The Reconstruction of An Hermitian Toeplitz Matrices with Prescribed Eigenpairs*

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Abstract Toeplitz matrices have been found important applications in bioinformatics and computational biology [5-9, 11-12]. In this paper we concern the reconstruction of an hermitian Toeplitz matrices with prescribed eigenpairs. Based on the fact that every centrohermitian matrix can be reduced to a real matrix by a simple similarity transformation, we first consider the eigenstructure of hermitian Toeplitz matrices and then discuss a related reconstruction problem. We show that the dimension of the subspace of hermitian Toeplitz matrices with two given eigenvectors is at least two and independent of the size of the matrix, and the solution of the reconstruction problem of an hermitian Toeplitz matrix with two given eigenpairs is unique.

Key words Centrohermitian matrix, hermitian Toeplitz matrix, reconstruction, inverse eigenproblems

1 Introduction

Hermitian Toeplitz matrices play an important role in the trigonometric moment problem, the Szegö theory, the stochastic filtering, the signal processing, the biological information processing and other engineering problems, see for example, [1, 3, 5, 6,7, 9, 11, 12], and references therein. Many properties of hermitian Toeplitz matrices have been studied for decades, see for example, [13-15].

Recall that a matrix $A \in \mathbb{C}^{n\times n}$ is said to be centrohermitian [10], if $JAJ = \overline{A}$, where $\overline{A}$ denotes the element-wise conjugate of the matrix and J is the exchange matrix with ones on the cross diagonal (lower left to upper right) and zeros elsewhere. Hermitian Toeplitz matrices are an important subclass of centrohermitian matrices and have the following form

$$H = \begin{bmatrix} h_0 & h_1 & \cdots & h_{n-1} \\ \bar{h}_1 & h_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & h_1 \\ \bar{h}_{n-1} & \cdots & \bar{h}_1 & h_0 \end{bmatrix}. \tag{1}$$

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A vector $\mathbf{x} \in \mathbb{C}^{\mathbf{n}}$ is said to be hermitian if $J\mathbf{x} = \bar{\mathbf{x}}$. Let $A \in \mathbb{C}^{n\times n}$ be a hermitian centrohermitian matrix and $\mathbf{x} \in \mathbb{C}^{\mathbf{n}}$ be an eigenvector of A associated with the eigenvalue λ, then $A\mathbf{x} = \lambda\mathbf{x}$ implies $AJ\bar{\mathbf{x}} = \lambda J\bar{\mathbf{x}}$, which means that $\mathbf{x} + J\bar{\mathbf{x}}$ is also an eigenvector of A associated with the eigenvalue λ, and $\mathbf{x} + J\bar{\mathbf{x}}$ is hermitian. So we claim that an hermitian centrohermitian matrix A has an orthonormal basis consisting of n hermitian eigenvectors. Naturally, an hermitian Toeplitz matrix also has an orthonormal basis consisting of n hermitian eigenvectors.

In this paper we consider the following reconstruction problem:

Problem A *Given a set of hermitian unitary vectors $\{\mathbf{x}^{(j)}\}_{j=1}^{k} \in \mathbb{C}^n$ $(k < n)$, and a set of scalars $\{\lambda_j\}_{j=1}^{k} \in \mathbb{R}$, find an $n \times n$ hermitian Toeplitz matrix H such that*

$$H\mathbf{x}^{(j)} = \lambda_j \mathbf{x}^{(j)}, \quad \text{for} \quad j = 1, \ldots, k,$$

where $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers respectively.

Remark In fact, the Problem A is a so called inverse eigenvalue problem which concerns the reconstruction of a structured matrix from prescribed spectral data. Such an inverse problem arises in many applications where parameters of a certain physical system are to be determined from the knowledge or expectation of its dynamical behaviour. Spectral information is entailed because the dynamical behaviour is often governed by the underlying natural frequencies and normal modes. Structural stipulation is designated because the physical system is often subject to some feasibility constraints. The spectral data involved may consist of complete or only partial information on eigenvalues or eigenvectors. The structure embodied by the matrices can take many forms. The objective of an inverse eigenvalue problem is to construct a matrix that maintains both the specific structure as well as the given spectral property. There exist many researches in literature dealing with the structured inverse eigenvalue problems, see for example [1, 3], and references therein.

Also, the Problem A is actually one of the partially described inverse eigenvalue problems (PDIEPs) [1]. On this topic of PDIEPs, the earlier study can be found for real symmetric Toeplitz matrices [1, 2] and Jacobi matrices in [1, 16], and some of the recent works can be found for anti-symmetric matrices in [17], anti-persymmetric matrices in [18], centrosymmetric matrices in [19], symmetric anti-bidiagonal matrices in [20], K-symmetric matrices in [21], and K-centrohermitian matrices in [10]. This is by far not a complete list, see [3] for a recent review, a number of applications and an extensive list of references.

