups whose regular $\mathcal{J}$-classes are subsemigroups.

Using Green's Lemma [13], one can easily prove that $\mathbf{DS}$ is defined by the pseudoidentity

$$\left[(xy)^\omega(yx)^\omega(xy)^\omega\right]^\omega=(xy)^\omega\ , \tag{8}$$

and that $\mathbf{DS}$ is the class of all finite semigroups whose regular elements are group elements.

The following lemma will be useful in the characterization of the implicit operations on $\mathbf{DS}$.

Lemma 4.4.

i) *A finite semigroup* S *belongs to* $\mathbf{DS}$ *if and only if, for any idempotent* e, *the set of elements* $\mathcal{J}$*-above* e *is a subsemigroup.*

ii) *If* π *and* ρ *are implicit operations on* $\mathbf{DS}$ *such that* $c(\rho)\subseteq c(\pi)$ *then* $(\pi^\omega\rho\,\pi^\omega)^\omega=\pi^\omega$.

iii) $\mathbf{DS}$ *is the largest pseudovariety* $\mathbf{V}$, *such that, for regular implicit operations* $\pi,\rho\in\overline{\Omega}_n\mathbf{V}$, π *and* ρ *are* $\mathcal{J}$*-equivalent if and only if they have the same content.*

iv) *For any subpseudovariety* $\mathbf{V}$ *of* $\mathbf{DS}$ *containing* $\mathbf{Sl}$, $\overline{\Omega}_n\mathbf{V}$ *has* 2^n-1 *regular* $\mathcal{J}$*-classes.*

v) $\mathbf{DS}$ *is defined by the pseudoidentity* $[(xy)^{\omega+1}x]^{\omega+1}=(xy)^{\omega+1}x$.

Proof: i) Let S be a finite semigroup of $\mathbf{DS}$, e an idempotent of S, a,b elements of S $\mathcal{J}$-above e and $x,y,z,t\in S$ such that

$$e=x\,a\,y\,,\qquad e=z\,b\,t\ .$$

Using the pseudoidentity (8), we have

$$e = e^\omega = (z\,b\,t)^\omega = \left((zbt)^\omega\,(tzb)^\omega\,(zbt)^\omega\right)^\omega \leq_J b(z\,b\,t)^\omega = be \leq_J e$$

and, in a similar way,

$$e = e^\omega = (x\,a\,y)^\omega \leq_J ea \leq_J e\ .$$

Hence ea and be are $\mathcal{J}$-equivalent to e. As the $\mathcal{J}$-class of e is a subsemigroup, we have $(ea \cdot be)\mathcal{J}e$, and so $a\,b$ is $\mathcal{J}$-above e.

ii) Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of explicit operations converging to ρ. By Lemma 4.1, we may assume that $c(u_n) \subseteq c(\pi)$ for every n so that, by i), we have

$$\pi^\omega\, u_n\, \mathcal{J}\pi^\omega\ .$$

Thus, by compactness, there exist convergent sequences of implicit operations $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$ such that

$$\pi^\omega = \alpha_n\, \pi^\omega\, u_n\, \beta_n\ .$$

As the semigroups of implicit operations are topological semigroups, we have

$$\pi^\omega = \alpha\, \pi^\omega\, \rho\, \beta\ ,$$

where α and β are the limits of the sequences $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$, and so $\pi^\omega \mathcal{J} \pi^\omega\rho$.

From i), it follows that $(\pi^\omega\rho\,\pi^\omega)^\omega = \pi^\omega$.

iii) The first part is a consequence of ii) and the continuity of the content function.

Suppose now that $\mathbf{V}$ is a pseudovariety such that for any regular implicit operations $\pi, \rho \in \overline{\Omega}_n\mathbf{V}$, π and ρ are $\mathcal{J}$-equivalent if and only if they have the same content. Let $\pi = (xy)^\omega$ and $\rho = ((xy)^\omega\,(yx)^\omega\,(xy)^\omega)^\omega$.

If $\mathbf{Sl} \subseteq \mathbf{V}$, then, by 4.1, the content function defined in $\overline{\Omega}_n\mathbf{V}$ is continuous. Then, as π and ρ have the same content, they are $\mathcal{J}$-equivalent idempotents. As $\pi\rho = \rho\pi = \rho$, we conclude that π and ρ are $\mathcal{H}$-equivalent and so $\pi = \rho$, which means that $\mathbf{V} \subseteq \mathbf{DS}$.

If $\mathbf{Sl}$ is not contained in $\mathbf{V}$ then one can easily show that $\mathbf{V} \subseteq \mathbf{LG}$, and so $\mathbf{V} \subseteq \mathbf{DS}$.

iv) It is an immediate consequence of iii).

v) Let $\mathbf{V} = [\![(xy)^{\omega+1}x = ((xy)^{\omega+1}x)^{\omega+1}]\!]$. If $S \in \mathbf{DS}$ and $a, b \in S$, then, by i), $(ab)^{\omega+1}a$ is regular, because it is $\mathcal{J}$-equivalent to $(ab)^\omega$. But, for semigroups of $\mathbf{DS}$, regular is equivalent to group element, and so $(ab)^{\omega+1}a = ((ab)^{\omega+1}a)^{\omega+1}$.

Suppose now that S is a semigroup of $\mathbf{V}$. To prove that S is a semigroup of $\mathbf{DS}$ we will prove that every regular element of S is a group element. Let a be a regular element of S, e be an idempotent of S $\mathcal{R}$-equivalent to a, and $b \in S$ such that $e = ab$. Then we have

$$a = ea = (ab)a = (ab)^2 a = \ldots = (ab)^{\omega+1} a \ ,$$

and, as $S \in \mathbf{V}$, $a = (ab)^{\omega+1}a = ((ab)^{\omega+1}a)^{\omega+1}$. ∎

For our characterization of regular implicit operations on $\mathbf{DS}$, we need the following definition.

Definition 4.5. Let $\mathbf{V}$ be a pseudovariety such that $\Omega_n\mathbf{V}$ is equal, as a semigroup, to $\{x_1, \ldots, x_n\}^+$. For $w, u \in \Omega_n\mathbf{V}$, define

$$\begin{bmatrix} w \\ u \end{bmatrix} = \max\Big\{ r\colon\ u^r \text{ is a subword of } w \Big\}$$

where, viewing words as finite sequences of letters, we use the term "subword" in the usual sense of "subsequence".

Lemma 4.6 [5]. *If $\mathbf{V}$ is a pseudovariety containing $\mathbf{J}$ and $u \in \Omega_n\mathbf{V}$, then the mapping*

$$\begin{array}{ccc} \Omega_n\mathbf{V} & \longrightarrow & \mathbb{N} \\ w & \longmapsto & \left[\begin{smallmatrix} w \\ u \end{smallmatrix}\right] \end{array}$$

is uniformly continuous, and so it extends to a continuous mapping

$$\begin{array}{ccc} \overline{\Omega}_n\mathbf{V} & \longrightarrow & \mathbb{N} \cup \{\infty\} \\ \pi & \longmapsto & \left[\begin{smallmatrix} \pi \\ u \end{smallmatrix}\right] \end{array} .$$

Thus, in the situation of the lemma, if $\pi = \lim_{k\to\infty} w_k$ with $w_k \in \Omega_n\mathbf{V}$, then $\left[\begin{smallmatrix} \pi \\ u \end{smallmatrix}\right]$ may be computed by taking $\lim_{k\to\infty} \left[\begin{smallmatrix} w_k \\ u \end{smallmatrix}\right]$.

The following lemma is crucial for the characterization of regular implicit operations on $\mathbf{DS}$. It extends the case of $\mathcal{J}$-trivial semigroups considered in [5].

Lemma 4.7. *If S is a semigroup of $\mathbf{DS}$ and w and u are words with the same content such that $\left[\begin{smallmatrix} w \\ u \end{smallmatrix}\right] > |S|$, then w assumes only regular values in S, that is $S \models w = w^{\omega+1}$.*

Proof: Let $c(w) = \{x_1, \ldots, x_n\}$ and $w = u_1 \cdots u_k\, t$, where $t, u_1, \ldots, u_k$ are words such that $k = \left[\begin{smallmatrix} w \\ u \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} u_i \\ u \end{smallmatrix}\right] = 1$.

