Combining the regularization strategy and the SQP to solve MPCC - a MATLAB implementation

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Abstract

Mathematical Program with Complementarity Constraints (MPCC) plays a very important role in many fields such as engineering design, economic equilibrium, multilevel game, and mathematical programming theory itself. In theory its constraints fail to satisfy a standard constraint qualification such as the linear independence constraint qualification (LICQ) or the Mangasarian-Fromovitz constraint qualification (MFCQ) at any feasible point. As a result, the developed nonlinear programming theory may not be applied to MPCC class directly. Nowadays, a natural and popular approach is try to find some suitable approximations of an MPCC so that it can be solved by solving a sequence of nonlinear programs.

This work aims to solve the MPCC using nonlinear programming techniques, namely the SQP and the regularization scheme. Some algorithms with two iterative processes, the inner and the external, were developed. A set of AMPL problems from MacMPEC database [7] were tested. The algorithms performance comparative analysis was carried out.

Key words: Mathematical Program with Complementarity Constraints, Sequential Quadratic Programming, nonlinear programming, regularization scheme

1 Introduction

Mathematical Program with Complementarity Constraints is an exciting new application of nonlinear programming techniques working like a challenge for the scientific community. There

exist several MPCC application areas like Engineering, Economics, Ecology among others. In Engineering one can distinguish the contact, obstacle and friction problems, process modeling, deformation and traffic congestion. MPCC problems arises in Economics related to game theory models like Nash and Stackelberg equilibrium, finances and taxes, markets competition. In Ecology the Quioto protocol is a real situation that can be modelled as a MPCC problem. Ralph [10] presents some MPCC applications like toll design in traffic networks or communication networks. The researchers have been spent lots of efforts studying the MPCC theory and proposing different algorithms to solve MPCC efficiently. One can emphasize the work of Fukushima and Pang [6], Scholtes [12], Anitescu [1], Scheel and Scholtes [11], Ralph and Wright [9] and Fletcher *et al.* [4]. The interior point method (IPM), the sequential quadratic programming (SQP), the smooth nonlinear programming, the penalty technique and regularization scheme are some strategies that have been studied to implement numerical algorithms. There also exist a growing collections of test problems.

An important reason why complementarity optimization problems are so pervasive in Engineering and Economics is because the concept of complementarity is synonymous with the notion of system equilibrium. This optimization problem is very difficult to solve because the usual constraint qualifications, necessary to guarantee the algorithms convergence, fail in all feasible points. This complexity is caused by the disjunctive constraints which lead to some challenging issues that typically are the main concern in the design of efficient solution algorithms. From the geometric point of view, its feasible region is not convex and not connected even in general. Recently, it has been shown that MPCC can be solved efficiently and reliably as nonlinear program (NLP). However this reformulation still continues to violate at any feasible point the same constraint qualifications (MFCQ), *ie*, has no feasible point that satisfies the inequalities strictly.

Recent studies of Scheel and Scholthes [11] have proved that the strong stationarity of an MPCC is equivalent to the first order optimality conditions of the NLP equivalent. This fact motivated the cientifique community to use NLP approaches to deal with MPCC. Fletcher *et al.* [4] complements these numerical observation giving a theoretical explanation for the good performance of the SQP method - they show that SQP is guaranteed to converge quadratically near a stationary point under relatively mild conditions. Ralph and Wright [9] described some properties of penalized and regularized nonlinear programming formulations of MPCC. Based on these results we propose a general algorithm, and some alternatives, combining the SQP and the regularization strategy.

This paper is organized as follows. The next section defines the MPCC problem. Some concepts related to the optimality conditions are presented in Section 3. Section 4 introduces the regularization scheme, some alternative regularized formulations and some convergence issues. The implemented algorithms in MATLAB environment are detailed in Section 5. Section 6 reports the MATLAB-AMPL interface, some numerical experiments to test the algorithms, a performance profiles analysis and some conclusions and future work ideas.