Since H is required to be an hermitian Toeplitz matrix, thus the description of the given eigenpairs cannot be totally arbitrary. The study of the distribution of eigenvalues of hermitian Toeplitz matrix attracted many mathematicians [22, 23] etc, but it is not clear for us that if the interlacing condition for the symmetric Toeplitz matrices is still hold for hermitian Toeplitz matrices. Apparently, it is another interesting and difficult problem to identify an orthogonal matrix (and each column is hermitian) so that its columns are eigenvectors of some hermitian Toeplitz matrices.

2 Preliminaries

We begin with a brief overview on the reducibility of centrohermitian matrices. All the formulae become slightly more complicated when n is odd. For simplicity, we restrict our attention to the case of even $n = 2m$ throughout this paper.

A centrohermitian matrix of order n can be partitioned as follows:

$$A = \begin{bmatrix} B & J\overline{C}J \\ C & J\overline{B}J \end{bmatrix}, \quad n = 2m. \tag{2}$$

We define

$$Q = \frac{\sqrt{2}}{2} \begin{bmatrix} I & iI \\ J & -iJ \end{bmatrix}, \quad n = 2m. \tag{3}$$

We then have the following well known theorems (see [10]).

Theorem 1 *Let Q be defined as in (3). Then $A \in \mathbb{C}^{n\times n}$ is centrohermitian if and only if $Q^H AQ \in \mathbb{R}^{n\times n}$, that is $Q^H AQ$ (denoted by R_A) is real and has the following form*

$$R_A := Q^H AQ = \begin{bmatrix} Re(B + JC) & -Im(B + JC) \\ Im(B - JC) & Re(B - JC) \end{bmatrix}, \quad n = 2m.$$

Corollary 1 *Let Q be defined as in (3). Then a vector $\mathbf{x} \in \mathbb{C}^n$ is hermitian if and only if $Q^H \mathbf{x} \in \mathbb{R}^n$.*

We note that an $n \times n$ hermitian Toeplitz matrix H can be completely characterized by the real and imaginary parts of its first row (or column).

Let

$$\mathbf{h} = (h_0, Re(h_1), Im(h_1), \ldots, Re(h_{n-1}), Im(h_{n-1}))^T,$$

which is a $(2n-1)$-dimensional vector; and let

$$S_j = \begin{bmatrix} 0_{n-j,j} & I_{n-j} \\ 0_{j,n-j} & 0_{j,j} \end{bmatrix}, \quad j = 0, 1, \cdots, n-1,$$

which is an $n \times n$ matrix, where $0_{p,q}$ denotes the $p \times q$ zero matrix. Then H in (1) can be parameterized as follows:

$$H = \phi_0 I + \sum_{j=1}^{2n-2} \phi_j H_j, \quad \text{(denoted by } H(\mathbf{h})), \tag{4}$$

where

$$\phi_0 = h_0,\ \phi_{2p-1} = Re(h_p),\ \phi_{2p} = Im(h_p)$$

and

$$H_{2p-1} = S_p + S_p^T, \quad H_{2p} = i(S_p - S_p^T),$$

for $p = 1, \ldots, n-1$.

Eq. (4) gives an one-to-one correspondence between complex hermitian Toeplitz $n \times n$ matrices and real $(2n-1)$-vectors.

Applying Theorem 1 to (4) gives

$$R_{H(\mathbf{h})} = \phi_0 I + \sum_{j=1}^{2n-2} \phi_j R_{H_j} \tag{5}$$

where all R_{H_j}, for $j = 1, \ldots, 2n-2$, are real symmetric, and their matrix structures for the case $n = 2m$ are given as follows:

(i) For $1 \le j \le m-1$,

$$R_{H_{2j-1}} = \left[\begin{array}{cc} \hat{T}_{2j-1} & \\ & \tilde{T}_{2j-1} \end{array} \right], \quad R_{H_{2j}} = \left[\begin{array}{cc} & \check{T}_{2j} \\ \check{T}_{2j}^T & \end{array} \right], \tag{6}$$

where

$$\hat{T}_{2j-1} = T(\mathbf{e}_{j+1}) + \left[\begin{array}{cc} 0 & 0 \\ 0 & J_j \end{array} \right],$$

$$\tilde{T}_{2j-1} = T(\mathbf{e}_{j+1}) + \left[\begin{array}{cc} 0 & 0 \\ 0 & -J_j \end{array} \right],$$

and

$$\check{T}_{2j} = \left[\begin{array}{cc} 0 & -I_{m-j} \\ I_{m-j} & J_j \end{array} \right].$$