Let $b_p = u_1 \cdots u_p$ with $p \leq k$. For $a_1, ..., a_n \in S$, as $k > |S|$, there exist $r, s \in \mathbb{N}$ such that $r < s < k$ and $b_r(a_1, ..., a_n) = b_s(a_1, ..., a_n)$. If, for simplicity, we denote $u_i(a_1, ..., a_n)$ by u_i, then we have

$$\begin{aligned} b_s &= b_r\, u_{r+1} \cdots u_s = b_s\, u_{r+1} \cdots u_s \\ &= b_r (u_{r+1} \cdots u_s)^2 = ... = b_r (u_{r+1} \cdots u_s)^{\omega+1} , \end{aligned}$$

and so

$$\begin{aligned} w &= b_r (u_{r+1} \cdots u_s)^{\omega+1}\, u_{s+1} \cdots u_k t \\ &= \left[b_r (u_{r+1} \cdots u_s)^{\omega+1}\, u_{s+1} \cdots u_k t\right]^{\omega+1} \qquad \text{by Lemma 4.4 v)} \\ &= w^{\omega+1} . \blacksquare \end{aligned}$$

As a consequence we have the following corollary in view of Corollary 3.4 i).

Corollary 4.8. *If* $\mathbf{V}$ *is a subpseudovariety of* $\mathbf{DS}$ *containing* $\mathbf{J}$ *and* $\pi \in \overline{\Omega}_n\mathbf{V}$, *then* π *is regular if and only if, for any word* u, $\left[{\pi \atop u}\right] \in \{0, \infty\}$. $\blacksquare$

For the factorization of implicit operations on $\mathbf{DS}$, in terms of explicit and regular ones, we need some remarks.

Let $A = \{x_1, ..., x_n\}$, let B be the set of all $u \in A^+$ without repeated letters such that $c(u) = A$, and consider the mapping $\varphi : \overline{\Omega}_n\mathbf{DS} \to \mathbb{N}$ where, for $\pi \in \overline{\Omega}_n\mathbf{DS}$, $\varphi(\pi)$ denotes the number of words $u \in B$ such that $\left[{\pi \atop u}\right] \notin \{0, \infty\}$. This mapping enjoys the following properties:

- if $\varphi(\pi) = 0$ then π is regular;
- if $\varphi(\pi) \neq 0$, then, for $u \in B$ such that $\left[{\pi \atop u}\right] = r \notin \{0, \infty\}$, there exists a sequence $(v_k)_{k \in \mathbb{N}}$ in $\Omega_n\mathbf{DS}$ such that
  - $\pi = \lim\limits_k v_k$;
  - $v_k = v_{k,0}\, a_1 \cdots v_{k,s-1}\, a_s\, v_{k,s}$ where $u^r = a_1 \cdots a_s$ and $a_i \notin c(v_{k,i-1})$;
  - $\lim\limits_k v_{k,i} = \pi_i$;
  - $\varphi(\pi_i) < \varphi(\pi)$.

This is a straightforward generalization of a similar result proved in [5] for $\mathbf{J}$ instead of $\mathbf{DS}$. Using induction and the pointwise characterization of regular implicit operations, we obtain the following theorem.

Theorem 4.9. *Let* $\mathbf{V}$ *be a subpseudovariety of* $\mathbf{DS}$. *If* $\pi \in \overline{\Omega}_n\mathbf{V}$, *then* π *can be decomposed as* $\pi = u_0\, \pi_1\, u_1 \cdots \pi_k\, u_k$ $(k \in \mathbb{N})$ *such that*

- $u_i \in \{x_1, ..., x_n\}^*$,

- *π_i is regular,*
- *last letter of $u_i \notin c(\pi_{i+1})$,*
- *first letter of $u_i \notin c(\pi_i)$,*
- *if $u_i = 1$, then $c(\pi_i)$ and $c(\pi_{i+1})$ are incomparable under inclusion.* ∎

For **DO**, the class of semigroups of **DS** such that the regular $\mathcal{J}$-classes are orthodox, we know more about regular implicit operations. We begin with a lemma.

Lemma 4.10. *Let $S \in \mathbf{DO}$ and $e, a, b \in S$ be such that $e = e^2$ and $e \leq_{\mathcal{J}} a, b$. Then $eabe = eaebe$.*

Proof: As ea and be are group elements (since they lie in the $\mathcal{J}$-class of e), we have

$$eabe = ea \cdot (ea)^{\omega}\, (be)^{\omega} \cdot be\ .$$

As the $\mathcal{J}$-class of e is orthodox, $(ea)^{\omega}\,(be)^{\omega}$ is an idempotent. But, as $(ea)^{\omega}\,(be)^{\omega}$ is $\mathcal{H}$-equivalent to e, $(ea)^{\omega}\,(be)^{\omega} = e$ and so $eabe = eaebe$. ∎

Theorem 4.11. *If $\mathbf{V}$ is a subpseudovariety of $\mathbf{DO}$, then every regular implicit operation on $\mathbf{V}$ is determined by its restriction to $\mathbf{V} \cap \mathbf{G}$.*

Proof: Let $\pi \in \overline{\Omega}_n\mathbf{V}$ be regular ($\pi = \pi^{\omega+1}$) and let $(u_m)_{m\in\mathbb{N}}$ be a sequence in $\Omega_n\mathbf{V}$ with limit π. For $S \in \mathbf{V}$, let $k \in \mathbb{N}$ be such that $S \models \pi = u_k$.

For $a_1, \ldots, a_n \in S$ and $e = \pi_S^{\omega}(a_1, \ldots, a_n)$, let H be the $\mathcal{H}$-class of e. Then $H \in \mathbf{V} \cap \mathbf{G}$ and

$$\begin{aligned}
\pi_S(a_1, \ldots, a_n) &= u_k(a_1, \ldots, a_n)\\
&= e\, u_k(a_1, \ldots, a_n)\, e\\
&= u_k(e\, a_1\, e, \ldots, e\, a_n\, e) \quad \text{by Lemma 4.10}\\
&= \pi_H(e\, a_1\, e, \ldots, e\, a_n\, e)\ . \blacksquare
\end{aligned}$$

At the level of **J**, a complete description of implicit operations has been obtained. In particular, one can establish uniqueness of the factorization given by Theorem 4.9. This result may also be interpreted as the solution of the word problem for a certain presentation of $\overline{\Omega}_n\mathbf{J}$.

Theorem 4.12 [5]. *Let $\pi, \rho \in \overline{\Omega}_n\mathbf{J}$. Then:*

i) $\mathbf{J} \models \pi = \rho$ *if and only if, for all $u \in \Omega_n\mathbf{J}$, $\binom{\pi}{u} = \binom{\rho}{u}$;*

ii) *if* $\pi = u_0 \pi_1 u_1 \cdots \pi_k u_k$ *and* $\rho = u_0 \rho_1 v_1 \cdots \rho_s v_s$ *are factorizations as in Theorem 4.9, then* $\mathbf{J} \models \pi = \rho$ *if and only if* $k = s$, $u_i = v_i$ $(i = 0, \ldots, k)$ *and* $c(\pi_i) = c(\rho_i)$ $(i = 1, \ldots, k)$.

Moreover, $\overline{\Omega}_n\mathbf{J}$ *is the free semigroup with a unary operation* $(x \mapsto x^\omega)$ *over the set* $\{x_1, \ldots, x_n\}$ *in the variety defined by the identities* $(xy)^\omega = (yx)^\omega = (x^\omega y^\omega)^\omega$ *and* $x^\omega x = x x^\omega = x^\omega = (x^\omega)^\omega$.

5 – A representation of $\overline{\Omega}_n(\mathbf{V} * \mathbf{D}_k)$

For $k \in \mathbb{N}$, let $\mathcal{D}_k$ be the class of all semigroups such that every product of k elements is a right zero. Let $\mathcal{D} = \bigcup_{n \in \mathbb{N}} \mathcal{D}_k$ and let $\mathbf{D}_k = (\mathcal{D}_k)^F$ be the class of all finite semigroups of $\mathcal{D}_k$. The definition of $\mathcal{K}_k$, $\mathcal{K}$ and $\mathbf{K}_k$ is dual.

