Centro de Matemática da Universidade do Minho

Research Seminar on

History and Epistemology of Mathematics

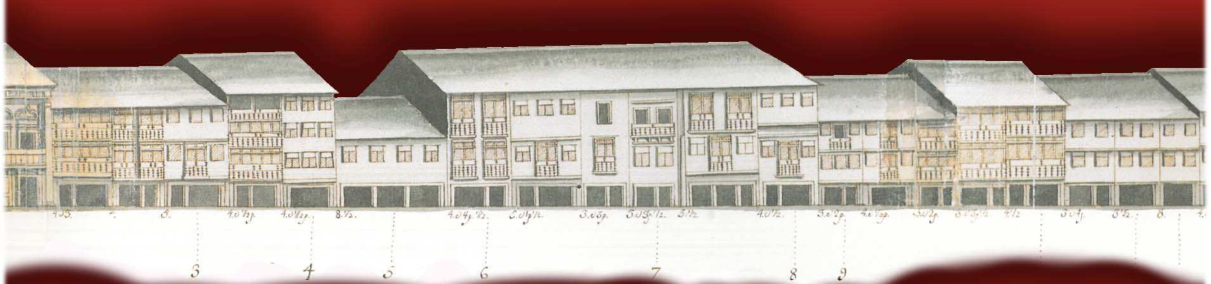
Proceedings

Assis Azevedo

M. Elfrida Ralha

Lisa Santos

Editors



Maria Fernanda de Oliveira Gonçalves Estrada

Licenciou-se em Ciências Matemáticas pela Faculdade de Ciências da Universidade do Porto e doutorou-se em História da Matemática pela Universidade de Birmingham, no Reino Unido.
É Professora Associada aposentada do Departamento de Matemática, na Universidade do Minho.
Em 1994 foi autora de "*Leitura, Introdução e Notas*" de um manuscrito setecentista intitulado *Ensayo sobre as Minas*, atribuído a José Anastácio da Cunha e que esteve, até então, inédito.
Em 2000 coordenou a publicação, pela Universidade Aberta (Lisboa), de um texto didáctico sobre *História da Matemática.*
Presentemente mantém-se activa, enquanto investigadora, no Centro de Matemática da Universidade do Minho.

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ASSIS AZEVEDO

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Ficha Técnica

Título	Research Seminar on *History and Epistemology of Mathematics* Proceedings
Edição	Centro de Matemática da Universidade do Minho – cmat
Organização	Ana Isabel Filipe M. Elfrida Ralha Lisa Santos M. Joana Soares
Editores	Assis Azevedo M. Elfrida Ralha Lisa Santos
Capa	Ana Isabel Filipe
Tiragem	200 exemplares
isbn	978-989-95122-4-5
Execução gráfica	Tipografia Abreu, Sousa e Braga Rua Areal de Cima, Montariol Apartado 1060, 4711-098 Braga abreusbraga@mail.telepac.pt

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Preface

Maria Fernanda Estrada was born on a Spring day, May the 20th, the eldest child of a large family of two sisters and four brothers at Escariz (S. Martinho) a small rural village in Minho, near Braga.

She started her school education in Freiriz because the primary school in her own village was closed for rebuilding after a fire. There, in a different village, she proved, from the very beginning, to be a diligent pupil who became so attached to her teacher that, when the village school building was completed, she decided to keep on walking for four kilometers each day, from home to school and back, just to be taught by that schoolteacher.

Later on, in Liceu Nacional Sá de Miranda, Braga - where she concluded her secondary education, in 1949, with the classification of 19 (out of 20) - she probably developed her interest in mathematics but she was also very much interested in history and literature and read widely. Maria Fernanda won a school prize for her translation, from French to Portuguese, of *Ivanhoe*, after having been asked, as well as all her colleagues in the class, by her French teacher to translate just a few paragraphs of a page in the book.

She entered University and, in 1953, Maria Fernanda finished her first degree in Mathematical Sciences, at the University of Porto, where she acknowledges having been led by some of her Professors (whom she often recalls with much appreciation) to enjoy the beauty of mathematics. She was certainly due to achieve a brilliant academic career. However, having finished her degree with a classification of 18 (certainly very rare in those days standards and for which she was to be awarded a prize from the Rotary Club) did not guarantee her admission, by the Faculty, as an assistant-lecturer, "simply" because she was a woman.

In 1958, Maria Fernanda finished her degree in Pedagogical Sciences, at the University of Coimbra, and in 1959 she completed the in-service teaching practice.

Twenty years after and having already influenced, as an excellent maths teacher, a large community of Secondary School pupils, both in Portugal and in Africa (Mozambique), Maria Fernanda was finally hired, in 1979, as an invited assistant-lecturer, to the University of Minho.

Several events may have influenced Maria Fernanda decision to pursue an academic career in the Department of Mathematics at our University:

- the justification given to the Rectory for obtaining permission to invite her to come from the Secondary School to the University: *to lecture History, as well as Didactics, of Mathematics, (...) fundamental courses in the Teaching of Mathematics degree.*

- Prof. Maria Raquel Valença, who became a dear friend and an important advisor on the process of her working towards a PhD degree.

At the University of Birmingham, Maria Fernanda's Master as well as Doctoral supervisor was Stella Mills. By 1987, Maria Fernanda required continuing her postgraduate studies on History of Mathematics; justifications included sentences such as *there is no PhD specialist on the History of Mathematics in Portugal* as well as *it is a domain of great interest not only for the university of Minho but also for all the other Portuguese universities.*

Maria Fernanda was awarded a PhD degree with a thesis entitled "A Study of the discovery and early representations of the 27 lines of a cubic surface". She became the first Portuguese mathematician, together with Carlos Correia de Sá (from the University of Porto, with a thesis on "Projective Geometry"), to qualify with a doctorate in History of Mathematics.

Known, by her students, to be a lecturer who could drop some real tears just for teaching Euclid, Arquimedes or, and above all, José Anastácio da Cunha, Maria Fernanda progressed to become an associated professor with recognised scientific skills, as well as rare human qualities. Sympathy, honor, patience, rigor, real pedagogical concerns or, simply, true friendship are just a few of the countless attributes which, in our modest opinion, fit Maria Fernanda's attitude towards life.

During all this time Maria Fernanda was always seen as someone with a wonderful character who was simultaneously easily accepted by her students as well as by her colleagues. In particular, during her time at the University of Minho, Maria Fernanda touched deeply numerous students not only because of her scientific knowledge but also because of her humanity towards her students and their own careers.

She started being interested on the post-graduation of school teachers back in 1991, when she became director of the first Master degree on *Teaching of Mathematics* to exist in Portugal. From this experience she often reports having been able to introduce three courses on History of

Mathematics: one on the "History of Analysis", taught by Stella Mills, one on the "History of Geometry", taught by herself, and one on the "History of Mathematics in Portugal", for which she invited many colleagues from various Portuguese universities to present course modules. Many of them, both students and lecturers, still refer to this wonderful opportunity and experience. It was by then that she supervised her first MSc thesis on the *Almagest* of Ptolemy and, from then on, she supervised and/or co-supervised almost fifteen MSc theses and one PhD thesis on the life and work of the Portuguese mathematician Francisco Gomes Teixeira. Her retirement in 1997 did not slow down her supervision of post-graduate students; occasionally, she returned to the university to lecture some courses on the History of Mathematics to those students.

Maria Fernanda's passion for the Portuguese History of Mathematics (and/or Portuguese Mathematicians) has grown over the years: Pedro Nunes or Álvaro Tomás, Gaspar Nicolás or Bento Fernandes and Gomes Teixeira have been the object of her attention and profound study on different occasions and/or for different reasons. However, José Anastácio da Cunha is, perhaps and on this domain, one of her favorite personalities; in 1994 she annotated and published an inedit manuscript found in the Archives of the University of Minho, *Ensaio sobre as Minas* and, a few years later (2005), she collaborated very actively in the study of another set of inedit manuscripts of the same author found in the same Archive.

Maria Fernanda is still an active collaborator and researcher of Centro de Matemática of the Universidade do Minho (CMAT).

For all of these factors we CONGRATULATE and THANK YOU, Maria Fernanda, on your 80th birthday, hoping that you may allow us to be around you for many years and also hoping to be able to keep in pace with your inexhaustible energy.

A. Filipe
Maria Elfrida Ralha
Lisa Santos
Maria Joana Srs

Prefácio

Maria Fernanda Estrada, a filha mais velha de uma família numerosa, de duas irmãs e quatro irmãos, nasceu, num dia de primavera, a 20 de Maio, em Escariz (S. Martinho), uma aldeia rural perto de Braga.

Iniciou a sua instrução primária em Freiriz porque a escola da sua aldeia estava a ser reconstruída, após um incêndio. Aí, numa aldeia diferente, provou, desde o início, ser uma aluna diligente, que rapidamente se afeiçoou à sua professora. Quando se completaram as obras de reconstrução da escola, preferiu continuar a percorrer os quatro quilómetros diários, para poder continuar a ser ensinada por ela.

Mais tarde frequentou o Liceu Nacional de Sá de Miranda, em Braga, que terminou em 1949 com a classificação de 19 valores - durante este período desenvolveu, provavelmente, o seu gosto pela matemática mas apreciava também, com entusiasmo, a aprendizagem da história e da literatura, lendo bastante. Obteve, nos seus primeiros anos do ensino liceal, um prémio atribuído pela tradução, do francês para português, do clássico *Ivanhoe*, após uma sugestão da professora de francês à sua turma para que traduzissem apenas alguns parágrafos do livro.

Frequentou o Curso de Ciências Matemáticas, na Universidade do Porto, que terminou, em 1953, aprendendo, também, com alguns dos seus professores (que recorda frequentemente, com muito apreço) a apreciar a beleza da matemática. Era previsível que prosseguisse para uma brilhante carreira académica mas, apesar de ter terminado o curso com a classificação final de 18 valores (certamente rara para a época e pela qual veio a receber um prémio, atribuído pelo Rotary Club de Portugal), não foi admitida a um lugar na universidade, simplesmente porque era "mulher".

Em 1958, Maria Fernanda obteve o grau em Ciências Pedagógicas, na Universidade de Coimbra e, um ano mais tarde, completou o seu estágio profissional.

Vinte anos depois, tendo já influenciado, como uma excelente professora de Matemática, uma extensa comunidade de alunos do ensino liceal, quer em Portugal Continental quer em Moçambique, a Maria Fernanda foi finalmente contratada, em 1979, como assistente da Universidade do Minho, requisitada ao Ensino Secundário.

Vários acontecimentos influenciaram a Maria Fernanda a prosseguir uma carreira académica no Departamento de Matemática da nossa Universidade:

- a justificação enviada à Reitoria solicitando permissão para proceder à sua requisição do Ensino Secundário para o Ensino Superior: *ensinar História, bem como Didáctica, da Matemática, (...) disciplinas fundamentais na Licenciatura de Matemática e Desenho.*

- a Prof. Maria Raquel Valença, que se tornou uma grande amiga e importante conselheira no seu percurso para a obtenção do grau de doutora.

Na Universidade de Birmingham, a Maria Fernanda teve como orientadora Stella Mills, para a obtenção do grau de Mestre e, em 1990, do de Doutoramento. Em 1987, Maria Fernanda requereu prosseguir os seus estudos pós-graduados em História da Matemática; como justificação escreveu-se: *não havendo em Portugal especialistas doutorados em História da Matemática* e ainda *trata-se de uma área de grande interesse não só para a Universidade do Minho (...) mas para as outras universidades portuguesas.*

A Maria Fernanda doutorou-se com uma tese intitulada "A Study of the discovery and early representations of the 27 lines of a cubic surface" tendo sido, simultaneamente com Carlos Correia de Sá (da Universidade do Porto, com uma tese sobre "Geometria Projectiva"), os primeiros portugueses a obter o grau de doutor na área de História da Matemática.

Conhecida pelos seus alunos como alguém que se emocionava ao ponto de verter verdadeiras lágrimas quando falava de Euclides, de Arquimedes ou, muito especialmente, de José Anastácio da Cunha, a Maria Fernanda passou a professora associada, vendo reconhecidos os seus conhecimentos científicos, mas também qualidades humanas raras. Simpatia, honra, paciência, rigor, preocupações pedagógicas ou, simplesmente, verdadeira amizade, ajudam a descrever a atitude da Maria Fernanda perante a vida.

A Maria Fernanda foi sempre vista como alguém com um excelente carácter, simultaneamente bem aceite pelos seus alunos e pelos seus colegas. Em particular, no período em que leccionou na Universidade do Minho, ela influenciou profundamente numerosos estudantes e os seus percursos ficaram claramente marcados, não apenas pelos conteúdos científicos que ensinava, mas também pelas suas qualidades humanas.

O seu interesse pela pós-graduação de estudantes recua ao ano de 1991, quando se tornou directora do primeiro Mestrado português em

Matemática - Área de Especialização em Ensino. Desta experiência a Maria Fernanda relata a introdução das disciplinas “História da Análise”, leccionada por Stella Mills, “História da Geometria”, leccionada por ela própria e, ainda, “História da Matemática em Portugal”, uma disciplina por módulos, para cuja leccionação a Maria Fernanda convidou vários colegas portugueses. Muitos, tanto professores como alunos, classificam, ainda hoje, esta experiência como excelente. Por essa altura, a Maria Fernanda orientou a sua primeira tese de mestrado, sobre o *Almagesto* de Ptolomeu e, desde então, orientou ou co-orientou perto de quinze teses de mestrado e ainda uma tese de doutoramento, sobre a vida e a obra do matemático português Francisco Gomes Teixeira. A sua aposentação em 1997 não esmoreceu a sua orientação de teses; inclusivamente, em certas situações, regressou à universidade para leccionar disciplinas de História da Matemática.

A paixão da Maria Fernanda pela História da Matemática Portuguesa (e/ou de matemáticos portugueses) cresceu com os anos: Pedro Nunes ou Álvaro Tomás, Gaspar Nicolás ou Bento Fernandes e Gomes Teixeira têm sido o fulcro da sua atenção e de estudos profundos em diferentes ocasiões e por diferentes razões. Contudo, José Anastácio da Cunha é, porventura, uma das suas personalidades preferidas; em 1994 a Maria Fernanda anotou e publicou um manuscrito inédito encontrado no Arquivo da Universidade do Minho, “Ensaios sobre as Minas” e, alguns anos mais tarde (2005), colaborou activamente no estudo de um outro conjunto de manuscritos inéditos do mesmo autor, encontrados no mesmo Arquivo.

A Maria Fernanda mantém-se actualmente uma investigadora activa do Centro de Matemática da Universidade do Minho (CMAT).

Por todas estas razões CONGRATULAMO-NOS E AGRADECEMOS do fundo do coração, à Maria Fernanda, no ano do seu 80º aniversário, esperando que nos permita permanecer perto de si por muitos anos e que sejamos capazes de a acompanhar com a sua energia inesgotável.

A. Filipe
Maria Elfrida Ralha
Lisa Santos
Maria Joana Sns

Research Seminar
History and Epistemology of Mathematics, 1–14

Historical comments about the memoir "On the development of functions in series" by Gomes Teixeira

Maria da Graça Ferreira Alves
Faculdade de Ciências Sociais
Centro Regional de Braga
Universidade Católica Portuguesa
graca@mgalves.com

Abstract

The memoir "On the development of functions in series", by Francisco Gomes Teixeira (1851-1933), is a remarkable study that allows the reader acquiring knowledge, at the same time global and detailed, because it gives an historical perspective of the theories development and presents detailed bibliographical references. In this issue Gomes Teixeira shows knowledge of the fundamental classical texts of contemporary authors of his time and also former ones, related to the development of functions in series, which reveals a profound scientific and historical knowledge on the subject.
The memoir was rewarded by the Royal Academy of Exact Physical and Natural Sciences of Madrid, published, in Portuguese, in the *Colección de Memorias de la Academia*, and later it was inserted in volume I of *Obras sobre Mathematica* (1904).

Keywords: Francisco Gomes Teixeira, Portuguese Mathematics, Royal Academy of Exact Physical and Natural Sciences of Madrid Prize, Teixeira's series.

1 Introduction

Within the framework of my doctorate, I studied, even though in a brief way, the memoir of Francisco Gomes Teixeira, entitled "On the development of functions in series". The content of the analysis to this work is inserted in the thesis[1].

When I was asked to make a brief communication at the "History of Mathematics Meeting", in homage to Professor Maria Fernanda Estrada, I thought over the subject to discuss. It was obvious that it had to be a content related to my research work, since I had the happiness of having my thesis guidance magisterially conducted by Professor Fernanda Estrada, with competence, serenity and a lot of abnegation. From that work relationship a big friendship has emerged, which prevails until present and will continue in the future.

The selection of the subject had an aim to propose to Professor Fernanda Estrada to make, together with me, a more in depth study on the referred memoir, since the one that already existed was a more generic one. Such a challenge was publicly accepted by my great Friend, to my own satisfaction and that of all the others present. The study to be produced will complete, not only the already existing one, but also some other research that meanwhile had been done.

Consequently, what I propose to present is a text, on the referred memoir, based on the thesis, which doesn't add anything else to the research already done, but that may be useful to those who had not read the thesis.

Even though Francisco Gomes Teixeira (1851-1933) is well-known, it seems to me opportune to make a brief reference to this imminent mathematician and master.

In the second half of the 19th century, Portugal was scientifically isolated from the rest of Europe and world, which was extremely harmful to the scientific development of our country. That fact didn't leave Gomes Teixeira indifferent and, therefore, he undertook the opening of Portugal to the international scientific community, reflected in the numerous papers that he published in the most prestigious scientific journals at the time, and in the vast correspondence exchanged with other scholars from several areas of knowledge.

[1] Alves, Maria Graça, Francisco Gomes Teixeira - The man, the scientist, the pedagogue, Doctorate Thesis, Minho University, 2004.

In his research, Gomes Teixeira, in fact, didn't develop new mathematical theories, but gave a major contribute to their development, presenting generalizations, clarifying and increasing research works published by other authors, covering in the pedagogical perspective of the presentation of mathematical texts, among others.

Besides dozens of articles on analysis and geometry, Gomes Teixeira published two books: *Curso de Analyse Infinitesimal - Calculo Differencial* and *Curso de Analyse Infinitesimal - Calculo Integral.* The fist one had its first edition in 1887, and the last one, its fourth, in 1906; the second treatise was published, for the first time, in 1889, with the third and last edition being from 1910. Gomes Teixeira also dedicated himself to the History of Mathematics. In those books, he introduced numerous historical notes, which were increased from one edition to the other; he wrote the book *História das Matemáticas em Portugal*; he inserted hundreds of bibliographic references and the history of curves in his treatise, entitled *Tratado de las curvas especiales notables* and *Traité drs Courbes Spéciales Remarquables planes et gauches.* The first was rewarded by the Real Academia de Ciencias Exactas Fisicas y Naturales, de Madrid and the second awarded a prize from the Académie de Sciences de Paris, with the Binoux prize, precisely due to the historical component inserted in the study of curves.

Besides being a mathematician, Gomes Teixeira was the director of the Porto Polytechnical School and the first rector of the University of Porto, performing management activities with competence and dedication. He was also an excellent professor, respected and considered by his students.

He participated in international congresses that took place in European countries and also beyond Europe. He was also honoured with the title of "Doutor honoris causa" by the Central University of Madrid and by the Toulouse University.

Among the papers he published, we can find the memoir "On the development of functions in series", which we are going to talk about, in a brief way, as referred.

Throughout his study life, Gomes Teixeira kept investigating, examining thoroughly and updating his knowledge on series. A good example of this are the more than thirty papers on series that were published in international magazines, the different approaches on this subject, in the several editions of *Curso de Analyse Infinitesimal - Calculo Differencial*, and even the bibliographic and historical references, each time more complete, that he kept introducing in those publications.

The memoir "On the development of functions in series" is a remarkable study that allows the reader to acquire knowledge, at the same time global and detailed, because it gives an historical perspective of the theories development and presents detailed bibliographical references. In this issue Gomes Teixeira shows knowledge of the fundamental classical texts of contemporary authors of his time and also former ones, related to the development of functions in series, which reveals a profound scientific and historical knowledge on the subject.

Even though the research fundamental aim that I carried out until then had been the analysis of some subjects of *Curso de Analyse- Calculo Differencial*, and the approached subjects in the memoir were not related, I decided, nevertheless, to make a sketch about the memoir to make it known and present a subject that deserved to be more deeply studied.

2 Memoir "On the development of functions in series"

The memoir was rewarded by the Royal Academy of Exact Physical and Natural Sciences of Madrid. Next, a brief analysis of the process of the given prize attribution is made.

Figure 1: Cover of volume XVIII from Memorias de la Academia de Ciencias Exactas Físicas y Naturales, 1897, where the paper "On the development of functions in series", by Gomes Teixeira, was published.

2.1 The prize of Madrid

The Exact Sciences section in the Royal Academy of Sciences of Madrid, in a contest opened from January the 1st, 1892 till December the 31st, 1893, proposed as a prize subject:

> Appropriate and methodical exposition of the series developments of mathematical functions. Their general theory. Meaning of the so called divergent series. Research of a typical series, from which, if possible, might be derivated, as particular cases, the theories of main importance and use in the analysis, such as those of Taylor, Lagrange and any other similar[2].

Two anonymous memoirs where in competition; one, written in Castilian, as asked in the competition announcement, and other, in Portuguese. This circumstance would exclude this text[3], immediately and without any exam, but the Academy decided, even though, to examine the paper, which was explained in the following way:

> [...] The Academy, agreeing without difficulties to the manifested author's desires of this second memoir, in a different paper, annexed to that, and being inspired by the justified reasons of prudence and scientific convenience, decided that, out of the competition and after appreciating the Castilian memoir, as if conveniently determined, proceeded the mentioned Section to the exam of the Portuguese to emit, on its content, the judgment, that also considered consequent[4].

[2]Exposición razonada y metódica de los desarrollos en serie de las funciones matemáticas. Teoría general de los mismos. Significación de las llamadas series divergentes. Investigación de una serie típica, de la cual, á ser posible, se deriven como casos particulares las series de mayor importancia y uso en la análisis, como las de Taylor, Lagrange, y cualquiera otra análoga. Anuario de la Real Academia de Ciencias Exactas Fisicas y Naturales de Madrid, 1893, p. 312.

[3]Anuario de la Real Academia de Ciencias Exactas Fisicas y Naturales de Madrid, 1893, p. 261.

[4][...] a Academia, accediendo sin dificulta á los deseos manifestados por el autor de esta segunda memoria, en papel independiente, á la misma adjunto, é inspirando-se además en justificadas razones de prudencia y de conveniencia científica, dispuso que, fuera de concurso, y después de apreciada la memoria en castellano como se estimase conveniente, procediese la mencionada Sección al examen de la portuguesa, y á emitir sobre su contenido el dictamen que también juzgare procedente. Anuario de la Real Academia de Ciencias Exactas Fisicas y Naturales de Madrid, 1893, p. 261.

The report on the memoir was at the expense of the academic Becerra, who made a report on it. This report was presented to the Academy by the Exact Sciences Section[5], in which he declared:

> From this succinct text and very short analysis on the work that we are concerned with, we concluded, without effort, that the author, a remarkable mathematician, without any doubts, gathers the two conditions of didactic relater of the most abstruse scientific truths that rarely are found together: copious abundant knowledge; and prudence, to give up all those subordinate questions, that are not absolutely necessary to the achievement of the aims that he proposed to reach. [...] it would be difficult would be to point out in the memoir only one paragraph whose suppression wouldn't seriously harm the integrity and good harmony of the whole text[6].

At the end of that report we can read:

> [...] the Section proposes to the Academy, even though leaving complete freedom, as usual, to determine whatever thinks to be most adequate:
>
> 1st To agree with the printing of this interesting scientific work in the Collection of Its Memoirs;
>
> 2nd That as a token of honoured and deserved esteem, should be given the author two hundred copies of his work, so that he can use them as he finds convenient[7].

[5]Anuario de la Real Academia de Ciencias Exactas Fisicas y Naturales de Madrid, 1896, p. 262-294.

[6]De esta sucinta noticia y muy somero análisis del trabajo que nos ocupa, conclúyese sin esfuerzo que el autor, eximio matemático, sin duda, reúne dos condiciones de expositor didáctico de las más abstrusas verdades científicas, que rara vez se encuentran juntas: copioso caudal de conocimientos; y prudencia, para prescindir de todas aquellas cuestiones subalternas, que non le son absolutamente necesarias para la consecución de los fines que se ha propuesto alcanzar. [...] difícil sería señalar en la memoria un solo párrafo cuya supresión no perjudicase gravemente á la integridad y buena armonía del conjunto. Anuario de la Real Academia de Ciencias Exactas Fisicas y Naturales de Madrid, 1896, p. 288-289.

[7][...] la Sección propone á la Academia, aunque dejando á ésta en completa libertad, como siempre, de resolver lo que más acertado juzgue: 1^o Que acuerde la impresión de esta interesante producción científica en la Colección de sus Memorias Y 2^o Que, en señal de honrosa y merecida estima, se le entreguen, después de impresa, al autor doscientos ejemplares de su obra, para que de ellos haga el uso que crea conveniente. Anuario de la Real Academia de Ciencias Exactas Físicas y Naturales de Madrid, 1896, p. 295-296.

The Academy accepted this proposal of the Exact Sciences Section, in the session of May the 29th, and decided the publication of the memoir, without any need of translation into Castilian, as shown in the statement:

> It is to bear in mind that the memoir printing should be done, not in Castilian, which would delay its publication and would make it difficult the printing revisions by the author, which is indispensable in works of this kind, of complicated typographic composition and which requires perfect and detailed correction, but in the language in which it is written, intelligible, without effort, by the people, that wishes or need to assure its content, in Spain, for those to whom the subject isn't strange at all. Exception in our usual procedures of publication, that no one should find strange, by such a singular case, that doesn't compromise anything, and shouldn't even compromise us in the future[8].

By opening the envelope that accompanied the memoir, closed until then, the Academy verified that its author was "the distinct Portuguese mathematician, Head of the Porto Polytechnical School, our correspondent D. Francisco Gomes Teixeira[9]". Later, in 1898, we can read, in the Academy Anuary, in some news entitled "By special decision of the Academy, out of contest[10]" that Gomes Teixeira's memoir had been presented in the ordinary competition to prizes in 1893 and excluded from it, due to being written in Portuguese, but, besides that, should be printed, in that language, in the Colección de Memorias de la Academia, and 200 copies should be delivered to the author. Between parentheses we can

[8]De advertir es, además, que la impresión da la memoria deberá hacerse, no en castellano, lo cual retrasaría demasiado su publicación, y dificultaría la revisión de las pruebas de imprenta por el autor, indispensable en trabajos de esta índole, de composición tipográfica complicada, y que pide corrección muy esmerada y hasta nimia, sino en el idioma en que de halla escrita, inteligible sin esfuerzo por cuantas personas deseen, ó necesiten, enterarse de su contenido en España, para quienes no sea de todo punto extraña la materia á que se refiere. Excepción ésta en nuestros habituales procedimientos de publicación, que nadie puede extrañar, por lo singular del caso de que se trata, y que á nada compromete, ni debe comprometernos tampoco, para lo sucesivo Anuario de la Real Academia de Ciencias Excatas Fisicas y Naturales, 1896, p. 296-297.

[9]distinguido matemático lusitano, Director de la Escuela Politécnica de Porto, nuestro Corresponsal D. Francisco Gomes Teixeira. Anuario de la Real Academia de Ciencias Excatas Fisicas y Naturales, 1896, p. 135.

[10]Por decisión especial de la Academia, fuera de concurso Anuario de la Real Academia de Ciencias Exactas Fisicas y Naturales, 1898, p. 286.

read: "Published in C. of M., volume XVIII, part I". This volume corresponds to 1897. Later it was inserted in volume I (1904) of *Obras*[11].

2.2 Memoir Structure

The memoir starts with the introduction, where the contents sequence of the six chapters, in which it is divided, is explained.

In the first chapter[12], Gomes Teixeira started by the developments ordered by the positive and full powers of a real variable, exposing the method presented by J. Bernoulli and Taylor and completed by Lagrange and Cauchy, to the development of real variable functions.

In chapters II and III the author develops an extension of that method to the case of the imaginary variable functions, that he stated as being sketched by Cauchy and completed by Darboux. He exposed the method of Cauchy, based on the theory of the curvilinear integrals; presented Riemann method, based in the theory of the harmonical functions; described the Weierstrass method, based on the theory of the full series; and deducted the Laurent series.

In chapter IV, Gomes Teixeira continues the study of Taylor's series in the case of the complex variable functions, presenting the Riemann's method.

In chapter V, he proceeds with the study of Taylor's series in the case of the complex variable functions and develops the Weierstrass and Mittag-Leffler's method.

In chapter VI, the last one, Gomes Teixeira demonstrates Burmann's formula, which gives the development of the functions in ordered series according to the full and positive powers of a given function, retrieving from it the Lagrange formula. After this exposition, Gomes Teixeira presents some personal contributions, as he says in his own words:

> In the sixth chapter we will demonstrate Burmann's formula, which gives the development of series functions ordered by the full and positive powers of a given function, from which we retrieve the Lagrange one, that only differs from the previous in the notation.

[11] The Portuguese government ordered the publication of the colection Obras de Mathematica, constituted by seven volumes, where we can see Gomes Teixeira's published papers, the two books of Analysis and the *Traité des Coubes Spéciales Remarquables planes et gauches.*

[12] We have followed the original writing, as far as the chapters numbers are concerned.

> Next we will do, in num. 61 and 62, an application, that we think is new, of the same formula to the development of series functions ordered by the $\sin x$ powers and also to the demonstration of two formulas attributed to Euler. Finally, to answer to the last part of the program, we will give a formula which gives the development of series functions ordered by the full, positive and negative powers of a given function. This formula, that we think is new and that we have studied in num. 64 and 65, includes the Burmann one, and also the Taylor's and Lagrange's, as well as the Laurent's[13].

After the last chapter, the memoir, published in volume XVIII of *Memorias de la Real Academia de Ciências Exactas Físicas y Naturales*, from Madrid, contains what Gomes Teixeira calls "Note to n. 61", in the beginning of which he wrote:

> After presenting the precedent memoir to the Academy, we spent some time especially in the development of series functions ordered by the $\sin x$ powers in a memoir published in *Jornal de Crelle* (Berlin, volume 116, p. 14), where we presented a formula to obtain this development, with a much easier application than the one that results from the Baumann's formula[14].

The memoir, referred in the text that we have just transcribed, has as title "Sur le développement des fonctions en série ordonnée suivant les puissances du sinus et du cosinus de la variable" and was published in 1896. Later, in 1900, Gomes Teixeira, published in volume CXXII of the same journal, another memoir, called "Sur les séries ordonnées suivant les puissances d'une fonction donnée[15]".

It should be noticed that the "Note to n. 61", transcript above, inserted in the Madrid publication, isn't the same as the existing "Notes" at the end of the text inserted in the *Obras*, probably because the context was different. In fact, from 1897 to 1904, Gomes Teixeira had already published other papers on series[16].

[13]Teixeira F. G., 1897, p. 3.

[14]*Ibidem*, p. 111.

[15]The memoir of 1896 is reproduced in volume I of Obras, from page 103 to page 125, and correctly referenced. The one from 1900, is also inserted in volume I of Obras, and reproduced from page 126 to page 161, referenced as having, also, been published in *Journal de Crelle*, 1896, Band 116 which isn't correct. In fact, by consulting *Journal de Crelle* in this reference, we have observed that this memoir doesn't exist there, but in *Journal de Crelle* from 1900, Band CXXII.

[16]It is noticeable that "Note to n. 61", transcript above, inserted in Madrid's publi-

2.3 Historical Component

In parallel to the scientific development, Gomes Teixeira enriched his work introducing detailed historical references on each one of the subjects that he kept approaching. With these historical texts Gomes Teixeira gives the reader a perspective of the scientific evolution of the several approached subjects, the mathematicians that have studied them and even the detailed indication of the original sources[17]. He, himself, enlightens us on this point, stating:

> We should even say that we accompanied each subject with the bibliographic and historical indications that appeared convenient[18].

In fact, the reader can take, from these precious indications, the most important sources on each subject. In each one of the chapters, parallel to the theoretical development, Gomes Teixeira introduces texts on the historical evolution of the method, indicating the authors, the respective works and even the publication dates. In some cases he even refers to correspondence among mathematicians, in which they had presented their results.

Due to being out of the temporal sphere of action of our work, we didn't analyze these references. Since we think that the knowledge of these references is very interesting for possible researches, we decided to produce a summary table (attached). This seems to be a way for the reader to have a quick, clear and generic perception of the introduced references in this memoir by Gomes Teixeira.

3 "Teixeira's Series"

We discussed the memoir "On the development of functions in series", not only because it was awarded, but by the scientific and historical views; global and, at the same time, detailed that the reader might have over what was published as most relevant, as far as the development of function

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cation isn't the same as the one in "Notes" that are at the end of the text inserted in Obras. Maybe because the context was different. In fact, from 1897 to 1904, Gomes Teixeira had already published other works on series.

[17] We have maintained the authors' designation given by Gomes Teixeira and their respective writing.

[18] Teixeira F. G., 1897, p. 3.

series is concerned. Gomes Teixeira's explanation is remarkable by its clarity, synthesis and personal contributions. Those contributions are still referenced today, mainly by his generalization of Burmann's series. This generalization is specially treated in *A Course of Modern Analysis*, from E. T. Whittaker & G. N. Watson and *Applied Analysis and Force Fields*, by Luís Manuel Braga da Costa Campos[19]. The first of these essays had its first edition in 1915 and a fourth reprint edition in 1969, with other intermediate editions. In paragraph 7.31 from the 1969 edition, entitled *Teixeira's extended form of Bürmann's theorem*, we can read:

> [...] in the last paragraph we have only considered the expansion of a function of *positive* powers from another function, and now we are going to discuss the expansion of a function of positive and negative powers from the second function.
>
> The general statement of the theorem is owed to Teixeira* whose exposition we will follow in this section[20].