2   Problem definition

We consider Mathematical Program with Complementarity Constraints (MPCC) of the form

$$
\begin{aligned}
min \quad & f(x) \\
s.t. \quad & c_i(x) = 0, \ i \in E, \\
& c_i(x) \geq 0, \ i \in I, \\
& 0 \leq x_1 \perp x_2 \geq 0,
\end{aligned}
\qquad \text{(MPCC)}
$$

where f and c are the nonlinear objective function and the constraint functions, respectively, assumed to be twice continuously differentiable. E and I are two disjoined finite index sets with cardinality p and m, respectively. $x = (x_0, x_1, x_2)$ is a decomposition of the variables into $x_0 \in \mathbb{R}^n$ (control variables) and $(x_1, x_2) \in \mathbb{R}^{2q}$ (state variables). $0 \leq x_1 \perp x_2 \geq 0 : \mathbb{R}^{2q} \rightarrow \mathbb{R}^q$ are the q complementarity constraints. The notation $x_1 \perp x_2$ means that $x_{1j}x_{2j} = 0$, for $j = 1, \ldots, q$, ie, the complementarity condition owns the disjunctive nature - $x_{1j} = 0$ or $x_{2j} = 0$, for $j = 1, \ldots, q$. This formulation doesn't exclude complementarity constraints like $0 \leq G(x) \perp H(x) \geq 0$. With this kind of complementarity constraints, the problem can be reformulated, by introducing the slack variables x_1 and x_2. Grouping all the equality constraints in $c_i(x) = 0$, the complementarity constraints have the form $0 \leq x_1 \perp x_2 \geq 0$ and the problem presents the formulation (MPCC). In this formulation all the properties like constraint qualifications or second order conditions are persevered. This formulation makes easy the properties theoretical study.

3   Optimality conditions

This section introduces some concepts related to stationarity and second order conditions. The optimality concepts follow the development of [4] and the corresponding proves can be consulted in this work. One attractive way of solving (MPCC) is to replace the complementarity constraints by a set of nonlinear inequalities, such as $x_{1j} x_{2j} \leq 0, j = 1, \ldots, q$, and then solve the equivalent nonlinear program (NLP):

$$
\begin{aligned}
min \quad & f(x) \\
s.t. \quad & c_i(x) = 0, \ i \in E, \\
& c_i(x) \geq 0, \ i \in I, \\
& x_1 \geq 0, \ x_2 \geq 0, \\
& x_{1j}x_{2j} \leq 0, \ j = 1, \ldots, q.
\end{aligned}
\tag{1}
$$

It has been shown [11] that (1) violates the Mangasarian-Fromovitz constraint qualification (MFQC) at any feasible point. This failure of MFQC implies that multiplier set is unbounded, the central path fails to exist, the active constraints normals are linearly dependent, and the NLP linearizations can became inconsistent arbitrarily close to a solution. Recently, new developments motivated the interest in the analysis of NLP solvers applied to (1) based on the success of SQP methods - the simple observation of Scholtes is that strong stationarity is equivalent to the KKT conditions of (1). This fact implies the existence of bounded multipliers.

Consider two index sets: $X_1, X_2 \subset \{1, \ldots, q\}$ with $X_1 \cup X_2 = \{1, \ldots, q\}$, denoting the corresponding complements in $\{1, \ldots, q\}$ by $X_1^\perp$ e $X_2^\perp$. For each pair of index one define the relaxed NLP corresponding to (MPCC):

$$
\begin{aligned}
min \quad & f(x) \\
s.t. \quad & c_i(x) = 0, \ i \in E, \\
& c_i(x) \geq 0, \ i \in I, \\
& x_{1j} = 0, \ \forall j \in X_2^\perp, \\
& x_{2j} = 0, \ \forall j \in X_1^\perp, \\
& x_{1j} \geq 0, \ \forall j \in X_2, \\
& x_{2j} \geq 0, \ \forall j \in X_1.
\end{aligned}
\tag{NLP-rel}
$$

Concepts like constraints qualification, stationarity and second order conditions of the MPCC problem will be defined in terms of (NLP-rel). The linear independence constraint qualification, LICQ, is extended to MPCC that is MPCC-LICQ:

Definition 1 *MPCC-LICQ*

Consider $x_1, x_2 \geq 0$ and define:

$$
\begin{aligned}
X_1 &= \{j : x_{1j} = 0\}, \\
X_2 &= \{j : x_{2j} = 0\}.
\end{aligned}
$$

The MPCC problem verifies the MPCC-LICQ at x if the corresponding (NLP-rel) verifies the LICQ.

If x^* is a local solution of (NLP-rel) and satisfies $x_1^{*T} x_2^* = 0$, then x^* is also a local solution of original MPCC.