For pseudovarieties $\mathbf{V}$ and $\mathbf{W}$ we define their semidirect product $\mathbf{V} * \mathbf{W}$ to be the pseudovariety generated by the semidirect products of elements of $\mathbf{V}$ with elements of $\mathbf{W}$. The definition of the semidirect product of varieties is similar. The definition of semidirect product of semigroups that we use here is the one Eilenberg [12] calls "left unitary" semidirect product.

The following result is the motivation for the remainder of this paper.

Theorem 5.1 [3].

i) $\mathbf{Com} * \mathbf{D}_k = (\mathcal{C}om * \mathcal{D}_k)^F$.

ii) *The variety* $\mathcal{C}om * \mathcal{D}_k$ *is generated by* $\mathbf{Com} * \mathbf{D}_k$.

iii) $\mathbf{Com} * \mathbf{D}_k \models u = v$ *if and only if* $N_k(u) = N_k(v)$.

Here, for a word u, $N_k(u)$ represents the *network* with: set of *vertices* the set $c_k(u)$ of all factors of u of length k; set of *arrows* $c_{k+1}(u)$; an arrow $w_1 \xrightarrow{w} w_2$ with *capacity* $|u|_w$ if $w_1 = i_k(w)$, $w_2 = t_k(w)$ are, respectively, the longest initial and terminal segments of w of length $\leq k$ and w occurs $|u|_w$ times as a factor of u; *source* $i_k(u)$; and *sink* $t_k(u)$. In case the length $|u|$ of u is less than k, we define $N_k(u)$ to be u.

Consider a finite alphabet $A = \{x_1, \ldots, x_n\}$ and let

$$A_k = \left\{ w \in A^+ \colon |w| = k \right\}, \quad B_k = \left\{ w \in A^+ \colon |w| < k \right\}.$$

Let S be a semigroup and let $f \colon A^+ \to S^1$ be a mapping. Consider the set

$M_kS = B_k \cup (A_k \times S \times A_k)$ endowed with the following operation:

$$u \cdot v = \begin{cases} uv & \text{if } |uv| < k \\ \big(i_k(uv), f(uv), t_k(uv)\big) & \text{otherwise} \end{cases} \qquad (u, v \in B_k)$$

$$(w_1, s, w_2) \cdot v = \big(w_1, sf(w_2v), t_k(w_2v)\big)$$

$$u \cdot (w_1, s, w_2) = \big(i_k(uw_1), f(uw_1)s, w_2\big)$$

$$(w_1, s, w_2) \cdot (z_1, t, z_2) = \big(w_1, sf(w_2z_1)t, z_2\big) .$$

A simple calculation shows that the following condition on f ensures that M_kS is a semigroup:

$$\forall\, u, v \in A^+, \quad f(uv) = f(u)\, f(t_k(u)v) = f(u\, i_k(v))\, f(v) . \tag{9}$$

Then, $A_k \times S \times A_k$ forms a Rees matrix semigroup over S under this operation and an ideal of M_kS.

Also using (9) and assuming that $f(B_k) = \{1\}$, another easy calculation shows that the following mapping is a homomorphism:

$$\begin{array}{rccl} \varphi_S: & A^+ & \longrightarrow & M_kS \\ & w & \longmapsto & \begin{cases} w & \text{if } w \in B_k \\ \big(i_k(w), f(w), t_k(w)\big) & \text{otherwise .} \end{cases} \end{array}$$

Example 5.2. Let S be the free commutative monoid on the set A_{k+1} and let $f\colon A^+ \to S^1$ be given by $f_w(u) = |u|_w$ where, for $w \in A_{k+1}$, f_w denotes the w-component of f. Clearly each f_w satisfies (9), whence so does f. Thus, we have a semigroup M_kS and a homomorphism $\varphi_S\colon A^+ \to M_kS$. By Theorem 5.1, for two words u and v in A^+, $\mathbf{Com} * \mathbf{D}_k \models u = v$ if and only if $N_k(u) = N_k(v)$, i.e., if and only if $\varphi_S(u) = \varphi_S(v)$. Hence φ_S factorizes through $\Omega_A(\mathbf{Com} * \mathbf{D}_k)$

$$\begin{tikzcd} A^+ \arrow[r, "\varphi_S"] \arrow[dr, "\text{can.}"'] & M_kS \\ & \Omega_A(\mathbf{Com} * \mathbf{D}_k) \arrow[u, "\psi_k"'] \end{tikzcd}$$

and the mapping ψ_k is an embedding.

As the next proposition shows, the construction $M_k{-}{-}$ is intimately related with the operator $-- * \mathcal{D}_k$. We will show later that the situation portrayed by Example 5.2 is quite general.

Proposition 5.3. *Let $\mathcal{V}$ be a variety of semigroups, let S be a monoid in $\mathcal{V}$ and let $f\colon A^+ \to S$ be any function satisfying (9) such that $f(B_k) = \{1\}$. Then*

$$M_k S \in \mathcal{K}_k \vee (\mathcal{V} * \mathcal{D}_k) .$$

Proof: Consider the partial function

$$\begin{aligned}\varphi\colon\quad B_{k+1} \times S \times B_{k+1} &\longrightarrow M_k S\\ (w_1, s, w_2) &\longmapsto \begin{cases} (w_1, s, w_2) & \text{if } (w_1, s, w_2) \in A_k \times S \times A_k\\ w & \text{if } w = w_1 = w_2 \in B_k \text{ and } s = 1\\ \emptyset & \text{in the remaining cases .}\end{cases}\end{aligned}$$

We claim φ defines a covering

$$M_k S \prec_\varphi B^{(l)}_{k+1} \times (S \circ B^{(r)}_{k+1}) ,$$

where $B^{(l)}_{k+1}$ (resp. $B^{(r)}_{k+1}$) denotes the set B_{k+1} endowed with the operation $u \cdot v = i_k(uv)$ (resp. $u \cdot v = t_k(uv)$) and $S \circ B^{(r)}_{k+1}$ denotes the semigroup of the wreath product of the transformation semigroups (S, S) and $({B^{(r)}_{k+1}}^1, B^{(r)}_{k+1})$ [12]. Since $B^{(l)}_{k+1} \in \mathcal{K}_k$ and $B^{(r)}_{k+1} \in \mathcal{D}_k$, the proposition will follow (cf. Eilenberg [12]).

Indeed, cover (w_1, s, w_2) by (w_1, g, w_2) and $w \in B_k$ by (w, h, w) where $g, h\colon B_{k+1} \to S$ are given by $vg = f(vw_1)\, s$ and $vh = f(vw)$. Then, for $v, w \in B_k$, $s, t \in S$ and $v_1, v_2, w_1, w_2 \in A_k$,

$$(v, 1, v)\, \varphi\, w = \begin{cases} vw & \text{if } |vw| < k\\ \big(i_k(vw), f(vw), t_k(vw)\big) & \text{otherwise ,}\end{cases}$$

$$\begin{aligned}(v, 1, v)(w, h, w)\, \varphi &= \big(i_k(vw), f(vw), t_k(vw)\big)\, \varphi\\ &= \begin{cases} vw & \text{if } |vw| < k\\ \big(i_k(vw), f(vw), t_k(vw)\big) & \text{otherwise ,}\end{cases}\end{aligned}$$

where the first case uses the hypothesis $f(B_k) = \{1\}$;

$$(v, 1, v)\, \varphi\, (w_1, s, w_2) = \big(i_k(vw_1), f(vw_1)s, w_2\big) = (v, 1, v)\, (w_1, g, w_2)\, \varphi$$

$$(v_1, t, v_2)\, \varphi\, w = \big(v_1, tf(v_2 w), t_k(v_2 w)\big) = (v_1, t, v_2)\, (w, h, w)\, \varphi$$

$$(v_1, t, v_2)\, \varphi\, (w_1, s, w_2) = \big(v_1, tf(v_2 w_1)s, w_2\big) = (v_1, t, v_2)\, (w_1, g, w_2)\, \varphi .$$

Hence $\varphi w \subseteq (w, h, w)\varphi$ and $\varphi(w_1, s, w_2) \subseteq (w_1, g, w_2)\varphi$. ∎

Corollary 5.4. *Let $\mathcal{V}$ be a variety containing a nontrivial monoid, let S be a monoid in $\mathcal{V}$ and let $f\colon A^+ \to S$ be a function satisfying (9) and $f(B_k) = \{1\}$. Then $M_kS \in \mathcal{V} * \mathcal{D}_k$.*

Proof: Under the assumption on $\mathcal{V}$, we have $\mathcal{K}_k \subset \mathcal{V} * \mathcal{D}_k$ (see, e.g., Straubing [16]). ∎

We call a function $f\colon A^+ \to S^1$ satisfying (9) and $f(B_k) = \{1\}$ *good*.