In this transcription, the character "*" leads the reader to a footnote, where "Journal für Math, CXXII, (1900), pp 97-123", can be read, which is a reference to Gomes Teixeira's memoir, "Sur les séries ordonnées suivant les puissances d'une fonction donnée". The same text can be found in all the previous editions.

In the paragraph entitled as "Hierarchy of developments in powers of series", Professor Costa Campos calls "Teixeira's Series" to the series that Gomes Teixeira obtained in the article of 1900, putting it on the top of the hierarchy of the powers series. This book presents an interesting scheme that we reproduced in the next page.

We established contact with Professor Costa Campos[21], who enlightened us, by saying that he himself had made a generalization of Teixeira's series to fractional calculus. He also informed us that "something similar to Teixeira's series had been used before by Lavoie, Tremblay e Osler" and that he had used "other power series of Teixeira's type, for instance, exponential series, in the resolution of fractional differential equations".

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[19] Professor António Leal Duarte's kindeness.

[20] [...] in the last paragraph we were concerned only with the expansion of a function in positive powers of another function, whereas we shall now discuss the expansion of a function in positive and negative powers of the second function. The general statement of the theorem is due to Teixeira* whose exposition we shall follow in this section Whittaker E. T. & Watson G. N., 1969, p. 131.

[21] Full Professor at Instituto Superior Técnico, Universidade Técnica de Lisboa.

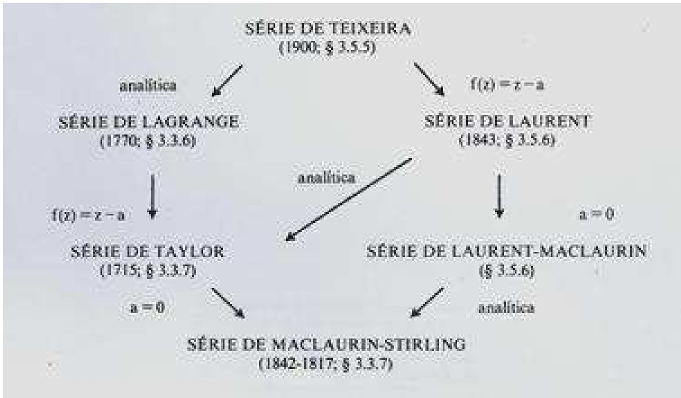

Figure 2: Hierarchy of the power series scheme Luís M. B. da Costa Campos, Applied analysis and force fields, 1988, p. 181.

In the National Library we found the *Transactions of the American Mathematical Society Magazine*, from several authors. In H. Bateman's article, entitled "An Extension of Lagrange's Expansion", of 1926, p. 346-356, there is also a reference to Gomes Teixeira's generalization.

4 Conclusions

The future research that we are prone to accomplish will have, among several others, as possible aims: to analyse the bibliographic and historical references inserted in the memoir; to check out the original sources, when feasible, with the respective memoir subject; to find out if there were influences from the referred sources in the memoir; to compare "Note n. 61" from the memoir with "Notes" from volume I of Obras; to analyse Gomes Teixeira's scientific contribution and his influence in subsequent applications.

Observations:

- A detailed bibliography can be found in the thesis *Francisco Gomes Teixeira - the man, the scientist, the pedagogue*, which can be found in Minho's University Repository, on pages 599 to 652.

- Attached to this article one may find, as an example, a list of some references in the studied memoir and related to chapter 1. A full list for chapter 1, itself and for other chapters may also be seen in the same thesis and it includes tens of other authors such as Peano, Cauchy, Laurent, Riemann, Weierstrass, Wronski and Gomes Teixeira himself, just to name a few.

Attachment
List of Gomes Teixeira's references in Chapter 1 of the memoir "On the development of functions in series"

Author	**Book/ Publication**	**Volume /Page /Year**	**Notes taken from the text**
Gregory	*Exercitationes geometricae*	1668	1st development in $\arctan x$ series
Mercator	*Logarithmotechnia*	1668	1st development in $\log(1+x)$ series
Newton	Letters to Leibniz	1676	Development in series of the binomial, sine, co-sine and exponential
João Bernoulli	*Acta eruditorum*/ (Opera omnia, t. I, p. 125)	1694	Very general formula to the development of function series
Taylor	*Methodusincrementorum*	1715	Taylor's formula
Euler	*Institutiones calculi differentialis*		Taylor's formula
Maclaurin	Treatise of Fluxions	1742	Maclaurin's formula
Lagrange	Memoir presented to Academia de Sciencias (de Berlim/Œuvres, t. III, p. 441)	1772	Taylor's formula demonstration
	Théorie des fonctions analitiques/Œuvres, t. IX, p.69)	1797	Taylor's formula
	Leçons sur le calcul des fonctions/Œuvres, t. X, p. 85)		Lagrange's surplus
Cauchy	*Exercices de Mathématiques* (Œuvres, t. VI, 2.ª série)	Vol.I/ p.29/ 1826	Cauchy's surplus

Research Seminar
on History and Epistemology of Mathematics, 15–26

A look through continued fractions

Assis Azevedo[1]
Centro de Matemática
Universidade do Minho
assis@math.uminho.pt

Abstract

The continued fraction representation of real numbers has some desirable properties: it is finite if and only if the number is rational; it is short, for "simple" rational numbers (like 1/3); it is unique for irrational numbers and almost unique for rational ones; the truncation of the continued fraction representation of a number yields "the best possible" rational approximations of it. In this work, we present the basic definitions and properties of finite and infinite continued fractions. Then the periodicity of the continued fractions for quadratic numbers is studied. Finally, we briefly refer how continued fractions can be used to solve Pell equations.

Keywords: Continued fractions, Pell equation.

[1]Work supported by the Research Centre of Mathematics of the University of Minho through the FCT Pluriannual Funding Program.

1 Preliminaries

The history of continued fractions started with the Euclidean algorithm, which introduced the idea of successive divisions to find the greatest common divisor of two natural numbers. Late in the seventeenth century John Wallis [3] introduced the term "continued fraction" into the mathematical literature.

In 1748 Leonhard Euler [1] published a very important theorem showing that a particular type of continued fraction is equivalent to a certain general infinite series.

Continued fractions are applied to problems in number theory, specifically in the study of Diophantine equations. In the eighteenth century Joseph Louis Lagrange [2] used continued fractions to solve Pell equations, answering a question with more than a thousand years.

An expression of the type

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}, \quad a_0 \in \mathbb{Z},\ a_1, \ldots, a_n \in \mathbb{N} \tag{1}$$

represents a rational number. Reciprocally, if $\frac{p}{q}$, with $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ is a rational number then, by the Euclides's algorithm, there exist $a \in \mathbb{Z}$ and $r \in \mathbb{N}$ with $0 \leq r < q$ such that $p = qa + r$, *i.e.*, $\frac{p}{q} = a + \frac{r}{q}$. Note that $a = \left[\frac{p}{q}\right]$. If $r = 0$ the process stops and if $r > 0$ we write $\frac{p}{q} = a + \cfrac{1}{\frac{q}{r}}$. The conclusion now follows from an induction argument on the size of the denominator, also noting that $\frac{q}{r} > 1$. For example

$$\frac{80}{29} = 2 + \cfrac{1}{\cfrac{29}{22}} = 2 + \cfrac{1}{1 + \cfrac{1}{\cfrac{22}{7}}} = 2 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{7}}}.$$

What happens with the irrational numbers? Let us consider the number $\sqrt{13}$ and try to adapt the argument used above. As the integer part of

$\sqrt{13}$ is 3, we have $\sqrt{13} = 3 + \underbrace{\left(\sqrt{13} - 3\right)}_{\in]0,1[}$ and then, after some calculations,

$$\begin{aligned}\sqrt{13} &= 3 + \cfrac{1}{\cfrac{1}{\sqrt{13}-3}} = 3 + \cfrac{1}{\cfrac{\sqrt{13}+3}{4}} \\ &= 3 + \cfrac{1}{1 + \cfrac{1}{\cfrac{\sqrt{13}+1}{3}}}, \quad \text{repeating the process,}\end{aligned}$$

and we can be doing this forever. We will return to this example later.

We are thus led to consider a more general type of fraction than the one presented in (1).

Definition 1. *A* **continued fraction** *is an expression of the form*

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots + \cfrac{1}{a_n}}}}, \quad \textit{with } a_0 \in \mathbb{Z}, a_1, \ldots, a_{n-1} \in \mathbb{N} \textit{ and } a_n \in]1, +\infty[.$$

If $a_n \in \mathbb{N}$ we will say the fraction is **simple***.*

We denote the above expression by $[a_0, a_1, \ldots, a_n]$. For example $\frac{80}{29} = [2, 1, 3, 7]$ and $\sqrt{13} = [3, \frac{\sqrt{13}+3}{4}] = [3, 1, \frac{\sqrt{13}+1}{3}]$.

If $x = [a_0, a_1, \ldots, a_n]$, $i = 0, 1, \ldots, n-1$ and $x_i = [a_i, \ldots, a_n]$ then:

- $x_i = [a_i, x_{i+1}] = a_i + \frac{1}{x_{i+1}}$;
- a_i is the integer part of x_i, as $\frac{1}{x_{i+1}} \in]0, 1[$.

Therefore, if $[b_0, b_1, \ldots, b_m]$ is another continued fraction then

$$[a_0, a_1, \ldots, a_n] = [b_0, b_1, \ldots, b_m] \iff \forall i = 0, 1, \ldots, n \quad a_i = b_i.$$

Sometimes we accept a_n to be equal to 1. In those case, we lose the unicity in the representation of rational numbers as continued fractions, as

$$[a_0, a_1, \ldots, a_n, 1] = [a_0, a_1, \ldots, a_n + 1].$$

Returning to the number $x = \sqrt{13}$ we have, using the above,

$$
\begin{array}{llll}
a_0 = 3, & x_1 = \frac{\sqrt{13}+3}{4}, & a_1 = 1, & x_2 = \frac{\sqrt{13}+1}{3} \\
a_2 = 1, & x_3 = \frac{\sqrt{13}+2}{3}, & a_3 = 1, & x_4 = \frac{\sqrt{13}+1}{4}, \\
a_4 = 1, & x_5 = \sqrt{13}+3, & a_5 = 6, & x_6 = \frac{\sqrt{13}+3}{4}.
\end{array}
$$

This reasoning can be used for every real number.

Theorem 2. *If $x \in \mathbb{R} \setminus \mathbb{Q}$ and $n \in \mathbb{N}$ then x can be written (in a unique way) in the form $[a_0, a_1, \ldots, a_{n-1}, x_n]$ where $a_0 \in \mathbb{Z}$, $a_1, \ldots, a_{n-1} \in \mathbb{N}$ and $x_n \in]1, +\infty[$.*

The effective calculation of some terms of the continued fraction of a given number can be very difficult due to the errors that can be accumulated by the successive inversions used in the process.

2 Convergents

If we develop the expression $[a_0, a_1, \ldots, a_n]$ we obtain a fraction whose numerator and denominator can be easily found, because they obey a simple recurrence rule. If $k \leq n$ then $[a_0, a_1, \ldots, a_k] = \frac{p_k}{q_k}$ where

$$
\left\{ \begin{array}{l} p_0 = a_0 \\ q_0 = 1 \end{array} \right. \quad \left\{ \begin{array}{l} p_1 = a_0 a_1 + 1 \\ q_1 = a_1 \end{array} \right. \quad \cdots \quad \left\{ \begin{array}{l} p_i = a_i p_{i-1} + p_{i-2} \\ q_i = a_i q_{i-1} + q_{i-2}, \quad \text{if } i \geq 2. \end{array} \right. \tag{2}
$$

It follows that the sequences $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ are strictly increasing (tending to $+\infty$) and, if the continued fraction is simple, then $(p_i, q_i) = 1$ for all i.

Definition 3. *With the above notation, we call* **convergents** *to the fractions $\frac{p_k}{q_k}$, with $k \geq 0$.*

Let us see some properties of the convergentes.

Theorem 4. *Let $a_0 \in \mathbb{Z}$ and $(a_n)_{n\in\mathbb{N}}$ be a sequence of positive integers and, for $n \in \mathbb{N}_0$, $\gamma_n = [a_0, a_1, \ldots, a_n] = \dfrac{p_n}{q_n}$. Under these conditions:*

a) the sequence $(\gamma_{2n})_{n\in\mathbb{N}_0}$ is strictly increasing;

b) *the sequence* $(\gamma_{2n+1})_{n\in\mathbb{N}_0}$ *is strictly decreasing;*

c) $\gamma_{2n} < \gamma_{2m+1}$, *for* $n, m \in \mathbb{N}_0$*;*

d) $p_n q_{n+1} - q_n p_{n+1} = (-1)^{n+1}$, *for* $n \in \mathbb{N}_0$*;*

e) $\gamma_n - \gamma_{n+1} = \dfrac{(-1)^{n+1}}{q_n q_{n+1}}$, *if* $n \in \mathbb{N}_0$.

In particular the sequence $(\gamma_n)_{n\in\mathbb{N}}$ *converges.*

This result gives us, not only the convergence of the sequence $(\gamma_n)_{n\in\mathbb{N}}$, but also the speed of convergence. On the other hand "the bigger the $a_i's$, the bigger the $q_i's$" and faster is the convergence of the sequence. In particular the convergence is the slowest possible in the case $a_i = 1$ for all $i \in \mathbb{N}$ (see page 20).

Consider the following example: let $\gamma = \lim\limits_{n\in\mathbb{N}} ([1, 2, 3, \ldots, n+1])_{n\in\mathbb{N}}$. As

$$[1, 2, 3, 4, 5] = \frac{225}{157} \sim 1,433121 \quad \text{and} \quad [1, 2, 3, 4, 5, 6] = \frac{1393}{972} \sim 1,433128,$$

we conclude that γ_4 and γ_5 are γ approximations with 5 correct decimals (as $\gamma_4 < \gamma < \gamma_5$).

Essentially as a consequence of Theorem 4 we have the following result.

Theorem 5. *If* $a_0 \in \mathbb{Z}$ *and* $(a_n)_{n\in\mathbb{N}}$ *is a sequence of positive integers then the sequence* $([a_0, a_1, \ldots, a_n])_{n\in\mathbb{N}}$ *converges for an irrational number.*

Reciprocally, every irrational number is the limit of a unique such sequence of continued fractions.

Notation: If $x = \lim\limits_{n\in\mathbb{N}} [a_0, a_1, \ldots, a_n]$ we write $x = [a_0, a_1, \ldots, a_n, \ldots]$. This expansion is called an **infinite simple** continued fraction. For example, $\sqrt{13} = [3, 1, 1, 1, 1, 6, 1, 1, 1, 1, 6, 1, 1, 1, 1, 6, \ldots]$.

3 Periodic continued fractions

We saw in page 17 that, if $x = \sqrt{13}$, then $x_6 = x_1$. From this we obtain $x_i = x_{i+5}$ and $a_i = a_{i+5}$ for $i \in \mathbb{N}$.

Definition 6. *An infinite continued fraction* $[a_0, a_1, \dots, a_n, \dots]$ *is* **periodic** *if*

$$\exists k \in \mathbb{N}_0 \; \exists r \in \mathbb{N} \; \forall i \in \mathbb{N} \quad [i \geq k \Rightarrow a_i = a_{i+r}].$$

If $k = 0$*, we say that the fraction is* **purely periodic**.

Under the conditions of this definition, we have $a_{k+nr+j} = a_{k+j}$, for $n \in \mathbb{N}$ and $0 \leq j < r$. Thus, the continued fraction $[a_0, a_1, \dots, a_n, \dots]$ is determined by the knowledge of $a_0, \dots, a_{k+r-1}$. We use the notation $[a_0, a_1, \dots, \dot{a}_k, \dots, \dot{a}_{k+r-1}]$.

The smallest r in the referred conditions is called the **period** of the fraction. For example, $\sqrt{13} = [3; \dot{1}, 1, 1, 1, \dot{6}]$ is a fraction with period 5.

Remark 1. *With the preceding notation, a number* x *has a periodic continued fraction if and only if there exist* $k \in \mathbb{N}_0$ *and* $r \in \mathbb{N}$ *such that* $x_k = x_{k+r}$.

3.1 Characterization of periodic and purely periodic continued fractions

We intend now to characterize the real numbers whose continued fraction is periodic or purely periodic. Of course those number are irrationals.

Let us consider two examples:

- $x = [\,\dot{1}\,]$. As $x = 1 + \frac{1}{x}$ we have $x = \frac{1 \pm \sqrt{5}}{2}$. But, as $x > 1$, then $x = \frac{1+\sqrt{5}}{2}$, the golden number, *the more irrational of the irrational number, from the continued fractions point of view* (see page 19).

- $x = [1, 2, \dot{3}, 4, \dot{5}]$. Notice that $x = [1, 2, y]$ where $y = [\dot{3}, 4, \dot{5}] = [3, 4, 5, y]$. Then

$$y = 3 + \cfrac{1}{4 + \cfrac{1}{5 + \cfrac{1}{y}}} = 3 + \cfrac{1}{4 + \cfrac{y}{5y+1}} = \frac{5y+1}{21y+4} = \frac{68y+13}{21y+4}.$$

  From this we obtain $21y^2 + y - 64y - 13 = 0$, which gives $y = \frac{32+\sqrt{1297}}{21}$. Finally, as $x = [1, 2, y]$, we obtain $x = \frac{103+\sqrt{1297}}{97}$.

Let us introduce some notations.

Definition 7. *An irrational number x is* **quadratic** *if it is a zero of o polinomial of degree two with integers coefficients. To the other zero of the polinomial we call* **conjugate** *of x and denote it by $\overline{x}$. Moreover, if $x > 1$ and $-1 < \overline{x} < 0$, x is called* **reduced**.

Note that every quadratic number can be written on the form $\frac{c+\sqrt{d}}{e}$ where $c \in \mathbb{Z}$, $e \in \mathbb{Z} \setminus \{0\}$ and $d \in \mathbb{N}$ with $\sqrt{d} \notin \mathbb{N}$. We can also suppose that $e|d - c^2$ (multiplying the numerator and the denominator by e, if necessary). Of course $\frac{c-\sqrt{d}}{e}$ is the conjugate of $\frac{c+\sqrt{d}}{e}$.

The *golden number* $\frac{1+\sqrt{5}}{2}$ and the number $\frac{32+\sqrt{1297}}{21}$, referred above, are both reduced quadratic number.

Using the same argument as in the cases $[\ \dot{1}\ ]$ and $[1, 2, \dot{3}, 4, \dot{5}]$, we have the following.

Proposition 8. *All periodic continued fraction represents a quadratic number.*

The following are elementary results that will be used in the next theorem.

Lemma 9. *Let $x = \frac{c+\sqrt{d}}{e}$, with $c \in \mathbb{Z}$, $e \in \mathbb{Z} \setminus \{0\}$ and $d \in \mathbb{N}$ with $\sqrt{d} \notin \mathbb{N}$ and $e|d - c^2$, be a quadratic number and $(x_n)_{n \in \mathbb{N}}$ be defined as in Theorem 2. Then:*

a) for $n \in \mathbb{N}$, there exist $c_n \in \mathbb{Z}$, $e_n \in \mathbb{N}$ such that $x_n = \frac{c_n+\sqrt{d}}{e_n}$ with $e_n|d - c_n^2$;

b) if x_k is reduced then x_n is reduced for $n \geq k$;

c) if x_n is reduced then $0 < c_n < \sqrt{d}$ and $0 < e_n < e\sqrt{d}$.

The following was first proved by Lagrange.

Theorem 10. *If x is an irrational number then x is represented by a periodic continued fraction (respectively, purely periodic continued fraction) if and only if it is quadratic (respectively, quadratic and reduced).*

Proof. (Just one implication) Suppose that $x = \frac{c+\sqrt{d}}{e}$ is quadratic. One can prove that there exists $k_0 \in \mathbb{N}$ such that x_{k_0} is reduced. Using the previous lemma (notations and results), the set $\{(c_n, e_n) : n \geq k_0\}$ is finite because, for $n \geq k_0$, $0 < c_n < \sqrt{d}$ and $0 < e_n < 2\sqrt{d}$. In particular, we must have $r > s \geq k_0$ such that $(c_r, e_r) = (c_s, e_s)$ or equivalently, $x_r = x_s$. $\square$

Although the Euler number e is a transcendental number, its continued fraction has a surprising regularity (with elaborated but elementary proof):

$$e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, \ldots]$$

It is easy to find good rational approximations of e (or any other number whose continued fraction is known) with as many correct decimals as we want. To do this we can keep going calculating convergents of the number, using the recurrence formula (2), and if two convergents have the same k first decimals, then these decimals are correct, relatively to the initial number. For example, as

$$\begin{cases} [2,1,2,1,1,4,1,1,6] & = & \frac{1264}{465} \sim 2,7182796 \\ [2,1,2,1,1,4,1,1,6,1] & = & \frac{1457}{536} \sim 2,7182836 \end{cases}$$

we can conclude that these two rational numbers are approximations of e with 4 correct decimals, as $2,7182796 < e < 2,7182836$.

Regarding the number π (which is equal to $[3, 7, 15, 1, 292, 1, 1, 1, 2, \ldots]$) it is not known any regularity on its continued fraction. Note that $[3, 7, 15, 1]$ is an approximation of π with 6 correct decimals.

3.2 Continued fraction of $\sqrt{d}$

With the purpose of studying Pell's equations, that will be defined later, we now study the continued fractions of numbers of the form $\sqrt{d}$. We start with an observation: if $d \in \mathbb{Q}$, $\sqrt{d} \notin \mathbb{Q}$ with $d > 1$ and $x = \sqrt{d}$, then x is a non reduced quadratic number but x_1 $(= \frac{1}{\sqrt{d}-[\sqrt{d}]})$ is reduced. Thus, using Theorem 10 we can conclude that the continued fraction of $\sqrt{d}$ is of the form $[a_0, \dot{a}_1, \ldots, \dot{a}_r]$.

Some simple calculations show that:

$$\begin{array}{lll} \sqrt{2} = [1, \dot{2}] & \sqrt{77} = [8, \dot{1}, 3, 2, 3, 1, \dot{16}] & \sqrt{\frac{11}{7}} = [1, \dot{3}, 1, 16, 1, 3, \dot{2}] \\ \sqrt{\frac{111}{13}} = [2, \dot{1}, 11, 1, \dot{4}] & \sqrt{34} = [5, \dot{1}, 4, 1, \dot{10}] & \sqrt{13} = [3, \dot{1}, 1, 1, 1, \dot{6}]. \end{array}$$

All these continued fractions have the form $[a_0, \dot{a}_1, a_2, \ldots, a_2, a_1, 2\dot{a}_0]$. In fact this is quite general. Before "proving" it, let us state a preliminary result, concerning the continued fraction of the symmetric of the inverse of the conjugate of a reduced number.

Proposition 11. *If* $y = [\dot{a}_0, a_1, \ldots, \dot{a}_{r-1}]$ *then* $-1/\overline{y} = [\dot{a}_{r-1}, \ldots, a_1, \dot{a}_0]$.

Theorem 12. *If x is an irrational positive number then, there exists $d \in \mathbb{Q}$ with $d > 1$ (respectively $d < 1$) such that $x = \sqrt{d}$ if and only if the continued fraction of x is of the form $[a_0, \dot{a}_1, a_2, \ldots, a_2, a_1, 2\dot{a}_0]$ (respectively, $[0, a_0, \dot{a}_1, a_2, \ldots, a_2, a_1, 2\dot{a}_0]$) with $a_0, a_1, \ldots \in \mathbb{N}$.*

Proof. (Just one implication) If $x = \sqrt{d}$ with d a rational number greater than 1 then $\sqrt{d}$ if of the form $[a_0, \dot{a}_1, \ldots, \dot{a}_r]$, being $[\dot{a}_1, \ldots, \dot{a}_r] = \frac{1}{\sqrt{d} - [\sqrt{d}]}$.

Then,

$$\sqrt{d} + \left[\sqrt{d}\right] = \begin{cases} [\dot{a}_r, \ldots, \dot{a}_1] & \text{by the previous proposition} \\ [2a_0, \dot{a}_1, \ldots, \dot{a}_r] & \text{obviously,} \end{cases}$$

and the result follows.

If $0 < d < 1$ then $\sqrt{d} = \left[0, \sqrt{\frac{1}{d}}\right]$ and we can use the first part of the theorem, as $\frac{1}{d} > 1$. $\square$

In page 26 we present a table with the continued fractions of the numbers of the form $\sqrt{d}$, where d is any non square number between 2 and 52.

The continued fraction of numbers of the form $\sqrt{d}$ with d a non square positive number has some more properties, easy to prove, that will be the core in the study of Pell's equations.

Theorem 13. *Let d be a non square positive integer, $n \in \mathbb{N}_0$ and r the period of the continued fraction of $\sqrt{d}$. Then, with the usual notations:*

a) $e_n = 1$ *if and only if* r *divides* n.

b) $(-1)^n e_n = -1$ *if and only if* r *is odd and* $n = rk$ *for some positive odd number* k.

4 Pell's equation

Let us see an application of the continued fractions to the study of the Pell's equations, which are equations, on the positive integer variables x and y, of the type $x^2 - dy^2 = m$ where $d \in \mathbb{N}$ and $m \in \mathbb{Z}$. Note that, if $d = k^2$, with $k \in \mathbb{N}$, then the equation is equivalent to $(x-ky)(x+ky) = m$, which has trivial solution if we have a factorization of m. By this reason we consider only the case where d is not a square number.

Using the natural bijection from $\mathbb{Z}\times\mathbb{Z}$ to $\mathbb{Z}[\sqrt{d}]$ $(=\{x+y\sqrt{d} : x, y \in \mathbb{Z}\})$ we can consider the solutions of the equation on $\mathbb{Z}[\sqrt{d}]$ instead of $\mathbb{Z} \times \mathbb{Z}$.

Theorem 14. *Let d be a non square positive integer and $m, m_* \in \mathbb{Z}$. Then, with the usual notations we have:*

a) if there exists $n \in \mathbb{N}$ such that $m = (-1)^n e_n$, the equation $x^2 - dy^2 = m$ has a nontrivial solution. More specifically, $(p_{n-1})^2 - d(q_{n-1})^2 = (-1)^n e_n$;

b) the equation $x^2 - dy^2 = 1$ always admits a nontrivial solution;

c) if the period of the continued fraction of $\sqrt{d}$ is odd, then the equation $x^2 - dy^2 = -1$ has a solution;

d) if $a + b\sqrt{d}$ is a solution of $x^2 - dy^2 = m$ and $a_* + b_*\sqrt{d}$ is a solution of $x^2 - dy^2 = m_*$ then $(a + b\sqrt{d})(a_* + b_*\sqrt{d})$ is a solution of $x^2 - dy^2 = mm_*$;

e) if the equation $x^2 - dy^2 = m$ admits a solution, then it admits infinitely many solutions.

Proof. (Just some ideas) a), b) and c) are consequences of Theorem 4 d) and Theorem 13. For d), just note that $(a + b\sqrt{d})(a_* + b_*\sqrt{d}) = (aa_* + dbb_*) + (ab_* + a_*b)\sqrt{d}$ and

$$\begin{aligned} (aa_* + dbb_*)^2 - d(ab_* + a_*b)^2 &= a^2a_*^2 + d^2b^2b_*^2 - da^2b_*^2 - da_*^2b^2 \\ &= a^2\left(a_*^2 - db_*^2\right) - db^2\left(a_*^2 - db_*^2\right) \\ &= \left(a_*^2 - db_*^2\right)\left(a^2 - db^2\right) = m_*m. \end{aligned}$$

Alinea e) is a consequence of b) and d) (using $m_* = 1$). $\square$

We wish to make some considerations about this theorem.

- If p, q are positive integers such that $p^2 - dq^2 = m$, $|m| < \sqrt{d}$ and m is square free then there exists $n \in \mathbb{N}$ such that $m = (-1)^n e_n$ and there exists $k \in \mathbb{N}$ such that $p = p_k$ and $q = q_k$.
- Let r be the period of the continued fraction of $\sqrt{d}$. Then:
  - if r is even, the equation $x^2 - dy^2 = -1$ has no solution and the solutions of $x^2 - dy^2 = 1$ are all of the form (p_{kr-1}, q_{kr-1});
  - if r is odd then equation $x^2 - dy^2 = -1$ admits as solutions exactly the pairs of the form (p_{kr-1}, q_{kr-1}) with odd k and the equation $x^2 - dy^2 = 1$ admits as solutions exactly the pairs of the form (p_{kr-1}, q_{kr-1}) with even k.

As an application of the last theorem we have,

⋆ $(p_4, q_4) = (18, 5)$ is a solution of equation $x^2 - 13y^2 = -1$;

⋆ $(p_9, q_9) = (649, 180)$ is a solution of equation $x^2 - 13y^2 = 1$;

⋆ equation $x^2 - 77y^2 = -1$ has no solution.

5 Khinchin's theorem

Among many other results about continued fractions we decided to present, without proof, the very surprising Khinchin's theorem.

We start with some notation: if $x \in \mathbb{R}$, $[a_0, a_1, a_2, \ldots]$ is its continued fraction and $n \in \mathbb{N}$ we denote by $G_n(x)$, the geometric mean of the set $\{a_1, a_2, \ldots, a_n\}$.

Theorem 15 (Khinchin's Continued Fraction Theorem)**.** *There is a constant K_0 such that, for* <u>*almost all*</u> *real numbers, x*

$$\lim_{n\to\infty} G_n(x) = K_0,$$

Here "almost all" means "except for a set of measure zero". For example, the rational, the quadratic numbers and the Euler number e do not satisfy this property. In fact, <u>no number</u> has been showed to satisfy the condition of this theorem.

Among the numbers whose continued fraction expansions apparently do have this property (based on numerical evidence) are π, the Euler-Mascheroni constant γ, andthe Khinchin's constant itself.

The constant K_0 is known to the equal to $\prod_{k=1}^{\infty}\left[1+\frac{1}{k(k+2)}\right]^{\log_2 k}$ and is approximately equal to $2,68545$. It is not known if K_0 is rational.

d	$\sqrt{d}$	d	$\sqrt{d}$	d	$\sqrt{d}$
2	$[1,\dot{2}]$	20	$[4,\dot{2},\dot{8}]$	37	$[6,\dot{12}]$
3	$[1,\dot{1},\dot{2}]$	21	$[4,\dot{1},1,2,1,1,\dot{8}]$	38	$[6,\dot{6},\dot{12}]$
5	$[2,\dot{4}]$	22	$[4,\dot{1},2,4,2,1,\dot{8}]$	39	$[6,\dot{4},\dot{12}]$
6	$[2,\dot{2},\dot{4}]$	23	$[4,\dot{1},3,1,\dot{8}]$	40	$[6,\dot{3},\dot{12}]$
7	$[2,\dot{1},1,1,\dot{4}]$	24	$[4,\dot{1},\dot{8}]$	41	$[6,\dot{2},2,\dot{12}]$
8	$[2,\dot{1},\dot{4}]$	26	$[5,\dot{10}]$	42	$[6,\dot{2},\dot{12}]$
10	$[3,\dot{6}]$	27	$[5,\dot{5},\dot{10}]$	43	$[6,\dot{1},1,3,1,5,1,3,1,1,\dot{12}]$
11	$[3,\dot{3},\dot{6}]$	28	$[5,\dot{3},2,3,\dot{10}]$	44	$[6,\dot{1},1,1,2,1,1,1,\dot{12}]$
12	$[3,\dot{2},\dot{6}]$	29	$[5,\dot{2},1,1,2,\dot{10}]$	45	$[6,\dot{1},2,2,2,1,\dot{12}]$
13	$[3,\dot{1},1,1,1,\dot{6}]$	30	$[5,\dot{2},\dot{10}]$	46	$[6,\dot{1},3,1,1,2,6,2,1,1,3,1,\dot{12}]$
14	$[3,\dot{1},2,1,\dot{6}]$	31	$[5,\dot{1},1,3,5,3,1,1,\dot{10}]$	47	$[6,\dot{1},5,1,\dot{12}]$
15	$[3,\dot{1},\dot{6}]$	32	$[5,\dot{1},1,1,\dot{10}]$	48	$[6,\dot{1},\dot{12}]$
17	$[4,\dot{8}]$	33	$[5,\dot{1},2,1,\dot{10}]$	50	$[7,\dot{14}]$
18	$[4,\dot{4},\dot{8}]$	34	$[5,\dot{1},4,1,\dot{10}]$	51	$[7,\dot{7},\dot{14}]$
19	$[4,\dot{2},1,3,1,2,\dot{8}]$	35	$[5,\dot{1},\dot{10}]$	52	$[7,\dot{4},1,2,1,4,\dot{14}]$

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Research Seminar
on History and Epistemology of Mathematics, 27–38

A geometrical proof by Anastácio da Cunha in 19th-century English and American textbooks

João Caramalho Domingues[1]
Centro de Matemática
Universidade do Minho
jcd@math.uminho.pt

Abstract

José Anastácio da Cunha (1744–1787) is usually recognized as one of the most important Portuguese mathematicians ever. Nevertheless, his work does not seem to have had much repercussion abroad. His *Principios Mathematicos* (Lisbon, 1790) were translated into French (*Principes Mathématiques*, Bordeaux, 1811), but with limited impact — until recently all that was known were a few reviews, passing mentions in biographical dictionaries, and a positive reference by Gauss in private correspondence.