There are several kinds of stationarity defined for MPCC problem. Among them, the strong stationarity is the following one:

Definition 2 *Strong stationarity*

x^* *is a strong stationary point if exist Lagrange multipliers* λ, $\widehat{\nu}_1$ *and* $\widehat{\nu}_2$ *so that:*

$$\nabla f^* - [\nabla(c_i^*), i \in E \ : \ \nabla(c_i^*), i \in I]\lambda - \begin{pmatrix} 0 \\ \widehat{\nu}_1 \\ \widehat{\nu}_2 \end{pmatrix} = 0,$$

$$
\begin{aligned}
& c_i^* = 0, i \in E, \\
& c_i^* \geq 0, i \in I, \\
& x_1^* \geq 0, \\
& x_2^* \geq 0, \\
& x_{1j}^* = 0 \ or \ x_{2j}^* = 0, \\
& \lambda_i \geq 0, i \in I, \\
& c_i \lambda_i = 0, \\
& x_{1j}^* \widehat{\nu}_{1j} = 0, \\
& x_{2j}^* \widehat{\nu}_{2j} = 0, \\
& if \ x_{1j}^* = x_{2j}^* = 0 \ then \ \widehat{\nu}_{1j} \geq 0 \ and \ \widehat{\nu}_{2j} \geq 0.
\end{aligned}
\tag{2}
$$

Note that (2) are the first order optimality conditions of the (NLP-rel) at x^*.

At x^* consider

$$A = \begin{bmatrix} & & & 0 & & 0 \\ \nabla(c_i^*), i \in E & : & \nabla(c_i^*), i \in I \cap A^* & : & I_1^* & : & 0 \\ & & & 0 & & I_2^* \end{bmatrix} =: [a_i^*]_{i \in A^*},$$

where $I_1^* := [e_i]_{i \in X_1^*}$ and $I_2^* := [e_i]_{i \in X_2^*}$ are part of the $q \times q$ identity matrix corresponding to the active simple bound constraints. The set of feasible directions with null curvature of (NLP-rel) is defined by:

$$S^* = \{s | s \neq 0, \nabla(f^*)^T s = 0, \ (a_i^*)^T s = 0, \ i \in A_+^*, \ (a_i^*)^T s \geq 0, \ i \in A^* \backslash A_+^* \}.$$

where A^* is the index set of active constraints and $A_+^* \subset A^*$ is the index set of nondegenerated active constraints.

The second-order sufficient condition (SOSC) for MPCC is given as follows:

Definition 3 *MPCC-SOSC*

A strong stationary point x^* with multipliers $(\lambda^*, \widehat{\nu}_1^*, \widehat{\nu}_2^*)$ verifies the a MPCC-SOSC if all direction $s \in S^*$ satisfies $s^T \nabla^2 L^* s > 0$ where $\nabla^2 L^*$ represents the Hessian matrix of the Lagrangean function of (NLP-rel) at $(x^*, \lambda^*, \widehat{\nu}_1^*, \widehat{\nu}_2^*)$.

4 Regularization scheme

The complementarity constraints are responsible for the main difficulties of an MPCC. In order to overcame this hard problem, some parameters are introduced to smooth or relax these constraints. Ralph and Wright [9] present several regularization schemes and the corresponding properties in order to solve MPCC. The same authors study a regularization scheme that is analyzed by Scholtes [12] where (MPCC) is approximated by the following NLP problem with a non negative scalar parameter t decreasing to zero:

$$\begin{aligned} Reg(t): \quad &min \quad f(x) \\ &s.t. \quad c_i(x) = 0, i \in E, \\ &\quad\quad c_i(x) \geq 0, i \in I, \\ &\quad\quad x_1 \geq 0, \ x_2 \geq 0, \\ &\quad\quad x_{1j} x_{2j} \leq t, \quad j = 1, ..., q. \end{aligned} \tag{3}$$

The solution of this problem is denoted by $x(t)$. The idea is to find a local minimum x_k of $Reg(t_k)$ where $0 < t_k \to 0$. Suppose x^* is a limit point of $\{x_k\}$, then x^* is feasible for $Reg(0)$ hence for the (MPCC). As the $Reg(0)$ is equivalent to (1), the regularization scheme can be used by applying a NLP algorithm to $Reg(t)$ for a sequence of problems where t is positive and tends to zero. In this sequence, the result of each minimization, *ie*, the approximate minimizer of the original problem, is the initial approximation of the next minimization. This minimization sequence represents the external iterative process.