Note that Proposition 5.3 and Corollary 5.4 remain valid if, throughout, we replace varieties by pseudovarieties ($\mathcal{D}_k$ and $\mathcal{K}_k$ being replaced by $\mathbf{D}_k$ and $\mathbf{K}_k$, respectively).

A variety $\mathcal{V}$ of semigroups is said to be *monoidal* if $S^1 \in \mathcal{V}$ whenever $S \in \mathcal{V}$. In other words, $\mathcal{V}$ is monoidal if and only if $\mathcal{V}$ is generated by its monoids.

Theorem 5.5. *Let $\mathcal{V}$ be a nontrivial monoidal variety and let $S = \Omega^1_{A_{k+1}}\mathcal{V}$. Let $f\colon A^+ \to S$ where $f(w)$ is the product in S of the successive factors of w of length $k+1$, by order of appearance from left to right. Then f is good and the homomorphism $\varphi_S\colon A^+ \to M_kS$ induces an embedding $\Omega_A(\mathcal{V} * \mathcal{D}_k) \hookrightarrow M_kS$ via the canonical mapping $A^+ \to \Omega_A(\mathcal{V} * \mathcal{D}_k)$. Hence $\varphi_S(A^+) \simeq \Omega_A(\mathcal{V} * \mathcal{D}_k)$.*

Proof: It is immediately verified that f is good. Moreover, since $M_kS \in \mathcal{V}*\mathcal{D}_k$ by Corollary 5.4 and $\Omega_A(\mathcal{V}*\mathcal{D}_k)$ is freely generated by A in the variety $\mathcal{V}*\mathcal{D}_k$, we do indeed have an induced homomorphism $\psi\colon \Omega_A(\mathcal{V}*\mathcal{D}_k) \to M_kS$. The claim is that ψ is injective. To establish it, we use the description of $\Omega_A(\mathcal{V}*\mathcal{D}_k)$ given by [3, Thm. 2.2]: $\Omega_A(\mathcal{V}*\mathcal{D}_k)$ is isomorphic to the subsemigroup of $\Omega_{(\Omega^1_A\mathcal{D}_k)\times A}\mathcal{V} * \Omega_A\mathcal{D}_k$ generated by the set $\{\bar{a} : a \in A\}$ where $\bar{a} = ((1,a),a)$ and the action in the semidirect product is given by $t(s,a) = (ts,a)$.

Now, let $a_1 \cdots a_p, b_1 \cdots b_q \in A^+$ be such that $\varphi_S(a_1 \cdots a_p) = \varphi_S(b_1 \cdots b_q)$. We verify that $\bar{a}_1 \cdots \bar{a}_p = \bar{b}_1 \cdots \bar{b}_q$ thereby showing that, if the words $a_1 \cdots a_p$ and $b_1 \cdots b_q$ have the same image under φ_S, then they are also identified under the canonical mapping $A^+ \to \Omega_A(\mathcal{V} * \mathcal{D}_k)$.

By the definition of φ_S, if $p < k$ then $q < k$ and $a_1 \cdots a_p = b_1 \cdots b_q$, so that $\bar{a}_1 \cdots \bar{a}_p = \bar{b}_1 \cdots \bar{b}_q$. Thus, we may assume that $p, q \geq k$, and so $\varphi_S(a_1 \cdots a_p) = \varphi_S(b_1 \cdots b_q)$ yields

$$
\begin{aligned}
i_k(a_1 \cdots a_p) &= i_k(b_1 \cdots b_q) \\
f(a_1 \cdots a_p) &= f(b_1 \cdots b_q) \qquad (10)\\
t_k(a_1 \cdots a_p) &= t_k(b_1 \cdots b_q)\,.
\end{aligned}
$$

In view of the above description of $\Omega_A(\mathcal{V} * \mathcal{D}_k)$, we have

$$\begin{aligned}\overline{a}_1 \cdots \overline{a}_p = \Big(&(1, a_1)\,(a_1, a_2) \cdots (a_1 \cdots a_{k-1}, a_k) \cdot \\ &\cdot (a_1 \cdots a_k, a_{k+1})\,(a_2 \cdots a_{k+1}, a_{k+2}) \cdots (a_{p-k} \cdots a_{p-1}, a_p), \\ &t_k(a_1 \cdots a_p)\Big)\end{aligned} \tag{11}$$

and a similar expression for $\overline{b}_1 \cdots \overline{b}_q$. By (10), it follows that, line by line, $\overline{a}_1 \cdots \overline{a}_p = \overline{b}_1 \cdots \overline{b}_q$ since the first line on the right side of (11) is determined by $i_k(a_1 \cdots a_p)$, the third by $t_k(a_1 \cdots a_q)$ and the middle one by $f(a_1 \cdots a_p)$. Hence ψ is injective. ∎

Corollary 5.6. *If* $\mathbf{V}$ *is a nontrivial monoidal pseudovariety, then* $\mathbf{V} * \mathbf{D}_k$ *is generated by all* $M_k S$ *with* $S \in \mathbf{V}$ *and* $f\colon A^+ \to S^1$ *good.*

Proof: By Corollary 5.4, all such $M_k S$ lie in $\mathbf{V} * \mathbf{D}_k$. Since $\mathbf{V}$ is the union of a chain of monoidal pseudovarieties, each generated by a single semigroup, we may assume $\mathbf{V}$ itself has this property. Then $\mathbf{V} = \mathcal{V}^F$ where $\mathcal{V}$ is a variety generated by a finite monoid. By Theorem 5.5, $\Omega_A(\mathcal{V} * \mathcal{D}_k)$ embeds in $M_k \Omega^1_{A_{k+1}} \mathcal{V}$. Hence the semigroups $M_k S$ suffice to generate $\mathbf{V} * \mathbf{D}_k$. ∎

Suppose $\mathcal{V}$ and $\mathcal{W}$ are two nontrivial monoidal varieties with $\mathcal{V} \subseteq \mathcal{W}$. Then we have homomorphisms

$$\alpha_{\mathcal{W}\mathcal{V}} \colon \Omega^1_{A_{k+1}} \mathcal{W} \longrightarrow \Omega^1_{A_{k+1}} \mathcal{V}$$

leaving A_{k+1} pointwise fixed and

$$\beta_{\mathcal{W}\mathcal{V}} \colon \Omega_A(\mathcal{W} * \mathcal{D}_k) \longrightarrow \Omega_A(\mathcal{V} * \mathcal{D}_k)$$

leaving A pointwise fixed. The first of these induces a homomorphism

$$\begin{array}{rccll} \alpha'_{\mathcal{W}\mathcal{V}} \colon & M_k\, \Omega^1_{A_{k+1}} \mathcal{W} & \longrightarrow & M_k\, \Omega^1_{A_{k+1}} \mathcal{V} & \\ & w & \longmapsto & w & \text{for } w \in B_k \\ & (w_1, s, w_2) & \longmapsto & (w_1, \alpha_{\mathcal{W}\mathcal{V}}\, s, w_2)\,. & \end{array}$$