However, thanks to the mainly positive review by John Playfair in the *Edinburgh Review*, English author John Radford Young (1799–1885) used in his *Elements of Geometry* (London, 1827) a proof by Anastácio da Cunha (of a proposition on parallels). This proof also made its way to an American textbook (Benjamin Greenleaf, *Elements of Geometry*, Boston, 1858). Young's and Greenleaf's are, so far, the only known cases of actual *use* (instead of mere reference) of Cunha's work outside Portugal.

Keywords: José Anastácio da Cunha, John Playfair, John Radford Young, Benjamin Greenleaf, theory of parallels, English textbook, American textbook.

[1]Work supported by the Research Centre of Mathematics of the University of Minho through the FCT Pluriannual Funding Program.

1 José Anastácio da Cunha's *Principios Mathematicos* and their reception abroad

José Anastácio da Cunha (1744–1787) is probably, with Pedro Nunes (1502–1578), the best-known Portuguese mathematician; see [Cunha *1987*, *2006a*; Queiró *1992*; Youschkevitch *1973*]. Here we need only to recall a few facts.

The only work by Cunha published in his own lifetime (and then, only partially) was *Principios Mathematicos* [Cunha *1790*]. This was also his only work published in any language other than Portuguese before the 20th century: a French translation, by Cunha's friend João Manuel d'Abreu, appeared in 1811 in Bordeaux and was reissued in 1816 in Paris[2].

The purpose of Cunha's *Principios* was to be a sort of elements (in the Euclidean sense) for the whole of mathematics, as rigorous as possible. It is an incredibly concise work, covering from elementary geometry to variational calculus, through arithmetic, analytical geometry, differential and integral calculus, finite differences, and so on, in just over 300 pages, divided into 21 "books". But its claim to international fame — that is, its historical relevance beyond the Portuguese context — rests just on three aspects:

1. The first rigorous treatment of convergent series, in book 9, using what is now called the Cauchy criterion as the definition of convergence — and actually using this definition in proofs [Oliveira *1988*, Queiró *1988*]. Cunha has appeared in a very successful American textbook on History of Mathematics [Katz *1993*] precisely because of this.
2. A surprisingly "modern"–looking treatment of powers (a^b defined as a power series, in a way equivalent to $e^{b \log a}$, thus including in one definition the cases of integer, rational, real, and even complex exponent), logarithms and exponentials, also in book 9 [Youschkevitch *1973*, 10–16]. Gauss himself had praised Cunha's definition of power in a letter to Bessel dated 11 November 1811 [Youschkevitch *1978*].

[2] This reissue does not mean that the book was successful: on the contrary, it appears to be a "false" reprint, making use of the remainders of the 1811 edition with only new cover and title pages printed.

3. A definition of what Cunha, in Newtonian fashion, calls "fluxion", described by Youschkevitch as the first "définition analytique rigoureuse de la différentielle"[3] [*1973*, 19]. This merited a footnote passage in [Grattan-Guinness *1980*, 112]: "Cauchy's definition of the differential was anticipated by the obscure Portuguese mathematician J. A. da Cunha".

At the time of its publication, [Cunha *1811*] went largely unnoticed. Three reviews are known[4]: one in the French newspaper *Moniteur Universel*, by Anastácio Joaquim Rodrigues, who had been a friend and student of Cunha's; one in the German literary journal *Göttingische gelehrte Anzeigen*; and one in the British literary journal *Edinburgh Review* (this last one is a fundamental piece connecting Cunha to John Radford Young). Besides these reviews, [Duarte & Silva *1987*] cite Gauss's private reference, mentioned above, criticizing the *Gelehrte Anzeigen* review, and passing references in Cunha's entries in a couple of biographical dictionaries published in France. It is noteworthy that these reviews and references exist, but they show only that Cunha's book was noticed by a few people in France, Britain and Germany (in the case of the review in the *Moniteur Universel*, not even that, since the reviewer was Portuguese).

Was Cunha's book ever *used* outside Portugal[5]? That is, was it ever cited, or did it influence in any way some non-Portuguese mathematician? As we will see, the answer is yes, even if the influence was not profound and was not concerned with the three aspects for which the book is nowadays renowned.

2 John Playfair

The name of the Scotsman John Playfair (1748–1819) is familiar among mathematicians because of the so-called "Playfair's axiom" – probably the most popular substitute for Euclid's parallel postulate: "two straight lines, which intersect one another, cannot be both parallel to the same straight line" [Playfair *1795*, 2nd ed, 7]. This had been used by others

[3] "rigorous analytic definition of differential".

[4] They are collected in [Cunha *1987*].

[5] In [Duarte & Silva *1987*] there are several examples of its influence on 19th-century Portuguese mathematicians.

[*1795*, 4th ed, 422], but its usual name indicates that it was Playfair who popularized it.

Actually, Playfair was a very respected and well-read mathematician, but not a highly creative one. He did publish a few research articles, but he is not remembered for them. What influence he may have had in mathematics resulted from his textbook [*1795*] and the reviews he wrote for the *Edinburgh Review*.

Reviews in the literary journal *Edinburgh Review* were published anonymously but, according to Ackerberg–Hastings [*2008*, 82], 39 of them can be certainly attributed to Playfair, 22 of which of works in the exact sciences. In several of these reviews Playfair expounded his views on the British decline in mathematics, when compared to the great advances made by Continental European mathematicians in the second half of the 18th century [Ackerberg–Hastings *2008*]; this decline was associated to the British clinging to synthetical methods and to fluxions, as opposed to Continental analytical methods and to the differential and integral calculus.

As for [Playfair *1795*], it is a textbook based for the most part on Robert Simson's edition of Euclid's *Elements*, with some changes introduced for pedagogical reasons, but trying to keep Euclid's order [*1795*, 369–371] (which is probably why Playfair did not introduce Cunha's proof about parallels, in post-1812 editions).

Playfair's connection with Anastácio da Cunha comes from the review he wrote of Cunha's *Principios* [Playfair *1812*][6]. This review is, globally, quite positive, although with several negative criticisms. The most important defect applied to the book as a whole: Cunha had used synthetical methods in too many situations ("even in Algebra"), thus deviating from "the path of discovery" and making some parts less simple than they could be[7].

Playfair did not see any of Cunha's three innovations listed at the

[6] Actually, as usual, the review is anonymous, but it has been systematically attributed to Playfair, at least since the publication of a Portuguese translation in *O Investigador Portuguez em Inglaterra* (a Portuguese periodical published in London), in February 1813; John Radford Young also attributed this review to Playfair. Here it will also be assumed that he is the author.

[7] British influence on Cunha is a well-known fact (Newton, with d'Alembert, was one of his mathematical heroes; he had English and Scottish friends; he even wrote a mathematical paper in English [Cunha *1778*], something quite unusual in 18th-century Continental Europe). He may have been too "British" for Playfair.

beginning of this paper as positive: he regarded Cunha's definitions of power and fluxion as too complicated, and simply ignored his treatment of convergence of series (marred, anyway, by a defective French translation). The strongest obstacle for Playfair appreciating Cunha's innovations lied probably in their lack of *utility*. Playfair saw two possible uses for Cunha's book: as a textbook (but one that required a very intelligent and skilful teacher, who should "furnish many illustrations, and supply many steps of the reasonings"); or as a "portable compendium for reminding [those already instructed in mathematics] of those formulas and demonstrations which may have escaped their recollection" [*1812*, 426, 433]. For these purposes, he preferred the "equally comprehensive" text of the abbé de La Caille: not only it was much clearer, as it did not "so much affect originality of method as the Portuguese; and on that account perhaps [it was] the more useful".

Despite all these objections, Playfair's global judgement was positive, as has already been said. He praised the conciseness and rigour of the work. To comprehend so much in such a small book had been "no doubt an undertaking of considerable difficulty"; and "the book forms a very useful and concise digest of Mathematical learning" [Playfair *1812*, 425].

The particular passages most praised by Playfair were three. One was the extraction of roots [Playfair *1812*, 429–430]. Another was the definition of proportion used by Cunha, which had "great merit", being equivalent, but simpler, easier to understand, and easier to remember than Euclid's [*1812*, 429]; this will not be examined here, but it may be observed that this definition is not due to Cunha, but rather to the Jesuit Andreas Tacquet (1612–1660). Finally, Cunha's simplification of the theory of parallels, by means of a new proof that straight lines making equal alternate angles with a third line are parallel, was "a considerable improvement in elementary geometry" [*1812*, 428]; this is the proof in the title of this article, and will be examined in section 5.

3 John Radford Young

John Radford Young (1799–1885) was an English mathematician and textbook author. His most important employment was as professor of mathematics at Belfast College, from 1833 to 1849 (he lost this appointment, apparently for religious reasons, when it was replaced by the Queen's

College, Belfast). He published some original research work, and several textbooks, helping to familiarize English students with continental methods of analysis [Carlyle *1900*].

Young's connection with Anastácio da Cunha occurs in his second textbook — *Elements of Geometry* [*1827*]. Young had read [Playfair *1812*] and agreed that Cunha's proof that straight lines making equal alternate angles with a third line are parallel was "superior to every other that has been given of the same proposition" as, unlike in Euclid and most modern author, it did not depend upon "a subsidiary theorem, which is of no other use whatever (Prop. XVIX. Euc.)" [Young *1827*, 165]; therefore, naturally, he adopted it [Young *1827*, 12–13][8].

[Young *1827*] had an American edition in 1833, "revised and corrected, with additions, by M. Floy, Jun. A. B.". But the passages related to Cunha remained unchanged.

4 Benjamin Greenleaf

Benjamin Greenleaf (1786–1864) was a very sucessful author of elementary mahematics textbooks from Massachussets. Having graduated from Dartmouth College in 1813, he was a preceptor at Bradford Academy from 1814 to 1836, a member of the Massachussets Legislature from 1837 to 1839, and afterwards he founded the Bradford Teacher's Seminary. "For a third of a century his works were almost universally used in the schools of New England" [*Essex 1900*, 56].

In his *Elements of Geometry* [*1858*], Greenleaf also used Cunha's proof that straight lines making equal alternate angles with a third line are parallel — stressing this not only in a note to the proof, but also in the preface [Greenleaf *1858*, iii–iv, 31]; moreover, in an advertisement for [Greenleaf *1858*], included at the end of [Greenleaf *1860*], one of the featured qualities is that

> "The acknowledged improvements of M. DA CUNHA, and other distinguished modern mathematicians, have been carefully incorporated into the work".

[8]He also used a definition of proportion given in [Playfair *1812*] and seemingly based on Cunha's; and, inspired by Cunha, he decided not to use *ratios* — only *proportions* (an aspect of Cunha's book that Playfair had not mentioned).

Since Greenleaf [*1858*, 31] invokes the authority of Young and Playfair for the importance of Cunha's proof, it seems clear that his source was [Young *1827*], possibly in its American edition[9]. There is no indication that he may have seen Cunha's book.

Like other textbooks by Greenleaf, [*1858*] had multiple "editions" (or maybe printings): in 1868 it had reached its "nineteenth electrotype edition".

5 Straight lines making equal alternate angles with a third line are parallel

What Playfair regarded as a "considerable improvement", due to Cunha, in the deduction of the properties of parallel lines, was the disentanglement of a "circuitous route" present in Euclid's *Elements*. To understand this circuitous route, let us look at the statements of three propositions from book I of the *Elements*[10]:

I, 16: In any triangle, if one of the sides be produced, the exterior angle is greater than either of the interior and opposite angles.

I, 17: In any triangle two angles taken together in any manner are less than two right angles.

I, 32: In any triangle, if one of the sides be produced, the exterior angle is equal to the two interior and opposite angles [taken together], and the three interior angles of the triangle [taken together] are equal to two right angles.

It is immediately obvious that the facts stated in I, 16 and I, 17 are less precise versions of those stated in I, 32. Why are they not corollaries of I, 32? Because Euclid's proof of the latter uses I, 31 (through a given point to draw a straight line parallel to a given straight line); the proof of this one uses I, 27, namely

I, 27: if a straight line falling on two straight lines make the alternate angles equal to one another, the straight lines will be parallel to one another;

[9]Nevertheless, Greenleaf's chapter on "ratio and proportion" shows no influence whatsoever from Young or Cunha — he uses ratios as quotients [Greenleaf *1858*, 43].

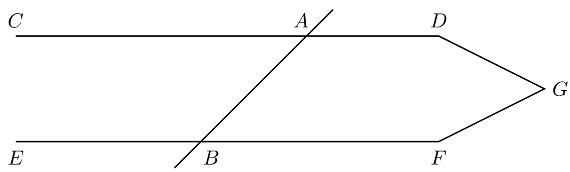
[10]Following Heath's version, which in this respect is close to Simson's.

and the proof of this one uses I, 16[11].

It should be said that Cunha was not the first to try to avoid this circuitous route. Tacquet [*1654*] avoided it too, but in a very different manner, much more confused, and assuming the existence of a parallel through a given point before giving the construction of such a line.

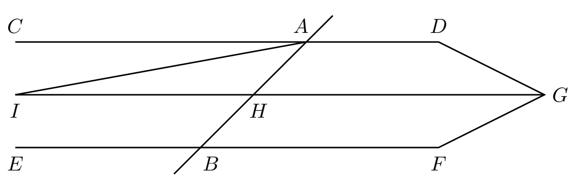
Cunha's path is very simple. Actually, it seems quite trivial — he simply merged Euclid's proof of I, 16 into the proof of I, 27.

Euclid's proof of the latter goes along the following lines:



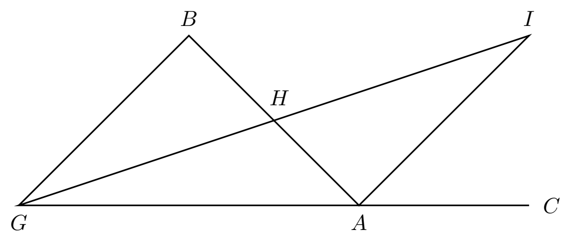
Suppose AB meets CD and EF making the alternate angles CAB, ABF equal. If CD, EF were not parallel, they would meet (when produced), say at G; but then ABG would be a triangle, with an exterior angle, namely CAB, equal to *one* interior and opposite angle, namely ABG — and this is impossible, by I, 16.

Cunha's proof [*1790*, 8–9; *1811*, 9–10] starts similarly, but he does not yet have results about external angles; so, he bisects AB at H, draws the straight line GHI with HI equal to GH, and joins I to A, in order to compare triangles AHI and BHG; in modern terms, they would be congruent (by the side-angle-side criterion) and angles IAH and $HBG (= HBF)$ would be equal; but this cannot be, because the latter is, by assumption, equal to CAH.



[11] Actually, I, 17 is not used before I, 32, so that it *could* be a corollary to the latter. But it is a very easy consequence of I, 16.

But this is precisely the main argument involved in Euclid's proof of I, 16: given any triangle BGA, with GA produced to C, exterior angle BAC is greater than interior angle GBA because the latter is equal to angle HAI (where H and I are constructed in the same way).



The rest of Cunha's path, up to the sum of the interior angles of a triangle being equal to two right angles, and an exterior angle being equal to the sum of the two interior and opposite angles, is not very different from Euclid's.

6 Final remarks

Cunha's proof is definitely not a major originality. Nevertheless, it does avoid the "circuitous route" mentioned by Playfair, allowing for a more economical arrangement than Euclid's. This was a typical concern of Cunha's. The numbering of Cunha's propositions has been avoided so far (to avoid confusion with Euclid's), but the proof we have seen is of his proposition *8*, of Book I. Indeed, Cunha's Book I has a total of only 16 propositions (going as far as the equivalent of Euclid's I, 34). As a further example of his economical style, it may be added that Pythagoras' theorem, in the usual version, is not included in Cunha's *Principios* — what is included, as prop. 15 of Book V, is its more general version, with arbitrary similar polygons (instead of squares) described on a right-angled triangle [Cunha *1790*, 70–71; *1811*, 81]; Euclid, as is well known, has both versions — propositions 47 of Book I and 31 of Book VI.

Nevertheless, as we have seen, Young and Greenleaf used Cunha's proof — and neither of them is so economical; rather, Young regarded the avoidance of Euclid's Prop. I, 16 as advantage enough (and Greenleaf simply followed Playfair's and Young's recommendations).

Two further remarks should be made. One relates to Cunha's lack of comments on this proof. In the *Principios*, comments are markedly absent (apart from a few technical scholia); but recently a set of manuscripts by Cunha has been discovered and published [Cunha *2006a*, II], and one of them is clearly a foreword to an early version of the geometrical books of Cunha's *Principios* [Cunha *2006b*]. There, Cunha discusses several issues where he departed from Euclid; he does not mention at all his proof that equality of alternate angles implies parallelism; either this was a later change or, much more likely, he did not regard it as an important issue.

The other remark is that, nowadays, Euclid's propositions I, 16 and I, 17 are relevant by themselves, as they are valid in absolute geometry, while I, 32 is not.

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Research Seminar
on History and Epistemology of Mathematics, 39–44

A Note on Mathematics in the Eighteenth Century Portugal

António Leal Duarte
Dep. Matemática/CMUC[1]
Universidade de Coimbra
leal@mat.uc.pt

Abstract

In 1713 was published in Lisbon the *Tratado de Arithmetica e Algebra* by António Pereira. The algebra that is presented in this book is a sixteenth-century, rhetorical algebra. We will discuss this Algebra comparing it with what was done in the rest of Europe.

Keywords: Portuguese mathematics, Algebra, eighteenth century, António Pereira.

1 Introduction

It is well known that, after Pedro Nunes (1502-1578) and until the middle of eighteenth century, the level of mathematical activity in Portugal was low, with a few exception in some applied subjects (see e. g. [3], [4], [7], [12]). According to F. Gomes Teixeira [12, p. 215-216]:

> *No período de pobreza cientifica a que nos estamos referindo, apareceram em Portugal alguns escritos sobre Aritmética, Geometria elementar e Astronomia, mas são apenas trabalhos*

[1]project FCT HC/0119/2009, História da Ciência na Universidade de Coimbra (1547-1933)

> *didáticos, mais ou menos bem compostos, sem originalidade apreciável, e que não concorreram para se introduzir as descobertas dos grandes matemáticos europeus.*
>
> *Dormem nas estantes das bibliotecas; e não serei eu quem os irá acordar*[2].

In this note, we will precisely *wake up* one of *works* which is, in our opinion, a good representative of those *didactic* works. We think that in order to understand the scientific activity of any epoch we need not only to know its major works bur also the minor works.

2 The *Tratado de Aritmethica e Algebra* by António Pereira

According to D. Barbosa Machado, [2, p. 346, 347], António Pereira was born in Lisbon (Machado does not furnish the birthday date) and became a member of the Congregation of Oratory in 13 of June, 1686, already an adult, although he was never ordered priest. He died in Extremoz in the Convent of the Congregation, on the 30th of October of 1698. In 1713 it was published, posthumously, in Lisbon, his only known work the *Tratado de Arithmetica e Algebra*[3].

The biographical notice in Inocencio F. da Silva, [10, p. 221], is based on Barbosa Machado. There is a second edition[4] in 1760, mentioned by R. Guimarães ([5, p. 191], who does not make any comment on the book.

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[2] *During the period of scientific poverty that we have been mentioning, some works on Arithmetic, elementary Geometry and Astronomy were published in Portugal, but they were merely didactic, more or less well written, without any appreciable originality, and they did not contribute to the introduction of the discoveries of the great European mathematicians in Portugal.*

They lie dormant in library shelves, and I will not wake them up.

[3] The full title is *Tratado de Arithmetica e Algebra em o qual com muita clareza se explica tudo o que pertence a esta Arte e se descrevem as Regras principais da Geometria, e as proporções que as distinguem, com a noticia dos pezos, de ouro, e de prata, e muitas questoens curiosas que se movem para sua inteligencia, nam so necessario aos contadores que a profeçam, mas tambem aos q̃ seguem a milicia, pilotos, navegantes, ourives, e aos que exertitam mercancia, ou de qualquer modo negoceam.*

[4] I will like to thank João Caramalho Domingues for having called my attention to this second edition.

In modern times there are also several references to it in [1] but in connection to the first part of the book devoted to the Arithmetic. The second part (*Livro segundo*), pages 230–392 of the book is devoted to algebra: it begins with an introduction, having the title: *Da Arte Mayor chamada Algebra ou Regra da Cousa, que trata das Conjugações simples*[5]. Of course this title is a sixteenth-century title and this is what the book is. Despite the fact of being published in 1713 its main sources are, according to the author, Pedro Nunes (1502–1578) and Moya:

> *A arte de Algebra, o primeiro que a escreveu em forma que se possa aprender, foy o famoso Pedro Nunes portugues no anno de 1567... Depois de Pedro Nunes fez Moya seu tratado della no seu livro de arithmetica, que de um e de outro tiraremos hum extrato que baste para a inteligencia de todas as suas questoens*[6].

Moya is certainly Juan Pérez de Moya (Santisteban del Puerto, Jaén, c. 1512 – Granada, 1597), author, among other things, of *Arithmetica practica, y speculatiua*, Salamanca 1562 (first edition); see e. g. [8, 105-108], [13] (in fact the Moya published his *Arithmetica* before the publications of the *Libro de Algebra* of Nunes.

We will not discuss here the relation of the Algebra of Pereira with its sources: we just want to stress that Pereira seems to have ignored all the evolution of Algebra since (at least) Vieta, namely the several Latin editions of Descartes' Geometry, prepared by F. van Schooten. If fact, the algebra of Pereira is a rhetorical, cossist, algebra. After the introduction we find a chapter on roots including the extraction of square and cubic roots, one on proportions, another corresponding to what we will now call polynomials operations (the *polynomials* being written in the sixteenth-century notation of co (cousa), ce (censo) ce.ce.); the last chapter is on conjugations that is equations; this includes the several cases of a second degree equation, and cases of second degree equations in powers of the unknown.

Now the question is: can the Algebra of Pereira and its popularity be a proof of the low mathematical activity and also of the isolation of the

[5] On the great Art or Rules of the cosa, about simple conjugations.

[6] The art of algebra, the first to write it in a way that could be learned was the famous portuguese on 1567. ... After Pedro Nunes, Moya composed a Treaty on the subject in his book on Arithmetic, and from one and the other we will take an extract enough for all its questions.

Portuguese mathematical community during the seventeenth-century and the first half of eighteenth century? To answer this question we have to answer first another question: is the Algebra of Pereira unique in Europe during this period or are there more examples? And the answer to this question is yes, there are! According to Jacqueline Stedall, [11, p. 44]:

> *Elsewhere in Western Europe cossist Algebra continue to appear well into the seventeenth century. The later text, however, lack the freshness and vigor of of their mid sixteen-century predecessors and are often ponderous or confused.*

In fact we think that cossist Algebra continue to appear (at least in Iberia and France) well into the eighteenth century: the *Arithmetica practica, y speculatiua* of Perez de Moya had several edition in the eighteenth century: see [13] were several eighteenth century editions are mentioned, the last one from 1798 (some of these editions are available through Google Books). In France the book *L'Arithmetique en sa Perfection*[7] was extremely popular with several editions in seventeen and eighteen centuries (again some editions are available through Google Books); the book has a section also on algebra but rhetorical algebra.

Of course there is another question: What makes those kinds of works so popular? Possibly it is not the chapters on Algebra but those on commercial and practical Arithmetics; and the same may be true for the *Tratado* of António Pereira. Moreover, we think that what may prove the low level of mathematical activity in Portugal, in this period, is not the existence of the the *Tratado* of Pereira, but the nonexistence of more advanced works.

For instance there is another work around the middle of seventeen century with a chapter on Algebra: the second volume of the *Compendio dos Elementos de Mathematica* by the Jesuit Inácio Monteiro[8] published in 1756. The Chapter on Algebra is the last one of the two volume set (pages 299–343) and it is even more elementary than the *Tratado* of

[7] The full title is *L'Arithmetique en sa Perfection, mise en pratique selon l'usage des financiers, banquiers, et marchands. Contenant une ample et familiere explication de ses principes, tant en nombres entiers qu'en fractions. Avec Un Traite de Geometrie pratique appliquee a L'Arpentage et au Toise, tant des superficies que des Corps solides. Et Un Abrege d'Algebre, suivi de quantitr de Questions tres-curieuses,* by François Le Gendre.

[8] On Inácio Monteiro see[6], [9].

Pereira: Monteiro does not go beyond first degree equations! Nevertheless there are two major differences: besides the modern aspect of Monteiro's Algebra, he was well aware of the major works of his time and he recommended them to his readers. But the the *Compendio dos Elementos de Mathematica* was addressed to a very different public: the curious people that want to cultivate themselves, namely in Natural Philosophy and not, as Pereira book, to merchants, accountants, pilots and sailors.

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Research Seminar
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Mathematics and Architecture: from conception to constructing architectonic elements, through time

Ana Isabel Filipe
Departamento de Matemática e Aplicações
Universidade do Minho
afilipe@math.uminho.pt

Abstract

Mathematics and Architecture have been always related. The constructive methods are much related to mathematics since calculations have to be done. However, the design of the architectural element is as well very much related with mathematics, since the geometrical forms and proportions have meanings. In order that an architectural element to be constructed, it has to be planned. The project consists in defining the form, the materials to be used in construction and the methods to raise the element, for as long as historical documents and buildings show us.
All these phases include the mathematics thoughts and thus depend on the development of science.
In this article, some examples of architectural elements that demonstrate this relation will be shown. In particular, a last example on the "Hospital da Irmandade de Riba de Ave" will show, with some detail, the actual relationship that may still be needed between Architects and Mathematicians.

Keywords: Architecture and Mathematics, geometrical forms, proportions.

Dedicated to my friend Maria Fernanda, with thanks for her patience and enseignements.

1 Introduction

It is difficult to fully define Architecture, since it can be addressed to many different fields. Here we will be concerned with the architecture of edifices. So we will define architecture as the art and science of designing and erecting buildings and other physical structures. The roman architect Marcus Vitruvius (born c. 80-70 bc, died after c. 15 bc) is a reference in this field.

Vitruvius is the author of the famous treatise “De architectura”, known nowadays as *The Ten Books on Architecture*, written in Latin and Greek and about architecture; it was, by then, dedicated to the emperor Augustus. Vitruvius is particularly famous for asserting in his work that *in a structure must exhibit the three qualities of firmitas, utilitas, venustas* - that is, it must be solid (it should stand up robustly and remain in good condition), useful (it should be useful and function well for the people using it), beautiful (it should delight people and raise their spirits).

The first property, solid, is related with the construction, the other two properties, useful and beautiful, are related with the conception. The construction evolves the materials and the techniques used.

The conception is related to the usefulness and the beauty, but defines as well the materials and techniques to be used. From what was said it seems that it easy to design a building since each function has its form, it is far from being true. When a construction is projected for a specific function the architect introduces his dreams, ambitions and ideals. All this depends on the period and the society the individual is placed in. The achievement of his goals depends on the knowledge of the individual and of the society.

Still according to Vitruvius architecture is an imitation of nature, humans construct buildings, with natural elements, to give them protection against weather and predators. In a simple way we can consider that a construction consists in raising a platform above the soil level using walls, columns or pillars. It has been always the men aspiration to create spaces wider and higher but the materials and the knowledge of the epoch has been an obstacle to their aspirations. Some examples of constructions will be given that evidence the connection between the ideals and the knowledge of the epoch.

2 Egyptian pyramids

The Egyptian religion considered that life in earth was temporary but the spiritual life was eternal [Cole 2003]. Consequently the housing of the eternal life, tumuli, should last forever. This explains the reason of the effort they apply in the conception and construction of these structures. The pyramids, the tomb for the pharaoh, where constructed in stone in order to perpetuate, since was the housing for the eternal life. They believe that their form facilitate the approach to Ré, sun god, with whom the pharaoh should cross the sky. The examples of the pyramids, shown in the Figure 1, are presented in chronological order. It seems that the perfect desired form was achieved by trial error, what means that the knowledge was an obstacle to the aim goal.

Figure 1: Egyptians pyramids - Pyramid of Djoser, Bent Pyramid, Khufu-Pyramid.

The Pyramid of Djoser, or step pyramid is placed in the Saqqara necropolis, Egypt, northwest of the city of Memphis. It was built for the burial of Pharaoh Djoser by his vizier Imhotep, during the 27th century bc. The Bent Pyramid, located at the royal necropolis of Dahshur, approximately 40 kilometers south of Cairo, of Old Kingdom Pharaoh Sneferu, is a unique example of early pyramid development in Egypt, about 2600 bc. The lower part of the pyramid rises from the desert at a 55-degree inclination, but the top section is built at the shallower angle of 43 degrees, lending the pyramid its very obvious “bent” appearance. The Great Pyramid of Giza (also called the Pyramid of Khufu and the Pyramid of Cheops) is the oldest and largest of the three pyramids in the Giza Necropolis bordering what is now El Giza, Egypt. It is believed the pyramid was built as a tomb for fourth dynasty Egyptian Pharaoh Khufu (Cheops in Greek) and constructed over a 20-year period concluding around 2560 bc. Initially at 146.5 meters, the Great Pyramid was the tallest man-made structure in the world for over 3,800 years, unsurpassed until the 160-metre-tall spire of Lincoln Cathedral was completed 1300, the longest period of time ever held for such a record. Originally, the Great Pyramid was covered by casing stones that formed a smooth outer surface. Some of the casing stones that once covered the structure can still be seen around the base. The sides of the square base are closely aligned to the four cardinal compass points based on true north, not magnetic north, and the finished base was squared to a mean corner error of only 12 seconds of arc. The completed design dimensions, as suggested by Petrie's survey and subsequent studies, [Wikipedia] are estimated to have originally been 280 cubits high by 440 cubits long at each of the four sides of its base. The ratio of the perimeter to height of 1760/280 cubits equals to 2π to an accuracy of better than 0.05% (corresponding to the well-known approximation of π as 22/7). Some Egyptologists consider this to have been the result of deliberate design proportion. Verner wrote, "We can conclude that although the ancient Egyptians could not precisely define the value of π, in practice they used it" [Verner 2003].

3 The Parthenon

Ancient Greece is the civilization belonging to the period of Greek history lasting from the archaic period of the 8th to 6th centuries bc to 146 bc and the Roman conquest of Greece after the Battle of Corinth. The art of ancient Greece has exercised an enormous influence on the culture of many countries from ancient times until the present.

Pythagoras, Euclid, and Archimedes were Greek mathematicians which made important developments in the mathematics field, including the basic rules of geometry, the idea of formal mathematical proof, and discoveries in number theory, mathematical analysis. Pythagoras' religious and scientific views were, in his opinion, inseparably interrelated. For him all could be interpreted by numbers.

The temple was the most common and best-known form Greek public architecture. The temple function was to serve as a storage place for the treasury associated with the cult of the god in question, as the location of a cult image, and as a place for devotees of the god to leave their votive offerings. The inner room of the temple, the cella, served mainly as a strong room and storeroom. Therefore, the temple was not to be visited in the sense of the temple of modern church, the altar stood under the open sky, often directly before the temple. This means that the temple is to be seen and not to be visited. The ideal of beauty does not allow the use the arch in temples. The materials to be used have to be the stone. The function associated with the ideal of beauty explains its form.

Figure 2: Parthenon.

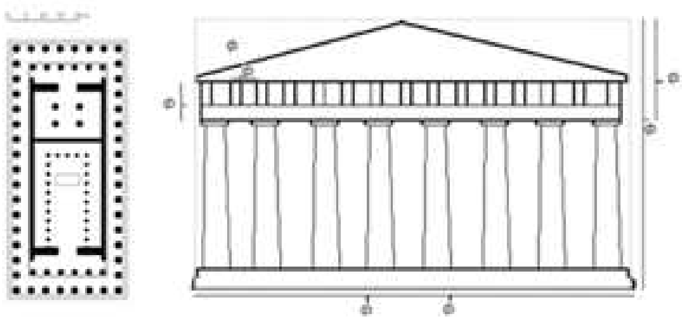


Figure 3: Parthenon; floor plan and facade scheme.

The Parthenon, temple dedicated to the Greek goddess Athena, placed in the Athenian Acropolis, Greece. Its construction began in 447 bc and was completed in 438 bc. In the Figure 3, is represented some of the mathematics involved in the planning of the temple. The floor plan of the Parthenon presents various symmetries; in fact the rear door only exists in order to give a full symmetry. The proportion of the Parthenon's facade as well as its elements can be circumscribed by golden rectangles (Φ). Although Greek do not knew the rules of perspective, the distances between columns were not equal in order to correct the perspective when view from far.

4 The Pantheon of Rome

Ancient Rome was a civilization that grew out of a small agricultural community, founded on the Italian Peninsula as early as the 10th century bc. Located along the Mediterranean Sea, and centered at the city of Rome, it became one of the largest empires in the ancient world.

The Architecture of Ancient Rome adopted the external Greek architecture for their own purposes; however the Romans didn't feel restricted by Greek aesthetic axioms, creating a new architectural style.

Social elements such as wealth and high population densities in cities forced the ancient Romans to discover new (architectural) solutions of their own. The use of vaults and arches together with a sound knowledge of building materials, for example, enabled them to achieve unprecedented successes in the construction of imposing structures for public use. Examples include the aqueducts of Rome, the Baths of Diocletian and the Baths of Caracalla, the basilicas and Coliseum.

Figure 4: The Pantheon Rome - outside view, model.

Political propaganda demanded that these buildings should be made to impress as well as perform a public function, free from the Greek aesthetic axioms and the invention of concrete, amongst other, allows them to fulfill their goals. The Pantheon of Rome, Figure 4, is a supreme example of this, particularly in the version rebuilt by Hadrian in about 126 ad.