Beyond this previous formulation, other equivalent formulations of (MPCC) are implemented in this work. The first one replaces the complementarity constraints by only one constraint, for this reason it is named *Reg-one*:

$$
\begin{aligned}
\text{\textit{Reg-one}}(t): \quad &\min \quad f(x) \\
&\text{s.t.} \quad c_i(x) = 0, i \in E, \\
&\qquad\;\; c_i(x) \geq 0, i \in I, \\
&\qquad\;\; x_1 \geq 0, \quad x_2 \geq 0, \\
&\qquad\;\; x_1^T x_2 \leq t,
\end{aligned}
\tag{4}
$$

where $x_1^T x_2 \leq t \Leftrightarrow \sum_{k=1}^{q} x_{1k} x_{2k} \leq t$. This formulation is of interest in computation because it has fewer constraints than (MPCC). This formulation was studied by Scholtes [12] and Anitescu [1] and we refer the reader to these sources for a detailed theoretical analysis.

Another plausible regularization was proposed by Ralph [9] where the inequalities of (3) are replaced by equalities:

$$
\begin{aligned}
\text{\textit{Reg-eq}}(t): \quad &\min \quad f(x) \\
&\text{s.t.} \quad c_i(x) = 0, i \in E, \\
&\qquad\;\; c_i(x) \geq 0, i \in I, \\
&\qquad\;\; x_1 \geq 0, x_2 \geq 0, \\
&\qquad\;\; x_{1j} x_{2j} = t, \quad j = 1, ..., q.
\end{aligned}
\tag{5}
$$

Based on the previous formulations, another regularization scheme is proposed in this work:

$$
\begin{aligned}
\text{\textit{Reg-eq-one}}(t): \quad &\min \quad f(x) \\
&\text{s.t.} \quad c_i(x) = 0, i \in E, \\
&\qquad\;\; c_i(x) \geq 0, i \in I, \\
&\qquad\;\; x_1 \geq 0, \quad x_2 \geq 0, \\
&\qquad\;\; x_1^T x_2 = t,
\end{aligned}
\tag{6}
$$

in which the complementarity constraints have been replaced by only one equality constraint

$$x_1^T x_2 = t \Leftrightarrow \sum_{k=1}^{q} x_{1k} x_{2k} = t.$$

5   MATLAB Algorithms

General algorithm

Initialization: t_0, $k = 0$;

Tolerances: t_{min}, k_{max}, ϵ_1, ϵ_2;

Inner iterations counter: $it\_int = 0$;

Problem information (amplfunc): x_0, *lb*, *ub*, *cl*, *cu*, *cv*;

Problem dimension: n, m, p, q;

REPEAT

   Step 1 - Built the constraints;

   Step 2 - Run the MATLAB function:

   $[x, f, LAMBDA, output]$=`fmincon('function',....'constraint')`;

   Step 3 - $it\_int \leftarrow it\_int + output.iterations$;

   Step 4 - Approximation update: $x_{k+1} \leftarrow x$;

   Step 5 - Lagrange multipliers update;

   Step 6 - Update t $(0 < \rho_2 < 1)$: $t_{k+1} \leftarrow t_k \times \rho_2$;

   Step 7 - $k \leftarrow k + 1$;

UNTIL Stop criterium

Four algorithms were implemented, *Reg*, *Reg-one*, *Reg-eq* and *Reg-eq-one*, corresponding to (3), (4), (5) and (6) regularized formulations, respectively. The diference between them is the complementarity constraints treatment. For this reason only the general algorithm is reported. The algorithm has two iterative processes - the external one, performs a sequence of minimization problems. Each external iteration executes the `fmincon` MATLAB subroutine, which implements the SQP strategy. The `fmincon` call is the inner iterative process.

Next, the meaning of some parameters and procedures is reported. The initial value of the regularization parameter is t_0; it_int and k are the inner and external iteration counter, respectively; t_{min} and k_{max} are the regularization parameter limit and the external iterations limit, respectively; ϵ_1 and ϵ_2 are small positive constants. The specific information about the

test problem is x_0, *lb*, *ub*, *cl*, *cu*, *cv* and will be explained in the next section. Using this information the final dimensions are calculated - n, m, p, q.