Denote by $f_{\mathcal{V}} : A^+ \to \Omega^1_{A_{k+1}} \mathcal{V}$ the mapping defined in the statement of Theorem 5.5 and let

$$\iota_{\mathcal{V}} \colon \Omega_A(\mathcal{V} * \mathcal{D}_k) \longrightarrow M_k\, \Omega^1_{A_{k+1}} \mathcal{V}$$

represent the embedding given by Theorem 5.5. Then, for $\mathcal{V}$ and $\mathcal{W}$ as above, we have a commutative diagram of homomorphisms

$$\begin{array}{ccc} \Omega_A(\mathcal{W} * \mathcal{D}_k) & \xrightarrow{\beta_{\mathcal{W}\mathcal{V}}} & \Omega_A(\mathcal{V} * \mathcal{D}_k) \\ \big\downarrow{\scriptstyle \iota_{\mathcal{W}}} & & \big\downarrow{\scriptstyle \iota_{\mathcal{V}}} \\ M_k\, \Omega^1_{A_{k+1}} \mathcal{W} & \xrightarrow{\alpha'_{\mathcal{W}\mathcal{V}}} & M_k\, \Omega^1_{A_{k+1}} \mathcal{V} \end{array} \tag{12}$$

Consider now a nontrivial monoidal pseudovariety $\mathbf{V}$ and let Δ denote the set of all monoidal varieties which are generated by a single semigroup of $\mathbf{V}$. The set Δ is directed when ordered by inclusion. By [2, Thm. 2.2], the projective limit

$$\varprojlim_{\mathcal{V}\in\Delta} \Omega_A(\mathcal{V} * \mathcal{D}_k)$$

(under the mappings $\beta_{\mathcal{W}\mathcal{V}}$) is isomorphic to the semigroup $\overline{\Omega}_A(\mathbf{V} * \mathbf{D}_k)$ since each $\Omega_A(\mathcal{V} * \mathcal{D}_k)$ lies in $\mathbf{V} * \mathbf{D}_k$ and every A-generated member of $\mathbf{V} * \mathbf{D}_k$ is a homomorphic image of one of these relatively free semigroups. In view of (11), we deduce that $\overline{\Omega}_A(\mathbf{V} * \mathbf{D}_k)$ embeds in $\varprojlim_{\mathcal{V}\in\Delta} M_k\Omega^1_{A_{k+1}}\mathcal{V}$.

Lemma 5.7. *There is a good mapping $f_{\mathbf{V}}\colon A^+ \to \overline{\Omega}^1_{A_{k+1}}\mathbf{V}$ such that*

$$\varprojlim_{\mathcal{V}\in\Delta} M_k\Omega^1_{A_{k+1}}\mathcal{V} \simeq M_k\overline{\Omega}^1_{A_{k+1}}\mathbf{V}\ .$$

Proof: It is well known that the given projective limit consists of all

$$(e_{\mathcal{V}})_{\mathcal{V}\in\Delta} \in \prod_{\mathcal{V}\in\Delta} M_k\,\Omega^1_{A_{k+1}}\mathcal{V}$$

such that $\alpha'_{\mathcal{W}\mathcal{V}}\, e_{\mathcal{W}} = e_{\mathcal{V}}$ whenever $\mathcal{V},\mathcal{W}\in\Delta$, $\mathcal{V}\subseteq\mathcal{W}$. From this observation, it is easy to see that, as a set, the given projective limit is naturally identified with $T = B_k \cup (A_k \times \overline{\Omega}^1_{A_{k+1}}\mathbf{V} \times A_k)$ since $\overline{\Omega}^1_{A_{k+1}}\mathbf{V} = \varprojlim_{\mathcal{V}\in\Delta} \Omega^1_{A_{k+1}}\mathcal{V}$ where the "$\mathcal{V}$-component" of $w\in B_k$ is w and the "$\mathcal{V}$-component" of $(w_1,\pi,w_2)\in A_k\times\overline{\Omega}^1_{A_{k+1}}\mathbf{V}\times A_k$ is (w_1,s,w_2) where s is the value of π in $\Omega^1_{A_{k+1}}\mathcal{V}$ when evaluated at the generators $a\in A_{k+1}$. The operation in the projective limit gives an associative operation on T which agrees with the definition of $M_k\overline{\Omega}^1_{A_{k+1}}\mathbf{V}$ with $f_{\mathbf{V}}\colon A^+ \to \overline{\Omega}^1_{A_{k+1}}\mathbf{V}$ given by: $f_{\mathbf{V}}(w)$ is the A_{k+1}-ary implicit operation on $\mathbf{V}$ (or 1) corresponding to the member $(f_{\mathcal{V}}(w))_{\mathcal{V}\in\Delta}$ of the projective limit $\varprojlim_{\mathcal{V}\in\Delta} \Omega_{A_{k+1}}\mathcal{V}$. ∎

The above considerations together with Lemma 5.7 yield the following.

Theorem 5.8. *Let $\mathbf{V}$ be a nontrivial monoidal pseudovariety. There is a good mapping $f_{\mathbf{V}}\colon A^+ \to \overline{\Omega}^1_{A_{k+1}}\mathbf{V}$ and a continuous embedding $\iota_{\mathbf{V}}\colon \overline{\Omega}_A(\mathbf{V}*\mathbf{D}_k) \hookrightarrow M_k\overline{\Omega}^1_{A_{k+1}}\mathbf{V}$. In particular, if we let*

$$\Big(i_k(\pi), \|\pi\|_k, t_k(\pi)\Big) = \begin{cases} (\pi,1,\pi) & \text{if } \iota_{\mathbf{V}}(\pi)\in B_k \\ (w_1,\pi',w_2) & \text{if } \iota_{\mathbf{V}}(\pi) = (w_1,\pi',w_2) \end{cases}$$

the mappings $\| \|_k \colon \overline{\Omega}_A(\mathbf{V} * \mathbf{D}_k) \to \overline{\Omega}^1_{A_{k+1}}\mathbf{V}$ *and* $i_k, t_k \colon \overline{\Omega}_A(\mathbf{V} * \mathbf{D}_k) \to B_{k+1}$ *are continuous. Via the canonical projection* $\overline{\Omega}_A\mathbf{W} \to \overline{\Omega}_A(\mathbf{V} * \mathbf{D}_k)$, *these mappings may be extended to* $\overline{\Omega}_A\mathbf{W}$ *for any* $\mathbf{W} \supseteq \mathbf{V} * \mathbf{D}_k$. ∎

Of course, we do not need pseudovarieties "as large as" $\mathbf{V} * \mathbf{D}_k$ to get continuity of the mappings i_k and t_k defined above.

There are several interesting consequences of Theorem 5.8. Here are some of them.

Corollary 5.9. *Let* $\mathbf{V}$ *be a nontrivial monoidal pseudovariety and let* $\pi, \rho \in \overline{\Omega}_A\mathbf{S}$. *Then* $\mathbf{V} * \mathbf{D}_k \models \pi = \rho$ *if and only if*

$$i_k(\pi) = i_k(\rho)\,, \qquad \|\pi\|_k = \|\rho\|_k\,, \qquad t_k(\pi) = t_k(\rho)\ . \blacksquare$$

Corollary 5.10. *Let* $\mathbf{V}$ *be a nontrivial monoidal pseudovariety and let* $\pi, \rho \in \overline{\Omega}_A\mathbf{S}$. *Then* $\mathbf{V} * \mathbf{D} \models \pi = \rho$ *if and only if, for all* $k \geq 1$,

$$i_k(\pi) = i_k(\rho)\,, \qquad \|\pi\|_k = \|\rho\|_k\,, \qquad t_k(\pi) = t_k(\rho)\ . \blacksquare$$

Corollary 5.10 is somewhat unsatisfactory since in effect it is not portraying the structure of $\overline{\Omega}_A(\mathbf{V} * \mathbf{D})$ explicitly, rather it is giving this semigroup as a projective limit

$$\varprojlim_{k \geq 1} \overline{\Omega}_A(\mathbf{V} * \mathbf{D}_k)\ .$$

It would be interesting to have a representation of $\overline{\Omega}_A(\mathbf{V} * \mathbf{D})$ of the same kind as the one provided by Theorem 5.8 for $\overline{\Omega}_A(\mathbf{V} * \mathbf{D}_k)$.

6 – The structure of $\overline{\Omega}_n(\mathbf{Com} * \mathbf{D}_k)$

We now return to the pseudovariety $\mathbf{Com} * \mathbf{D}$.

The main simplification produced from the general case when we take $\mathbf{V} = \mathbf{Com}$ comes from the existence of an isomorphism

$$\overline{\Omega}^1_X\mathbf{Com} \simeq \left(\overline{\Omega}^1_1\mathbf{Com}\right)^X \qquad \text{(direct power)}$$

for any set X.