(ii) For $m \le j \le n-1$,

$$\begin{aligned} R_{H_{2j-1}} &= \left[\begin{array}{cc|cc} J_{2m-j} & 0 & & \\ 0 & 0 & & \\ \hline & & -J_{2m-j} & 0 \\ & & 0 & 0 \end{array} \right], \\ R_{H_{2j}} &= \left[\begin{array}{cc|cc} & & J_{2m-j} & 0 \\ & & 0 & 0 \\ \hline J_{2m-j} & 0 & & \\ 0 & 0 & & \end{array} \right]. \end{aligned} \tag{7}$$

Here $T(\mathbf{e}_{j+1})$ denotes the Toeplitz matrix generated by the m-dimensional unit vector $\mathbf{e}_{j+1}$; I_s and J_s denote the identity matrix and exchange matrix of order s, respectively.

Based on the above analysis, Problem A can be restated as follows:

Problem B *Given a set of orthonormal vectors* $\{\mathbf{y}^{(j)}\}_{j=1}^{k} \in \mathbb{R}^n$ $(n > k)$ *and a set of scalars* $\{\lambda_j\}_{j=1}^{k} \in \mathbb{R}$*, find a symmetric matrix* $R_{H(\mathbf{h})} \in \mathbb{R}^{n\times n}$ *in the form* (5) *such that*

$$R_{H(\mathbf{h})}\mathbf{y}^{(j)} = \lambda_j \mathbf{y}^{(j)}, \quad \text{for} \quad j = 1, \ldots, k.$$

In this paper, we mainly concentrate our study on the eigenpairs for the cases $k = 1$ and $k = 2$.

3 Hermitian Toeplitz matrices with a given eigenvector

Suppose that $\mathbf{x}$ is an eigenvector of two matrices A and B, with associated eigenvalues λ and μ, respectively, then $\mathbf{x}$ is also an eigenvector of matrix $A+B$ with associated eigenvalue $\lambda+\mu$. Hence, given any vector, the space of matrices with that vector as an eigenvector is a linear subspace. Since there is an one-to-one correspondence between complex hermitian Toeplitz $n \times n$ matrices H and real $(2n-1)$-vectors $\mathbf{h}$, then the collection of these $(2n-1)$-vectors form a linear subspace of $\mathbb{R}^{(2n-1)}$.

Assume that $\mathbf{x} \in \mathbb{C}^n$ is an arbitrary hermitian vector. Let

$$S(\mathbf{x}) = \{\mathbf{h} \in \mathbb{R}^{(2n-1)} \mid H(\mathbf{h})\mathbf{x} = \lambda\mathbf{x}, \quad \text{for some} \quad \lambda \in \mathbb{R}\}$$

be this linear subspace. It is evident that $S(\mathbf{x})$ is nonempty. In fact, the standard basis $(2n-1)$-vector $\mathbf{e}_1 = (1, 0, \ldots, 0)^T \in S(\mathbf{x})$ for all $\mathbf{x}$. This means that the dimension of $S(\mathbf{x})$ is at least 1. Furthermore, let

$$S_0(\mathbf{x}) = \{\mathbf{h} \in \mathbb{R}^{(2n-1)} \mid H(\mathbf{h})\mathbf{x} = 0\}$$

denote the linear subspace consisting of all hermitian Toeplitz matrices for which $\mathbf{x}$ is an eigenvector associated with eigenvalue 0.

Clearly, $H(\mathbf{h})\mathbf{x} = \lambda\mathbf{x}$ if and only if $\mathbf{h} - \lambda\mathbf{e_1} \in \mathbf{S_0}(\mathbf{x})$. So

$$S(\mathbf{x}) =< \mathbf{e}_1 > \oplus S_0(\mathbf{x}).$$

The following result gives the precise dimension of $S_0(\mathbf{x})$ for general hermitian vector $\mathbf{x}$.

Lemma 1 *Let $\mathbf{x} \in \mathbb{C}^{\mathbf{n}}$ be hermitian. Then*

$$\text{dimension}(S_0(\mathbf{x})) = n - 1.$$

Proof From the hypothesis, we know that $H(\mathbf{h})$ is centrohermitian and $\mathbf{x}$ is hermitian. By Theorem 1 and Corollary 1, we have that

$$H(\mathbf{h})\mathbf{x} = 0$$

is equivalent to

$$R_H(\mathbf{h})\mathbf{z} = 0,$$

where $R_{H(\mathbf{h})} \in \mathbb{R}^{n \times n}$ is defined as in (5) and $\mathbf{z} = Q^H\mathbf{x} \in \mathbb{R}^n$.