The external iterative procedure starts at the Step 1 with the constraints treatment routine - each algorithm performs its own strategy replacing the complementarity constraints. Step 2 refers to the inner iterative procedure - the `fmincon` call performs the SQP and its input parameters will be explained in the next section. Step 3 updates the inner iterations using an output `fmincon` parameter. The iteration k of the external iterative procedure updates the approximation of the solution x_{k+1} with the solution of the inner iterative procedure. In Step 5 the Lagrange multipliers of the original problem, MPCC, are updated using the structure LAMBDA which is a `fmincon` output parameter. Step 6 updates the regularization parameter with a smaller value and Step 7 increments the external iteration counter. This external iterative procedure is controlled by a stop criterium that is the disjunction of the four conditions:

$$t \leq t_{min} \quad \vee \quad k = k_{max} \quad \vee \quad \frac{\|x_{k+1} - x_k\|}{\|x_{k+1}\|} \leq \epsilon_1 \quad \vee \quad \|\nabla \mathcal{L}(x, \lambda)\| \leq \epsilon_2 \tag{7}$$

where $\frac{\|x^{k+1} - x^k\|}{\|x^{k+1}\|}$ is the relative error estimation in x and $\|\nabla \mathcal{L}(x, \lambda)\|$ is he stationarity evaluation.

5.1   Convergence issues

In this work, the proposed algorithm is based on the ideas presented by Scholtes [12] and Ralph and Wright [9]. Scholtes ([12], Theorem 4.1) shows that in the neighborhood of the solution x^* of (MPCC), satisfying certain assumptions, exists only a stationary point $x(t)$ for $Reg(t)$ for all t positive value sufficiently small and furthermore verifies $\|x(t) - x^*\| = O(t)$. The convergence behavior of a sequence of stationary points of a parametric NLP which regularizes an MPCC in the form of complementarity conditions is presented. Some important convergence properties are also proved: accumulation points are feasible points of the MPCC; they are M-stationary if, in addition, an approaching subsequence satisfies second order necessary conditions, and they are B-stationary if, in addition an upper level strict complementarity condition holds. The same article shows that every local minimizer of the MPCC which satisfies the linear independence, upper level strict complementarity, and a second order optimality condition can be embedded into a locally unique piecewise smooth curve of local minimizers of the parametric NLP.

Based on this work, Ralph and Wright [9] show that $Reg(t)$ has a local solution, possibly not only, such that $O(t^{\frac{1}{2}})$. They also prove that the Lagrange multipliers of the $Reg(t)$ solution are bounded and satisfies $O(t)$. The authors describe some properties of the solutions to the regularized formulations $Reg(t)$ (3) to the MPCC (MPCC): distance between solutions of (3) and (MPCC), boundedness of Lagrange multipliers, local uniqueness and smoothness of the

solution mapping, under some assumptions on (MPCC) at a local solution x^*.

6 Computational experiments

This section summarizes the numerical experiences using AMPL test problems from MacM-PEC [7]. Details on problem size and characteristics can be found there. The computational experiences were made on a *centrino* with 504MB of RAM, Windows operating system and 95 problems are used to test the four algorithms.

6.1 AMPL-MATLAB interface

The proposed algorithms were implemented in MATLAB language using the `fmincon` routine from the MATLAB Optimization toolbox (version 7.0.1.). This routine finds a constrained minimum of a several variables function starting at an initial estimate using a SQP method.

The test problems [7] are in AMPL language [5] whose files have the *.mod* extension. As the MATLAB only recognizes the data in the *.nl* shape the files have to be converted from *.mod* to *.nl* format. An interface between AMPL and MATLAB was developed. The algorithms use the objective and constraints derivatives provided by AMPL.

The MEX amplfunc function allows the MATLAB accessing to *.mod* data. In order to make uniform the amplfunc function parameters two M-files (function.m and constraint.m) were developed to prepare the `fmincon` input parameters related to the objective and constraints functions. The AMPL-MATLAB connection scheme is in Figure 6.1.
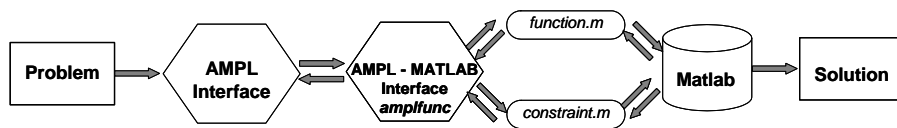

Fig. 1. AMPL-MATLAB interface

To test a problem it is necessary the following input in the MATLAB command window:

```
>>[x0,lb,ub,v,cl,cu]=amplfunc('problem')
```

The output information is: $x0$ - initial estimate, lb and ub - lower and upper limit of x variable, v dual problem initial estimate, cl and cu - constraints lower and upper limit, cv - complementarity

Algorithm	Reg	Reg-one	Reg-eq	Reg-eq-one
failures	3	1	54	29
%	3.2	1.1	56.8	30.5

Table 1

Failures

identifier array. The *cv* array allows to identify the complementarity constraints:

$$cv(i) = \begin{cases} > 0 & \text{if the } i \text{ constraint complements with } x(cv(i)) \\ 0 & \text{otherwise.} \end{cases}$$

6.2 Numerical tests

The algorithms were tested using two different update values for the regularization parameter: $t = 0.1 \times t$ performed better than $t = 0.05 \times t$. The Table 1 reports the results of the four algorithms with respected to their robustness.