We define, for a pseudovariety $\mathbf{W}$ containing $\mathbf{Com} * \mathbf{D}_r$, $\pi \in \overline{\Omega}_A\mathbf{W}$ and $u \in A_{r+1}$, $|\pi|_u$ to be the u-component of $\|\pi\|_r$. Then we may extend the definition of network of a word given at the beginning of section 5 so that, for a pseudovariety $\mathbf{W}$ containing $\mathbf{Com} * \mathbf{D}_k$ and $\pi \in \overline{\Omega}_A\mathbf{W}$, $N_k(\pi)$ is described by: let

$$c_k(\pi) = \left\{u \in A_k \colon |\pi|_u \neq 0\right\}$$

(as usually, we think of commutative semigroups as additive structures and so 0 stands for the identity element of $\overline{\Omega}_1^1\mathbf{Com}$); then $N_k(\pi)$ has set of vertices $c_k(\pi)$, set of arrows $c_{k+1}(\pi)$, an arrow $w_1 \xrightarrow{w} w_2$ with capacity $|\pi|_w$ if $w_1 = i_k(w)$, $w_2 = t_k(w)$, source $i_k(\pi)$ and sink $t_k(\pi)$. From Corollary 5.9, we deduce the following result which extends Theorem 5.1 iii) to the case of pseudoidentities.

Theorem 6.1. *Let $\pi, \rho \in \overline{\Omega}_A\mathbf{S}$. Then $\mathbf{Com} * \mathbf{D}_k \models \pi = \rho$ if and only if $N_k(\pi) = N_k(\rho)$.* ∎

For the rest of this paper, we analyze the local structure of $\overline{\Omega}_A(\mathbf{Com} * \mathbf{D}_k)$ using Theorem 6.1.

Theorem 6.2. *Let $\pi, \rho \in \overline{\Omega}_A(\mathbf{Com} * \mathbf{D}_k)$. Then:*

i) *π is regular if and only if, for all $u \in A_{k+1}$, $|\pi|_u$ is regular;*

ii) *if π and ρ are regular, then $\pi \mathcal{J} \rho$ if and only if $c_{k+1}(\pi) = c_{k+1}(\rho)$;*

iii) *if π is regular, then $\pi = \pi^{\omega+1}$ (i.e., π lies in a group) if and only if $c_{k+1}(t_k(\pi)\, i_k(\pi)) \subseteq c_{k+1}(\pi)$.*

Proof: (We think of π as a "traversal" of $N_k(\pi)$ where, for each $w \in c_{k+1}(\pi)$, the capacity $|\pi|_w$ of the arrow w counts the "number of times" π goes through w. In this interpretation, we identify each explicit operation $x^m \in \overline{\Omega}_1^1\mathbf{Com}$ with the natural number m, where x denotes the identity unary operation symbol.)

Since $\mathbf{Com} = \bigcup_{m,l\geq 1} \mathbf{Com}_{m,l}$ where $\mathbf{Com}_{m,l} = [\![xy = yx,\, x^{m+l} = x^m]\!]$, by the results of section 3 it suffices to work with each $\mathbf{Com}_{m,l} * \mathbf{D}_k$ instead of $\mathbf{Com} * \mathbf{D}_k$. We may then define, for $\pi \in \overline{\Omega}_A(\mathbf{Com} * \mathbf{D}_k)$, a network $N_k^{m,l}(\pi)$ in which the capacities $|\pi|_u$ of arrows are reduced by the relation $x^{m+l} = x^m$. Then $|\pi|_u$ is regular if and only if $|\pi|_u \geq m$ or $|\pi|_u = 0$. Moreover, Corollary 5.9 again yields that, for $\pi, \rho \in \overline{\Omega}_A\mathbf{S}$, $\mathbf{Com}_{m,l} * \mathbf{D}_k \models \pi = \rho$ if and only if $N_k^{m,l}(\pi) = N_k^{m,l}(\rho)$.

i) Now, let $|\pi|_u \geq m$ or $|\pi|_u = 0$ for all $u \in A_{k+1}$. For our present purposes, it suffices to consider the case $m \geq 2$ (the case $m = 1$ has to be phrased in a somewhat different manner since then $|\pi|_u$ regular modulo $x^{1+l} = x$ is always satisfied). Then, each arrow of $N_k^{m,l}(\pi)$ may be traversed at least twice and hence there must be, for each such arrow, a directed path in $N_k^{m,l}(\pi)$ from the arrow's end to its beginning. It follows that, for any two vertices v and w in $N_k^{m,l}(\pi)$ (i.e., $v, w \in c_k(\pi)$), there is a directed path from v to w. In particular, there is a directed path P in $N_k^{m,l}(\pi)$ from the sink to the source which passes through enough vertices so that the path described by π followed by P followed again by the path described by π is described by a product $\pi\rho\pi$ (for this, it suffices to assume that P has length k). Then, for each $u \in A_{k+1}$, $|\pi|_u = |(\pi\rho)^l\pi|_u$ modulo $x^{m+l} = x^m$ and so $\pi = (\pi\rho)^l\pi$, proving that π is regular in $\overline{\Omega}_A(\mathbf{Com}_{m,l} * \mathbf{D}_k)$.

The converse in i) is obvious in view of Theorem 5.8.

ii) Here, for the nontrivial part of the statement, suppose π and ρ are regular (in $\overline{\Omega}_A(\mathbf{Com}_{m,l} * \mathbf{D}_k)$ with $m \geq 2$, $l \geq 1$ fixed) and $c_{k+1}(\pi) = c_{k+1}(\rho)$. Then π and ρ have the same underlying digraph in their network $N_k^{m,l}$ and this digraph has the property that, for any two vertices v and w, there is a directed path from v to w. Thus, there are long enough paths P_1 and P_2 respectively from $t_k(\pi)$ to $i_k(\rho)$ and from $t_k(\rho)$ to $t_k(\pi)$ so that we can go from $i_k(\pi)$ to $t_k(\pi)$ following $\pi\sigma_1\rho\sigma_2$, whence $\pi = \pi(\sigma_1\rho\sigma_2)^l$ and so $\pi \leq_{\mathcal{J}} \rho$.

iii) Suppose $m \geq 2$ and π is a regular element of the semigroup $\overline{\Omega}_A(\mathbf{Com}_{m,l} * \mathbf{D}_k)$ such that $c_{k+1}(t_k(\pi)\, i_k(\pi)) \subseteq c_{k+1}(\pi)$. Then $c_{k+1}(\pi^2) = c_{k+1}(\pi)$ and so $\pi^2 \mathcal{J} \pi$ by ii). Since $\overline{\Omega}_A(\mathbf{Com}_{m,l} * \mathbf{D}_k)$ is finite, it follows that π lies in a group. The converse is again immediate. ∎

The argument given above for the proof of ii) also yields the following.

Proposition 6.3. *Let $\pi, \rho \in \overline{\Omega}_A(\mathbf{Com} * \mathbf{D}_k)$ be regular elements. Then:*