Note that $R_{H(\mathbf{h})}\mathbf{z}$ is a linear function of both entries of $\mathbf{z}$ and $\mathbf{h}$. So we can write

$$R_{H(\mathbf{h})}\mathbf{z} = A(\mathbf{z})\mathbf{h}, \tag{8}$$

where $A(\mathbf{z})$ is an $n \times (2n-1)$ matrix whose entries depend linearly on the n-vector $\mathbf{z} = (z_1, z_2, \ldots, z_n)^T$. Thus the dimension of $S_0(\mathbf{x})$ is the nullity of $A(\mathbf{z})$.

Note that $A(\mathbf{z})\mathbf{h} = 0$ is a homogeneous linear system of n equations in $2n-1$ unknowns, so the nullity of $A(\mathbf{z})$ is at least $n-1$.

We now show that the nullity of $A(\mathbf{z})$ is exactly $n-1$. Note that

$$A(\mathbf{z}) = [\ \mathbf{z} \quad R_{H1}\mathbf{z} \quad \ldots \quad R_{Hj}\mathbf{z} \quad \ldots \quad R_{H2n-2}\mathbf{z}\ ],$$

where R_{Hj}, $j = 1, \cdots, 2n-2$, are defined as in (6) and (7), respectively. Note also that the R_{Hj}, $j = 1, \cdots, 2n-2$, are direct sum (j is odd) or anti-direct sum (j is even) of two matrices with the same structure, so the first m rows and the last m rows of $A(\mathbf{z})$ have also the same structure. Now we exchange the order of rows, we put together the rows whose right side has the same number of zeros, then we will get a block echelon matrix like,

$$
\left[\begin{array}{ccccccc|cc|cc}
z_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & z_2 & z_{m+2} & z_1 & z_{m+1} \\
z_{m+1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -z_{m+2} & z_2 & -z_{m+1} & z_1 \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & z_2 & z_{m+2} & z_1 & z_{m+1} & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & -z_{m+2} & z_2 & -z_{m+1} & z_1 & 0 & 0 \\
\hline
\vdots & \vdots & & & & \cdots & \cdots & 0 & 0 & & \\
\vdots & \vdots & & & \cdots & \cdots & & & & & \\
\vdots & \vdots & z_{m+1}+z_{m+2} & z_1 & z_{m+1} & 0 & 0 & & & & \\
z_n & \vdots & -z_1+z_2 & -z_{m+1} & z_1 & 0 & 0 & & & &
\end{array}\right]
$$

By a simply observation of this matrix, we have that in case $z_1^2 + z_{m+1}^2 \neq 0$, the rank of this matrix is n. When both z_1 and z_{m+1} are zero, we get another block echelon form on the left, the block is

$$
\begin{bmatrix} z_2 & z_{m+2} \\ -z_{m+2} & z_2 \end{bmatrix}.
$$

In case z_2 and z_{m+2} are not both zero, then the rank of that matrix is n, when they are both zero, we go on to another block on the left, go on with this process, since $\mathbf{z}$ is a nonzero vector, so at least one of the block is nonsingular, which guarantees the rank of this matrix is n. So the nullity of $A(\mathbf{z})$ is $n-1$, which means dimension$(S_0(\mathbf{x})) = n-1$. We complete the proof. ∎

For the reader's convenience, we illustrate our strategy with a 6×6 example. Assume that