The Algorithms *Reg-eq* and *Reg-eq-one* present a significant number of failures and for this reason they are not considered in the following detailed results analysis.

The Tables 2 and 3 show the problem name, n is the number of variables, m, p and q are the number of inequality, equality and complementarity constraints, respectively. The next three columns report the Algorithm *Reg* results - the minimum found (f^*), *it_int* and *it_ext* are the iteration counter for the inner and external iterations, respectively. The last three columns concern to the Algorithm *Reg-one* and have the same meaning that the previous one. When the algorithm doesn't converge we denote by NC. To confirm the robustness of algorithms a comparison with two codes (MacMPEC and Biegler [13]) was performed.

6.3 Performance profiles

Dolan and Moré [3] present a tool to analyse the relative performance of the optimization codes with respect to a specific metric. An easy interpretation of this graphic is that for any given metric a solver is the best when its graphic is tending faster to 1. The performance metrics considered are the number of internal and external iterations, respectively. The graphics of performance profiles are in Figure 2 and Figure 3 and a *log* scale is used.

From the Figure 2 one can conclude that the Algorithm *Reg* and *Reg-one* have similar per-
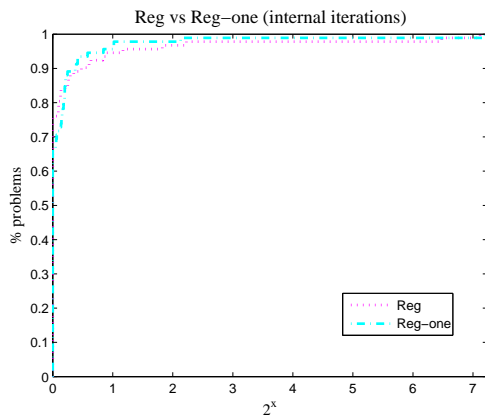
Fig. 2. Inner iterations performance profile
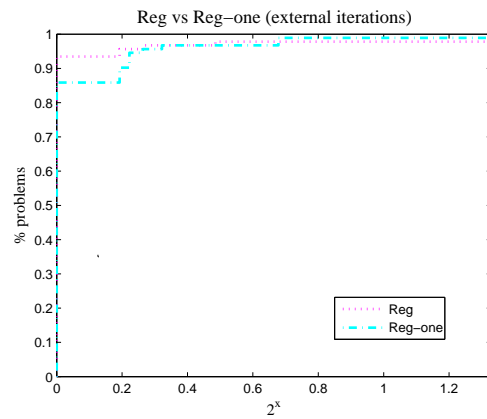
Fig. 3. External iterations performance profile

formance with respect to the inner iterations - both algorithms present very good robustness. Concerning to the external iterations, one can attest that the Algorithm *Reg* presents a little better performance.

6.4   Final remarks and future work

The first conclusion of this work is that it is possible to solve MPCC using nonlinear techniques. Four algorithms were implemented in MATLAB, combining the SQP philosophy with a regularization scheme. The numerical results of Algorithms *Reg* and *Reg-one* are very promising - both algorithms present robustness and efficiency. The other two algorithms, *Reg-eq* and *Reg-eq-one*, did not perform well. This behavior was already expected since the equalities in the complementarity constraints still continue in both formulations. As future work a very careful analysis must be performed in order to implement some strategies to prevent this bad performance. Some important procedures like testing the algorithms with larger dimension problems and studying their convergence properties are on going.