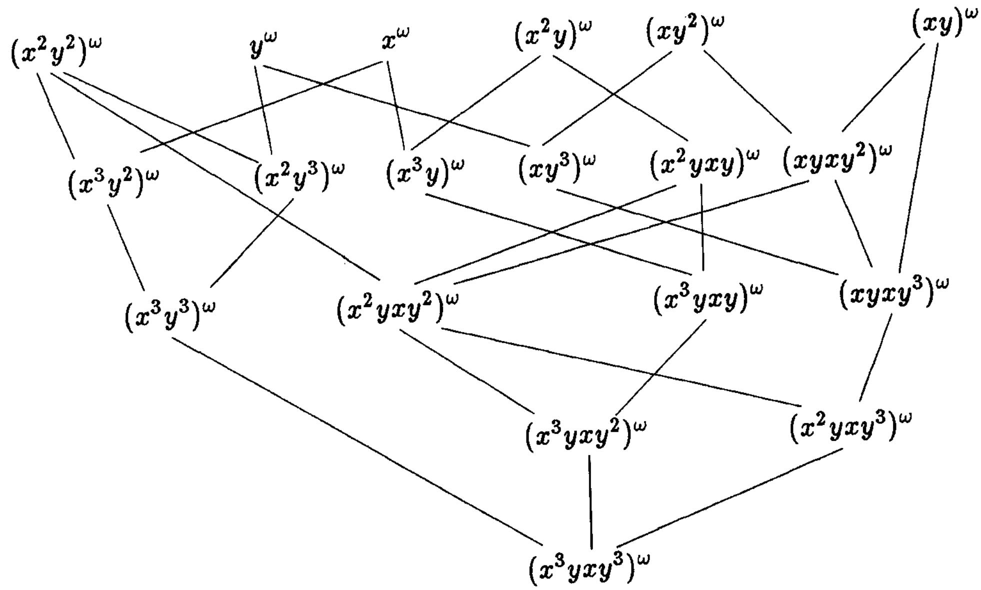
i) *$\pi \leq_{\mathcal{J}} \rho$ if and only if $c_{k+1}(\pi) \supseteq c_{k+1}(\rho)$;*

ii) *$\pi \leq_{\mathcal{R}} \rho$ if and only if $c_{k+1}(\pi) \supseteq c_{k+1}(\rho)$ and $i_k(\pi) = i_k(\rho)$;*

iii) *$\pi \leq_{\mathcal{L}} \rho$ if and only if $c_{k+1}(\pi) \supseteq c_{k+1}(\rho)$ and $t_k(\pi) = t_k(\rho)$.* ∎

We illustrate these results with an example.

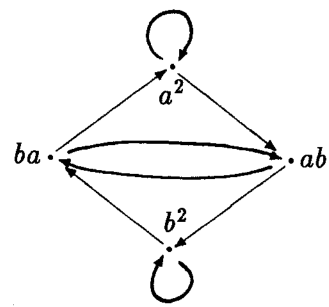
Example 6.4. The partial order of the regular $\mathcal{J}$-classes of $\overline{\Omega}_2(\mathbf{Com} * \mathbf{D}_2)$ can be represented by the following diagram:



For example, the $\mathcal{J}$-class of $(x^2y^2)^\omega$ has 4 $\mathcal{R}$-classes and 4 $\mathcal{L}$-classes. The following picture gives us an element of each $\mathcal{H}$-class of $J_{(x^2y^2)^\omega}$. The star means that the corresponding $\mathcal{H}$-class is a group. This group is isomorphic to $\overline{\Omega}_1\mathbf{G}$ since the underlying digraph of the corresponding networks is a simple cycle.

$*\ (x^2y^2)^\omega$	$(x^2y^2)^\omega x$	$(x^2y^2)^\omega x^2$	$(x^2y^2)^\omega x^2y$
$y(x^2y^2)^\omega$	$y(x^2y^2)^\omega x$	$y(x^2y^2)^\omega x^2$	$*\ y(x^2y^2)^\omega x^2y$
$y^2(x^2y^2)^\omega$	$y^2(x^2y^2)^\omega x$	$*\ y^2(x^2y^2)^\omega x^2$	$y^2(x^2y^2)^\omega x^2y$
$xy^2(x^2y^2)^\omega$	$*\ xy^2(x^2y^2)^\omega x$	$xy^2(x^2y^2)^\omega x^2$	$xy^2(x^2y^2)^\omega x^2y$

As we move down on the partial ordering of the $\mathcal{J}$-classes, the underlying digraphs of the corresponding networks become more complicated. For instance, we have the following digraph for the elements of the minimal ideal of $\overline{\Omega}_2(\mathbf{Com} * \mathbf{D}_2)$.



As $\mathbf{Com} * \mathbf{D} = \bigcup_{k\geq 1} \mathbf{Com} * \mathbf{D}_k$, the following corollary is a consequence of the previous results.

Corollary 6.5. *Let $\pi, \rho \in \overline{\Omega}_A(\mathbf{Com} * \mathbf{D})$. Then π is regular if and only if, for all $k \in \mathbf{N}$ and $u \in A_{k+1}$, $|\pi|_u$ is regular. Moreover, in case π and ρ are regular, we have:*

i) $\pi \leq_{\mathcal{J}} \rho$ *if and only if, for all $k \in \mathbf{N}$, $c_{k+1}(\pi) \supseteq c_{k+1}(\rho)$;*

ii) $\pi \leq_{\mathcal{R}} \rho$ *if and only if, for all $k \in \mathbf{N}$, $c_{k+1}(\pi) \supseteq c_{k+1}(\rho)$ and $i_k(\pi) = i_k(\rho)$;*

iii) $\pi \leq_{\mathcal{L}} \rho$ *if and only if, for all $k \in \mathbf{N}$, $c_{k+1}(\pi) \supseteq c_{k+1}(\rho)$ and $t_k(\pi) = t_k(\rho)$;*

iv) $\pi = \pi^{\omega+1}$ *if and only if, for all $k \in \mathbf{N}$, $c_{k+1}(t_k(\pi)\, i_k(\pi)) \subseteq c_{k+1}(\pi)$.* ∎

The following theorem gives a factorization of implicit operations on $\mathbf{Com} * \mathbf{D}_k$ in terms of a product of "words" and regular implicit operations.

Theorem 6.6. *Every implicit operation on $\mathbf{Com} * \mathbf{D}_k$ is a finite product of explicit operations and regular ones.*

Proof: For $\pi \in \overline{\Omega}_A(\mathbf{Com} * \mathbf{D}_k)$ let $\nu(\pi)$ represent the number of words $u \in A_{k+1}$ such that $|\pi|_u = 0$. Note that, if $\nu(\pi)$ is maximal, then π is an explicit operation.

We proceed by induction on $\nu(\pi)$. Suppose that all $\pi \in \overline{\Omega}_A(\mathbf{Com} * \mathbf{D}_k)$ such that $\nu(\pi) > r$ are products of explicit operations and regular ones. Let $\pi \in \overline{\Omega}_A(\mathbf{Com} * \mathbf{D}_k)$ be such that $\nu(\pi) = r$. If, for every u, $|\pi|_u$ is not an explicit operation then, by Proposition 2.4, $|\pi|_u$ is regular for every u and so, by Theorem 6.2, π is regular. So, we may assume that there exists $u \in A_{k+1}$ such that $|\pi|_u$ is explicit.

Since $\Omega_A(\mathbf{Com} * \mathbf{D}_k)$ is dense in $\overline{\Omega}_A(\mathbf{Com} * \mathbf{D}_k)$ and $|\pi|_u$ is a continuous function of π, there is a sequence $(w_m)_{m\in\mathbf{N}}$ of words converging to π such that $|w_m|_u = |\pi|_u$ for every $m \in \mathbf{N}$. Isolating the occurrences of u in each of the words w_m, for which there may be overlappings but whose possible configurations depend only on u and $|\pi|_u$, by taking a subsequence of $(w_m)_{m\in\mathbb{N}}$, if necessary, we may assume that there are factorizations $w_m = w_{0m}\, u_1\, w_{1m}\, u_2 \cdots u_p\, w_{pm}$ such that $|w_{im}|_u = 0$ for all m, i and the sequences $(w_{im})_{m\in\mathbb{N}}$ converge. To conclude, just apply the induction hypothesis to $\pi_i = \lim_{m\to\infty} w_{im}$. ∎

The following example shows that the analog of this theorem for $\mathbf{Com} * \mathbf{D}$ is not true.

Example 6.7. Let π be an accumulation point, in $\overline{\Omega}_2(\mathbf{Com} * \mathbf{D})$, of the sequence $(ab\, ab^2\, ab^3 \cdots ab^n a)_{n\in\mathbb{N}}$. Then π is not a finite product of regular implicit operations and explicit operations.

To prove this result, start by observing that, for any $k, s \in \mathbb{N}$, $|\pi|_{ab^k a \cdots ab^{k+s-1} a} = 1$. Thus, if $\pi = \pi_1 \cdots \pi_l$ is a factorization of π in terms of regular and explicit operations, then, for each $m \in \mathbb{N}$, there exist $p, i \in \mathbb{N}$ with $p \geq m$ such that $|\pi_i|_{ab^p a} = 1$. If we take m large enough, π_i must be regular and so $|\pi_i|_{ab^p a} = \infty$, which is absurd since $|\pi|_{ab^p a} = 1$.