$$
\mathbf{z} = (z_1, \ldots, z_6)^T \in \mathbb{R}^6
$$

and

$$
\mathbf{h} = (\phi_0, \phi_1, \ldots, \phi_{10})^T \in \mathbb{R}^{11}
$$

are nonzero real vectors. We now check the equality

$$
R_{H(\mathbf{h})}\mathbf{z} = A(\mathbf{z})\mathbf{h}.
$$

A 6x6 hermitian Toeplitz matrix is like,

$$
\begin{aligned}
H &= \begin{bmatrix}
\varphi_0 & \varphi_1 + i\varphi_2 & \varphi_3 + i\varphi_4 & \varphi_5 + i\varphi_6 & \varphi_7 + i\varphi_8 & \varphi_9 + i\varphi_{10} \\
\varphi_1 - i\varphi_2 & \varphi_0 & \varphi_1 + i\varphi_2 & \varphi_3 + i\varphi_4 & \varphi_5 + i\varphi_6 & \varphi_7 + i\varphi_8 \\
\varphi_3 - i\varphi_4 & \varphi_1 - i\varphi_2 & \varphi_0 & \varphi_1 + i\varphi_2 & \varphi_3 + i\varphi_4 & \varphi_5 + i\varphi_6 \\
\varphi_5 - i\varphi_6 & \varphi_3 - i\varphi_4 & \varphi_1 - i\varphi_2 & \varphi_0 & \varphi_1 + i\varphi_2 & \varphi_3 + i\varphi_4 \\
\varphi_7 - i\varphi_8 & \varphi_5 - i\varphi_6 & \varphi_3 - i\varphi_4 & \varphi_1 - i\varphi_2 & \varphi_0 & \varphi_1 + i\varphi_2 \\
\varphi_9 - i\varphi_{10} & \varphi_7 - i\varphi_8 & \varphi_5 - i\varphi_6 & \varphi_3 - i\varphi_4 & \varphi_1 - i\varphi_2 & \varphi_0
\end{bmatrix} \\
&= \varphi_0 I_6 + \textstyle\sum_{j=1}^{10} \varphi_j H_j.
\end{aligned}
$$

It can be reduced into a real matrix per $Q^H H Q$ (we denote this real matrix by $R_{H(h)}$), where Q is defined in (3).

$$
\begin{aligned}
R_{H(h)} &= \begin{bmatrix}
\varphi_0 + \varphi_9 & \varphi_1 + \varphi_7 & \varphi_3 + \varphi_5 & \varphi_{10} & -\varphi_2 + \varphi_8 & -\varphi_4 + \varphi_6 \\
\varphi_1 + \varphi_7 & \varphi_0 + \varphi_5 & \varphi_1 + \varphi_3 & \varphi_2 + \varphi_8 & \varphi_6 & -\varphi_2 + \varphi_4 \\
\varphi_3 + \varphi_5 & \varphi_1 + \varphi_3 & \varphi_0 + \varphi_1 & \varphi_4 + \varphi_6 & \varphi_2 + \varphi_4 & \varphi_2 \\
\varphi_{10} & \varphi_2 + \varphi_8 & \varphi_4 + \varphi_6 & \varphi_0 - \varphi_9 & \varphi_1 - \varphi_7 & \varphi_3 - \varphi_5 \\
-\varphi_2 + \varphi_8 & \varphi_6 & \varphi_2 + \varphi_4 & \varphi_1 - \varphi_7 & \varphi_0 - \varphi_5 & \varphi_1 - \varphi_3 \\
-\varphi_4 + \varphi_6 & -\varphi_2 + \varphi_4 & \varphi_2 & \varphi_3 - \varphi_5 & \varphi_1 - \varphi_3 & \varphi_0 - \varphi_1
\end{bmatrix} \\
&= \varphi_0 I_6 + \textstyle\sum_{j=1}^{10} \varphi_j R_{H_j}
\end{aligned}
$$

which is symmetric.

$R_{H(h)}z =$

$$\begin{bmatrix} z_1(\varphi_0+\varphi_9)+z_2(\varphi_1+\varphi_7)+z_3(\varphi_3+\varphi_5)+z_4\varphi_{10}+z_5(-\varphi_2+\varphi_8)+z_6(-\varphi_4+\varphi_6) \\ z_1(\varphi_1+\varphi_7)+z_2(\varphi_0+\varphi_5)+z_3(\varphi_1+\varphi_3)+z_4(\varphi_2+\varphi_8)+z_5\varphi_6+z_6(-\varphi_2+\varphi_4) \\ z_1(\varphi_3+\varphi_5)+z_2(\varphi_1+\varphi_3)+z_3(\varphi_0+\varphi_1)+z_4(\varphi_4+\varphi_6)+z_5(\varphi_2+\varphi_4)+z_6\varphi_2 \\ z_1\varphi_{10}+z_2(\varphi_2+\varphi_8)+z_3(\varphi_4+\varphi_6)+z_4(\varphi_0-\varphi_9)+z_5(\varphi_1-\varphi_7t)+z_6(\varphi_3-\varphi_5) \\ z_1(-\varphi_2+\varphi_8)+z_2\varphi_6+z_3(\varphi_2+\varphi_4)+z_4(\varphi_1-\varphi_7)+z_5(\varphi_0-\varphi_5)+z_6(\varphi_1-\varphi_3) \\ z_1(-\varphi_4+\varphi_6)+z_2(-\varphi_2+\varphi_4)+z_3\varphi_2+z_4(\varphi_3-\varphi_5)+z_5(\varphi_1-\varphi_3)+z_6(\varphi_0-\varphi_1) \end{bmatrix},$$