The building is circular with a portico, of three ranks of huge granite Corinthian columns under a pediment, opening into the rotunda. A coffered concrete dome, with a central opening (oculus) to the sky covers the entire rotunda. The oculus is the only opening that gives light to the interior, being 9 meters wide. The height to the oculus and the diameter of the interior circle are the same, 43.3 meters, so the whole interior would fit exactly within a cube, also, the interior could house a sphere 43.3 meters in diameter.

The dome represents the geocentric universe as described by Ptolemy. In the center is placed the sun, represented by the oculus, surrounded by the trajectories of the five planets, represented by the five rows of sunken panels (coffers) [Stierlin 1997].

Figure 5: The Pantheon Rome - interior view, dome from interior.

Placed in seven apses used to be the seven gods linked to the worship of planets, or considered to be such: the Sun, the Moon, Venus, Saturn, Jupiter, Mercury and Mars. Pantheon means temple to all gods.

The massive dome required an advanced technology for the time, it was the wider dome built for long centuries. In fact, it is only compared with Brunelleschi's 42 meter dome of Santa Maria del Fiore in Florence, completed in 1436. The coffers help to diminish the height of the dome.

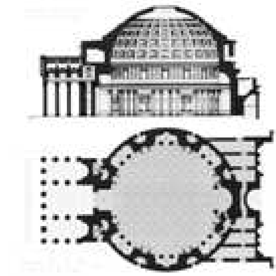


Figure 6: The Pantheon Rome - Floor plan and transversal plan.

5 Cathedral Hagia Sofia

Flavius Valerius Aurelius Constantinus (272-337), commonly known as Constantine I, was Roman Emperor from 306 to 337, best known for being the first Christian Roman emperor. Constantine transformed the ancient Greek colony of Byzantium into a new imperial residence, Constantinople, which would be the capital of the Eastern Roman Empire for over one thousand years. Among the several changes he set to the capital it was the cathedral of Hagia Sophia, Church of the Holy Wisdom.

Figure 7: Cathedral of Hagia Sophia.

On 23 February 532, only a few days after the destruction of the second basilica, Emperor Justinian I elected to build a third and entirely different basilica, larger and more majestic than its predecessors, that served as the cathedral of Constantinople and the spiritual heart of Eastern Christianity until the final extinction of Byzantium in 1453. It is also regarded as the greatest monument of Byzantine architecture.

Justinian chose physicist Isidore of Miletus and mathematician Anthemius of Tralles as architects.

Anthemius of Tralles (474 - before 558) was a Greek professor of Geometry in Constantinople and architect. He described the string construction of the ellipse and he wrote a book on conic sections, which was excellent preparation for designing the elaborate vaulting of Hagia Sophia. He compiled a survey of mirror configurations in his work on remarkable mechanical devices.

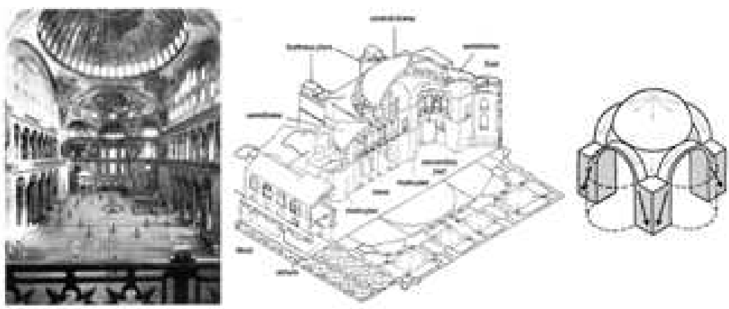


Figure 8: Cathedral of Hagia Sophia, interior view and structural schemes.

The first Hagia Sophia was a typical Roman basilica, timber-roofed and entered through an atrium. It was virtually destroyed by fire in 532. The emperor Justinian I immediately ordered the construction of the present building, which was substantially complete by 537, when it was dedicated, although work continued until 563. The architects, Anthemios of Tralles and Isidorus of Miletus, designed the new cathedral as a huge, almost square interior surmounted by a vast central dome. The dome, which rises some 56 m from ground level, appears to be dramatically poised over a circle of light radiating from the windows that pierce the drum on which it rests. Four curved triangles, or pendentives, support the rim and are in turn locked into the corners of a square formed by four huge arches. The transition between the circular dome and the square base of the building, achieved through the use of the pendentives, was a major advance in building technology. To the east a vast semi-dome surmounts the three large vaulted niches of the sanctuary below. Arcades that recall the arcaded naves of basilica churches occupy the ground level on the northern and southern sides of the central square [Parkyn 2002].

6 Cenotaph for Isaac Newton

Neoclassicism was a movement that began after 1765, as a reaction against both the surviving Baroque and Rococo styles, and as a desire to return to the apparent purity of roman and Greek arts.

Étienne-Louis Boullée (1728-99) was a visionary French neoclassical architect whose work greatly influenced contemporary architects and is still influential today. It was as a teacher and theorist at the École Nationale des Ponts et Chaussées between 1778 and 1788 that Boullée made his biggest impact, developing a distinctive abstract geometric style inspired by Classical forms. His work was characterized by the removal of all unnecessary ornamentation, inflating geometric forms to a huge scale and repeating elements such as columns in huge ranges. Boullée's fondness for grandiose designs has caused him to be characterized as both a megalomaniac and a visionary.

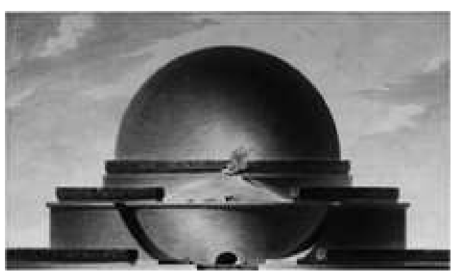


Figure 9: Boullee's Cenotaph for Isaac Newton.

His style was most notably exemplified in his proposal for a cenotaph for the English scientist Isaac Newton, which would have taken the form of a sphere 150 m high embedded in a circular base topped with cypress trees. Though the structure was never built, its design was engraved and circulated widely in professional circles. Boullee's Cenotaph for Isaac Newton is a funerary monument celebrating a figure interred elsewhere; the hollow sphere foreshadows the death of Newtonian Physics. For Boullée symmetry and variety were the golden rules of architecture, he considered the sphere the perfect form, since no perspective rule could alter its look [Janson 1998].

7 Hospital da Irmandade de Riba de Ave

The work presented here was designed by the architect Jorge Pinheiro Rodrigues in 2009. In this building of linear forms an element of curvilinear form is attached. From the outside of the building the element can be perceptible, in fact it is placed in the open area inside the edifice, s shown in Figure 10. The architect pretended that this element symbolizes the beginning of life, what gives a new message to an hospital.

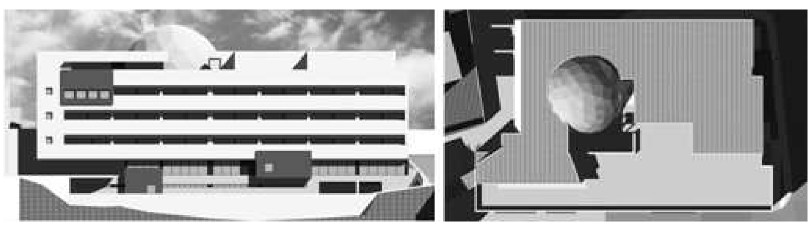
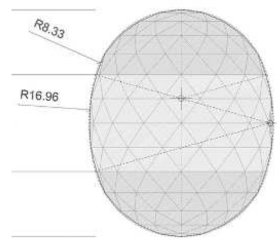


Figure 10: Hospital da Irmandade de Riba de Ave, external view, aerial view.

The beginning of life should be an element of prestige and so the construction should reflect it, the construction element should be high tech. So new forms need new technology, and the architect feels the need of a mathematician to help solving the arisen problems.

The architectonic element form it is composed by the junction of two spherical caps by a revolution surface. The radius of the spherical caps is 8,33 m. The joining revolution surface is defined by the rotation of a circular segment of radius 16,96 and the rotation axis is coincident with the axis of the spherical caps. Only the superior spherical cap is complete and self supported, the remaining part of the form is attached to the building, in fact only about a quarter of the element is constructed. Two aims were taken in consideration to create such a surface: i) to be build with GRC pre-constructed elements, ii) to facilitate the construction in site.



The use of pre-constructed elements enforces the possibility to transport and establish the maximum size of the elements. With this aims the problem of how to divide the surface in a number of elements that facilitate the construction in site arises, and the solution is to find the smallest number of different elements possible. Since part of the total surface is a spherical cap, then the Buckminster Fuller Dome was a starting point. But some modifications are due to the fact that the total surface is a junction of two different types of surfaces (sphere and a revolution surface) and the fact that the minimum of different elements were required. Thus, the surface is sliced in fourteen rings, with a size that each constructive element (triangle) could be carried out into the construction site. Since the curvilinear element is symmetric, thus the seven first rings are symmetrical to the last seven rings. The first three top rings follow the Buckminster Fuller Dome idea. The first ring is divided into 5 equal triangles, the second ring is formed by fifteen triangles, being of three types, and third top ring has twenty five triangles, of five types. This ring is the one that connects to the new structure type. The next four remaining rings are composed by 30 triangles of two types in each row. Next figure explain the assemblage of this triangles in order to form the surface.

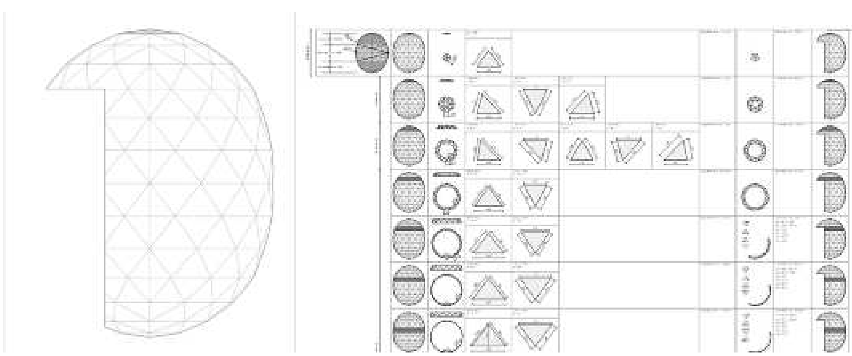


Figure 11: Hospital da Irmandade de Riba de Ave, structural scheme.

8 Conclusion

The examples presented show that the form and the function are related, but above all the form is a function of the ideals of the epoch and of the architect.

In order to be able to fulfil their goal technology has to be developed. Mathematics is involved in the technology, but as well in the definition of the forms. Regular solids are preferred since they translate the perfection that is desired.

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Research Seminar
on History and Epistemology of Mathematics, 59–72

It all began with publications in Teixeira's Journal: some remarks on August Gutzmer

Helmuth R. Malonek
Departamento de Matemática
Universidade de Aveiro
hrmalon@ua.pt

Vera I. Kharlamova
Departamento de Matemática
Universidade de Aveiro
vera.kharlamova@ua.pt

Abstract

This article emphasizes some moments in the life of the German mathematician August Gutzmer (1860-1924), who started his scientific career 1887 with 3 publications in the *Jornal de Sciências Matemáticas e Astronómicas* (Teixeira's Journal) created by the famous Portuguese mathematician Francisco Gomes Teixeira in 1877, and continued to publish therein until the volume of 1897. A brief analysis of the correspondence between both scientists makes clear that among all foreign authors of this journal Gutzmer played a singular role as link between Gomes Teixeira and Germany, probably also motivated by some coincidence of interests and areas of their academic work. Becoming later the successor of Georg Cantor at the *Vereinigten Friedrichs-Universität Halle-Wittenberg* and a closed collaborator of Felix Klein, Gutzmer kept contact with Gomes Teixeira until the end of his life and contributed significantly to the recognition of the Portuguese mathematician in Germany.

Keywords: August Gutzmer, Teixeira's Journal, international scientific collaboration in the 19th century.

Dedicated to Fernanda Estrada on the occasion of her 80th birthday.

Introduction

Today we understand better than ever that Mathematics can only be successfully developed in an open world, although we know that in the past as well as in the present Mathematics also depends on the general political situation. For example, the problems between French and German mathematicians after 1871 and again between World War I and World War II are well known ([PR2002]). The distortions in the international scientific cooperation as consequence of the Cold War, particularly in the field of Mathematics, are more recent and a comprehensive analysis as part of research in the History of Mathematics is still missing.

Nevertheless, looking in detail after some concrete examples of international cooperation in the past, the picture can sometimes become different. Of course, the therefore needed search for historical sources in different countries is very difficult. But, if available, those sources often show us that against all the difficulties in communication and all the political frontiers the exchange of scientific results or questions of academic character continued to flow due to the civil courage and scientific devotion of the involved actors. Two of the most famous mathematicians of all times, the French Henri Poincaré and the German Georg Cantor, essentially contributed at the end of the 19th century to the new foundation of a mutually respected relationship between French and German mathematicians, promoting also the organization of the first International Congress of Mathematics 1894 in Zurich. It is known that the choice of this city in a neutral country and not Paris or Berlin was not by chance, but the expression of real existing political conflicts. The same happened again in 1932, when the shadow of Hitler's cohorts already poisoned the international cooperation. A well known example of the return to a friendly relationship between France and Germany after World War II was given by the French Henri Cartan and the German Heinrich Behnke[1]. In 1949 Cartan was the first French mathematician after 1945, who visited a German university by invitation of Behnke in Münster ([GR1981]).

But the History of Mathematics during the 19th century was not only

[1]Behnke served from 1955 until 1958 as President of the Executive Committee of the International Commission on Mathematical Instruction (ICMI). ICMI was established at the Fourth International Congress of Mathematicians held in Rome in 1908. The founding President of ICMI was the distinguished German mathematician Felix Klein (1849-1925), for whom mathematics education was a deep and career-long interest.

written by such giants like Cauchy, Gauss, Riemann, Weierstrass or the already mentioned Poincaré and Cantor. Too often we tend to forget, that for a comprehensive picture also the contributions of those mathematicians are of importance, which have been disciples of this group of sounding names or simply followed in their steps as part of the main current of that time. Their work widened the new theories and methods, extended the fields of applications and helped to transform all in, what we call now, the common knowledge of mankind.

It is well known, that in the 19th century first Paris, than Berlin, and later also Göttingen developed into, proverbially saying, Meccas of Mathematics with pilgrims from whole Europe and also the United States. An astonishing number of them became later on the leading figures in their own countries. But, of course, it happened also in geographical distance to those centers, that very remarkable mathematicians grew up, based on their own talent, the help of dedicated teachers, and, may be, nearby a well equipped library. The Portuguese mathematician Francisco Gomes Teixeira bears therefore eloquent witness.

Sometimes it seems difficult to estimate the true scientific value of the work of those scientists from countries in the periphery of Europe, particularly if they stood alone as exceptional figures and their recognition in their own country is mainly based on fame from abroad, confirming the proverb, that the prophet does not have value in its own country. If they are not forgotten at all and gained some name also in other spheres of action more closed to the general understanding, then they are sometimes considered as national heros. But does this mean that doubts about their real merits are vanishing for ever? It seems to be not so. In this aspect Gomes Teixeira is not an exception. We feel that the singular role of this Portuguese mathematician from the end of the 19th and beginning 20th century, whose scientific work is world wide known, is often put in question. Of course, here we are not able to discuss all aspects of this problem. First of all it would be necessary to agree about the meaning of *scientific work*, which is also now a hot theme in the everyday discussion about the role of sciences in the modern society in general and the role of Universities, in particular. Based on a common understanding of scientific work as work doing research, teaching and educating on university level, or being responsible for other academic duties like acting as editor of journals or writing books, the main problem will be to understand him today without prejudices, but really as person working under the specific conditions of his time and under the limitations of a peripherally country.

A possible approach to answer these questions could be by means of comparative studies with foreign mathematicians of the same time. The well documented ([V1935]) contacts of Gomes Teixeira with a large spectrum of foreign mathematicians of similar scientific interests seem to be very useful for such attempts. As example, in this short note we report about the life and work of August Gutzmer, who *started his scientific*

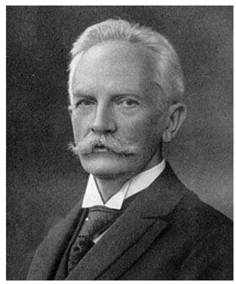


Figure 1: August Gutzmer 1860-1924

career 1887 with 3 publications in the *Jornal de Sciências Matemáticas e Astronómicas.* The simple fact, that this happened with a mathematician from a central country of Europe could, in our opinion, already create some curiosity. On the other hand it also confirms, together with other cases, that Teixeira's Journal really found international recognition after some years. But after some detailed study we found many indicators for the fact that Gutzmer has been the most important person for Teixeira's contacts to German mathematicians. Even more, the activities in which Gutzmer became involved later on show several parallels with Teixeira's live. Directly, like in the case of Teixeira, and indirectly, like in the case of Gutzmer, both contributed by several activities to the development of a European community of mathematicians at the end of the 19th, beginning of the 20th century. We try here in this short note to stress this comparative point of view on one man from a central country and one from a peripherally country, without being able to go into details, particularly what concerns the life of Gomes Teixeira which we consider as well known in its main parts. In particular, we are not able to consider here the actions of both mathematicians under the conditions of the general political development in Europe, as it should be. We mentioned in the beginning the problems between French and German mathematicians for showing that also mathematics does not go on in isolation from the surrounding world. Here we remember only, that Gutzmer and Teixeira most probably have been in contact since 1886 until 1924, which was a time - besides the disastrous time caused by World War I with all its problems before and after, marked in both countries by the Industrial Revolution in the 2nd half of the 19th century. What concerns particularly Portugal, we have to remind the time of the *Fontismo* between 1886 and 1889, which, in general, is characterized by the attempt to modernize the public sector and specially the infra-structures of Portugal in continuation of the *Regeneration* period. The construction of railways and roads, the installation of a well working post service and other measures by the government of Fontes Pereira de Melo had the objective to overcome the backlog and stagnation of Portugal compared with other European countries. It is evident, that these new possibilities of a rapid and secure communication have been an indispensable condition for the success of Teixeira's Journal as one of the few well recognized mathematical journals edited in a peripherally country of that time. We can also not mention the merits of Gomes Teixeira's participation as representative of Portugal in several European projects which emerged in the second half of the 19th century. We mention only that Teixeira maintained an extensive correspondence with the German E. Lampe , who served from 1885 until 1918 as editor of the *Jahrbuch über die Fortschritte der Mathematik* (1868-1942)[2], and participated actively in the work of the *International Congress on Bibliography of Mathematical Sciences* in 1889 under the direction of H. Poincaré with the aim of founding the *Répertoire Bibliographique des Sciences Mathématiques*(1894-1912).

August Gutzmer

August Gutzmer was born 150 years ago at 2nd of February 1860 in Neu-Roddahn, a small village in the north of Prussia, situated closed to Schwerin, the capital of the Herzogtum Mecklenburg-Schwerin. His father was a carpenter. In 1868, his family went to Berlin where he attended the Friedrichswerdersche Gymnasium and finished the high-school in 1881 without having studied Latin or other foreign languages. That's why he

[2]The well known *Zentralblatt* is considered as its successor.

could not obtain together with his final examination the university entrance permission. Nevertheless he succeeded to attend mathematics lectures at Berlin University, but was officially not registered as a student. Only after the study of Latin with a private teacher and the corresponding examination in 1884 he could finally be matriculated at the end of April of 1884. Among those who lectured to Gutzmer were L. Kronecker, K. Weierstrass and L. Fuchs. With the retirement of Kummer, replaced by Fuchs in 1883, the "golden period" for mathematics in Berlin had ended. Kronecker died in 1891 and H. A. Schwarz succeeded Weierstrass in 1892. Gutzmer, therefore studied with two of the three great Berlin mathematicians in the last years of their careers. He finished his studies in Berlin in 1887 and began his doctorate under the supervision of Albert Wangerin (1844-1933) at Halle-Wittenberg. He submitted his theses *On certain partial differential equations of higher order* to the University of Halle-Wittenberg and was awarded his doctorate on 13 January 1893. The fact that up to this moment already five publications of Gutzmer appeared in Teixeira's Journal (in the volume VIII, IX and X; [G1886a], [G1886b], [G1887], [G1889], [G1890]) permits at least two conclusions. First of all, Gutzmer started serious research before he finished the University, the same as Teixeira had done following the advise of Daniel de Silva ([V1935]). Secondly, he knew about the existence of Teixeira's Journal and did not hesitate to submit his articles, written in French, to a Journal edited in Portugal, far away from Germany. The analysis of the foreign authors in Teixeira's Journal indicates some probable reasons for his decision. M. Lerch, later also becoming a very active and famous mathematician, was also a student of Kronecker and Weierstrass at almost the same time. He published already in volume VII the article *Remarque sur la théorie des séries.* As a remark to this paper Gutzmer sent his first paper, entitled *Sur une série considérée par M. Lerch.* Remarkable that also the well known Italian mathematician E. Cesaro after a note in volume VI has published in volume VII his paper *Remarques sur la théorie des séries.* It is well known, that Teixeira himself was very much interested and working in the theory of series. Without going too far, we believe that all this shows how Teixeira's Journal had already earned some international recognition in this field, very actual in that time. In this context we would also like to mention the general analysis concerning *Portuguese Mathematical Journals: Some Aspects of (almost) Periodical Research Publications* by J. F. Rodrigues ([R2004]), which relies only on some well known foreign authors of Teixeira's Journal. Among a very reasonable number of foreign authors Lerch occupied the first place with 8 papers, followed by Ocagne with 7, Gutzmer with 6, and Cesaro with 5 papers. But let us come back to Gutzmer. After a short interruption for one year as administrator of a manor of his wife, the couple sold the manor and moved to Berlin. Gutzmer became an assistant at the Technischen Hochschule of Charlottenburg. Being in the army between October 1894 and January 1895, but dismissed as being unfit for duty, he returned to his position at the Technischen Hochschule. In 1896 Gutzmer already was able to submit his habilitation thesis *On the theory of adjoint differential equations* to the University of Halle-Wittenberg and worked there as a Privatdozent until March 1899. From that time on he was appointed as professor of mathematics at the University of Jena, holding this position until 1905. Nevertheless, the contacts to Albert Wangerin and also to Georg Cantor[3] as one of the most influential mathematicians

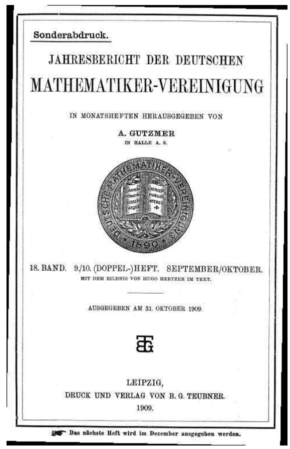
Figure 2: Georg Cantor (1845-1918)

from Gutzmer's former University Halle-Wittenberg led naturally to his support of the German Mathematical Society. He produced for the German Mathematical Society yearly reports, for example about the annual meeting in

München, 17-23 September 1899; the annual meeting in Aachen, 16-23 September 1900; and also in Breslau, vom 18 bis 24 September 1904.

[3]Georg Cantor was the first president of the German Mathematical Society (DMV) and served as its President for three years until 1893.

These activities culminated in the first document on its history, written by Gutzmer ([G1909]).


Sonderabdruck.
JAHRESBERICHT DER DEUTSCHEN
MATHEMATIKER-VEREINIGUNG
IN MONATSHEFTEN HERAUSGEGEBEN VON
A. GUTZMER
IN HALLE A. S.

18. BAND. 9./10. (DOPPEL-)HEFT. SEPTEMBER/OKTOBER.
MIT DEM BILDNIS VON HUGO HERTZER IM TEXT.
AUSGEGEBEN AM 31. OKTOBER 1909.
LEIPZIG,
DRUCK UND VERLAG VON B. G. TEUBNER.
1909.
Das nächste Heft wird im Dezember ausgegeben werden.

Figure 3: Frontispiece of the Jahresbericht

Finally, on 3rd of August 1905 Gutzmer was appointed to an additional full professorship for mathematics at Georg Cantor's chair. He had to substitute Cantor's lectures in case of Cantor's illness. In 1911 Georg Cantor gave his last lecture and in 1913 Gutzmer succeeded on Cantor's chair.

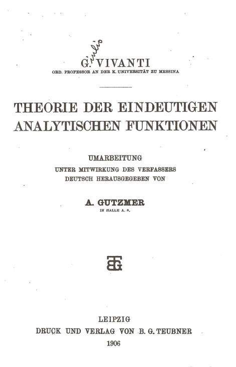
Figure 4: Felix Klein (1849-1925)

Already one year before he was elected as President of the *Commis-*

sion for Mathematical and Scientific Instruction from 1904 to 1907 and President of the German sub-committee of ICMI from 1908 to 1913, in closed contact with the President of ICMI, Felix Klein ([G1908], [G1912], [G1914], [G1917]).

Already on the occasion of his visit to Heidelberg for participating in the Third International Congress of Mathematicians Gomes Teixeira had the opportunity to meet both of them personally ([G1904]).

Remarkable that Teixeira and Gutzmer shared also another field of academic activity. Gomes Teixeira became Rector of the new created University of Porto in 1911 and Gutzmer shared four years later the same experience. During session 1914-1915 he was Rector of the University Halle-Wittenberg. He was asked to continue as rector for a second session, but because at this time his health was deteriorating he declined.


G. VIVANTI
ORD. PROFESSOR AN DER K. UNIVERSITÄT ZU MESSINA

THEORIE DER EINDEUTIGEN
ANALYTISCHEN FUNKTIONEN

UMARBEITUNG
UNTER MITWIRKUNG DES VERFASSERS
DEUTSCH HERAUSGEGEBEN VON

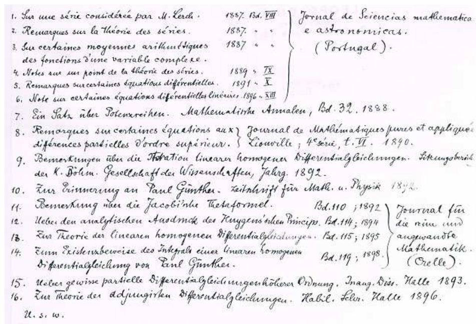
A. GUTZMER
IN HALLE A. S.

LEIPZIG
DRUCK UND VERLAG VON B. G. TEUBNER
1906

Figure 5: The famous Vivanti-Gutzmer textbook

Gutzmer was highly gifted as a teacher, showing infectious enthusiasm for mathematics. He taught courses on differential and integral calculus, analytic geometry, ordinary differential equations, analytic mechanics, calculus of variations, number theory, higher algebra, function theory, and the theory of algebraic curves. He did not wrote a textbook from the beginning alone as Teixeira did, but the book of G. Vivanti on the *Theory of Analytic Functions*, edited in German after a revision in collaboration with the author by Gutzmer became an excellent and unique textbook in the beginning of the 20th century, highly appreciated by students and researches due to the first appearance of set theoretic basics in the function theoretic context.

Being elected to the German Academy of Scientists Leopoldina in 1900, Gutzmer sent his curriculum vitae which proves very clearly how he estimated his first six publications in Teixeira's Journal.



1. Sur une série considérée par M. Lerch. 1887. Bd. VIII
2. Remarques sur la théorie des séries. 1887. " "
3. Sur certaines moyennes arithmétiques des fonctions d'une variable complexe. 1887 " "
4. Notes sur un point de la théorie des séries. 1889 " IX
5. Remarques sur certaines équations différentielles. 1891 " X
6. Note sur certaines équations différentielles linéaires 1896 " XIII

Jornal de Sciencias mathematicas e astronomicas. (Portugal).

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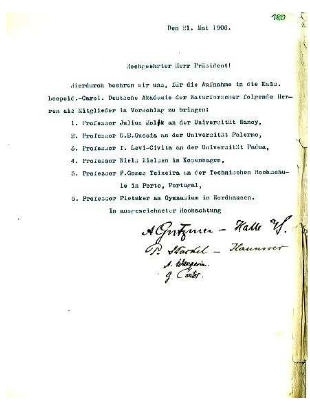
16. Zur Theorie der adjungirten Differentialgleichungen. Habil. Schr. Halle 1896.

u. s. w.

Figure 6: Part of Gutzmer's CV

In 1906 and together with Georg Cantor, Albert Wangerin, and P. Stäckel from the University of Hannover, he proposed the election of Francisco Gomes Teixeira to this famous academy, together with T. Levi-Civita, for example.



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Den 21. Mai 1906.

Hochgeehrter Herr Präsident!

Hierdurch beehren wir uns, für die Aufnahme in die Kais. Leopold.-Carol. Deutsche Akademie der Naturforscher folgende Herren als Mitglieder in Vorschlag zu bringen:

1. Professor Julius Molk an der Universität Nancy,
2. Professor G.B. Guccia an der Universität Palermo,
3. Professor T. Levi-Civita an der Universität Padua,
4. Professor Niels Nielsen in Kopenhagen,
5. Professor F. Gomes Teixeira an der Technischen Hochschule in Porto, Portugal,
6. Professor Pietzker am Gymnasium in Nordhausen.

In ausgezeichneter Hochachtung

A. Gutzmer – Halle a/S.
P. Stäckel – Hannover
A. Wangerin.
G. Cantor.

Figure 7: Letter supporting the election of G. Teixeira

Like before Wangerin also Gutzmer has later on in 1921 been elected as President of this academy ([Sch1998], [Sch1999]), a position which recognized and crowned his dedication to all areas of his scientific work.

Final remarks

Our aim was to call attention to some facts in the live of the German author of Teixeira's Journal August Gutzmer, which could be useful for a comparative study of the work of Francisco Gomes Teixeira with other foreign mathematicians, i.e. on international level. Of course, we are conscious that we mentioned only few details, among them those which are directly related to Gomes Teixeira and based on some unknown material, available only in German. That's why we consider this short remarks only as the beginning of a longer lasting research on aspects in the live of Gomes Teixeira, not studied so far in detail. It was a great pleasure to got the opportunity for publishing it in honor of our colleague Fernanda Estrada.

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Research Seminar
on History and Epistemology of Mathematics, 73–90

Construction of a numerical system for the octave according to Jean Edme Gallimard in *"Aritmétique des musiciens"*

Ana Pereira do Vale[1]
Centro de Matemática
Universidade do Minho
avale@math.uminho.pt

Abstract

The determination of the frequencies of the pitches in musical scales has been a problem that goes back to ancient Greece. The problem is in itself simple. In modern language it could be stated as follows: if you have an octave *i.e.* two pitches defined such that the frequency of one pitch is twice the frequency of the other, what values should have the frequencies of the pitches in between them in order to have a musical scale? A musical scale is considered, within this article, to be a set of pitches whose frequencies satisfy predefined proportions. There are many possible systems but there is not a solution that fulfills all the demands of the problem. Gallimard presents a solution through a numerical system defined with progressions and tempered with an accuracy possible by the use of logarithms.

Keywords: arithmetic, geometric and harmonic progressions, musical intervals, musical scales, musical temperament, logarithm.

[1] Work supported by the Research Centre of Mathematics of the University of Minho through the FCT Pluriannual Funding Program.

Maria Fernanda ensinou-me História da Aritmética, retribuo com um pouco de História da Música.

1 Introduction

The definition of proportions between the frequencies of the notes in musical scales in western music is related with the study of the vibration of chords. If two vibrating chords are absolutely equal except in length, and one is half as long as the other, the long one will produce a pitch that is one octave lower than the shorter one. If we want to state this fact through the frequencies of the sounds we would have the inverse, *i.e.*, the frequency of the higher pitch is twice the frequency of the lower. The study of the vibrating chords made it possible to determine the ideal proportion (pure intervals) between the frequencies of the pitches (and the length of the chords that produce them). The pitches that form a musical scale in western music should have frequencies with values such that any two of them played simultaneously should produce a pure interval. All musical subjects in [Gallimard1754] follow the work of [Rameau1750]. For further details the reader should also see [D'Alembert1772] and [Smith1749]. The list of pure intervals can be found on Table 1.

proportion	musical interval	example
1	unisonus	C - C (do - do)
9/8	second	C - D (do - re)
5/4	third (major)	C - E (do - mi)
4/3	fourth	C - F (do - fa)
3/2	fifth	C - G (do - so)
5/3	sixth	C - A (do - la)
15/8	seventh	C - B (do - ti)
2	octave	C - C' (do - do')

Table 1: Pure intervals (the value of the pitch of C' is twice the one of C).

Table 2 lists the musical notes that correspond to the first overtones of C. The numerical index in each note specifies the octave. For example D_4 is the note D four octaves higher that the note D_1.

The rules of composition allow that even when the tonality is chosen

1 C_1	2 C_2	3 G_2	4 C_3	5 E_3	6 G_3	7	8 C_4	9 D_4	10 E_4
11	12 G_4	13	14	15 B_4	16 C_5	17 $C_5\sharp$	18 D_5	19 $D_5\sharp$	20 E

Table 2: C overtones.

(which means essentially the scale used), there is a certain amount of freedom to use other scales that are near to the one chosen. Near means that the new scale will have few notes (one or two) that are different from the original one[2].

For example if the scale chosen is C major, then Gallimard admits that the composer may also use the notes of major scales of G , F and B♭ and from the minor scales G, D, A and E. If we look at the notes that form these scales we see that to play a nice music written in C major we would probably need thirteen notes C, C♯, D, D♯, E♭[3], E, F, F♯, G, G♯, A, B♭, B and C. For further details on scales see [Gallimard1754], [Rameau1750], [Smith1749] or [D'Alembert1772]. It is important to notice that Gallimard only uses melodic minor scales.