Much more structural information concerning $\overline{\Omega}_A(\mathbf{Com} * \mathbf{D}_k)$ may be read off the representation provided by Theorem 5.8. We proceed by describing the maximal subgroups of the minimal ideal of that semigroup.

First, an observation from graph theory. Let Δ be a digraph with ν vertices, ε arrows and ω connected components in its underlying undirected graph. The *incidence matrix* M of Δ is the $\nu \times \varepsilon$-matrix with entries 0 or ± 1 in which the column corresponding to an arrow has entries 1 and -1 in the positions corresponding to its beginning and its end, respectively, and 0 elsewhere. The following result is elementary and well-known.

Lemma 6.8. *With the above notation,* rank $M = \nu - \omega$.

Proof: Since the sum of the entries in each column is 0, we deduce that, for each connected component, the sum of the rows corresponding to its vertices is 0. Moreover, if the sum of certain rows is 0, then the corresponding vertices must be isolated from the remaining vertices. On the other hand, since the only way to eliminate a particular entry performing row operations is to add to the corresponding row another row, if there is a relation of linear dependence between certain rows (all with nonzero coefficients), then the sum of those rows must be zero. The lemma follows easily. ∎

Theorem 6.9. *The maximal subgroups of the minimal ideal $K_{m,l}$ of $\overline{\Omega}_A(\mathbf{Com}_{m,l} * \mathbf{D}_k)$ are isomorphic to $\mathbb{Z}_l^r$ where $r = n^k(n-1)+1$, $n = |A|$, $m \geq 2$, and $l \geq 1$.*

Proof: By Theorem 6.2 and Proposition 6.3 (or, better, their analogs for $\Omega_A(\mathbf{Com}_{m,l} * \mathbf{D}_k)$, $K_{m,l}$ consists of all π such that $c_{k+1}(\pi) = A_{k+1}$ and $|\pi|_u \geq m$. The $\mathcal{H}$-class of such an element π of $K_{m,l}$ is determined by the pair $(i_k(\pi), t_k(\pi))$. Choose such an $\mathcal{H}$-class H for which $i_k(\pi) = t_k(\pi) = w_0$. Then, the elements π of H are determined by $\|\pi\|_k \in \Omega_A(\mathbf{Com}_{m,l} \cap \mathbf{G}) = \mathbb{Z}_l^{n^{k+1}}$. However, not all elements of this group are realized as some $\|\pi\|_k$ with $\pi \in H$.

For $u \in A_{k+1}$, let $\alpha\colon H \to \mathbb{Z}_l$ be defined by $\alpha_u(\pi) = |\pi|_u + t_u$ where $t_u = |w_0^2|_u$. Then, for $\pi, \rho \in H$, $|\pi\rho|_u = |\pi|_u + |\rho|_u + t_u$, and so $\alpha_u(\pi\rho) = \alpha_u(\pi) + \alpha_u(\rho)$. Hence, each α_u $(u \in A_{k+1})$ is a homomorphism. It remains to be shown that r of these homomorphisms separate the points of H and their images are totally independent and arbitrary.

Since any $\pi \in H$ provides a traversal of the same underlying digraph Δ of the network $N_k^{m,l}(\pi)$, such a π describes a closed path starting at w_0. Hence, for each vertex of Δ, such a path must go through arrows which end at that vertex as many times as it goes through arrows which begin at it. Thus, the numbers $|\pi|_u$ $(u \in A_{k+1})$ must satisfy a system S of homogeneous equations in $\mathbb{Z}_l$ whose matrix of coefficients (a $0, \pm 1$-matrix) is the incidence matrix M of Δ. By Lemma 6.8, rank $M = n^k - 1$ since Δ is obviously connected. Moreover, since each column of M has precisely two nonzero entries which are 1 and -1, this matrix may be reduced to row echelon form by perfoming row operations which maintain all coefficients 0 or ± 1. Hence, there are $u_i \in A_{k+1}$ $(i = 1, \ldots, n^k - 1)$, independent of π, such that each of the numbers $|\pi|_{u_i}$ is determined by the $|\pi|_u$ with $u \in A_{k+1} \setminus \{u_i\colon i = 1, \ldots, n^k - 1\}$, so that the corresponding homomorphisms α_u separate points of H.

Let C be a closed path starting at w_0 and going through every arrow, and let σ denote a word obtained by traversing $m + t$ consecutive times the path C, where t is such that $m + t \equiv 0\,(l)$ and $m + t \geq m$. Then $\sigma \in H$ and $|\sigma|_u = m + t$ for $u \in A_{k+1}$.

Consider next a solution $(p_u)_{u \in A_{k+1}}$ of the system S. Since each vertex of Δ has the same in-degree as out-degree, $(p_u - m - t)_{u \in A_{k+1}}$ is also a solution of S. Although we view S as a system of equations in $\mathbb{Z}_l$, we may choose the p_u to be integers such that all $p_u - m - t$ are nonnegative and the system S is satisfied in $\mathbb{Z}$ (take the p_u with $u \in A_{k+1} \setminus \{u_i\colon i = 1, \ldots, n^k - 1\}$ sufficiently large by adding appropriate multiples of l and then adjust the p_{u_i} also by appropriate multiples of l so as to get equality in each equation in S — this is possible since we may solve for the p_{u_i} and adding the same number to all p_u produces another solution of the system).

We proceed by induction on $\sum_{u \in A_{k+1}} (p_u - m - t)$ to show that there is some $\pi \in H$ such that, for all $u \in A_{k+1}$, $|\pi|_u = p_u$. If the sum is zero, σ does the job. Otherwise, there is some $p_{v_0} > m + t$ and so there must be some cycle

$\langle v_1, \ldots, v_s \rangle$ with $p_{v_j} > m + t$ $(j = 1, \ldots, s)$. Applying the induction hypothesis to $(p_u - q_u)_{u \in A_{k+1}}$ where $q_u = 1$ for $u \in \{v_1, \ldots, v_s\}$ and $q_u = 0$ otherwise, we obtain some $\pi \in H$ with $|\pi|_u = p_u - q_u$ for all $u \in A_{k+1}$. To obtain π' such that $|\pi'|_u = p_u$ for all $u \in A_{k+1}$, just traverse Δ as indicated by π until reaching the vertex $i_k(v_1)$ and then make the detour $\langle v_1, \ldots, v_s \rangle$, proceeding afterwards as indicated by π. This completes the induction step. Hence $H \simeq \mathbb{Z}_l^r$. ■

Corollary 6.10. *The maximal subgroups of the minimal ideal K of $\overline{\Omega}_A(\mathbf{Com} * \mathbf{D}_k)$ are isomorphic to $(\overline{\Omega}_1\mathbf{G})^r$ (i.e., $\overline{\Omega}_r(\mathbf{Com} \cap \mathbf{G})$) where $r = n^k(n-1) + 1$, $n = |A|$.*

Proof: Here, also by Theorem 6.2 and Proposition 6.3, K consists of all π such that $c_{k+1}(\pi) = A_{k+1}$ and all $|\pi|_u$ are regular. We again consider an $\mathcal{H}$-class H consisting of all $\pi \in K$ such that $i_k(\pi) = t_k(\pi) = w_0$. For each $m, l \geq 1$, let $H_{m,l}$ denote the corresponding $\mathcal{H}$-class of $K_{m,l}$. Then it is obvious that the canonical mapping $\overline{\Omega}_A(\mathbf{Com} * \mathbf{D}_k) \to \overline{\Omega}_A(\mathbf{Com}_{m,l} * \mathbf{D}_k)$ induces an onto homomorphism $H \to H_{m,l}$ and these homomorphisms behave well with respect to the canonical $H_{m',l'} \to H_{m,l}$. Hence we have an onto homomorphism $H \to \varprojlim H_{m,l}$. Moreover, each $\pi \in H$ is characterized by the array $(|\pi|_u)_{u \in A_{k+1}}$ which in turn corresponds to the same kind of array interpreted in $\overline{\Omega}_{A_{k+1}}\mathbf{Com}_{m,l}$ in $H_{m,l}$. Hence H is isomorphic to $\varprojlim H_{m,l}$ and this projective limit is the desired group by Theorem 6.9. ■

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