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Table 2

Algorithm *Reg* and Algorithm *Reg-one*

Name	n	m	p	q	f* (*Reg*)	it_int	it_ext	f* (*Reg-one*)	it_int	it_ext
bard1	5	3	1	3	1.700000E+01	19	6	1.700000E+01	22	6
bard1m	6	3	1	3	1.700000E+01	20	7	1.700000E+01	23	7
bard2	12	4	5	3	6.163000E+03	4	2	6.163000E+03	4	2
bard2m	12	4	5	3	-6.598000E+03	6	2	-6.598000E+03	6	2
bard3	6	2	3	1	-1.267871E+01	7	6	-1,y267871E+01	7	6
bard3m	6	4	1	3	-1.267871E+01	28	7	-1.267871E+01	26	8
bar-truss-3	35	6	28	6	1.016657E+04	16	4	1.016657E+04	21	5
bilevel1	10	7	2	6	4.999999E+00	21	5	-2.812000E-07	25	6
bilevel2	16	9	4	8	-6.600000E+03	39	8	-6.600000E+03	21	8
bilevel3	11	4	6	3	-1.267871E+01	37	8	-1.267871E+01	42	8
bilin	8	7	0	6	-1.600000E-09	28	8	-9.400000E-09	18	8
dempe	3	1	1	1	3.125000E+01	60	8	3.125000E+01	60	8
design-cent-1	12	5	6	3	3.283210E-05	28	2	3.283210E-05	28	2
design-cent-2	13	9	6	3	NC			NC		
design-cent-3	15	5	6	3	3.208987E-01	24	3	1.691148E-01	53	2
design-cent-4	22	10	10	8	0.000000E+00	4	3	0.000000E+00	4	3
desilva	6	2	2	2	-1.000000E+00	5	4	-1.000000E+00	5	4
df1	2	3	0	1	0.000000E+00	2	2	0.000000E+00	2	2
ex9.1.1	13	5	7	5	-1.300000E+01	10	5	-1.300000E+01	11	6
ex9.1.2	8	2	5	2	-6.250000E+00	8	6	-6.250000E+00	9	7
ex9.1.3	23	6	15	6	-2.920000E+01	41	7	-2.920000E+01	23	7
ex9.1.4	8	2	5	2	-3.700000E+01	8	5	-3.700000E+01	8	5
ex9.1.5	13	5	7	5	-1.000000E+00	16	7	-1.000000E+00	15	8
ex9.1.6	14	6	7	6	-4.900000E+01	38	6	-4.900000E+01	31	6
ex9.2.1	10	4	5	4	1.700000E+01	40	6	1.700000E+01	81	7
ex9.2.2	9	4	4	3	9.999984E+01	60	8	9.999972E+01	63	8
ex9.2.3	14	5	8	4	5.000000E+00	5	2	5.000000E+00	5	2
ex9.2.4	8	2	5	2	5.000000E-01	37	8	5.000000E-01	29	8
ex9.2.5	8	3	4	3	9.000000E+00	37	8	9.000000E+00	48	8
ex9.2.6	16	6	6	6	-1.000000E+00	20	8	-1.000000E+00	20	8
ex9.2.7	10	4	5	4	1.700000E+01	40	6	1.700000E+01	81	7
ex9.2.8	6	2	3	2	1.500000E+00	15	7	1.500000E+00	14	7
ex9.2.9	9	3	5	3	2.000000E+00	8	3	2.000000E+00	8	3
ex9-1-7n	17	6	9	6	-2.300000E+01	35	7	-2.600000E+01	23	7
ex9-1-9n	12	5	6	5	3.111111E+00	24	7	3.111111E+00	25	7
ex9-1-10n	11	4	5	3	-3.250000E+00	9	6	-3.250000E+00	10	7
gauvin	3	2	0	2	2.000000E+01	22	6	2.000000E+01	20	6
gnash10	13	8	4	8	-2.308232E+02	25	7	-2.308232E+02	21	5
gnash11	13	8	4	8	-1.299119E+02	23	6	-1.299119E+02	21	5
gnash12	13	8	4	8	-3.693311E+01	19	6	-3.693311E+01	21	6
gnash13	13	8	4	8	-7.061784E+00	20	5	-7.061783E+00	23	8
gnash14	13	8	4	8	-1.790463E-01	30	8	-1.790463E-01	26	8
gnash15	13	8	4	8	-3.546991E+02	117	6	-3.546991E+02	33	6
gnash16	13	8	4	8	-2.414420E+02	117	6	-2.414420E+02	25	6
gnash17	13	8	4	8	NC			-9.074910E+01	36	8
gnash18	13	8	4	8	NC			-2.569822E+01	56	8
gnash19	13	8	4	8	-6.116708E+00	43	7	-6.116708E+00	51	8
hs044-i	20	10	4	10	1.561777E+01	34	8	2.061065E+01	29	8
incid-set1-8	117	70	49	49	0.000000E+00	7	2	0.000000E+00	7	2
incid-set1c-8	117	77	49	49	0.000000E+00	7	2	0.000000E+00	7	2
incid-set2-8	117	70	49	49	5.075396E-03	99	8	5.095247E-03	44	8
incid-set2c-8	117	77	49	49	5.642375E-03	59	8	5.640741E-03	106	8
jr1	2	1	0	1	5.000000E-01	3	2	5.000000E-01	3	2
jr2	2	1	0	1	5.000000E-01	21	8	5.000000E-01	21	8
kth1	2	1	0	1	0.000000E+00	3	2	0.000000E+00	3	2
kth2	2	1	0	1	0.000000E+00	3	2	0.000000E+00	3	2
kth3	2	1	0	1	0.000000E+00	1	1	0.000000E+00	1	1
liswet1-050	152	51	52	50	1.399428E-02	4	2	1.399428E-02	4	2
nash1	6	2	2	2	2.563700E-06	30	8	1.138420E-05	25	8