Gallimard will define a numerical system associating to each note a number. The proportions defined by these numbers would have to obey the proportions of pure intervals. This will allow the precise definition of the length of the chord that will produce the musical note the player wants.

2 Notations and definitions

Definition of **interval**, “**relation**” (French) or “**rapport**” (French) according to [Gallimard1754]. “When one compares two quantities considering in what way the small is contained in the large, their respective situation present to the mind some kind of “amplitude” (...) that changes according to how the small one is contained in the larger”; this amplitude

[2]C-D-E-F-G-A-B are the notes that form the C major scale. There are scales that are very similar to this one like A-B-C-D-E-F-G (A minor scale) or G-A-B-C-D-E-F♯ (G major scale).

[3]D♯ and E♭ are two different notes.

is what Gallimard call **interval** or "**relation** " or "**rapport**" between this two quantities.

A relation like 2 to 3 will be written as : $2 \bullet 3$. If one changes the order of the terms we get the inverse. The inverse will represent the relation between the lengths of the chords. From now on the word interval always designates this relation.

If two intervals are equal, *i.e.*, : $a \bullet b =: c \bullet d$ we obtain a **geometric proportion** and write will it as follows: $a \bullet b :: c \bullet d$. Each interval will be called a member of the proportion. If we have two intervals $a \bullet b :: b \bullet c$ then we can write $\div\!\!\div a \bullet b \bullet c$. It will represent the same as $a \bullet b :: b \bullet c$. This new sequence will be called a **geometric progression**.

A musical interval between two sounds is the interval (relation) between the two numbers that express this interval. For a complete list of the usual intervals please check the table in the appendix taken from [Smith1749]. Gallimard also gives a very complete table identifying the proportions with the musical intervals. His Table is not shown here because it too detailed and more difficult to understand.

Composition of intervals. If we consider an interval : $a \bullet b$ we can decompose it in several intervals : $a \bullet c$, : $c \bullet d$,..., : $z \bullet b$, called roots or parts of the composed interval.

Mathematically if we consider fractions these computations are clear, it all amounts to $\frac{b}{a} = \frac{c}{a} \times \frac{b}{c}$.

Let us look at composition of intervals in musical terms. Let us take, for example the interval : $1 \bullet 5$ and take the decomposition in the intervals : $1 \bullet 3$ and : $3 \bullet 5$.

These relations correspond to special musical intervals and this equality states that two octaves plus a major third is equal to an octave plus a perfect fifth plus a sixth. Or to put more easily if C-D-E-F-G-A-B-C-D-E-F-G-A-B-C-D-E is a sequence of two octaves plus a third to go from the first note to the last is the same as to go from C to the second G (C-D-E-F-G-A-B-C-D-E-F-G) and afterwards from G to E (G-A-B-C-D-E).

$$\underbrace{C-D-E-F-G-A-B-C-D-E-F-G-A-B-C-D-E}$$

is the same as

$$\underbrace{C-D-E-F-G-A-B-C-D-E-F-G} \text{ followed by } \underbrace{A-B-C-D-E}.$$

The multiplication becomes an addition on music.

For more details on this correspondence see the Table included in the appendix taken from [Smith1749] .

If we have a sequence of intervals like : $b \bullet a$, : $c \bullet b$, : $d \bullet c$ we can transform these intervals in a progression, using the rule described by Gallimard:

> *Pour renfermer distinctement les interv. qu'on voit en M dans une suite N, on formera le 3e. terme en N du produit des deux premiers conséquens divisé par le 2e. antécédent, et le 4e. terme du produit des trois conséquens divisé par celui des 2 derniers antécédents, et ansi à l'infinit, (un coup d'oeil remplit ici l'esprit de la regle de trois.*

Gallimard presents the rule but at the same time shows an example (see Table 3) to help the reader. Note how he calls the attention to the fact that this process is just an application of a simple rule of proportion he mentions earlier.

Here is Gallimard's example for the sequence of intervals : $5 \bullet 6$, : $4 \bullet 5$ and : $3 \bullet 4$ exactly how it is presented in Gallimard's text.

	: 5	$\bullet$	6				
M			: 4	$\bullet$	5		
					: 3	$\bullet$	4
N	5	$\bullet$	6	$\bullet$	$\frac{6\times5}{4}$	$\bullet$	$\frac{6\times5\times4}{4\times3}$

Table 3: Example given by Gallimard.

The sequence could be simplified and we would have:

$$5 \times 4 \times 3 \bullet 6 \times 4 \times 3 \bullet 6 \times 5 \times 3 \bullet 6 \times 5 \times 4$$

If two intervals are given : $a \bullet b : c \bullet d$ such that $(b-a) = (d-c)$ we will write $a \bullet b \therefore c \bullet d$. If $c = d$ we can write the **arithmetic progression**, $\div a \bullet b \bullet d$ meaning $a \bullet b \therefore b \bullet d$.

Let a b and c be numbers such that : $c \bullet a =: (b-c) \bullet (b-a)$ which is the same as $c \bullet a :: (b-c) \bullet b-a$. We will say the three numbers a, b and c make an **harmonic progression** and write $\backsim a \bullet b \bullet c$.

If $\div a \bullet b \bullet c \bullet d$ is an arithmetic progression it is easy to see that the sequence of the inverses of its members form an harmonic progression, *i.e.*, $\backsim \frac{1}{a} \bullet \frac{1}{b} \bullet \frac{1}{c} \bullet \frac{1}{d}$. This progression can also be obtained if we consider the intervals : $d \bullet c$, $c \bullet b$ and $b \bullet a$ and form a new sequence using the process described above.

It is important to remark that Gallimard gave a correct rule to obtain a harmonic progression from an arithmetic one. One of the most used books on theory of counterpoint called "Gradus ad Parnassum" written by Johann Fux has the following rule: if one takes an arithmetic progression $\div a \bullet b \bullet c$ one obtains an harmonic progression from this as follows $\backsim ab \bullet ac \bullet bc$. This only works if the ratio of the arithmetic progression is the unity. Since the arithmetic progression the musicians were interested in was the one mentioned above this rule worked.

3 Numerical system for an octave

Let us consider the following arithmetic progression $\div 1 \bullet 2 \bullet 3 \bullet 4 \bullet 5 \bullet 6...$ and the sequence of intervals : $6 \bullet 5$, : $5 \bullet 4$, : $4 \bullet 3$, ... that will generate the harmonic progression $\backsim 10 \bullet 12 \bullet 15....$

Let us now consider the geometric progression $\div\div\ 1 \bullet 3 \bullet 9$ on each of its terms we will construct an arithmetic progression similar to $\div 1 \bullet 3 \bullet 5$, and we will obtain $(\div 1 \bullet 3 \bullet 5)(\div 3 \bullet 9 \bullet 15)(\div 9 \bullet 27 \bullet 45)$.

In musical terms the geometric progression $\div\div 1 \bullet 3 \bullet 9$ is just a sequence of the intervals of fifth. The circle of fifths is a well known sequence in music (... $B\flat$ F C G D A E B $F\sharp$ $C\sharp$....). Gallimard just takes F C and G and produces over each of these notes the major chords.

1	3	9	F	C	G
3	9	27	C	G	D
5	15	45	A	E	B

Table 4: Progressions and musical notes 2.

Let us organize these numbers from the smaller to the larger we will get the sequence $1, 3, 5, 9, 15, 27, 45$. This musical scale is not all in the same octave. And what Gallimard does next is to rescale the notes to try to get all values within the interval : $1 \bullet 2$. He takes each one of these progressions $(\div 1 \bullet 3 \bullet 5)(\div 3 \bullet 9 \bullet 15)(\div 9 \bullet 27 \bullet 45)$ and multiplies the first one by the geometric progression $\div\div\ 16 \bullet 8 \bullet 4$, the second by $\div\div\ 8 \bullet 4 \bullet 2$

F	C	A	G	E	D	B
1	3	5	9	15	27	45

Table 5: Major scale 1.

and the third by $\div\div\, 4 \bullet 2 \bullet 1$. In doing so he reduces all the notes to the same octave and at the same time readjusts the numerical values to keep the proportions. He then obtains the Table 6:

F	A	C	E	G	B	D
16	20	24	30	36	45	54

Table 6: Major scale 2.

This Table will also give major chords of the numerical system. We will have C, E, G corresponding to ($\div 24 \bullet 30 \bullet 36$) F, A, C ($\div 16 \bullet 20 \bullet 24$), G, B, D ($\div 36 \bullet 45 \bullet 54$).

Notice that these notes are still not in the same octave, but if we take F and A one octave higher (multiply by 2), take D one octave lower (divide by 2) and consider another C one octave higher than the first C, we will obtain:

C	D	E	F	G	A	B	C
24	27	30	32	36	40	45	48

Table 7: Major scale (numerical system).

This Table corresponds to a numerical system for the major octave.

But it is not enough we still need more notes. Remember that for a C major scale the composer will have to be able to use thirteen notes. Gallimard does a similar process to obtain a numerical system for minor scales. Gallimard only considers melodic minor scales and this means, in the case of C minor for example, that we would have an ascending scale C, D, E♭, F, G, A, B, C and one descending scale C, D, E♭, F, G, A♭, B♭, C. The process is very similar to the previous one, but through an harmonic progression.

Let us consider the geometric progression $\backsim 5 \bullet 15 \bullet 45$ on each of its terms we will construct an harmonic progression similar to $\backsim 3 \bullet 5 \bullet 15$. The progression $\backsim 3 \bullet 5 \bullet 15$ represents a minor chord and the progression $\backsim 5 \bullet 15 \bullet 45$ continues to represent three intervals of fifth consecutive. So basically one takes three musical notes from the circle of fifths and over each one of them considers a minor chord. This produces the data in Table 8. If we multiply the numbers on Table 5 by 5 and join those with

A♭	F	E♭	C	B♭	G	D
3	5	9	15	27	45	135

Table 8: Minor scale (numerical system).

this last Table we will get Table 9.

A 25	E 75	B 225	
F 5	C 15	G 45	D 135
	A♭ 3	E♭ 9	B♭ 27

Table 9:

It is important to notice the numerical pattern we have here: From

25	75	225	
5	15	45	135
	3	9	27

Table 10:

left to right we have geometric progressions with ratio 3 from the bottom up we have geometric progressions of ratio 5. Let us expand Table 10 up and down and using the relations given in the Table in the appendix we will get Table 11[4].

Since he is only using progressions of ratio 3 and of ratio 5 he is actually only using two types of musical intervals the thirds and the fifths. It is

[4]Gallimard does not use fraction on this Table he just leaves the last row of the Table with only the names of the notes. He actually also does not fill the first row.

E $\sharp$ 625	B$\sharp$ 1875	F $\sharp\sharp$ 5625	C $\sharp\sharp$ 16875
C$\sharp$ 125	G$\sharp$375	D$\sharp$ 1125	A$\sharp$ 3375
A 25	E 75	B 225	F$\sharp$675
F 5	C 15	G 45	D 135
D$\flat$ 1	A$\flat$ 3	E$\flat$ 9	B$\flat$ 27
B$\flat$ $\frac{1}{5}$	F$\flat$$\frac{3}{5}$	C$\flat$$\frac{9}{5}$	G$\flat$$\frac{27}{5}$

Table 11:

important to remember here that these two intervals define the principal accord of a scale (major or minor depending on the type of third).

Merging columns and readjusting the notes multiplying the numbers by powers of two he actually obtains the lists of major and minor chords.

In order to reduce the notes in the previous Table to the same octave Gallimard multiplies the values for powers of two and obtains Table 12:

C$\sharp$ 125×8	G$\sharp$ 375×4	D$\sharp$ 1125×2	
A 25×32	E 75×16	B225×8	F$\sharp$ 675×4
F 5×128	C 15×64	G 45×32	D 135×16
D$\flat$ 1×512	A$\flat$ 3×256	E$\flat$ 9×128	B$\flat$ 27×64

Table 12:

If one multiplies the values for D$\flat$, F, A$\flat$ A by 2 and divide the values of D, D$\sharp$ and F$\sharp$ by 2, (this is the equivalent to adjust the notes taking them one octave higher or lower as needed) we will obtain what Gallimard calls the NUMERICAL SYSTEM FOR THE GENERAL OCTAVE OF C.

Having produced his system Gallimard organizes the notes in intervals of fifths and obtains the data organized in Table 14.

Looking at this Table one finds two geometric progressions. One with ratio 5 written in bold characters and the order with ratio $\frac{3}{2}$ that has four consecutive terms and interrupts at the fifth. Because of this behaviour the system has only one type of major third intervals but has two types of fifth intervals.

Gallimard will temper his system adjusting his defected fifth, dividing this difference (comma) through all four fifths. This adjustment is

2	C	1920
$\frac{15}{8}$	B	1800
$\frac{9}{5}$	B♭	1728
$\frac{5}{3}$	A	1600
$\frac{8}{5}$	A♭	1536
$\frac{25}{16}$	G♯	1500
$\frac{3}{2}$	G	1440
$\frac{45}{32}$	F♯	1350
$\frac{4}{3}$	F	1280
$\frac{5}{4}$	E	1200
$\frac{6}{5}$	E♭	1152
$\frac{75}{64}$	D♯	1125
$\frac{9}{8}$	D	1080
$\frac{16}{15}$	D♭	1024
$\frac{25}{24}$	C♯	1000
1	C	960

Table 13: Numerical system for the general octave

$\frac{75}{2}$	D♯	2250
25	G♯	1500
$\frac{50}{3}$	C♯	1000
$\frac{45}{4}$	F♯	675
$\frac{15}{2}$	B	450
5	E	300
$\frac{10}{3}$	A	200
$\frac{9}{4}$	D	135
$\frac{3}{2}$	G	90
1	C	60
$\frac{2}{3}$	F	40
$\frac{9}{20}$	B♭	27
$\frac{3}{10}$	E♭	18
$\mathbf{\frac{1}{5}}$	A♭	12
$\frac{2}{15}$	D♭	3

Table 14: Inner intervals

very precise and it is the reason he introduces logarithms. That subject has to be left for another occasion. It is important to notice that the temperament Gallimard is interested in here does not prevent C♯ and D♭ from being different notes. Most people think of temperament related to keyboard instruments where C♯ and D♭ are two notes that correspond to the same key, therefore are equal. He want to have coherence between intervals even with these different pitches.

Not satisfied with the results of the numerical system Gallimard obtains even after the adjustments, he tries other systems of temperament, either by dividing the octave into more parts (31 or 55 for example), the equal temperament, or even the Pythagorean temperament (leaving the intervals of fifth pure and adjusting the thirds).

4 Jean Edme Gallimard and *"Aritmétique des musiciens"*

Jean Edme Gallimard was born in 1685 in Paris and died on June the 12th, 1771 [Fetis1837], [Larousse1872], [Lacoarret1957]. According to both Larousse and Lacoarret, he must have dedicated part of his life to teaching. Lacoarret concludes through the analysis of his works that he must have thought children since he wrote a Latin method "*Alphabet raisonné*" (1757) that was never published, as well as some manuals like "*L' Arithmétique littérale démontré*" (1740) or "*Géométrie élémentaire d' Euclide*" (1746,1749). Lacoarret quotes Des Essarts in "*Les siècles Littéraires...*" Paris 1800-1803, Michaud, Hoefer for this data. Actually since her article is dedicated to the French translators of Euclides works, it was this later manual that called her attention to Gallimard. There are nevertheless other mathematical works mentioned in the "*Grand Dictionaire Universel XIX Siécle*" [Larousse1872] , like "*L'Algébre ou la science du calcul numérique*" (1750) or "*Méthode théorique et pratique d'arthmétique, d'álgébre et de géométrie*" (1753). None of this three works ([Fetis1837], [Larousse1872], [Lacoarret1957]) mentioned above includes"*Les sections coniques et autres courbes anciennes traitées profondement*" (1752) in their lists. However it is another work by Gallimard that is quoted in the XXI century. It can be found in digital format and is still available in old books sellers, but had a modern edition by Kessinger Pub Co. in 2009.

Fétis is a respected author in music. He wrote several major works that are still used as reference nowadays. In "*Biographie Universelle des Musiciens et Bibliographie Génerale de la Musique*" , Fétis includes Gallimard in the following terms:

> *GALLIMARD (Jean-Edme), mathématicien mediocre, né a Paris en 1685, morut dans la meme ville, le 12 juin 1771, à l'age de quatre-vingt-six ans. Il a publié un petit écrit sous ce titre: La théorie des sons applicable à la musique, où lón démontre, dans une exacte precision, les rapports de tous les*

> *intervalles diatoniques et chromatiques de la gamme, Paris, 1754 in 8^o, 16 pages.*
>
> *Le même ouvrage remanié a reparu dans la meme année sous ce titre, bien long pour si peu de choses: "Aritmétique des musiciens, ou Essai qui a pour object diverses espéces de calcul des intervalles; le développement de plusieurs systêmes de sons de la musique; des experiences pour aider à discerner quel est le veritable, c'est- à-dire celui de la voix: la description de celui qu'on suppose l'étre sur quelques instrumens, ses rencontres avec celui du clavecin, et leur disparité dans tous les modes imaginables; des soupçon sur le nombre que l'oreille aperçoit dans tous ou presque tous les accords de deux sons, notamment dans ceux qui forment des intervalles superflus ou diminués; une hypothèse relative aux sons harmoniques, et les moyens de faire rendre par une même chorde en même temps deux sons don't l'intervalle ne soit point une consonance. On y a ajouté une explication des propriétés les plus connues des logarithms par celle qu'ils ont de mesurer les intervalles." On voit par ce titre que Gallimard n'avait pas l'art d'exprimer ses idées avec simplicité, quoiqu'il eût mis pour épigraphe à sa brochure "cum veritate simplicitas et ordo"; toutefois, son petit ouvrage est un manuel qui n'est pas sans utilité.*

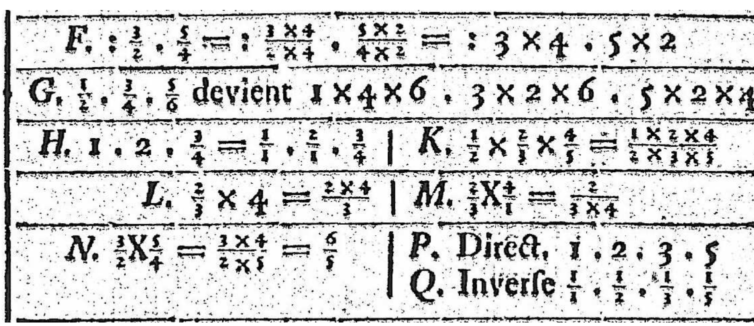
It is curious this last statement from Fétis because this work must have been important for theoretical musicians in order to remain a reference work in tuning and temperament (see [Balfour1948], [Balfour1951], [CristensenBent1993] or the web site of Huygens-Fokker Foundation Center for Microtonal music [website1]). Balfour, a respected specialist in tuning and temperament, refers to the work of Gallimard [Balfour1951] as follows:

> *Various writers have attempted to reduce this error - by dividing it between two fifths instead of having it concentrated upon one; by raising several of the preceding fifths also, as Mersenne and Rameau did; by carefully graduating all five black key fifths, as Gallimard did.*

Having placed Gallimard's work with Mersennes' "*Harmonicuru*" (1647), and Rameau's "*Noveau systême de musique théorique*" (1726), shows that Balfour did not share Fétis’s opinion.

5 *“Aritmétique des musiciens”*

The copy we had access to is a digital copy available from the French National Library through Gallica stamped by the Biblioteque Royale. It is written in French, and consists of 30 pages organized in 120 numbered articles. It includes 6 pages (pages 22 to 27) totally or almost totally filled with 29 Tables most with several columns (Table 16 has as many as eight different columns), 5 geometric figures with Euclidean constructions, (page 28) and three more figures that show the division of a chord to produce the sounds with the tuning systems described by the author. A curious characteristic of Gallimard presentation is the inclusion in almost every page of examples identified by a capital letter like for instance in Figure 1.



F. $: \frac{3}{2} \cdot \frac{5}{4} = : \frac{3\times4}{2\times4} \cdot \frac{5\times2}{4\times2} = : 3\times4 \cdot 5\times2$

G. $\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6}$ devient $1\times4\times6 \cdot 3\times2\times6 \cdot 5\times2\times4$

H. $1 \cdot 2 \cdot \frac{3}{4} = \frac{1}{1} \cdot \frac{2}{1} \cdot \frac{3}{4}$ | *K.* $\frac{1}{2}\times\frac{2}{3}\times\frac{4}{5} = \frac{1\times2\times4}{2\times3\times5}$

L. $\frac{2}{3}\times4 = \frac{2\times4}{3}$ | *M.* $\frac{2}{3}\times\frac{4}{1} = \frac{2}{3\times4}$

N. $\frac{3}{2}\times\frac{5}{4} = \frac{3\times4}{2\times5} = \frac{6}{5}$ | *P.* Direct. $1 \cdot 2 \cdot 3 \cdot 5$
Q. Inverse $\frac{1}{1} \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{5}$

Figure 1:

This data refers to examples that are explained to in the text. For example in the article 15:

> *Un entire pet outpours être censé multiplié ou divisé par 1. Voyez H’.*

(An integer can always be considered as multiplied or divided by 1).

These references have some particularities. They are ordered by capital letters but the order is not very common. The first one on page one is B, (there is no A in the beginning). In the second page comes D and E followed by the data on page 3 mentioned above. The letters "I", "J" and "O" do not occur ever. Maybe because they could be confused with "1" or "0".

In page 4 the letters are "M", "N", "P" and "Q" in one set of data. "H" in the next and "R", "S" and "T" in the third. The data in this "M" is not related with the previous one. In page 5 two more sets of data this time both named "A" and "B" (also not related with the previous ones). In page 6 three more sets the first one named "X", the second "G", "H", "L" and "K" by this order and the third "C", "D", "E", and "F". So apparently it started in some letter but the letters used in each set of data were sequential (except for the second set in page 6, which could be justified by reasons of economy of space), however in page 7 the letters used are "A", "P", "B", "Q", "X" and "Y". This choice of letters is probably made to emphasize the relation between the arithmetic progression in A and the arithmetic proportion in B. The same justification applies to the harmonic progression in P and the proportion in Q. or to the progressions in "X" and "Y".

The structure of "L' Arithmétique des Musiciens" is very well defined. The first 46 articles deal with proportions and progressions (arithmetic, geometric and harmonic), powers and roots. The author gives clear definitions and explains the basic rules he will need in the following discussions. An example of how clear he tries to be is given in :

> *13. Réduction des rapports et des suites de fractions à des entiers. On réduira un rapport de fractions à un rapport déntiers, en formant le produit du numérateur de la premié par de dénominateur de la segonde, et celui du numérateur de la segonde. Par de dénominateur de la premié. Voyez F.*

In articles 47 to 57 the author introduces a numerical system for scales in C, major and the minor. In 58 to 61 Gallimard introduces logarithms and from 62 till 98 applies logarithms to adjust temperaments in trying different systems. Among others he tries to divide the octave in 31 equal intervals (article 72) or in 55 (article 77), analyzes the Pythagorean system (article 74) in article 83 he introduces the last experience ("Derniere expér."). The following items are destined to apply the temperaments defined before to different instruments like Viola, Cello, Harpsichord, etc. The end of the work includes some explanation on the properties of logarithm and geometric Euclidean constructions needed to divide the chord of the instrument in the proportions defined previously.

6 Conclusions

As it is shown in Figure 1 Gallimard actually shows the position of the notes defined by the length of the chord. Not satisfied with the results of this numerical system Gallimard tries other systems of temperament It is quite obvious that Gallimard had a major concern in explaining some mathematics needed for musicians in the XVIII century. He had a greater concern explaining the arithmetic then explaining the music behind it, which shows that since he was writing for musicians he supposed they had some knowledge on the background needed to understand the musical part of the work. There is a great deal left to be explained in both subjects Mathematics and Music. This present article only covers partially the first 52 articles in Gallimard's book. Some of the musical background is not exactly what is taught to music students nowadays, even if one can see a pattern in what Gallimard says. This is the case of modulations. How exactly do harmonic progressions in music relate to harmonic progressions in arithmetics? It is clear that the harmonic division of the octave in an arithmetic sense is important in music because it even justifies some intervals being consonant or dissonant. For instance [Fux1725] the interval of fourths are considered dissonant if they result from the arithmetic division of the octave and are considered by some musicians consonant if they are obtained by the harmonic division (in the mathematical sense) of the chord. These two divisions lead to different chords.

7 Appendix

The table included in Figure 2 taken from [Smith1749] gives a list of the most common music intervals.

A table of the Order of the ſimplicity of conſonances of two ſounds.

Ratios of the vibrations.	Order of the ſimplicity.	Intervals of the ſounds.
1 : 1	1	o
1 : 2	2	VIII
1 : 3	3	VIII + V
1 : 4	4	2VIII
2 : 3	$4\frac{1}{2}$	V
1 : 5	5	2VIII + III
1 : 6	6	2VIII + V
2 : 5	$6\frac{1}{3}$	VIII + III
3 : 4	$6\frac{2}{3}$	4th
1 : 7	7	
3 : 5	$7\frac{2}{3}$	VI
1 : 8	8	3VIII
4 : 5	$8\frac{3}{4}$	III
1 : 9	9	3VIII + T
1 : 10	10	3VIII + III
2 : 9	$10\frac{1}{5}$	2VIII + T
3 : 8	$10\frac{2}{5}$	VIII + 4th
5 : 6	$10\frac{4}{5}$	3d
1 : 12	12	3VIII + V
3 : 10	$12\frac{1}{1}$	VIII + VI
4 : 9	$12\frac{1}{2}$	VIII + T
5 . 8	$12\frac{2}{3}$	6th
6 : 7	$12\frac{5}{6}$	

Continuation of the table.

1 : 15	15	3VIII + VII
1 : 16	16	4VIII
2 : 15	$16\frac{1}{8}$	2VIII + VII
5 : 12	$16\frac{1}{2}$	VIII + 3d
8 : 9	$16\frac{7}{8}$	T
1 : 18	18	4VIII + T
3 : 16	$18\frac{2}{9}$	2VIII + 4th
4 : 15	$18\frac{1}{1}$	VIII + VII
9 : 10	$18\frac{8}{9}$	t
1 : 20	20	4VIII + III
5 : 16	$20\frac{2}{5}$	VIII + 6th
1 : 22	22	
3 : 20	$22\frac{2}{11}$	2VIII + VI
5 : 18	$22\frac{4}{11}$	VIII + 7th
8 : 15	$22\frac{7}{11}$	VII
1 : 24	24	4VIII + V
9 : 16	$4\frac{2}{1}$	7th
1 : 28	28	
5 : 24	$28\frac{2}{7}$	2VIII + 3d
9 : 20	$28\frac{4}{7}$	VIII + t
1 : 30	30	4VIII + VII
15 : 16	$30\frac{14}{15}$	H
32 : 45	$76\frac{31}{38}$	IV
45 : 64	$108\frac{22}{27}$	5th

Figure 2: Proportion and Musical Intervals

8 Acknowledgement

The author would like to thank the organizers of the "Seminário de Investigação em História e Epistemologia da Matemática" and the organizers of the Homage to Maria Fernanda Estrada for the invitation to be an active participant. A special thank you to Maria Fernanda for introducing the author to the realm of the History of Arithmetics. In this contribution it was a fortunate event to bring together the History of Arithmetics and Music.

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Note on the use of History in the teaching of Mathematics

João Filipe Queiró
Departamento de Matemática
Universidade de Coimbra
jfqueiro@mat.uc.pt

Abstract

Some comments are made on the use of History in the teaching of Mathematics.

Keywords: History of Mathematics, teaching of Mathematics.

Dedicated to Maria Fernanda Estrada, with friendship and respect.

1 Mathematics and its History

The French philosopher of Mathematics Jean Cavaillès, in the Introduction to his 1938 book *Remarques sur la formation de la théorie abstraite des ensembles* [1], writes the following:

> "L'histoire mathématique semble, de toutes les histoires, la moin liée à ce dont elle est véhicule; s'il y a lien, c'est *a parte post*, servant uniquement pour la curiosité, non pour l'intelligence du résultat: l'après explique l'avant. Le mathématicien n'a pas besoin de connaître le passé, parce que c'est sa vocation de le refuser: dans la mesure où il ne se plie pas à

> ce qui semble aller de soi par le fait qu'il est, dans la mesure où il rejette autorité de tradition, méconnaît un climat intellectuel, dans cette mesure seule il est mathématicien, c'est-à-dire révélateur de nécessités."

It's true that he immediately adds:

> "Cependant avec quels moyens opère-t-il? L'œuvre négatrice d'histoire s'accomplit dans l'histoire."

And he uses the remainder of the Introduction to analyse the relations between mathematical creation and historical conditions.

But the insight contained in those first sentences, which I read while still a student, left me with an impression which, after many years, with more information and maturity, has never gone away. The creation and the study of Mathematics in the present can be carried out ignoring History. It is possible to conceive of a Fields Medal mathematician who knows absolutely nothing about the origins and historical evolution of his field of expertise. Of course, we can also consider him an uncultivated mathematician — or human being — but that is a kind of moral judgment, external to Mathematics itself.

It is a fact that intelligibility of Mathematics is independent of the knowledge of its past. One who studies Mathematics, be it at research level be it while learning the subject, does not need to know the History of what he is studying, apart from, possibly, in the case of the researcher, the recent contributions on the problem under investigation.

The History of Mathematics — understood as the history of mathematical ideas — is today an established academic field. Gone are the days when the history of a scientific subject was considered a part of the subject itself, and its knowledge a precondition to any attempt towards progress in the field. This classical justification for studying the history of any scientific discipline no longer exists (see [3] for a discussion).

As to Mathematics itself, its legitimacy is twofold: it is the area, among human discourses, which most radically and rigorously questions itself, its correctness and internal coherence; on the other hand, it is an essential component of all fields of knowledge with any aspiration to be called scientific.

Why Mathematics possesses the latter characteristic is a classical and fascinating question, analysed, for example, in the famous article "The

Unreasonable Effectiveness of Mathematics in the Natural Sciences", by the Nobel Prize winner Eugene Wigner. As noted by another Nobel Prize, Paul Dirac, the mathematician plays a game for which he invents the rules, while the physicist plays a game in which the rules are given by Nature; but it turns out, over time, that the rules are the same.

What is of interest to us here is that History plays no part in the legitimization of Mathematics, except in the broad sense that the subject has been around for a long time (plus, of course, the historical component of the twofold legitimization mentioned in the previous paragraphs).

History of Mathematics, on the other hand, cannot claim total independence from Mathematics, at least for the last few centuries, for obvious technical reasons. This is what makes it such a difficult field, beyond linguistic, cultural and methodological requirements.

2 History and teaching

If Mathematics, as a current scientific field, is independent of its history, as discussed, *a fortiori* the teaching of Mathematics does not formally necessitate the use of the History of Mathematics. Such use, therefore, can only be justified by pragmatic considerations, for example, bringing about a stronger student motivation and thus improving the quality of their learning.

My pragmatic claim is that the History of Mathematics should not be used in the teaching of Mathematics too early, surely not before the 6th grade. By "used" I mean really used, not giving the name of Pythagoras to a theorem and the like.

Until the 6th grade, students are in a very immature phase of their intellectual development. Teaching and learning should concentrate on the acquisition of basic knowledge and techniques. Any reference to the origins and historical evolution of introductory algorithms and results — even assuming that such origins and evolution are easy to unravel — would have, at these levels, a serious effect of distraction and confusion.

It is unproductive, for example, to try to teach children two or more algorithms for the arithmetical operations, mentioning different historical or cultural contexts. In the very imperfect environment of elementary schools, the question is not whether children will learn more than one algorithm, the alternative is between learning one or none at all.

The idea behind learning more than one algorithm seems to be that it enhances understanding of the underlying concept. The emphasis on "understanding" basic material in these age groups is misguided. In Mathematics, there are many examples of skills that have to be acquired before the corresponding concepts are fully understood.

Between the 6th and the 9th grade, there will occasionnaly be an opportunity for interesting and relevant use of historical material. A few times, with motivated groups of students, a reference to short biographies of mathematicians and to historical contexts for the appearance of certain results or techniques — including the re-enactment of famous calculations — may be justified and useful. But this will be the exception, not the rule, and it most certainly should not be mandated by the national curriculum.

Historical references should never crowd out the real purpose of mathematical study, which is the acquisition of important knowledge and techniques — either as ends in themselves or as prerequisites for further study — and the progressive development of a logical and rigorous mind. Time constraints, and student mental overload, are legitimate considerations here.

On the other hand, Mathematics is a human activity, which progresses and changes like any other. But truth criteria in Mathematics have been the same for centuries, even millennia, only progressively more refined. So one should not overuse historicism, nor make vague "historical" references as illustrating an alleged temporal contingency of mathematical activity. If this is to be the role of History of Mathematics in teaching, it's better to leave it out altogether.

In late secondary school, and especially in the university, the situation changes. The book [2] contains several examples of the use of history in teaching at those levels, and there are many other references on the subject.

This makes sense. A cultivated relation with Mathematics, which supposes an historical vision, can only be attained at university level, where contents already possess some sophistication. So it is clearly an option to use history to enhance and enrich teaching of Mathematics at that level.

For students enrolled in teacher training, a course dealing exclusively with the History of Mathematics should be mandatory. A Mathematics teacher should know some History of Mathematics, not because he is going to teach it, but because, unlike researchers — and even more so people who apply Mathematics to other fields such as Engineering — a teacher should have the above mentioned cultivated relation with Mathematics. To have a reasonable knowledge of the History of Mathematics makes for a more cultivated researcher (possibly even a better one), but for a Mathematics teacher it is an obligation, because a teacher must be a person of culture.