we can also view $R_{H(h)}z$ as a function of $\mathbf{h}$, rewrite it in function of $\mathbf{h}$, we got that $A(z)$ take the following form

$$A(z) = \left[\begin{array}{ccccc|cccccc} z_1 & z_2 & -z_5 & z_3 & -z_6 & z_3 & z_6 & z_2 & -z_5 & z_1 & z_4 \\ z_2 & z_1+z_3 & z_4-z_6 & z_3 & z_6 & z_2 & z_5 & z_1 & -z_4 & 0 & 0 \\ z_3 & z_2+z_3 & z_5+z_6 & z_1+z_2 & z_4+z_5 & z_1 & z_4 & 0 & 0 & 0 & 0 \\ \hline z_4 & z_5 & z_2 & z_6 & z_3 & -z_6 & z_3 & -z_5 & -z_2 & -z_4 & z_1 \\ z_5 & z_4+z_6 & z_3-z_1 & -z_6 & z_3 & -z_5 & z_2 & -z_4 & -z_1 & 0 & 0 \\ z_6 & z_5-z_6 & z_3-z_2 & z_4-z_5 & z_2-z_1 & -z_4 & z_1 & 0 & 0 & 0 & 0 \end{array}\right].$$

After exchange of the rows, we can get a matrix like

$$\left[\begin{array}{ccccc|cccccc} z_1 & z_2 & -z_5 & z_3 & -z_6 & z_3 & z_6 & z_2 & z_5 & z_1 & z_4 \\ z_4 & z_5 & z_2 & z_6 & z_3 & -z_6 & z_3 & -z_5 & z_2 & -z_4 & z_1 \\ z_2 & z_1+z_3 & z_4-z_6 & z_3 & z_6 & z_2 & z_5 & z_1 & z_4 & 0 & 0 \\ z_5 & z_4+z_6 & -z_1+z_3 & -z_6 & z_3 & -z_5 & z_2 & -z_4 & z_1 & 0 & 0 \\ z_3 & z_2+z_3 & z_5+z_6 & z_1+z_2 & z_4+z_5 & z_1 & z_4 & 0 & 0 & 0 & 0 \\ z_6 & z_5-z_6 & -z_2+z_3 & z_4-z_5 & -z_1+z_2 & -z_4 & z_1 & 0 & 0 & 0 & 0 \end{array}\right].$$

The elements z_1, z_4 form a block echelon form and the rank of this matrix is 6 if z_1 and z_4 are not both zeroes. If z_1 and z_4 are both zeroes, then the elements z_2, z_5 form another block echelon form, if z_2 and z_5 are both zero, then the block echelon form go to elements z_3, z_6. Because we assume that $\mathbf{z}$ is a nonzero vector, so at least one of z_i cannot be zero, that is at least one of the block is nonsingular, so $rank(A(\mathbf{z})) = 6$.

The following theorem gives the precise dimension of $S(\mathbf{x})$ and shows that for any hermitian vector $\mathbf{x} \in \mathbb{C}^{\mathbf{n}}$, there is a large collection of Hermitian Toeplitz matrices with $\mathbf{x}$ as an eigenvector.

Theorem 2 *Let $\mathbf{x} \in \mathbb{C}^{\mathbf{n}}$ be hermitian. Then*

$$\text{dimension}(S(\mathbf{x})) = n.$$

4 Hermitian Toeplitz matrices with two given eigenvectors

In this section, we consider the case that two eigenvectors are given. Assume that $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ ($\mathbf{x^T y = 0}$) are arbitrary hermitian vectors. Let

$$S(\mathbf{x}) = \{\mathbf{h} \in \mathbb{R}^{(2n-1)} \mid H(\mathbf{h})\mathbf{x} = \lambda\mathbf{x}, \quad \text{for some} \quad \lambda \in \mathbb{R}\}$$

$$S(\mathbf{y}) = \{\mathbf{h} \in \mathbb{R}^{(2n-1)} \mid H(\mathbf{h})\mathbf{y} = \gamma\mathbf{y}, \quad \text{for some} \quad \gamma \in \mathbb{R}\}$$

Our objective is find the dimension of $S(\mathbf{x}) \cap S(\mathbf{y})$. Since the standard basis $(2n-1)$-vector $\mathbf{e}_1 = (1,\, 0,\, \ldots,\, 0)^T \in S(\mathbf{x}) \cap S(\mathbf{y})$, so $S(\mathbf{x}) \cap S(\mathbf{y})$ is nonempty. That means that the dimension of $S(\mathbf{x}) \cap S(\mathbf{y})$ is at least 1.