Table 3

Algorithm *Reg* and Algorithm *Reg-one* (cont.)

Name	n	m	p	q	f* (*Reg*)	it_int	it_ext	f* (*Reg-one*)	it_int	it_ext
outrata31	5	4	0	4	3.207700E+00	23	7	3.207700E+00	23	7
outrata32	5	4	0	4	3.449404E+00	29	8	3.449404E+00	30	7
outrata33	5	4	0	4	4.604254E+00	25	7	4.604254E+00	29	8
outrata34	5	4	0	4	6.592684E+00	29	8	6.592684E+00	29	7
pack-comp1-8	107	72	49	49	6.000000E-01	19	8	6.000000E-01	18	8
pack-comp1c-8	107	79	49	49	6.000000E-01	15	8	6.000000E-01	17	8
pack-comp2-8	107	72	49	49	6.731171E-01	21	8	6.731171E-01	22	8
pack-comp2c-8	107	79	49	49	6.734582E-01	9	2	6.734582E-01	9	2
pack-rig1-8	87	40	46	32	7.879311E-01	24	8	7.879318E-01	25	8
pack-rig1c-8	87	47	46	32	7.882998E-01	20	8	7.883001E-01	23	8
pack-rig1p-8	105	55	49	47	7.879301E-01	24	8	7.879318E-01	28	8
pack-rig2-8	85	38	46	30	7.804042E-01	23	8	7.804027E-01	25	8
pack-rig2c-8	85	45	46	30	7.993058E-01	21	8	7.993051E-01	21	8
pack-rig2p-8	103	53	49	45	7.804042E-01	35	8	7.804027E-01	39	8
portfl-i-1	87	1	13	0	1.525510E-05	13	8	1.535040E-05	13	8
portfl-i-2	87	1	13	0	1.465720E-05	14	8	1.460560E-05	14	8
portfl-i-3	87	1	13	0	6.280600E-06	14	8	6.280000E-06	14	8
portfl-i-4	87	1	13	0	2.322300E-06	13	8	2.297200E-06	13	8
portfl-i-6	87	1	13	0	2.524600E-06	13	8	2.505700E-06	13	8
qpec1	30	20	0	20	8.000000E+01	3	2	8.000000E+01	4	2
qpec2	30	20	0	20	4.499888E+01	47	8	4.499842E+01	206	8
ralph1	2	1	0	1	-2.798950E-05	40	8	-2.798950E-05	40	8
ralph2	2	1	0	1	-2.000000E+00	1	1	-2.000000E+00	1	1
scholtes1	3	1	0	1	2.000000E+00	6	2	2.000000E+00	6	2
scholtes2	3	1	0	1	1.500000E+01	5	2	1.500000E+01	5	2
scholtes1n	3	1	0	1	2.000000E+00	6	2	2.000000E+00	6	2
scholtes2-n	3	1	0	1	1.500000E+01	5	2	1.500000E+01	5	2
scholtes3	2	1	0	1	5.000000E-01	26	7	5.000000E-01	26	7
scholtes4	3	3	0	1	-5.590180E-05	46	8	-5.590180E-05	46	8
scholtes5	3	2	0	2	1.000000E+00	3	2	1.000000E+00	4	2
sl1	8	3	2	3	1.000001E-04	12	8	1.000000E-04	11	8
stub	12	5	6	3	3.283210E-05	28	2	3.283210E-05	28	2
stackelberg1	3	1	1	1	-3.266667E+03	4	2	-3.266667E+03	4	2
tap-09	86	36	32	32	1.091310E+02	91	5	1.091311E+02	136	8
tap-15	194	99	68	83	1.843557E+02	255	8	1.842949E+02	183	8
water-net	66	14	36	14	9.272644E+02	1317	8	9.197094E+02	1511	8