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on History and Epistemology of Mathematics, 97–114

On a method proposed by J. Anastácio da Cunha

Rui Ralha[1]
Departamento de Matemática e Aplicações, Escola de Ciências
Universidade do Minho
r_ralha@math.uminho.pt

Abstract

The problem of finding the roots of polynomials has a long history. Throughout centuries XVII-XIX, eminent scientists like Newton (1643-1727), Lagrange (1736-1813), Fourier (1772-1837), Cauchy (1789-1857), Sturm (1803-1855), and many others, gave contributions for solving the problem. The Portuguese mathematician J. Anastácio da Cunha (1744-1787) envisaged to make his own contribution and in [Cunha1790] he describes his method. The same method is also presented in one of the manuscripts left by da Cunha but lost for more than 200 years, until they were found, a few years ago, at the Arquivo Distrital de Braga. The motivation of our work has been the analysis of the convergence of the method proposed by da Cunha.

Keywords: Portuguese mathematics, roots of polynomials, iterative methods, convergence.

[1] Work supported by the Research Centre of Mathematics of the University of Minho through the FCT Pluriannual Funding Program.

1 Introduction

The history of methods for approximating the roots of polynomials goes back to ancient civilizations. Egyptians, Greeks and Hindus looked for approximate roots of numbers and formal equations. Those interested in the topic will find a good survey in [Nordgaard1922] and [Chabert1999]. The seventeenth century saw a remarkable advance in the theory of equations. In [Nordgaard1922], p. 33, one can read

> *The symbolic notation introduced by Vieta ... made the symbolic equation a chief instrument of analysis...For purposes of numerical approximations one of the tools made most effective by the improved symbolism was the infinite series in an equation ....*

While finding the area under curves by his "method of fluxions", Isaac Newton discovered a new application of series. This was the starting point for a method that is still, at present, one of the most popular for solving equations (and much more). Raphson, Lagrange and others contributed to the development of the method as it is known today.

The Portuguese mathematician J. A. da Cunha, inspired by the works of Newton, Lagrange and others, proposed his own method to determine, with as many digits as required, the largest positive root of a polynomial of any degree. This method appears in one of the chapters (*Livro X*) of "Principios Mathematicos" (in Portuguese) which has been published in Lisbon, in 1790, a few years after the author's death. A French translation of "Principios Mathematicos" has been published in Bordeaux, in 1811, and a fac-simile reproduction was published at Universidade de Coimbra, in 1987. The same method is in the manuscript "Nouvelle Résolution Numérique des Équations de tous les Degrés" which has been annotated by Estrada et al. [Estrada2006].

In the manuscript, da Cunha claims that his own method is a serious competitor with the best known methods (those of Newton and Lagrange). However, in "Principios Mathematicos" he suggests its use to compute initial approximations for Newton's method. Unfortunately, there are polynomials for which the method proposed by da Cunha will fail to locate the largest positive root. The main purpose of the present paper is to clarify this issue; in particular, we will give a sufficient condition for the convergence of the method.

2 *Facile est inventis addere...*

As pointed out in [Estrada2006], A. da Cunha gets his inspiration for the new method mainly from Newton and Lagrange. Combined with Newton's result on the bounds of polynomial roots, A. da Cunha uses successive changes of variable. He gives credit to Lagrange for his own idea of using changes of variable. However, as observed in [Estrada2006], much before Lagrange, Newton has made an extensive use of changes of variables. This is also emphasized in [Nordgaard1922], p. 46:

> *The method of solving equations of higher degree by successive substitutions in derived equations was originated by Newton (1669) incidental to his getting integral expressions for his work in areas. He performed a transformation for every new supplement to the root. His method was first made public in the algebra of Wallis (1685).*

In [Estrada2006], the authors wrote that it would be interesting to study the speed of convergence of the method proposed by A. da Cunha. This was the motivation for our work whose conclusions we present here.

3 Newton's bounds for the roots

We said before that a key component of the method proposed by A. da Cunha is the result which gives an upper bound for the real roots of a polynomial with real coefficients. In a modern style, this result may be simply stated in the form

(Newton's theorem) Let p be a monic polynomial, of degree n, with real coefficients. If p and its derivatives $p^{(k)}$, for $k = 1, ..., n$, are all positive at some point M, then M is an upper bound for the real roots of p.

This result is stated, without proof, in [Newton1720], p. 208, as follows:

> *...Multiply every Term of the Equation by the Number of its Dimensions, and divide the Product by the Root of the Equation; then again multiply every one of the Terms that come out by a Number less by Unity than before, and divide the Product*

> *by the Root of the Equation, and so go on, by always multiplying by Numbers less by Unity than before, and dividing the Product by the Root, till at length all the Terms are destroy'd, whose Signs are different from the Sign of the first or highest Term, except the last; and that Number will be greater than any Affirmative Root...*

Derivatives were still not in use at the time, therefore Newton needs a verbose description of the process of getting the derivatives of p. Then, he uses his result to show that every real root of the polynomial $p(x) = x^5 - 2x^4 - 10x^3 + 30x^2 + 63x - 120$ has to be to the left of $M = 2$. This is how he proceeds: from p, he determines the sequence of polynomials (derivatives of p with coefficients divided by their gcd)

$$\begin{aligned} p(x) &= x^5 - 2x^4 - 10x^3 + 30x^2 + 63x - 120 \\ p1(x) &= 5x^4 - 8x^3 - 30x^2 + 60x + 63 \\ p2(x) &= 5x^3 - 6x^2 - 15x + 15 \\ p3(x) &= 5x^2 - 4x - 5 \\ p4(x) &= 5x - 2 \end{aligned}$$

and then tries successive integer numbers to find the smallest one for which all the polynomials are positive. Starting with 1, he finds $p4(1) = 3 > 0$ but $p3(1) = -4 < 0$, therefore the limit is larger than 1. Then, he finds that for $M = 2$ the polynomials are all positive, therefore $M = 2$ is an upper bound for the real roots.

Applying the result to the polynomial produced from the original one by replacing x with $-x$, Newton gets a lower bound for the negative roots:

> *...In like manner, if I would find the Limit of the Negative Roots, I try Negative Numbers. Or that which is all one, I change the Signs of every other Term, and try Affirmative ones...*

In this way, he finds out that all the real roots of the given polynomial are between -3 and 2. As already said, Newton did not include a proof of his "theorem" in [Newton1720]. In [Cunha1790], pp.132-133, A. da Cunha presents a proof using a monic polynomial of degree four for this purpose. He describes the process of forming the derivatives of a polynomial very much in the same way as Newton did. From

$$G(x) = x^4 + ax^3 + bx^2 + cx + d \tag{3}$$

he gets, with $x = z + e$,

$$\begin{aligned} F(z) \quad = \quad & z^4 + (4e + a)z^3 + (6e^2 + 3ae + b)z^2 + \\ & +(4e^3 + 3ae^2 + 2be + c)z + e^4 + ae^3 + be^2 + ce + d \end{aligned} \tag{4}$$

and concludes that if all the new coefficients are positive, then for z to be a root of F, it has to be $z < 0$, consequently if $x = z + e$ is a root of G, then $x < e$. Furthermore, the coefficients of F are the values of the polynomials G and those formed in the manner described, at the point e.

4 Lagrange's influence in the method proposed by A. da Cunha

In his manuscript, A. da Cunha gives Lagrange full credit for the discovery of a method that solves a polynomial equation (of any degree) which, he adds, can be found in two memoirs published by the Royal Academy of Sciences of Berlin. Full references for these memoirs are given in [Estrada2006]. A. da Cunha briefly exposes Lagrange's method that produces a number α which is smaller than the distance between any two consecutive real roots of a polynomial. Known α, the computation of the polynomial in $\alpha, 2\alpha, 3\alpha, etc.$ and also $-\alpha, -2\alpha, -3\alpha, etc.$ would find intervals containing one and only one real root. A. da Cunha praises Lagrange for this method but then observes that α is frequently too small for the method to be of practical value.

One of the changes of variable used by Lagrange in his process of finding α is simply a displacement $x = z + e$ and A. da Cunha, inspired in Lagrange's works, uses the same change of variable. As we will see later, such change of variable is not essential in da Cunha's method and its use obscures, to a certain extent, the essence of the process.

5 Evolving the digits of a root

We will see in the next section that da Cunha's method aims at finding the successive digits of the root, beginning with the most significant one.

This idea of evolving the digits of a root, order by order, had been used by other mathematicians before. This is the case of Vieta, for a good description of his method see [Nordgaard1922], chapter VI. Before Vieta, also Stevin computed roots of equations by evolving the digits, order by order. The following can be found in [Nordgaard1922], pp. 21-22, with a reference to the works of Stevin:

> *We can best explain his process by recording one of his illustrations. Suppose* $x^3 = 300x + 33915024$*:*
>
> $$\begin{aligned} &\textit{if } x = 1, \textit{ then } 1^3 < 300 \cdot 1 + 33915024;\\ &\textit{if } x = 10, \textit{ then } 10^3 < 300 \cdot 10 + 33915024;\\ &\textit{if } x = 100, \textit{ then } 100^3 < 300 \cdot 100 + 33915024;\\ &\textit{if } x = 1000, \textit{ then } 1000^3 > 300 \cdot 1000 + 33915024. \end{aligned}$$
>
> *Hence, the root lies between 100 and 1000. By trying x=200, x=300, x=400, he finds the root to lie between 300 and 400. Similarly, trying 310, then 320, then 330, he finds it to lie be between 320 and 330. A similar procedure for the units gives 324 as the exact value of the root. Stevin observes that an irrational root can be approximated to within any desired degree of accuracy by using this method and his scheme of computing decimal fractions. This method was advocated and used considerably by Albert Girard in his Invention Nouvelle en l'Algèbre (1629). Though laborious, the method is general. "Stevin's rule" was used by later algebraists...in connection with other methods.*

6 The method of A. da Cunha

A. da Cunha aims at finding the largest positive root of the polynomial, evolving the digits by order. For this, he uses the signs of the values of the polynomial and its derivatives at successive points. Unfortunately, there appears to exist a major flaw in the proposal of the Portuguese mathematician. We will clarify this in the next section. To illustrate his method, in the manuscript, A. da Cunha uses the following example

$$\begin{aligned} p(x) &= x^3 - 702x^2 + 160600x - 16080000 \\ p1(x) &= 3x^2 - 1404x + 160600 \\ p2(x) &= 6x - 1404 \end{aligned}$$

Then, using Newton's procedure, A. da Cunha immediately concludes, just by looking at $p2(x)$, that the bound M must be larger than 200 and then considers successively the numbers $300, 400, ...$ to find out that $M = 500$ is an upper bound for the largest positive root (if it exists). However, he also says that such root can not be smaller than 400...

> *Donc la racine cherchée est* < 500 *(par le théorém) elle ne peut être* < 400 *(par le même théorém) et on vient de trouver qu'elle n'est pas =400; donc le premier chiffre de la racine cherchée est 4 manquant des centaines.*

As observed in [Estrada2006], it looks like A. da Cunha is abusively using the reciprocal of Newton's theorem, and such reciprocal is not true, in general. The polynomial has in fact a root between 400 and 500, which is 402. Estrada et al. conjecture a possible explanation for the conclusion of A. da Cunha: x=400 fails the conditions of the Newton's theorem because $p(400) < 0$. Since $p(500) > 0$, it follows that there is a root between 400 and 500. Since this is such a simple argument (in fact this is essentially the old "Stevin's rule", *used by later algebraists in connection with other methods*, according to Nordgaard), can it be the case that A. da Cunha did use it in an automatic manner, without caring to mention it?

In another example, presented in [Cunha1790], p. 134, the polynomial is $p(x) = x^4 - 9x^3 + 15x^2 - 27x + 9$; after correctly finding that $M = 8$ is the smallest positive integer for which p and its derivatives are positive, A. da Cunha states that the first digit is 7. Again, 7 fails the conditions of Newton's theorem just because $p(7) < 0$ and, being $p(8) > 0$, this shows that there is, in fact, a root between 7 and 8.

7 A false reciprocal

Being clear to us that Newton's theorem can not guarantee lower bounds for the largest positive root, the method of A. da Cunha works well (in concluding that some x is a lower bound for such root) when $p(x) < 0$. This is the situation in the two examples mentioned before and given by A. da Cunha, one in the manuscript, the other one in [Cunha1790]. How would A. da Cunha proceed if $p(x) > 0$ but some of the other polynomials (derivatives) is negative? The answer is given by yet another example which appears in [Cunha1790], pp. 136-137, in the context of dividing $p(x)$ by $x - \alpha$, where α is some root already determined. In the following, we will try to be as precise as possible in our translation of the original text, in Portuguese.

> *Let* $x^3 - 3x^2 - 2x + 16$ *be the polynomial whose root is asked for. Write (according to what has been taught)*
>
> $$\begin{aligned} ((x-3)x-2)x+16 \\ (3x-6)x-2 \\ 3x-3 \end{aligned}$$
>
> *The smallest positive integer that makes all these expressions positive is 3: then 2 is the first digit of the largest, or not the smallest root of the proposed polynomial, if it exists.*

In the process of determining the upper bound $M = 3$, A. da Cunha has observed that the polynomial of second degree (the first derivative) is negative in 2. Although $p(2) > 0$, he concludes that if a positive root exists then it is between 2 and 3. This shows that A. da Cunha is, in fact, using the reciprocal of Newton's theorem. In continuation, A. da Cunha makes a change of variable, as he learned from Lagrange

> *Supposing that the root is* $x = z + 2$, *we will use be the polynomial (formed as taught before)* $z^3 + 3z^2 - 2z + 8$ *to find the second digit, that is, the first digit in the fractional part. Writing*
>
> $$\begin{aligned} ((z+3)z-2)z+8 \\ (3z+6)z-2 \end{aligned}$$
>
> *we find that that it is equal to* $0,2$.

Again using the false reciprocal, he takes $0,2$ to be a lower bound for the largest root of the polynomial $z^3 + 3z^2 - 2z + 8$, even if this is positive in

$0,2$. He does so because the first derivative is negative in $0,2$. The same happens for the polynomial $z^3 + 3,6z^2 - 0.68z + 7,728$ which is used to find the next digit and gives the approximation 2,28. At this point, A. da Cunha adds

> *This is all useless because $x^3 - 3x^2 - 2x + 16$ does not have positive roots; and the method does not say that there is a positive root; what it says is that there is no positive root that exceeds or equals 3, or that is equal or smaller than 2; or equal or smaller than 2,2; or equal or smaller than 2,28; and so on; and this is all true.*

It is true indeed for the given example, because the polynomial has no positive roots, as A. da Cunha shows. From $z^3 + 3z^2 - 2z + 8 = 0$, he writes[2]

$$z = \frac{8}{2 - 3z - z^2} \tag{5}$$

and notes that this is impossible for $z < 1$. He then concludes that the polynomial $x^3 - 3x^2 - 2x + 16$ has no positive roots, replaces x with $-x$ and uses his method to find the negative root which is equal to -2.

At this point, one may observe that A. da Cunha managed to get the correct answer for each one of the polynomials used, although the last example shows that one may have to carry out useless calculations trying to get an approximation for a root that does not exist. Worst than this is the fact that the method may wrongly conclude that there is no positive root. In modern words, the method may fail to converge.

8 A sufficient condition for the convergence of the method proposed by A. da Cunha

From what has been said before, it may be understood that the method of A. da Cunha will conclude, wrongly, that there are no positive roots whenever the largest positive root is to the left of a point x for which at least one of the derivatives is negative. This is because, using the false reciprocal of the Newton's theorem, the method will take such x as a lower bound for the largest positive root, if it exists. One such

[2] In the original text, this appears as $z = \frac{8}{2-3z-z^3}$.

example has been given in [Estrada2006], p. 262; here, the polynomial is $p(x) = x^3 - 7x^2 + 16x - 10$, the second derivative $6x - 14$ is negative in 1 and 2 and positive in 3, then it may be seen, in this order, that the first derivative and the polynomial itself are both positive in 3. The conclusion would be, using A. da Cunha's own words,

> *the method does not say that there is a positive root; what it says is that there is no positive root that exceeds or equals 3, or that is equal or smaller than 2.*

This is wrong since the polynomial has the root 1, the other two roots being $3 \pm i$. As observed in [Estrada2006], p. 261, the reciprocal of Newton's theorem holds if all the roots of the polynomial are real. In fact, it is not necessary that the roots are all real. We now show that a sufficient condition for the reciprocal of Newton's theorem to hold, that is, for the method of da Cunha to converge, is that for any pair of imaginary roots $a \pm ib$, the real part a is not larger than the largest positive root of the polynomial, which will be denoted by r_n. To prove this, let us write

$$p(x) = \prod_{j=1}^{n} (x - r_j) \tag{6}$$

where imaginary roots occur in conjugate pairs (the polynomial has real coefficients). Each derivative $p^{(k)}(x)$, for $k = 1, \cdots, n-1$, is a sum of products of factors of the form $(x - r_j)$. When the roots r_j are all real, such factors are all positive for $x > r_n$ and the derivatives are all positive. For those products involving a pair of imaginary roots $a \pm ib$, we have

$$[x - (a + ib)][x - (a - ib)] \prod (x - r_j) = [(x - a)^2 + b^2] \prod (x - r_j) \tag{7}$$

for some values of j, and see that the sign of the overall product is independent of the roots $a \pm ib$. Now, consider a product which is of the form

$$[x - (a + ib)] \prod (x - r_j) \tag{8}$$

where none of the r_j is the conjugate $a - ib$. In this case, there is another product, in the expression of the derivative, which is equal to

$$[x - (a - ib)] \prod (x - r_j) \tag{9}$$

for the same values of j, and the sum of these two products is

$$2(x-a)\prod(x-r_j). \tag{10}$$

Any derivative $p^{(k)}(x)$, for $k = 1, \cdots, n-1$, is the sum of expressions of the form (7) and (10) and is therefore positive for $x > a$ and $x > r_n$. When $a < r_n$, no derivative can be negative for $x > r_n$, therefore the reciprocal of Newton's theorem holds in this case.

Our condition $a < r_n$ is not necessary for the reciprocal of Newton's theorem to hold[3]. There are polynomials for which the condition is not true and, nevertheless, the polynomial and derivatives are all positive for $x > r_n$. This is the case of the polynomial with roots 0, 3 and $4 \pm 3i$. Therefore, da Cunha's method will work well in this case, in spite of being $r_n = 3 < a = 4$.

We conclude this section by noting that the polynomials used by A. da Cunha in the illustration of his method satisfy our condition, with the exception of the polynomial $x^3 - 3x^2 - 2x + 16$, mentioned in the previous section, which has no positive roots.

9 Change of variables and derivatives

As described before, the method proposed by A. da Cunha uses a change of variable and produces a new polynomial for each digit computed. In fact, unlike in the methods proposed by Lagrange, which inspired A. da Cunha, the change of variables is not essential in da Cunha's method, in the sense that it can be formulated in an equivalent manner without changing the variable.

If, for some displacement value e, we replace x with $z + e$ in the expression of $p(x)$, getting a new polynomial $q(z)$, then we have

$$q(z) = p(z+e) \tag{11}$$
$$q^{(k)}(z) = p^{(k)}(z+e), k = 1, \cdots, n. \tag{12}$$

The previous relations have a simple visualization: the graphic of q is simply a translation in the x-direction of the graphic of p and so the

[3]Interestingly, the condition is necessary in the case of cubic polynomials; when there is a pair of complex conjugate roots with real part a larger than the real root r, the second derivative is equal to $(x-r)+2(x-a)$, which is negative for $x < (r+2a)/3$.

same is true for the graphic of the derivatives. It is therefore clear that the original polynomial p can be used throughout the entire process, to compute all the digits in the approximation, not just the first digit.

At this point, we wish to emphasize that our observation on the change of variables (not being essential) is not a criticism to the method proposed by A. da Cunha. The purpose of our observation is only that of exposing the essence of da Cunha's method, in order to simplify our analysis of the efficiency of the method.

10 Efficiency of A. da Cunha's method

The method proposed by A. da Cunha converges to r_n, the largest positive root of a given polynomial p (if it exists), whenever no derivative of p has a real root larger than r_n. In such cases, the speed of convergence of the method is that of Stevin's rule: since, in each iteration, one more correct decimal place is added to the previous approximation, the error in each iterate is bounded by one tenth of the error in the previous iterate, that is,

$$r_n - x_{i+1} \leq \frac{r_n - x_i}{10} \tag{13}$$

where x_i denotes the approximation produced in the ith iteration (here no absolute value is required in the expression of the errors since it is always $x_i \leq r_n$).

A sequence $\{x_i\}$ is said to converge to the limit L with order of convergence equal to p when

$$\lim_{i \to +\infty} \frac{|x_{i+1} - L|}{|x_i - L|^p} = \mu \tag{14}$$

for some constant μ. From (13) it follows that the method of A. da Cunha converges to r_n with order $p = 1$, that is, the method converges linearly.

Now, if $p^{(k)}(x_i)$, for $k = 1, \cdots, m$, with $m \leq n - 1$, are all positive, it follows that the same is true for every $x > x_i$. An efficient implementation of the method should take this into account and avoid useless computations. A. da Cunha has done so, in an implicit manner. For instance, in the case of the polynomial $p(x) = x^4 - 9x^3 + 15x^2 - 27x + 9$, the derivatives are all positive in 7, and the change of variables $x = z + 7$ produces the new polynomal $z^4 + 19z^3 + 120z^2 + 232z - 131$. Looking at the signs of the coefficients of this polynomial, A. da Cunha concludes that the derivatives will be all positive for $z > 0$, therefore, there is no need to compute them. Next, in the determination of the third correct place, he gets the polynomial $z^4 + 20,6z^3 + 143,76z^2 + 337,376z - 17,7584$ and, again, observes that the derivatives do not need to be computed. Although the Portuguese mathematician has avoided the redundant computation of the derivatives, it is not clear that he came to see the general result.

The method of A. da Cunha can be understood as a method for finding an interval that contains the largest positive root of a given polynomial. It is for this end that the computations of the derivatives do play a role. Under our assumption that the derivatives do not have real roots larger than r_n, when this root has been isolated, the derivatives will all be positive and so it is the sign of $p(x)$ only that brings useful information for a more precise computation of the root.

It should be noted that in the iterations following the isolation of the root, bisection will be more efficient than evolving the decimal digits by order. In the bisection method (which uses the middle point of the current interval containing the root) the error is halved in each iteration, therefore the number of iterations required for each digit is $log_2(10) \approx 3,3$. In da Cunha's method, one decides which one of the digits $0, ..., 9$ comes next in the approximation of the root. If it is 0, then one single computation of the polynomial (which will be positive when we test 1 as the next digit) will be enough. This is the best possible case. In the worst possible case, the next digit is 8 or 9 and p will need to be computed for as many as 9 points to finish the iteration. The average number of times that p needs to be computed per iteration is therefore $(1+2+3+4+5+6+7+8+9+9)/10 = 5,4$ which is larger than $log_2(10)$ in bisection.

11 A. da Cunha and Newton's method

The popular iterative formula

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \tag{15}$$

for computing some root of the equation $f(x) = 0$ has its origins in Newton, incidental to his work in computing areas. J. Raphson developed the original method of Newton (the method is frequently referred to as the

Newton-Raphson's method) but, according to [Nordgaard1922], p. 15, the functional expression in (15) is due to Lagrange (Simpson used it before for solving simultaneous equations). Newton and Raphson computed the same approximations as formula (15) but using transformed equations. The method and its illustration with the classical example $y^3 - 2y - 5$, as given by Newton, are in [Newton1736], pp. 5-6. About Newton-Raphson's method, Nordgaard wrote (see [Nordgaard1922], p. 59)

> *... Its technique was improved by DeLagny, Halley, Taylor and Simpson, and its scientific basis was clarified and strengthened by Lagrange, Fourier, Budan and Sturm. No other method of approximation has come up to it in general popularity.*

The method was so popular that it is almost sure that A. da Cunha knew about it. In his manuscript, A. da Cunha wrote

> *On me permettra donc de proposer une autre méthode, non seulement générale e rigoureuse, mais d'une exécution expéditive et aisée, que souvent elle rend inutiles les meilleurs méthodes que nous ayons, celles de Sir Isaac Newton pour les racines rationelles.*

In our opinion, it is the popular Newton-Raphson's method that A. da Cunha has in mind when he writes this and it is that same method that the Portuguese mathematician wants to beat with his new method.

In [Cunha1790], p. 138, A. da Cunha, without any mention to the works of Newton, or of anyone else, presents Newton-Raphson's method in the following way. He takes the equation $A+Bx+Cx^2+Dx^3+\cdots=0$ and makes the substitution $x = r + z$ to produce

$$A + Br + Cr^2 + Dr^3 + \cdots + (B + 2Cr + 3Dr^2 + \cdots)z + \cdots \tag{16}$$

Then, assuming that r is the number of the first two digits of the root, he neglects the terms which involve powers of z larger than one, to write

$$r - \frac{A + Br + Cr^2 + Dr^3 + \cdots}{B + 2Cr + 3Dr^2 + \cdots} \tag{17}$$

as the new approximation for the root which he states to have twice as many the number of correct digits (this is to say that he expects the method to converge quadratically, which is in fact the case for simple roots).

A. da Cunha uses Newton-Raphson's method to compute the largest positive root of the polynomial $x^4 - 9x^3 + 15x^2 - 27x + 9$. Starting with the initial approximation $r = 7,4$, he gets the successive approximations $r = 7,45$, $r = 7,4514$, $r = 7,45149836$, etc. We emphasize that A. da Cunha uses the same expression (17) in every iteration and, doing so, he is in fact using the functional expression (15) due to Lagrange, that is, the Newton's method in its modern formulation (without transforming the polynomial through changes of variable).

A. da Cunha adds that praxis has shown that this method works well. From here, one can conclude that he is presenting a method which he reckons has been known for quite some time. However, the interesting point is the following: A. da Cunha says that one possibility is to use his own method to produce the initial approximation $7,4$ that is then improved upon with Newton's method. It looks like that, at this point in his work, the Portuguese mathematician has already understood that Newton's method converges faster than his own method and so, rather than trying to "beat the opponent", he looks for an ally! And it is quite true that Newton's method alone can not do the job since good initial approximations are required for the method to converge fast (if at all).

If our previous observations on Newton's and da Cunha's methods are correct, as we believe they are, then an immediate conclusion is that the manuscript has been written before the material in [Cunha1790] to which we have been referring. However, this conclusion does not agree with that in [Estrada2006], p. 256, and may deserve further discussion.

12 Conclusions

The method proposed by da Cunha for finding the largest positive root r_n of a polynomial equation $p(x) = 0$ works well if and only if the reciprocal of Newton's theorem, which gives an upper bound for the positive roots, is true. Such reciprocal holds if and only if no derivative of p has a real root larger than r_n. We have shown that this condition is certainly true whenever any pair of complex conjugate roots has real part smaller than r_n.

The derivatives play a role in da Cunha's method, in producing a lower bound for the root, only until one point x_i is found for which the derivatives are all positive. This is because for x such that $x > x_i$ the derivatives are all positive and if $p(x_i) < 0$ one may conclude that there is one and only one root larger than $x > x_i$. From here, the calculation of the root uses the value of the polynomial only. Therefore, da Cunha's method is in fact a criteria for locating the largest positive root (which may fail, if a pair of complex roots has real part larger than the largest positive root).

In the final stages (when the root is already isolated from the roots of the derivatives), da Cunha's method requires more iterations than the simple bisection method (even if both methods converge linearly). The Portuguese mathematician appears to have become aware of the slow convergence of his own method and has suggested its use for finding initial approximations to be used with the Newton-Raphson's method.

Acknowledgements I am grateful to Elfrida Ralha and Fernanda Estrada for challenging me with questions that raised my interest in da Cunha's work (in particular his method for solving polynomial equations) and also for helping me with their expertise in the History of Mathematics. In particular, Elfrida has recommended all the bibliography that has been used in this work. Also, my thanks to António Leal and João Queiró for a useful discussion on da Cunha's method.

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Research Seminar
on History and Epistemology of Mathematics, 115–126

Geometrical version (*à la* Nicole Oresme) of two propositions of Álvaro Tomás

Carlos Correia de Sá[1]
Centro de Matemática
Universidade do Porto
csa@fc.up.pt

Abstract

In the third chapter of the second treatise of the *Liber de triplici Motu*, printed in 1509, Álvaro Tomás presents some results concerning local movements that are a continuation of those given by Nicole Oresme in the second half of the 14th century in the *Tractatus de Configurationibus Qualitatum et Motuum.* Oresme had given diagrams depicting all the aspects of the question, from its statement to its solving procedure. It may seem strange that in the time span of one and a half centuries geometry has disappeared, but it should be taken into consideration that by 1509 the reproduction of geometrical diagrams posed very hard problems to the printing techniques. It is probable that the absence of diagrams in Tomás' treatise was just caused by the will to reduce the costs of printing.

[1]Work (partially) supported by the Centro de Matemática da Universidade do Porto (CMUP), financed by FCT (Portugal) through the programmes POCI 2010 (Programa Operacional "Ciência e Inovação 2010"). Pdf file available from http://www.fc.up.pt/cmup.

Anyway, there can be no doubt about the affiliation of these results of Tomás' in the last chapters of Oresme's treatise. The purpose of this short presentation is to propose geometrical versions of the rhetorical arguments with which Tomás proves two of the most important *conclusiones* of this chapter of the *Liber de triplici Motu*, namely, the second and the fourth ones. These versions are strictly based on Tomás' text and on the rules of the theory of configurations as practised by Oresme.

Keywords: Tomás, numerical series, scholastic kinematics, Oresme's theory of configurations.

Introduction

Álvaro Tomás published his *Liber de triplici Motu* in Paris in 1509. Some of the results about local movement in this treatise have caught the attention of Heinrich Wieleitner in 1914 for admitting an interpretation in terms of real number series[2]. The reader of this part of Tomás' treatise is immediately reminded of the last sections of Nicole Oresme's *Tractatus de Configurationibus Qualitatum et Motuum*, because of the identical (and indeed rather unexpected) way in which the subject line is subdivided into infinitely many *partes proportionales.*

Besides this striking similarity, there are also important differences between the two texts. Tomás restricts himself to the study of local motions, whereas Oresme had considered these as a particular instance of more general distributions of qualities. More importantly, Oresme's *configurationes* are totally absent from Tomás's treatise. There are very few geometrical diagrams in the *Liber*, and absolutely none in the part that will interest us here.

Álvaro Tomás[3] was a master of arts at Coqueret college in Paris, one and a half centuries after Nicole Oresme had taught at the same University. Tomás mentions Oresme in the *Liber* several times, although he spells this name as "Horem" or "Horen". Therefore, there can be no question as to whether Tomás knew the work of Oresme. Furthermore, it is highly

[2]H. WIELEITNER 1914: "Zur Geschichte der unendlichen Reihen im christlichen Mittelalter", in Bibliotheca Mathematica 14, pp. 150-168. A Portuguese translation by S. Gessner of this article is found in H. WIELEITNER 2010.

[3]A good summary of what is known about Álvaro Tomás' life can be found in H. LEITÃO 2000.

probable that he was well acquainted with the *theory of configurations*, which is believed to have been a creation of Oresme and is exposed in the above mentioned *De Configurationibus*.

Had Tomás chosen to use *configurationes*, they would have proved most helpful in the explanation of some of his questions and in the exposition of their solutions. I shall try to illustrate this statement taking the example of two of Tomás' kinematic *conclusiones*.

Prime partis Capitulum quintum

Tomás' kinematic examples and arguments are based on arithmetic foundations regarding a theory of proportions in the Nicomachean tradition. In the fifth chapter of the first part of his treatise[4] Tomás gives some results concerning magnitudes in *continuous proportion*. The number of such magnitudes being infinite, one obtains results about *geometrical progressions*.

To have an idea of the sort of prerequisite established by Tomás in this chapter, we only have to look at three of his statements, namely, the 3rd and 4th suppositions and the 1st conclusion. For the sake of brevity, they are given here in modern notation:

Let $p_1, p_2, \ldots, p_n, \ldots$ denote the parts into which a whole is subdivided. We shall consider the case $\frac{p_1}{p_2} = \frac{p_2}{p_3} = \frac{p_3}{p_4} = \cdots = g$. In other words, $p_1, p_2, \ldots, p_n, \ldots$ form a geometrical progression of ratio $\frac{1}{g}$.

Tomás' "third supposition" is the following implication:

$$\frac{p_1}{p_2} = \frac{p_2}{p_3} = \frac{p_3}{p_4} = \cdots = g \Longrightarrow \frac{p_1 - p_2}{p_2 - p_3} = \frac{p_2 - p_3}{p_3 - p_4} = \cdots = g.$$

The "fourth supposition" is merely a corollary of the previous one:

$$\frac{p_1 + p_2 + p_3 + \cdots}{p_2 + p_3 + p_4 + \cdots} = \frac{p_2 + p_3 + p_4 + \cdots}{p_3 + p_4 + p_5 + \cdots} = \cdots = g \Longrightarrow \frac{p_1}{p_2} = \frac{p_2}{p_3} = \frac{p_3}{p_4} = \cdots = g.$$

The "first conclusion" is the converse implication of the latter:

$$\frac{p_1}{p_2} = \frac{p_2}{p_3} = \frac{p_3}{p_4} = \cdots = g \Longrightarrow \frac{p_1 + p_2 + p_3 + \cdots}{p_2 + p_3 + p_4 + \cdots} = \cdots = g.$$

[4] *Capitulum quintum in quo agitur de diuisione corporis in partes porportionales qua proportione rationali quis voluerit* (Fifth chapter, in which one handles the division of a body into proportional parts according to the ratio one wishes).