As we did in previous section, we first transform our equations into real equations, that is we rewrite them as

$$\begin{aligned} R_{H(\mathbf{h})}\mathbf{z} &= \lambda \mathbf{z}, \\ R_{H(\mathbf{h})}\mathbf{w} &= \gamma \mathbf{w}. \end{aligned} \tag{9}$$

or

$$\begin{aligned} (R_{H(\mathbf{h})} - \lambda I)\mathbf{z} &= 0 \\ (R_{H(\mathbf{h})} - \gamma I)\mathbf{w} &= 0. \end{aligned} \tag{10}$$

where $R_{H(\mathbf{h})} \in \mathbb{R}^{n \times n}$ is defined as in (5) and $\mathbf{z} = Q^H \mathbf{x}$, $\mathbf{w} = Q^H \mathbf{y} \in \mathbb{R}^n$.

Let

$$\mathbf{t} = (\phi_0 - \gamma,\, \phi_0 - \lambda,\, \phi_1,\, \ldots,\, \phi_{2n-2})^T,$$

we then have that the system (10) is equivalent to

$$\mathbf{M}\mathbf{t} = 0 \tag{11}$$

where $\mathbf{M}$ is the $2n \times 2n$ matrix defined by

$$\mathbf{M} = \begin{bmatrix} 0 & \mathbf{z} & R_{H_1}\mathbf{z} & \cdots & R_{H_j}\mathbf{z} & \cdots & R_{H_{2n-2}}\mathbf{z} \\ \mathbf{w} & 0 & R_{H_1}\mathbf{w} & \cdots & R_{H_j}\mathbf{w} & \cdots & R_{H_{2n-2}}\mathbf{w} \end{bmatrix}.$$

Suppose that $\mathbf{s} = (\mathbf{s}_0',\, \mathbf{s}_0,\, \mathbf{s}_1,\, \mathbf{s}_2,\, ...,\, \mathbf{s}_{2n-2})^T$ is a solution of (11). For arbitrary λ and α, define

$$\begin{aligned} \phi_0 &:= \alpha s_0 + \lambda, \\ \phi_i &:= \alpha s_i, \qquad i = 1, ..., 2n-2. \end{aligned} \tag{12}$$

and

$$\gamma := \alpha(s_0 - s_0') + \lambda. \tag{13}$$

Then $\mathbf{z}, \mathbf{w}$ are eigenvectors of $R_{H(\mathbf{h})}$, or we say that $S(\mathbf{x}) \cap S(\mathbf{y})$ is the direct sum of the subspace spanned by $\mathbf{e}_1 = (1, 0, ..., 0)^T$ and the subspace obtained by deleting the first component from $ker(\mathbf{M})$ (see $\mathbf{h} = \alpha\bar{\mathbf{s}} + \lambda\mathbf{e_1}$, $\bar{s} = (\mathbf{s}_0, \mathbf{s}_1, \mathbf{s}_2, ..., \mathbf{s}_{2n-2})^T$). On the other hand, suppose that the two eigenvalues λ, γ are given, then by (13), the α in (12) must be

$$\alpha = \frac{\lambda - \gamma}{s_0' - s_0}$$

provided $s_0' \neq s_0$. This gives us the following lemma

Lemma 2 *Suppose that $\mathbf{s} = (\mathbf{s}_0',\, \mathbf{s}_0,\, \mathbf{s}_1,\, \mathbf{s}_2, ..., \mathbf{s}_{2n-2})$ is a solution of (11) with $s_0' \neq s_0$. Then corresponding to the direction of* $\mathbf{s}$*, there is only one solution to (9).*

Now we determine the null space of $\mathbf{M}$. We multiply on the left of $\mathbf{M}$ a nonsingular matrix $\mathbf{T}$ defined by

$$\mathbf{T} = [T_1,\, T_2,\, ...,\, T_{2n}]^T,$$

where

$$T_1 = (-w_1,\ -w_2, ..., -w_n,\ z_1,\ z_2, ...,\ z_n) = (-\mathbf{w^T},\ \mathbf{z^T}),$$

$$T_i = \mathbf{e}_i^T, \quad i = 2, ..., 2n.$$

We can see that the first row of the matrix $\mathbf{TM}$ is

$$[z^T w, -w^T z, -w^T R_{H_1} z + z^T R_{H_1} w, ..., -w^T R_{H_j} z + z^T R_{H_j} w, ..., -w^T R_{H_{2n-2}} z + z^T R_{H_{2n-2}} w],$$

which is identically zero because of the orthogonality condition $\mathbf{z^T w} = 0, \mathbf{w^T z} = 0$ and the symmetries of $R_{H_i}{}'s, i = 1, 2, ..., 2n-2$. So the rank of $\mathbf{M}$ is at most $2n-1$. On the other side, by the proof of lemma 1, we know that the last n row of $\mathbf{TM}$ can form an echelon block with rank n, so the rank of $\mathbf{M}$ is at least n.