The first and last of these three implications may look like mere restatements, for the special case of geometrical progressions, of Euclid's propositions *Elements* VII, 11 and 12 (for integers) or *Elements* V, 19 and 12 (for magnitudes). However it is important to remark that, whereas Euclid always restricts himself to a finite number of quantities, Tomás is admitting that the whole is subdivided into infinitely many parts

$$p_1, p_2, \ldots, p_n, \ldots.$$

These results are instrumental in the third chapter of the second treatise (*Secundi tractatus Capitulum tertium*), where Tomás studies movements taking place in a time interval which is subdivided into an infinity of *proportional parts*. As Wieleitner has noticed, Tomás' propositions in this part of the *Liber* may be interpreted in terms of infinite series of positive numbers[5].

Before we go into any technical detail, an observation should be made concerning the absence of algebraic symbolism in the *Liber*. Not only the three propositions above, but indeed all of Tomás' kinematic *conclusiones*, are stated rhetorically.

Another instance of this lack of algebraic notation is the following. Tomás denotes by the letter g the ratio of any proportional part to the next one. Thus, by the "first conclusion" above, g is also the ratio of the whole to the residue of the first proportional part. Tomás denotes by another letter, f in this instance, the ratio of the whole to its first proportional part. The obvious equality

$$\frac{1}{f} + \frac{1}{g} = \frac{p_1}{p_1 + p_2 + p_3 + p_4 + \cdots} + \frac{p_2 + p_3 + p_4 + \cdots}{p_1 + p_2 + p_3 + p_4 + \cdots}$$

then reduces to

$$\frac{1}{f} + \frac{1}{g} = 1.$$

Tomás is perfectly conscious of the relationship between the ratios f and g expressed by this equality, but he is unable to write it down in such a condensed way. All he can do is express it rhetorically and give some (more or less complicated) examples. We should keep it in mind, though, in order to understand Tomás arguments.

[5]Of course there are serious anachronisms in this interpretation, not the least of which is viewing geometrical or physical magnitudes as positive real numbers.

Secundi tractatus Capitulum tertium

The sort of movements that Tomás considers in the third chapter of the second treatise[6] may look rather strange to many, but not to those acquainted with the last sections[7] of Oresme's *De Configurationibus.* Time is subdivided into proportional parts and the mobile moves with constant velocity during each one of these parts. There are, therefore, infinitely many "jumps". What might be the degree of velocity at those instants is not said, but one should of course be cautious not to fall into the trap of reading (or rather: misreading) Tomás from an anachronistically "functional" point of view. Besides, Tomás' purpose does not seem to be the study of existing movements, either natural or artificial. He is much more interested in exploring imaginary situations that allow him to find infinite series of spaces or velocities that do have a sum, in spite of their infinitude.

We shall use a unified and simplified notation. Thus, let

T be the whole time;

V be the mean velocity;

S be the whole space traversed during T.

Now let T be subdivided into proportional parts according to ratio g and, for each $n \in \mathbb{N}$, let

t_n be the n^{th} proportional part of time;

v_n be the (constant) velocity during t_n;

s_n be the space traversed during t_n with the velocity v_n.

One should note that V is the velocity that would make the mobile traverse exactly the space S during the time T if the movement were

[6] *Capitulum tertium in quo ostenditur modus cognoscendi siue commensurandi motum uniformiter difformem et difformiter difformem quo ad tempus quo ad velocitatem et tarditatem in omni specie* (Third chapter, in which the way is shown to know or measure the uniformely difform and difformly difform movement, both as to time and as to velocity and slowness in all species).

[7] Sections III, 8, 9 and 10 are meant. Section III, 11, which is the very last one, deals with an infinitely long subject or time, subdivided into equal parts.

uniform[8].

We shall concentrate on the second and the fourth of Tomás' twelve *conclusiones* of this chapter[9]. As will presently be seen, they both deal with the same kinematic situation: a time interval divided into proportional parts and a mobile which in the n^{th} time subinterval moves with a velocity equal to n times the velocity it had in the first time subinterval. Symbolically, $\forall\, n \in \mathbb{N} \quad v_n = nv_1$.

This is a generalization of the situation studied by Nicole Oresme in *De Configurationibus* III, 8. In fact, Tomás admits subdivisions of time according to any ratio greater than unity[10], whereas Oresme considered only the case in which each time subinterval is the double of the next one.

In the 2^{nd} conclusion Tomás proves that the subdivision of velocities into proportional parts is analogous to the subdivision of times: the mean velocity, V, has the same ratio to v_1 as the whole time, T, to the first proportional part of time, t_1. Thus,

$$\big(\forall\, n \in \mathbb{N} \quad v_n = nv_1\big) \implies \frac{V}{v_1} = f.$$

The situation regarding spaces is quite different. In his 4^{th} conclusion Tomás proves that the ratio of whole space, S, to the first space, s_1, is the duplicate ratio of the ratio of corresponding times. Thus,

$$\big(\forall\, n \in \mathbb{N} \quad v_n = nv_1\big) \implies \frac{S}{s_1} = f^2.$$

As we have previously mentioned Oresme had considered only the particular case in which $f = g = \frac{1}{2}$, concluding of course that $V = 2v_1$ and that $S = 4s_1$. In order to construct his example, he started from two equal squares; each one of these squares has one side subdivided into proportional parts according to the ratio $\frac{1}{2}$; this subdivision of the side induces an analogous subdivision of the corresponding square into

[8]Tomás uses the expression *velocitas totalis* (as well as several other Latin variants) to express what would today be called the *mean velocity* (and what is represented by V in this text). Oresme, on the contrary, uses *velocitas totalis* to express the *space* traversed by the mobile. An interesting work on this topic is P. SOUFFRIN 1997.

[9]A more detailed analysis of the twelve conclusions may be found in C. CORREIA DE SÁ 2005.

[10]One ought to be aware that the terminology is different from the one used today. The ratio according to which time is subdivided is necessarily greater than the *ratio* 1:1, because it means the ratio of any proportional part of time to the next part.

proportional parts (which are rectangles), as in *Diagram 1* (a). Next Oresme moved the parts of the second square and places them on top of the first square and on top of one another, as shown in *Diagram 1* (b). The bottom horizontal line represents the subject (which is linear) and all the possible vertical lines inside the configuration are taken to represent the corresponding degrees of intensity of the quality under scrutiny. The top contour pictures the distribution of the quality along the subject and the area of the configuration represents the totality of the quality.

When interpreted kinematically, the configuration in *Diagram 1* (b) obviously corresponds to: $\forall\, n \in \mathbb{N} \quad v_n = nv_1$.

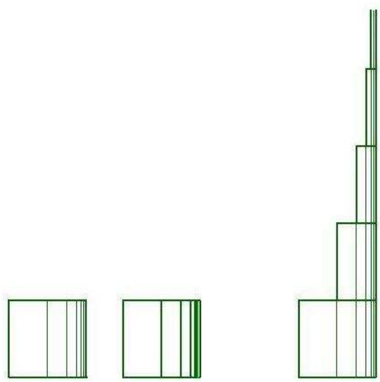

Diagram 1 (a) *Diagram 1* (b)

Secunda conclusio

One may distinguish two steps in Tomás' rhetorical proof of the second conclusion:

- In the 1st step Álvaro Tomás begins by extending velocity v_1 to the whole time interval; next he extends the excess of v_2 over v_1 to the residue of t_1; then he extendes the excess of v_3 over v_2 to the residue of t_1 and t_2; and so on indefinitely.

- In the 2nd step Tomás considers the velocity that, if it were extended to the whole time, would have the same effect[11] as the excess of v_2 over

[11]The *effect* of a certain degree of velocity extended to a certain time interval is of course the *space* (or the *length* of the space) traversed during that time by a mobile

v_1 has being extended to the residue of t_1; next he considers the velocity that, if it were extended to the whole time, would have the same effect as the excess of v_3 over v_2 has being extended to the residue of t_1 and t_2; and so on.

If we choose to use Oresme's *theory of configurations*, these two steps of Tomás' proof have very easy visualizations. The 1st step can be explained by the passage from *Diagram 2* (a) to *Diagram 2* (b). The 2nd step consists in the passage from *Diagram 2* (b) to *Diagram 2* (c).

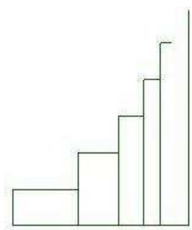

Diagram 2 (a)

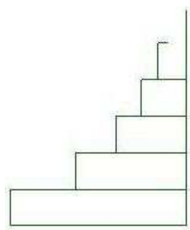

Diagram 2 (b)

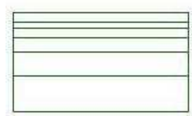

Diagram 2 (c)

The configurations in *Diagram 2* (a) and (b) are the same, only their subdivision into partial rectangles changes. The rectangles in *Diagram 2* (b) (all with equal heights) have the same area as the corresponding ones in *Diagram 2* (c) (all with equal bases); since the bases of the former decrease according to the ratio g, the heights of the latter must decrease exactly in the same ratio. Therefore, in *Diagram 2* (c), the height of the whole rectangle (which represents the degree of intensity of V) must be to the height of the first rectangle (which represents the degree of intensity of v_1) in the ratio f.

Quarta conclusio

The third conclusion works like a lemma for the proof of the next conclusion. It states that, when the velocity is constant, the traversed space is proportional to the time elapsed. In fact, Tomás says that, if the velocity v_1 be extended to the whole time, then $\frac{S}{s_1} = \frac{T}{t_1}$.

In the fourth conclusion Tomás considers once more the same subdivision of times and the same distribution of velocities as in the second one.

with such a velocity.

But now he is interested in the question of the spaces traversed, or rather, in the ratio between the space traversed during the whole time and the space traversed during the first proportional part of time. Tomás' proof is based on the second and third conclusions.

Just as before, let T, S, V, t_1, s_1 and v_1 denote the whole time, the whole space, the mean velocity, the first proportional part of time, the space traversed during t_1 and the velocity during t_1. Tomás also needs to consider one of the auxiliary movements which had already been useful in the proof of the second conclusion; namely, he considers the space that would be traversed by the mobile during the whole time T if it had the constant velocity v_1. We shall use the letter σ to denote this space[12].

Using once again configurations à la Oresme, the spaces s_1, σ, S correspond to the shaded rectangles in *Diagrams 3* (a), (b), (c), respectively. This makes the following argument by Tomás visually clear.

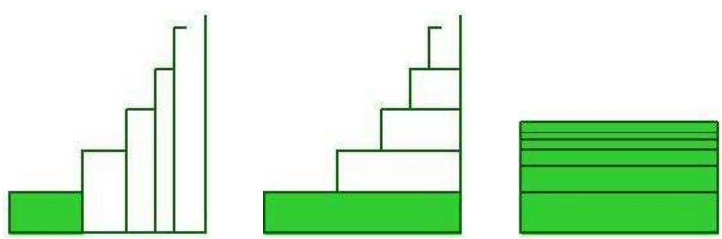

Diagram 3 (a) *Diagram 3* (b) *Diagram 3* (c)

The ratio of S to σ is the same as the ratio of V to v_1, because the two movements occur in the same time; but, by the second conclusion, we already know that this ratio is f. By the third conclusion, the ratio of σ to s_1 is the same as the ratio of T to t_1, because the velocity is the same; by definition, this ratio is also f. Compounding both ratios, S is to s_1 in the ratio f^2.

Symbolically:

$$\left(\frac{S}{\sigma} = \frac{V}{v_1} = f \ \wedge \ \frac{\sigma}{s_1} = \frac{T}{t_1} = f\right) \implies \frac{S}{s_1} = \frac{S}{\sigma} \cdot \frac{\sigma}{s_1} = f^2.$$

[12]Tomás denotes the spaces S, σ, s_1 by the letters a, b, c, respectively, but there is no need to introduce the two new letters a and c.

Concluding remark

It is not probable that Tomás had any sort of philosophical objections to the use of Oresme's *configurationes* in the analysis of kinematic questions. It is more likely that the absence of geometry in this part of the *Liber* is due to the financial cost that diagrams represented in book printing at the beginning of the sixteenth century. Whatever the reason may have been, Oresme's *theory of configurations* is not present in the *Liber* and we lack other documental evidences that might allow us to admit the possibility that Tomás' mental process of discovery of his kinematic *conclusiones* was based on a geometric reasoning. In the present state of development of Tomás' intellectual biography, we must renounce any unconditional statement in that direction, no matter how tempting it might appear. The translation of Tomás' rhetoric arguments into Oresmes's geometrical language is mathematically sound and pedagogically effective, but historically undocumented. One may only wish for the development of the studies on Renaissance mathematical physics in Paris, in order to be able to decide whether or not such a translation constitutes any treason either to the spirit of the *Liber de triplici Motu* or to the genius of Álvaro Tomás.

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Research Seminar
on History and Epistemology of Mathematics, 127–140

About the reality of the real numbers

Lisa Santos[1]
Centro de Matemática
Universidade do Minho
lisa@math.uminho.pt

Abstract

We briefly refer the kind of constructions of the irrational numbers that appeared in the last quarter of the nineteenth century.
We present the construction due to Dedekind, known as Dedekind cuts. In the set of Dedekind cuts we define the sum and the product of two cuts and we introduce a total order relation, providing a few technical details about the proof that this structure is a complete ordered field, unique up to isomorphism.

Key words: Construction of real numbers, Dedekind cuts.

Dedicated to Fernanda Estrada, on the occasion of her 80th birthday

1 Introduction

In 1821, in his "Cours d'Analyse" (see [5]), Cauchy revealed his concerns about the foundations of Analysis, raising the following questions:

[1]Work supported by the Research Centre of Mathematics of the University of Minho through the FCT Pluriannual Funding Program.

What is a derivative, really? Answer: a limit.

What is an integral, really? Answer: a limit.

What is an infinite series, really? Answer: a limit.

This leads to:

What is a limit, really? Answer: a number.

And finally, the last question:

What is a number?

In 1858, Dedekind wrote (see [2]): *As a professor in the Polytechnic School in Zürich I found myself for the first time obliged to lecture upon the elements of the differential calculus and felt more keenly than ever before the lack of a really scientific foundation for arithmetic.*

At that time, the rational numbers were well understood. This was not the situation regarding the non-rational algebraic numbers. And, for transcendental numbers, the first example was given by Liouville (see [8]), only in 1851.

Dedekind started thinking about a construction of the real numbers. In his book "Essays on the theory of numbers - continuity and irrational numbers; the nature and meaning of numbers" ([2]), he stated he had succeeded on the 24th of November of 1958 and he wrote , *... but I could not make up my mind to its publication, because in the first place, the presentation did not seem altogether simple, ...* This construction was published only in 1872, coinciding with two publications on the same subject, one of Heine and another of Cantor. These two last constructions had nothing in common with the one of Dedekind: while Dedekind introduced the notion of cut, the other two authors used the notion of Cauchy sequence. The construction due to Dedekind is of great beauty and elegance, using very few mathematical tools, in fact only elementary set theory.

Another construction, due to Weierstrass, using infinite decimals, was published as course notes by a student, Kossac, also in 1872. The most known version of this construction was published by Pincherle, in 1880.

The perception of the difficulties felt by the mathematicians in the last quarter of the nineteenth century can be slightly understood by the following sentences:

> $\sqrt{3}$ *is thus only a symbol for a number which has yet to be found, but is not its definition. This definition is, however, satisfactorily given by my method as, say*
>
> $$(1.7, 1.73, 1.732, \ldots).$$
>
> (G. Cantor)

> *Numbers are the free creation of the human mind.*
>
> (R. Dedekind)

> *I take in my definition a purely formal point of view, calling some given symbols numbers, so that the existence of these numbers is beyond doubt.*
>
> *(H. Heine)*

And, out of the context, but irresistible...

> *It is true that a mathematician who is not also something of a poet will never be a perfect mathematician.*
>
> (K. Weierstrass)

What was the situation in Portugal, at the end of the nineteenth century?

In the first edition of the *Curso de Análise Infinitesimal - Cálculo Diferencial* ([4]), Gomes Teixeira presented a brief construction of the set of real numbers in a note, at the end of the book, using the Dedekind cuts and he referred there the works of Dedekind, Dini, Heine and Tannery (for details about the life and work of Gomes Teixeira, see [1]). The construction presented in that first edition was not so rigorous as the constructions proposed by Dedekind and Weierstrass. Nevertheless, his didactic concerns are admirable.

In the three subsequent editions (1890, 1896 and 1906), Gomes Teixeira improved his presentation of the set of real numbers. The construction of the irrational numbers appears in the first chapter of the introduction and it is obvious the influence of the presentation of Dini, who also used the construction of Dedekind.

In the forth edition, Gomes Teixeira proved the completeness of $\mathbb{R}$ and provided a long list of bibliographic references, concerning the constructions of the real and complex numbers.

In this paper, we define the Dedekind cuts and present the definitions of sum and product of two cuts. We introduce an order relation in the set of cuts and we verify that such set, endowed with these operations and this order relation, becomes a totally ordered field, having the fundamental property of *completness* with respect to the order relation. Finally we prove the uniqueness of such a field, up to isomorphism.

In many calculus courses, the set of real numbers is introduced axiomatically as the unique complete ordered field. The construction of Dedekind is an efective tool to verify that this structure exists, in fact.

2 Dedekind cuts

It is well known that, starting from the Peano Axioms, we have the natural numbers well defined. Simple mathematical concepts leads us easily to the construction of integer and rational numbers. One aim of the first construction is to find an adequate set, containing the natural numbers, where the subtraction is a closed operation. The set of the rational numbers extends the set of the integer numbers in a way that the division operation is closed in this set apart zero. The construction of the rational numbers raises another questions:

1. Is it true that any subset of $\mathbb{Q}$ bounded from above has a least upper bound?
2. Is it true that every Cauchy sequence of rational numbers has a rational limit?

The answer to both questions is obviously no:

- $\left\{\left(1+\frac{1}{n}\right)^n : n \in \mathbb{N}\right\}$ is a bounded subset of $\mathbb{Q}$ with no rational least upper bound;
- $\left(\left(1+\frac{1}{n}\right)^n\right)_n$ is a Cauchy sequence not convergent in $\mathbb{Q}$.

We wish to find out a set $\mathscr{C}$, with two operations, the sum, $+$, and the product, $\cdot$, and an order relation, $\leq$, such that $(\mathscr{C}, +, \cdot, \leq)$ is an ordered field which is complete (i.e., all its subsets bounded from above have a least upper bound). And this set $\mathscr{C}$ must have a subset that may be identified with the ordered field of the rational numbers.

Definition 1. *A (Dedekind) cut in* $\mathbb{Q}$ *is a pair* $(A, B) \in \mathscr{P}(\mathbb{Q}) \times \mathscr{P}(\mathbb{Q})$ *such that:*

- $A \cup B = \mathbb{Q}$, $A \cap B = \emptyset$, $A \neq \emptyset$, $B \neq \emptyset$;
- *if* $a \in A$ *and* $b \in B$ *then* $a < b$;
- A *has no maximum.*

We denote such a cut by $A|B$ and the set of Dedekind cuts by $\mathscr{C}$.

We observe that, if $A|B$ is a cut, then $B = \mathbb{Q} \setminus A$ and, given $x \in \mathbb{Q}$, if there exists $a \in A$ such that $x < a$, then $x \in A$.

Definition 2. *A real number is a cut in* $\mathbb{Q}$.

Let us present two examples:

1. $A|B = \{x \in \mathbb{Q} : x < 2\}\big|\{x \in \mathbb{Q} : x \geq 2\}$;
2. $A|B = \{x \in \mathbb{Q} : x \leq 0 \text{ or } x^2 < 2\}\big|\{x \in \mathbb{Q} : x > 0 \text{ and } x^2 \geq 2\}$.

Informally, we understand that the cut defined in 1. represents the rational number 2 while the cut defined in 2. does not represent any rational number. In fact, it defines a new (irrational) number, which is called $\sqrt{2}$, by convention.

Below we make precise the above underlying idea:

- we say that a cut $A|B$ is a rational number if B has a minimum and we identify the cut $A|B$ with $\min B$, representing it by $(\min B)^*$; reciprocally, given any rational number c, we identify it with the cut $\{x \in \mathbb{Q} : x < c\}\big|\{x \in \mathbb{Q} : x \geq c\}$, which we simply denote by c^*;
- a cut $A|B$ such that B has no minimum shall be understood as a new number, called an irrational number.

In the set of cuts $\mathscr{C}$ we define the following order relation:

Definition 3. *Given two cuts $x = A|B$ and $y = C|D$ we say that $x \leq y$ if $A \subseteq C$ and that $x < y$ if $x \leq y$ and $x \neq y$.*

We say that a cut $x = A|B$ is positive if $0^* < x$.

It is easy to prove that this relation is a total order.

The sum of two subsets X and Y of $\mathbb{Q}$ is defined as follows

$$X + Y = \{x + y : x \in X, y \in Y\}$$

and the symmetric of a subset X of $\mathbb{Q}$ by

$$-X = \{-x : x \in X\}.$$

It is immediate to verify that:

- if X and Y are subsets of $\mathbb{Q}$, so is $X + Y$;
- if $A|B, C|D \in \mathscr{C}$ then $\emptyset \neq A + C \neq \mathbb{Q}$;
- if $\emptyset \neq X \neq \mathbb{Q}$, we also have $\emptyset \neq -X \neq \mathbb{Q}$.

Proposition 4. *Given two cuts $A|B$, $C|D$, the pair $(A+C)|\big(\mathbb{Q}\setminus(A+C)\big)$ is a cut.*

Proof. By the observations above, we know that $A + C$ and $\mathbb{Q} \setminus (A + C)$ are not empty.

To prove that if $x \in A+C$ and $y \in \mathbb{Q}\setminus(A+C)$ then $x < y$, we start by proving that any rational number below an element of $A + C$ belongs to $A+C$: in fact, if $x < a+c$ for some $a \in A$ and $c \in C$ then $x-a < c$ which implies that $x-a \in C$, because $C|D$ is a cut. So $x = a+(x-a) \in A+C$. It follows that if $x \in A + C$ and $y \in \mathbb{Q} \setminus (A + C)$ then, necessarily, $x < y$.

It only remains to prove that $A + C$ has no maximum: as $A|B$ and $C|D$ are cuts, then neither A nor C have maximum. So, given $x \in A+C$, $x = a + c$, for some $a \in A$ and some $c \in C$ and there exists $a' \in A$ and $c' \in C$ such that $a < a'$ and $c < c'$. So $x = a + c < a' + c'$ and x is not maximum of A. $\square$

Definition 5. *Given two cuts $A|B$, $C|D \in \mathscr{C}$ we define its sum by*

$$A|B + C|D = (A + C)|\big(\mathbb{Q} \setminus (A + C)\big).$$

Proposition 6. $(\mathscr{C}, +)$ *is an abelian group.*

Proof. The proofs of the commutativity and associativity of + are immediate.

We recall the notations

$$\mathbb{Q}^- = \{x \in \mathbb{Q} : x < 0\} \text{ and } \mathbb{Q}_0^+ = \{x \in \mathbb{Q} : x \geq 0\}.$$

Let us prove that the cut $0^* = \mathbb{Q}^-|\mathbb{Q}_0^+$ is the element zero of $\mathscr{C}$: given a cut $A|B$,

$$A + \mathbb{Q}^- = \{x + y : x \in A, y \in \mathbb{Q}^-\} = A.$$

In fact,

- $A + \mathbb{Q}^- \subseteq A$, because for $x \in A$ and $y \in \mathbb{Q}^-$ we have $x + y < x$;
- $A \subseteq A + \mathbb{Q}^-$, because given $x \in A$, since A has no maximum, there exists $x' \in A$ such that $x < x'$. Then $x = x' + (x - x') \in A + \mathbb{Q}^-$.

$$A|B + 0^* = A|B + \mathbb{Q}^-|\mathbb{Q}^+ = (A + \mathbb{Q}^-)|(\mathbb{Q} \setminus (A + \mathbb{Q}^-)) = A|(\mathbb{Q} \setminus A) = A|B.$$

In the case where $A|B$ represents the rational number a, the symmetric cut of $A|B$ is $\{x \in \mathbb{Q} : x < -a\}|\{x \in \mathbb{Q} : x \geq -a\}$, because

$$A + \{x \in \mathbb{Q} : x < -a\} = \{x + y : x < a,\ y < -a\} = \mathbb{Q}^-,$$

which means that $A|B + \{x \in \mathbb{Q} : x < -a\}|\{x \in \mathbb{Q} : x \geq -a\} = 0^*$.

In the case where $A|B$ is an irrational cut, the symmetric of $A|B$ is the cut $(-B)|(-A)$ (it is immediate to verify it is a cut). Let us prove that $A - B = \mathbb{Q}^-$:

- let $a \in A$ and $b \in B$. As $a < b$ then $a - b < 0$, so $A - B \subseteq Q^-$;
- to prove that $\mathbb{Q}^- \subseteq A - B$ is equivalent to prove that $\mathbb{Q}_0^+ \supseteq B - A$. Given $a \in A$ and $b \in B$, as $a < b$ then $0 < b - a$.

So $A|B + (-B)|(-A) = 0^*$. □

Notation: We represent by $-x$ the symmetric of a cut x.

We are now able to define the product of two cuts.

Definition 7. *Let $x = A|B$ and $y = C|D$ be two cuts.*

If $0^* < x$ and $0^* < y$, we define $x \cdot y = E|F$ by:

$$E = \{r \in \mathbb{Q} : \exists\, a \in A\ \exists\, c \in C \quad 0 < a,\, 0 < c,\, r < ac\}, \quad F = \mathbb{Q} \setminus E.$$

In the other cases, we define:

$$x \cdot y = \begin{cases} 0^* & \text{if } x = 0^* \text{ or } y = 0^*; \\ -\big(x \cdot (-y)\big) & \text{if } 0^* < x \text{ and } y < 0^*; \\ -\big((-x) \cdot y\big) & \text{if } x < 0^* \text{ and } 0^* < y; \\ (-x) \cdot (-y) & \text{if } x < 0^* \text{ and } y < 0^*. \end{cases}$$

Theorem 8. *$(\mathscr{C}, +, \cdot, \leq)$ is a totally ordered field.*

Proof. We have already observed that $(\mathscr{C}, +)$ is an abelian group and $(\mathscr{C}, \leq)$ is a totally ordered set.

We will only prove that the element one of $\mathscr{C} \setminus \{0^*\}$ is the rational cut 1^* and we will find out the inverse of any cut $x \neq 0^*$ (denoted, in what follows, by x^{-1}).

Given a positive cut $A|B$, the cut $A|B \cdot 1^*$ is, by definition, the cut $E|F$, where

$$E = \{r \in \mathbb{Q} : \exists\, a \in A\ \exists\, x \in \{y \in \mathbb{Q} : y < 1\}\ 0 < a,\, 0 < x,\, r < ax\},$$

and it is easy to understand that $E = A$. So, $E|F = A|B$, as we wanted to prove. If $A|B$ is a negative cut, then $-(A|B)$ is a positive cut and

$$A|B \cdot 1^* = -((-(A|B)) \cdot 1^*) = -(-(A|B)) = A|B.$$

To find the inverse of a positive cut $A|B$, we suppose, first, that $A|B$ represents the rational number $a > 0$. Then the cut

$$\{x \in \mathbb{Q} :\ x < \tfrac{1}{a}\}|\{x \in \mathbb{Q} :\ x \geq \tfrac{1}{a}\}$$

is the inverse of $A|B$. To prove our statement, we need to check that the product of these two cuts is 1^*. Let $E|F$ denote this product. Then, by definition,

$$\begin{aligned} E =& \{r \in \mathbb{Q} : \exists\, x \in \mathbb{Q}\ \exists\, y \in \mathbb{Q} \quad 0 < x < a,\ 0 < y < \tfrac{1}{a},\ r < xy\} \\ =& \{r \in \mathbb{Q} :\ r < 1\} \end{aligned}$$

and so $E|F = 1^*$.

Suppose now that $A|B$ is a positive not rational cut. Then $(A|B)^{-1}$ is the cut $C|D$, where $C = \{x \in \mathbb{Q} : x \leq 0 \text{ or } \frac{1}{x} \in B\}$ and $D = \mathbb{Q} \setminus C$. If $E|F = A|B \cdot C|D$, we need to prove that

$$E = \{r \in \mathbb{Q} : \exists\, a \in A\ \exists\, c \in C \quad 0 < a,\, 0 < c,\, r < ac\} = \{r \in \mathbb{Q} : r < 1\}.$$

If $r \in E$ and $r \leq 0$, then $r < 1$. Let $r \in E$ and $0 < r$. Then there exists $a \in A$ and $c \in C$ such that $0 < a$, $0 < c$ and $r < ac$. By the definition of C we have $\frac{1}{c} \in B$ and so $a < \frac{1}{c}$, implying that $r = ac < 1$.

Let $r < 1$. If $r \leq 0$ then $r \in E$. Suppose that $0 < r < 1$. We want to prove that $r \in E$. Defining $X = \{\frac{a}{b} : a \in A,\ 0 < a,\ b \in B\}$, we start by proving that $\sup X = 1$. If $\sup X = r_0 < 1$ then $r_0 \in X$: if not, we should have, for all $b \in B$, $r_0 b \notin A$ and, fixing any $b_0 \in B$, necessarily $r_0^n b_0 \in B$, for every $n \in \mathbb{N}$; but, as $r_0^n b_0 \to 0$ and $A|B$ is a positive cut, there would exist $n_0 \in \mathbb{N}$ for which $r_0^{n_0} b_0 \in A$, and this is a contradiction. The assumption $\sup X = r_0 < 1$ and $r_0 \in X$ also leads to a contradiction because, in this case, we have $r_0 = \frac{a_0}{b_0}$, for some $a_0 \in A$ and $b_0 \in B$; but, as A has no maximum, we can find $a_0' \in A$, $a_0 < a_0'$ and then $r_0 < \frac{a_0'}{b_0} \in X$, which is a contradiction. To conclude that $E|F = 1^*$ it is now enough to notice that

$$X = \{\tfrac{a}{b} : a \in A,\ 0 < a,\ b \in B\} = \{ac : a \in A,\, 0 < a,\ c \in C,\, 0 < c\}.$$

Given a negative cut $A|B$, then $(A|B)^{-1} = -((-(A|B))^{-1})$ since, by the definition of $\cdot$,

$$(A|B) \cdot (-((-(A|B))^{-1})) = (-(A|B)) \cdot (-(A|B))^{-1} = 1^*.$$

□

Definition 9. *We say that a partially ordered set $(P, \leq)$ is complete if any subset of P, bounded from above, has a least upper bound.*

Theorem 10. *The ordered field $(\mathscr{C}, +, \cdot, \leq)$ is complete.*

Proof. Let $\mathscr{D} = \{A_i|B_i : i \in I\}$ be a subset of $\mathscr{C}$, bounded from above. Let

$$A = \bigcup_{i \in I} A_i, \qquad B = \mathbb{Q} \setminus A. \tag{18}$$

Notice that

$$B = \mathbb{Q} \setminus \bigcup_{i\in I} A_i = \bigcap_{i\in I} \mathbb{Q} \setminus A_i = \bigcap_{i\in I} B_i.$$

We start by proving that $A|B$ is a cut:

- obviously, $A \neq \emptyset$, $B \neq \emptyset$, $A \cup B = \mathbb{Q}$ and $A \cap B = \emptyset$;
- let $a \in A$ and $b \in B$. We need to check that $a < b$. By (18), we know there exists $j \in I$ such that $a \in A_j$ and $b \in B_i$, for all $i \in I$. As $A_j|B_j$ is a cut, then $a < b$;
- let us see now that A does not have maximum. Let c be any upper bound of A. If $c \in A$ then $c \in A_j$ for some $j \in I$ and so, c would be maximum of A_j, which is impossible, since $A_j|B_j$ is a cut.

The cut $A|B$ is an upper bound of $\mathscr{D}$ since, because $A_i \subseteq A$, for all $i \in I$, by the definition of the order in $\mathscr{C}$, we have $A_i|B_i \leq A|B$, for any $i \in I$.

To prove that $A|B$ is the least upper bound of $\mathscr{D}$, let $C|D$ be any upper bound of $\mathscr{D}$. Then $A_i \leq C$, for all $i \in I$ and so $A = \bigcup_{i\in I} A_i \subseteq C$, which means that $A|B \leq C|D$. □

For further details concerning the Dedekind cuts construction, see [6, 7].

Definition 11. *We say that two ordered fields* $(\mathscr{A}, +, \cdot, \leq)$ *and* $(\mathscr{B}, \oplus, \odot, \preceq)$ *are isomorphic if there exists a function* $f : \mathscr{A} \longrightarrow \mathscr{B}$ *such that:*

a) f *is bijective;*

b) $\forall\, x, y \in \mathscr{A} \qquad f(x + y) = f(x) \oplus f(y);$

c) $\forall\, x, y \in \mathscr{A} \qquad f(x \cdot y) = f(x) \odot f(y);$

d) $x \leq y \Longleftrightarrow f(x) \preceq f(y).$

Theorem 12. *Any complete ordered field is isomorphic to* $(\mathscr{C}, +, \cdot, \leq)$.

Before proving this theorem, we need some auxiliary properties of the complete ordered fields.