In conclusion, we give the following theorem

Theorem 3 *Suppose that n is even, and $\mathbf{x}$ and $\mathbf{y}$ are two hermitian orthogonal vectors. Then*

$$2 \leq dim(S(\mathbf{x}) \cap S(\mathbf{y})) \leq n + 1.$$

5 Conclusions

In this paper we have exploited the facts that every centrohermitian matrix can be reduced to be a real matrix by a simple similarity transformation and that every $n \times n$ hermitian Toeplitz matrix H can be completely characterized by the real and imaginary parts of its first row (or column) (viz. there exists a one-to-one correspondence between complex hermitian Toeplitz $n \times n$ matrices and real $(2n-1)$-vectors) to show some theoretical results, which can be thought of extensions of the works in [4] and [2], from real symmetric Toeplitz matrices to complex hermitian Toeplitz matrices.

The main results are listed as follows.

- For an arbitrarily given hermitian vector x, the set

  $$S(\mathbf{x}) = \{\mathbf{h} \in \mathbb{R}^{(2n-1)} | H(\mathbf{h})\mathbf{x} = \lambda \mathbf{x}, \quad \text{for some} \quad \lambda \in \mathbb{R}\}$$

  is a nonempty linear subspace of $\mathbb{R}^{(2n-1)}$ with $\dim(S(\mathbf{x})) \geq 1$ (Note that the standard basis $(2n-1)$-vector $\mathbf{e}_1 = (1, 0, \ldots, 0)^T \in S(\mathbf{x})$ for all $\mathbf{x}$), which can be written as

  $$S(\mathbf{x}) = <\mathbf{e}_1> \oplus S_0(\mathbf{x}),$$

  where

  $$S_0(\mathbf{x}) = \{\mathbf{h} \in \mathbb{R}^{(2n-1)} | H(\mathbf{h})\mathbf{x} = 0\}$$

  denotes the linear subspace consisting of all hermitian Toeplitz matrices for which $\mathbf{x}$ is an eigenvector corresponding to eigenvalue 0. Furthermore,

  $$\dim(S_0(\mathbf{x})) = n - 1.$$

- For two arbitrarily given hermitian vectors $\mathbf{x}$ and $\mathbf{y}$ satisfying $\mathbf{x^H y} = \mathbf{0}$, the dimension of $S(\mathbf{x}) \bigcap \mathbf{S}(\mathbf{y})$ is at least 2.

- For almost all hermitian vectors $\mathbf{x}$ and $\mathbf{y}$ satisfying $\mathbf{x^H y = 0}$ and for any real scalars λ and μ, there exists a unique hermitian Toeplitz matrix H such that $H\mathbf{x} = \lambda\mathbf{x}$ and $H\mathbf{y} = \mu\mathbf{y}$.

For **Problem A** (or equivalently **Problem B)** in the cases $k = 1, 2$, we can therefore come to the following conclusions.

I Being hermitian is sufficient for a single vector to be an eigenvector of a hermitian Toeplitz matrix, and the collection of hermitian Toeplitz matrices with one given eigenvector is quite large.

II The set $S(\mathbf{x}) \cap S(\mathbf{y})$ contains all the hermitian Toeplitz matrices with two prescribed eigenvectors and its dimension is at least 2 in despite of the size of the problem.

III For each direction of $ker\mathbf{M}$, there is only one hermitian Toeplitz matrix with two prescribed eigenpairs.

In general case, suppose that $\{\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, ..., \mathbf{x}^{(k)}\}, k \geq 3$, is a set of hermitian orthonormal vectors for some eigenvalues. Then $\bigcap_{i=1}^{k} S(\mathbf{x}^{(i)})$ contains all hermitian Toeplitz matrices for which each $\mathbf{x}^{(i)}$ is an eigenvector. Evidently, $(2n-1)$-vector $\mathbf{e}_1 = (1, 0, \ldots, 0)^T \in \bigcap_{i=1}^{k} S(\mathbf{x}^{(i)})$ for all i. So $\bigcap_{i=1}^{k} S(\mathbf{x}^{(i)})$ is at least of dimension 1. We have attempted to work with the upper bound of $dim \bigcap_{i=1}^{k} S(\mathbf{x}^{(i)})$, but it is not trivial, we expect to study the upper bound of $dim \bigcap_{i=1}^{k} S(\mathbf{x}^{(i)})$ in the near future.

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