Proposition 13. *Let $(\mathscr{A}, \oplus, \odot, \preceq)$ be a complete ordered field. Denoting by $\overline{0}$ its element zero, by $\overline{1}$ its element one, by $-a$ the symmetric of $a \in \mathscr{A}$ and by a^{-1} the inverse of $a \in \mathscr{A} \setminus \{\overline{0}\}$, we have:*

a) if $\overline{n} = \underbrace{\overline{1} \oplus \cdots \oplus \overline{1}}_{n \text{ times}}$, for $n \in \mathbb{N}$, the set $\mathscr{N} = \{\overline{n} : n \in \mathbb{N}\}$ is not bounded from above;

b) given $a, b \in \mathscr{A}$, $a \prec b$, there exists $m \in \mathbb{Z}$, $n \in \mathbb{N}$, such that $a \prec \overline{m} \odot (\overline{n})^{-1} \prec b$, where, for $m \in \mathbb{Z}^-$, $\overline{m} = \underbrace{(-\overline{1}) \oplus \cdots \oplus (-\overline{1})}_{-m \text{ times}}$.

Proof. a) Suppose that $\mathscr{N}$ is bounded from above. As $\mathscr{A}$ is complete, then $\mathscr{N}$ has a least upper bound $m \in \mathscr{A}$. But then, as $m \oplus (-\overline{1}) \prec m$, then $m \oplus (-\overline{1})$ is not an upper bound of $\mathscr{N}$, which means that there exists $\overline{n}_0 \in \mathscr{N}$ verifying $m \oplus (-\overline{1}) \prec \overline{n}_0$. So, $m \prec \overline{n}_0 \oplus \overline{1} \in \mathscr{N}$ and this contradicts the fact that m is an upper bound of $\mathscr{N}$.

b) Given $a, b \in \mathscr{A}$, $a \prec b$, we have $\overline{0} \prec b \oplus (-a)$. So, there existe $\overline{n} \in \mathscr{N}$ such that $(b \oplus (-a))^{-1} \prec \overline{n}$ or, equivalently, $(\overline{n})^{-1} \prec b \oplus (-a)$. If $\overline{0} \prec a$, it is easy to understand that there exists $\overline{m} \in \mathscr{N}$ such that $a \prec \overline{m} \odot (\overline{n})^{-1} = \underbrace{(\overline{n})^{-1} \oplus \cdots \oplus (\overline{n})^{-1}}_{m \text{ times}} \prec b$. The case $a \preceq \overline{0}$ is treated similarly.

$\square$

Proof of the Theorem 12: Let $(\mathscr{A}, \oplus, \odot, \preceq)$ be a complete ordered field. We want to find an isomorphism $f : \mathscr{C} \longrightarrow \mathscr{A}$. Using the notations of the previous proposition, we define

- $f(0^*) = \overline{0}$;
- $f(n^*) = \overline{n}$, for $n \in \mathbb{N}$;
- $f(-n^*) = -f(n^*)$, for $n \in \mathbb{N}$;
- $f((\frac{m}{n})^*) = f(m^*) \odot f(n^*)^{-1}$, for all $m \in \mathbb{Z}$ and all $n \in \mathbb{N}$ such that $(m, n) = 1$.

So, we have f defined in the set of rational cuts and it is easy to verify that

$$\forall\, r_1, r_2 \in \mathbb{Q} \qquad \begin{array}{l} f(r_1^* + r_2^*) = f(r_1^*) \oplus (r_2^*), \\ f(r_1^* \cdot r_2^*) \;\; = f(r_1^*) \odot (r_2^*), \\ r_1^* < r_2^* \quad \Longrightarrow \; f(r_1^*) \prec f(r_2^*). \end{array} \tag{19}$$

Let $A|B$ be any cut. The set $\mathscr{A}_A = \{f(r^*) : r \in A\}$ is bounded from above because, given an element $b \in B$, by (19) we have

$$\forall\, r \in A \qquad f(r^*) \prec f(b^*).$$

As $\mathscr{A}$ is complete, the set $\mathscr{A}_A$ has a least upper bound. Defining

$$\overline{f}(A|B) = \sup \mathscr{A}_A \tag{20}$$

we want to verify that, if $A|B$ is the rational cut a^*, then $f(a^*) = \overline{f}(a^*)$:

- as, for all $r \in A$, we have $r < a$, then, by (19), $f(r^*) \prec f(a^*)$ and so $\overline{f}(a^*) = \sup \mathscr{A}_A \preceq f(a^*)$;
- suppose that $\overline{f}(a^*) \prec f(a^*)$. By Proposition 13, b), there exist $\overline{m}, \overline{n}$, with $m \in \mathbb{Z}$ and $n \in \mathbb{N}$, such that $\overline{f}(a^*) \prec \overline{m} \odot (\overline{n})^{-1} \prec f(a^*)$. But $\overline{m} \odot (\overline{n})^{-1} = f(r^*)$, for $r = \frac{m}{n}$ and, since $f(r^*) \prec f(a^*)$, we have $r < a$, i.e. $r \in A$, which is a contradiction, because $\overline{f}(a^*) = \sup \mathscr{A}_A$.

So, defining $f = \overline{f}$, we have f defined in all $\mathscr{C}$. It remains to prove that f is an isomorphism, i.e., to prove a), b), c) and d) of the Definition 11:

d) Suppose that $A|B \leq C|D$. Then $\mathscr{A}_A \subseteq \mathscr{A}_C$ and so

$$f(A|B) = \sup \mathscr{A}_A \preceq \sup \mathscr{A}_C = f(C|D).$$

a) Suppose that $A|B \neq C|D$ and assume, without any loss of generality, that $A|B < C|D$. Then, by applying the Proposition 13, b), there exists $r = \frac{m_1}{n_1}$, with $m_1 \in \mathbb{Z}$ and $m_2 \in \mathbb{N}$, such that

$$f(A|B) \prec f(r^*) = \overline{m}_1 \odot (\overline{n}_1)^{-1} \prec f(C|D),$$

so f is one-to-one. The set $B_y = \{r^* \in \mathscr{C} : r \in \mathbb{Q} \text{ and } f(r^*) \prec y\}$, for a fixed $y \in \mathscr{A}$, is obviously not empty and is bounded from above (it is enough to note that there exists $n \in \mathbb{N}$ such that $f(r^*) \prec \overline{n}$, by the Proposition 13, a)). Let $A|B = \sup B_y$. We want to prove that $f(A|B) = y$. Supposing that $f(A|B) \prec y$, by the Proposition 13, b), there exists $r_0 \in \mathbb{Q}$ such that $f(A|B) \prec f(r_0^*) \prec y$, which contradicts the fact that $A|B = \sup B_y$. Supposing now that $y \prec f(A|B)$ we can find $r_1 \in \mathbb{Q}$ such that $y \prec f(r_1^*) \prec f(A|B)$. But then we have $r_1^* < A|B$ (recall that f is one-to-one) and, being $A|B = \sup B_y$, there exists $s^* \in B_y$ such that $r_1^* < s^* < A|B$. But then $f(r_1^*) \prec f(s^*) \prec y$, which also contradicts the fact that $A|B = \sup B_y$. Then $f(A|B) = y$, as we wanted to prove.

b) Let $x, y \in \mathscr{C}$ and suppose that $f(x+y) \neq f(x) \oplus f(y)$. Then

$$f(x+y) \prec f(x) \oplus f(y) \qquad \text{or} \qquad f(x) \oplus f(y) \prec f(x+y).$$

In the first case, there exists $r_0 \in \mathbb{Q}$ such that $f(x+y) \prec f(r_0^*) \prec f(x) \oplus f(y)$ and so $x + y < r_0^*$. We can write $r_0^* = r_1^* + r_2^*$ with $x \prec r_1^*$, $y \prec r_2^*$. Using (19) we know that

$$f(x) \oplus f(y) \prec f(r_1^*) \oplus f(r_2^*) = f(r_0^*) \prec f(x) \oplus f(y),$$

which is a contradiction.

Analogously it can be proved that $f(x) \oplus f(y) \prec f(x+y)$ is also a contradiction.

c) The proof that, for $x, y \in \mathscr{C}$, we have $f(x \cdot y) = f(x) \odot f(y)$ is also similar, so we omit it. □

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Research Seminar
on History and Epistemology of Mathematics, 141–156

The beginnings of the Royal Military Academy of Rio de Janeiro and the teaching of mathematics during the colonial period (1810-1822)[1]

Luis Saraiva
CMAF/Universidade de Lisboa
mmff5@ptmat.fc.ul.pt

Abstract

In this paper we aim to analyze the Royal Military Academy of Rio de Janeiro in its beginnings. We outline the situation in Brazil before and after the arrival of King D. João VI to Rio de Janeiro in 1808. We discuss in detail the founding document of the academy, evidencing some of its more innovative aspects.

Keywords: Portuguese mathematics, Brazil, Royal Military Academy, 19th century.

1 Introduction

The aim of this paper is to give a general overview of the beginnings of the Royal Military Academy of Rio de Janeiro in the period 1810-1822, the year of Brazil's independence. To be able to realize the impact of the

[1] This text is an abridged version of the paper "The Beginnings of the Royal Military Academy of Rio de Janeiro", which was published in *Revista Brasileira de História da Matemática*, vol 1(2007), n° 13, pp. 19-41, and is here included with its publishers' kind authorization.

coming to Brazil of the Royal Court, we give a brief overview of Brazil as a Portuguese colony before the arrival of King D. João VI in 1808, and contrast it with the changes implied in the multitude of decrees issued by the King in Rio. The main part of the paper will be on the Royal Military Academy, showing how it was thought to function, what was the structure of its courses, the subjects taught, the textbooks used, the rules for students and teachers.

2 Brazil before 1808[2]

In order to maintain the intellectual dependence of the colony and to control the diffusion of knowledge, the Portuguese did not allow either the press or high education studies in Brazil before the establishment of the Portuguese court in Rio de Janeiro. In those times only the Church and the military had forums to teach scientific knowledge. The fact that Brazil's economy was mainly based on slave work in agriculture and on the export of goods did not encourage the implementation of new techniques and consequently it was not a factor for a demand of new scientific practices and knowledge.

Some academies were founded during the 18th century, but they did not last long and they had no influence in the diffusion of scientific knowledge. In this the great distances between towns, with none of them acting as a centralizing focus was also a factor that reduced their impact. Among the more important we have:

1724 - Academia Brazileira dos Esquecidos [Brazilian Academy of the Forgotten], in Salvador;

1736 - Academia dos Felizes [Academy of the Happy Ones], in Rio de Janeiro;

1752 - Academia dos Selectos [Academy of the Chosen Ones], also in Rio;

1771 - Academia Científica do Rio de Janeiro[3] [Scientific Academy of Rio de Janeiro].

[2]On this subject, and on the subject of the next chapter, our main reference is [de Oliveira 2005].

[3]Concerning this academy, there are records that show that its general secretary maintained correspondence with the Swedish Royal Academy of Sciences ; it is significant also that the Academy's regulations stated that at their meetings debates should

Probably the most important of all the pre-1808 institutions was the Seminary of Olinda[4], in Pernambuco. This was founded in 1800 by Bishop Azeredo Coutinho, who had studied in Coimbra's *Faculty of Law* and had returned to Brazil in 1798. The Seminary had the structure of Lisbon's *Colégio dos Nobres*. According to [de Oliveira 2005, p. 88] this was the first Brazilian institution which was founded under the influence of Pombal's reform. It added a new science component, with the introduction of topics of Mathematics, Physics and Natural Sciences. French and Greek were also included in the curricula. It is also significant of a new attitude towards science that the state supported the founding of the seminary. Among the subjects taught there were geometry, natural history, chemistry and natural philosophy. In the last of these matters it was stated that the teacher should explain experimental physics, including mechanics, hydrostatics, and the principles needed to understand the machines and their power[5].

In spite of not having a press in Brazil, the written word arrived from Europe regularly, either brought by people crossing the Atlantic or through bookshops. According to [Cavalcanti 2003], between 1754 and 1805 there were at least 23 bookshops in Rio de Janeiro. Although censorship was always present, Cavalcanti states that he did not see a single technical book in the lists of forbidden books. We can have some idea of the books available through the analysis of the inventory that he produced of a bookshop in Rio in 1794. He mentions that there were 6.540 books. Of this, the vast majority (about 86%) were religious books, but still there was a significant minority of books on cultural matters- art, theater, poetry and opera-, and on didactic themes, respectively 531 books (about 8.2%) and 214 (about 3.3%). Of this last category, 47 books were on scientific matters, including 10 trigonometry and multiplication tables, 7 books on the elements of arithmetic, 6 architecture books, 4 pharmacy books, and 3 botany textbooks.

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be based on clearly written texts which had been in some way scientifically approved [de Oliveira 2005, p. 95].

[4]For more detailed information on the Olinda Seminary, see [das Neves 1984].

[5]Among the Olinda staff was Friar José da Costa Azevedo, later a teacher at the Royal Military Academy of Rio de Janeiro.

3 D. João VI in Brazil, 1808-1810

Due to the invasion of Portugal by Napoleon's armies in 1807, King D. João VI decided to leave Europe and to go with his court to Rio de Janeiro, where they arrived on January 22, 1808. With the King 15.000 people left Lisbon for Brazil. Their arrival had an immense impact in Brazilian life. Not only the nobility at least doubled their numbers, but also the population in Rio increased between 20 and 30%. One month before the King, the Royal Academy of Ensigns left Lisbon for the same destination, arriving in Guanabara Bay on January 18, 1808. They chose S. Bento Monastery as their new home.

The war had unexpected good consequences in the field of diffusion of ideas. As a counterpart for England's help in the war against France, the Brazilian ports were declared open to what was labeled "friendly nations", and which in fact had very bad economic consequences for the Portuguese crown. However this made the diffusion of ideas simpler, as many foreigners entered Brazil between 1808 and 1822. In Rio alone there is a record of 4.234 arrivals, of which 1.600 are from Spanish South-America, 1000 French, 600 English and over 200 German [Manchester 1970, p. 216] quoted by [de Oliveira 2005, p. 122]. Until 1808 only people born in the Portuguese Empire were authorized to compile data on scientific expeditions in Brazil. From 1808 onwards foreign scientific expeditions were encouraged to do research in Portuguese controlled territory, a clear sign of a change in mentality in the Portuguese authorities.

With the King and his Court living permanently in Rio, it became urgent to create a set of structures which until then had been denied to Brazil in order to maintain its depending status on the European Portugal. Thus a series of decrees started to change the face of the colony. It began with a declaration on January 28, 1808, that Brazil was now the center of the Portuguese Empire; on March 1 was declared the right to the free establishment of factories; on March 7 the *Military Archive* was created, with the aim to preserve and assemble all maps and charts, to copy them for border reassessments, fortress plans, new roads designs. It was also decided to publish a book on Topography aimed at perfecting geodesy measures that could be the counterpart of an annual. French topography handbook. On May 13 was founded the powder factory, with a higher educated board and inspectors; the same day was created the *Imprensa Régia* [the Royal Press]. It printed the first textbooks of mathematics, chemistry, physics and others used in higher education. The year after, on April 23, 1809, there was the founding of *Colégio das Fábricas* [Factories College], the first establishment in Brazil with the aim of teaching the technical trade to people coming from Europe lured by the working possibilities in Brazil.

In the 1820s there were 13 periodical journals in Rio. O *Patriota* [The Patriot], although it lasted less than two years, was of paramount importance[6]. It started its publication in 1813 on a monthly basis, and in 1814 it was published twice a month. It ended its publication in December 1814. Its director was Manoel Ferreira Araujo Guimarães (1777-1838), a teacher at the Royal Military Academy of Rio de Janeiro, having been previously a lecturer at Lisbon's Navy Royal Academy. This was Brazil's first cultural journal, which included articles of pure and applied science side by side with literary and history memoirs, translations, poems, news. Some of the most prestigious members of Brazil's scientific community collaborated with this journal, among them José Bonifácio de Andrada e Silva (1763-1838), naturalist; Francisco de Borja Garção Stockler (1759-1829), mathematician, historian of Portuguese mathematics and a previous secretary of Lisbon's Academy of Sciences; José Saturnino da Costa Pereira (1773-1852), mathematician and lecturer at the Royal Military Academy.

Among journals which were published outside Rio it is worth mentioning *A Idade de Ouro do Brazil* (Brazil's Golden Age), the first periodical newspaper published in Bahia, and which had over 120 issues between 1811 and 1819. It was essentially of local interest, as it published mainly local economic news.

4 The founding of the Royal Military Academy of Rio de Janeiro

4.1 Introduction

By a Decree of December 4, 1810, the new military Academy was founded. In its preface, it is said that that the main reason for the founding of the academy is the need of trained staff at a higher level. It is the need of new learned military staff that would be able not only to provide adequate

[6] On this journal see [de Oliveira 2004].

leadership in military matters, but also be competent in administrative posts, mainly those concerning trade and communications:

> *[...] Having in consideration that it is very important to my Royal Service, to the public welfare of my subjects, and to the defense and safety of my vast domains, that it is established in Brazil, in my actual Court and in the city of Rio de Janeiro a regular course of exact sciences, and of observation sciences, as well as all those which are applications of these to military and practical studies [...] such that from these courses graduate skilful Artillery and Engineering Officers, and also Officers from the Class of Geographic and Topographic Engineers, who can have the useful job of directing administrative matters in mining, roads, ports, channels, bridges, fountains and pavements. I decide to establish in my actual Court and city of Rio de Janeiro a Royal Military Academy with a complete course of Mathematical Sciences, of Observation Sciences, such as Physics, Chemistry, Mineralogy, Metallurgy Natural History, which will include the vegetal and animal kingdoms, and of Military Sciences in all its range, as well of Tactics as of Fortification and Gunnery [...]*[7]

It is clear that there were high hopes that the success of the Academy would bring a new generation of highly qualified officers, and to encourage the enrolment in the Academy of those who wanted to follow a military career many privileges were given.

The Decree [Collecção 1826], a 13-page document in twelve chapters (in Portuguese, *Títulos*), describes in detail, how the Academy should

[7] "[...] Tendo consideração ao muito que interessa ao Meu Real Serviço, ao Bem Público dos Meus Vassallos, e á defensa e segurança dos Meus vastos Domínios, que se estabeleça no Brazil, e na Minha actual Corte e Cidade do Rio de Janeiro, hum Curso regular das Sciencias exactas, e de Observação, assim como de todas aquellas, que são applicações das mesmas aos Estudos Militares e Práticos [...] de maneira, que dos mesmos Cursos de estudos se formem habeis Officiaes de Artilharia, Engenharia, e ainda mesmo Officiaes da Classe da Classe de Engenheiros Geographos e Topographos, que possão tambem ter o util emprego de dirigir objectos administrativos de Minas, de Caminhos, Portos, Canaes, Pontes, Fontes e Calçadas; Hei por bem, que na Minha actual Corte e Cidade do Rio de Janeiro, se estabeleça huma Academia Real Militar para hum Curso completo de Sciencias Mathematicas, de Sciencias de Observação, quaes, a Physica, Chimica, Mineralogia, Metallurgia e Historia Natural, que comprehenderá o Reino Vegetal e Animal, e das Sciencias Militares em toda a sua extensão, tanto de Tactica, como de Fortificação, e Artilharia,[...]" [Collecção 1826, p. 995].

function. All is regulated, one feels that there is a will to let the minimum possible left to chance, to guarantee the good functioning of the institution. There is an implicit recognition that all involved need well defined guiding lines in order to well accomplish their tasks.

We are going to analyze this document in some of its most important features.

4.2 The Academy's Course

In the Coimbra reform of 1772 it was advised that the teachers should write their own textbooks. However this was not successful, the only original textbook written by a Coimbra Faculty of Mathematics teacher before 1808 was not approved for teaching in the Faculty, and was only published as part of a bigger work after his author's death (Principios Mathematicos by José Anastácio da Cunha). Maybe because of this, the legislators wrote in Chapter III of the decree:

> *The appointed teachers can neither progress in the career nor obtain rewards and favour unless they have organized and written their textbooks by the method stipulated in the Statutes, and their works are approved by the Military Board*[8].

This clearly seems to have been effective, as the majority of books stated in the Statutes of 1810 were translated in the period 1809-1814, frequently extensively annotated, and some original textbooks were also written during this period.

The course was made to last seven years, of which the first four corresponded to the Mathematics course, and the last three to the Military Sciences course. The matters to be taught, the recommended textbooks, the number of teachers and of substitutes were all regulated in Chapter II, the longest of the twelve chapters of the decree. We are going to describe in some detail the course's first four years.

Let us see in each year what were the subjects taught, the recommended textbooks, its translations by Academy teachers (between brackets are their years of publication), and the major guidelines for teachers.

[8] "Os Lentes, que forem nomeados, não poderão ser adiantados em Postos, nem obter recompensas, e Graças, sem que cada hum delles tenha organizado e feito o seu Compendio pelo methodo determinado nos Estatutos, e sem que o seu trabalho seja approvado pela Junta Militar" [Collecção 1826, pp. 940/941].

Titles of the original works will only be given if they are mentioned explicitly in the decree. There would be one mathematics teacher for each of the first four years.

First year

Subject	**Textbooks**	**Translations**
Arithmetic	S. F. Lacroix	F.C.S.T. Alvim: Tratado de Aritmetica (1810)
Algebra (up to 3rd and 4th order equations)	S. F. Lacroix L. Euler: Elementos de Algebra	F.C.S.T. Alvim: Elementos de Algebra (1812) M.F.A.Guimarães: Elementos de Algebra (1809) M.F.A.Guimarães: Complementos dos Elementos de Algebra de Lacroix (1809)[9]
Geometry	A. M. Legendre	M.F.A.Guimarães: Elementos de Geometria (1809)
Trigonometry (including basic notions Spherical Trigonometry)	A. M. Legendre	M.F.A.Guimarães: Tratado de Trigonometria (1809)

Table 1: Subjects of the mathematics class of the first year of the Royal Military Academy

There are two important guidelines that are emphasized since the first year of the course: firstly there is an explicit recommendation to show connections between the different parts of mathematics, its inner coherence (the decree mentions its *beauty*) and their application to the real world; in particular it mentions the connections between the Principles of Algebra and those of Geometry, and the applications of Trigonometry to Geodesy; secondly it encourages the teachers to stir up research among students.

> [...] to try hard to make them [the students] work on problems, and try to develop that spirit of invention which in Mathema-

[9]In the Preface it is stated that this book aims to compense the lack of the second volume of Euler's Elements of Algebra.

tical Sciences bring the great discoveries[10].

And it states that the textbook that each teacher in the Academy should write not only had to include the matters developed in the recommended textbooks but also any new methods and innovations that might be discovered. So it is not only the Academy student that is encouraged to be inventive in his work, but also the teacher, who must be aware of what new knowledge is regularly being brought by the international community. In the first year it is also stated that there will be a class of Drawing [11], immediately after the Mathematics class, and lasting the same length o time.

In this part it is again emphasized the connections between different parts of mathematics and its applications to the real world (Mechanics, Hydrodynamics, Optics). The second year students will have another class: in alternate days they will have to attend Descriptive Geometry (textbook: G. Monge[13]) and Drawing.

It is recommended that the school should gradually build models of the machines studied for the students use. Also all theoretical aspects of Ballistics should be studied, so when the students arrived to the Gunnery Class they had only to study the practical uses derived from the theoretical principles. Third year students also had a class of Drawing, twice a week.

Concerning Laplace's Celestial Mechanics, the decree advised the teacher not to go into its theories, because "time would not be enough", but instead recommended that he should use its results for practical problems, like computing latitudes and longitudes, or to obtain Geodesy results. Students of the fourth year had another two classes: Physics was taught every day except two: in those two days the students had a Drawing class,

[10] "[...] trabalhando muito em exercitallos nos diversos Problemas, e procurando desenvolver aquelle espirito de invenção, que nas Sciencias Mathematicas conduz às maiores descobertas´´ [Collecção 1826, p. 937].

[11] Roberto Ferreira da Silva (?-?), substitute teacher of the class of Drawing at the Academy, wrote a textbook, "Elements of Drawing, Painting, and General Rules of Pespective´´, which was published in 1817.

[12] Although the date 1811 is stated on the book, it was only published in 1812.

[13] Translated into Portuguese by José Vitorino dos Santos e Souza (1780-1852), teacher of Descriptive Geometry at the Academy, as "Elementos de Geometria Descriptiva com applicações às Artes" (1812).

Second year

Subject	**Textbooks**	**Translations**
Revision of the Calculus notions taught in the 1st year		
Methods of solving equations – Applications of Algebra to the Geometry of lines and curves (degrees 2 and higher)	S. F. Lacroix: Principles of Algebra	F.C.S.T. Alvim: Elementos de Algebra (1812)[12] J.V.S. Sousa: Tratado Elementar de Applicação da Algebra à Geometria (1812) – translation of another Lacroix book
Differential and Integral Calculus (and applications to Physics, Astronomy and Probability Calculus)	S. F. Lacroix: Differential and Integral Calculus	F.C.S.T. Alvim: Tratado Elementar de Calculo Differencial e Calculo Integral – Two volumes: (Part I: 1812; Part II: 1814)

Table 2: Subjects of the mathematics class of the second year of the Royal Military Academy.

where they drew figures and machinery they studied in the fourth year. For Physics the main recommended textbook was Abbé Hauy's *Elements of Physics*[14]; another reference given in the decree for the Physics course was Brisson's textbook of Physics.

To complete this first outline of the Mathematics course it is important to point out the huge task accomplished by the Academy's lecturers in translating textbooks. Up to 1814 there were 16 major works translated[15], five in the period 1809-1814, all published in Rio de Janeiro by the *Imprensa Regia*. The fact that five of them were published during 1809/1810

[14]This was translated into Portuguese by Francisco Cordeiro da Silva Torres e Alvim (1775-1856) , at one time teacher of the sixth year subject on Fortification, etc, as "Tratado Elementar de Physica´´ (1810).

[15]Francoeur's Mechanics is a set of four books on Statics, Dynamics, Hydrostatics and Hydrodynamics.

Third year

Subject	**Textbooks**	**Translations**
Principles of Mechanics (Statics and Dynamics)	L. B. Francoeur	J: S. C. Pereira: Tratado Elementar de Mechanica – 4 volumes (1812)
Principles of Hydrodynamics (Hydrostatics and Hydrolics)	L. B. Francoeur is the main reference; Prony, Abbé Bossut, Fabre and Gregory (for machines and their applications)	
Theory of Ballistics	E. Bezout, B. Robbins and L. Euler	

Table 3: Subjects of the mathematics class of the third year of the Royal Military Academy

Fourth year

Subject	**Textbooks**	**Translations**
Spherical Trigonometry	A. M. Legendre	M.F.A. Guimarães: Tratado de Trigonometria (1809)
Optics, Catoptrics, and Dioptrics	N. L. Lacaille	A. P. Duarte: Tratado de Óptica (1813)
System of the World	J. J. L. F. Lalande, P. S. Laplace – Celestial Mechanics	
Geographic maps and projection techniques; Globe Geography	P: S. Laplace, N. L. Lacaille, J. J. L. F. Lalande, J. Pinkerton's Geography	

Table 4: Subjects of the mathematics class of the fourth year of the Royal Military Academy.

suggests that this project could have started as early as 1807. At the end of 1815 all the textbooks mentioned in the decree for the first two years of the course had been translated, and all the major works mentioned in the decree were either translated or had Academy teachers writing textbooks on those matters. For instance, Manoel Ferreira de Araujo Guimarães published textbooks on Astronomy (1814) and on Geodesy (1815), and had a small text on spherical triangles (1812). Guimarães also published on military matters: he translated Jean de Briche's *Engineer Handbook or Practical Geometry for Encampment Fortification* published in Bahia, with a second edition out in 1815. In a future paper I intend to analyze the works and translations done by the Academy's teachers in the early 19th century.

4.3 On the Academy's teachers

Besides the appointed teachers, the decree considered that there should be five Substitute teachers, in order to guarantee that there would not be classes not functioning in subjects where there were registered students. Chapter II ends with a recommendation: it said that, when conditions would allow it, a Scientific and Military Library should be founded, and its librarian would be the teacher of Military History, a subject to be taught in a future eight year of the course. Chapter III mentions that the publishing of *Memoires* is a criterion for the nomination of teachers and their substitutes. And to encourage the military to apply to the Academy, it is stated that they have the same privileges as the teachers in the Lisbon Military Academies of the Navy and of the Army. Also it is guaranteed that after 20 years of teaching in the Academy they are entitled to retire. As for wages, teachers will have, besides the pay due to their rank, 400,000 *reis* per year. The substitutes will be paid 200,000 *reis* per year, but if they are required to some destination which makes impossible their contribution to the teaching of their subjects, they will have no pay. On Chapter X it is also stated that the Academy's teachers will also have the same privileges as the University of Coimbra teachers.

4.4 On the Academy's students

This is the subject of Chapter IV. In it it is established the conditions of admission: the students must know the four elementary operations and be at least 15 years old. Among those eligible, there will be a preference for those who are acquainted with Greek, Latin, or any living languages. There will be two classes of students: compulsory and voluntary. The Academy tried to provide encouragement for the students' work, and above all for the compulsory ones, who were by far the main set of students in the Academy. So the compulsory students were the only ones eligible to be given the so called "*partidos*", honors given to those who distinguished themselves in the Academy's studies. Also there were monetary penalties for those who did not do well at the exams. From the moment they entered the Academy they received the pay of Artillery sergeants. But those who in the annual final exam did not have a complete pass saw their pay reduced to that of a soldier (Chapter XI).Also the Military Board had the power to expel from the Academy those who fail the exams in two consecutive years, and are seen with no hope of improving their situation. The Royal Academy students were declared to have the same privileges of the University of Coimbra (Chapter X). Also, when applying for promotion officers of equal good services, it should be selected the one who has completed the Academy's seven year course with good marks. Thinking of bringing improvement to the military leadership, it is also stated that during peace time no officer can have the rank of General or higher without having completed the Military Course. This last rule only concerns those who engaged in the Army after the date of this decree (Chapter VII) As said above, research was also encouraged. There were three prizes of 250,000 reis each for those who, each year, presented the best memoirs which had to include some discovery or some useful application to science. The Military Board was said to be the jury for their appreciation, and it had the authority to make them publish if they felt they deserved it (Chapter XI). Good students would be favoured in promotions application, and it was said that 2/3 of the Officer's places should be filled from the ranks of Academy students who have completed their courses with good marks and with an exemplary behaviour in the King's Service (Chapter VII).

5 Concluding Remarks

The shortcomings of the functioning of the Royal Military Academy should not detract from its immense merits. It tried to bring the mathematics made in Europe to Brazil, and in a way that the Academy´s students, once finishing the Academy's course, could be useful to their country using the subjects learned in the Academy. It was organized in a way that also tried to promote research, as well by the lecturers as by the students, with its "partidos" and awards. It marks the beginning of mathematics courses at a higher level in Brazil, an introduction that was seconded by accomplishing an enormous task of translating textbooks of some of the most reputed European mathematicians and writers of textbooks, among them Euler, Lacroix, Legendre, Francoeur, and Lacaille. It provided the basis for the future development of mathematics in Brazil.

Acknowledgements

I want to thank Professor Elfrida Ralha and the organizers of the meeting in honour of Professor Fernanda Estrada for inviting me to be one of the speakers in it. I feel honoured to participate in this most deserved homage to a long-standing friend who has such outstanding (and rare) human qualities. I also would like to thank Professor Sérgio Nobre of UNESP, Rio Claro, and Professor Luís Miguel Carolino of MCUL, Lisbon, for providing the necessary conditions for my research work in Brazil. I also thank the librarians of Biblioteca Nacional and of Arquivo Nacional of Rio de Janeiro for their help during my research. The research for this paper was supported by Fundação para a Ciência e Tecnologia, Programa POCI (Portugal/FEDER-EU), ISFL-1-209.

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Research Seminar
on History and Epistemology of Mathematics, 157–172

Semigroups also have history

M. Paula Marques Smith[1]
Centro de Matemática
Universidade do Minho
psmith@math.uminho.pt

Abstract

The algebraic theory of semigroups is a relatively recent addition to the development of Mathematics. Historically, original development of the theory began in the first third of the twentieth century. For decades prior to this, inspired by existing results in both group and ring theory, researchers proved many important results on semigroups, thus providing a solid foundation for the theory of algebraic semigroups. In this article, we give a brief account of the development of the algebraic theory of semigroups up to the publication of the major textbooks in the 1960's. We begin with aspects of the theory which were analogous to existing results in both group and rings and we present a short description of the group-theoretic circumstances which led to the initial definition of a semigroup. Finally, we consider the first independent theorems on semigroups: theorems with no group or ring analogues.

Keywords: Semigroup, history, completely regular semigroup, Rees matrix semigroup.

To Fernanda Estrada, on the occasion of her 80^{th} birthday.

[1]Research supported by the Research Centre of Mathematics of the University of Minho through the FCT Pluriannual Funding Program.

1 Introduction

A semigroup is simply a non-empty set upon which an associative binary operation is defined. The natural numbers, $\mathbb{N}$, under either addition or multiplication, and the set of all mappings of a set X into itself, under composition of mappings, $T(X)$, are natural examples of semigroups. This makes the notion of semigroup extremely natural and, considering $\mathbb{N}$, we can even say that semigroups have been present in mathematics right from its earliest origins. The real theory of semigroups is, however, a much more recent development - the algebraic theory of semigroups is firmly rooted in the twentieth century, with most of the major developments taking place after the Second World War.

Paying a tribute to my colleague *Fernanda Estrada*, this paper gives a short and simple account of the history of the algebraic theory of semigroups, focussing on certain aspects of the early theory which had a profound effect on its subsequent development.

We will take the story as far as 1941, since the early 1940's are a landmark in the development of the theory:

- Three seminal papers were published around 1940, written by Rees (1940) [20], Clifford (1941) [4] and Dubreil (1941) [9].
- Accordingly to Kleiner [16], the term *semigroup* was first defined to correspond to the notion of an associative, cancellative magma - here, magma is the Bourbaki term meaning a set which is closed under a given binary operation. In spite of this, between 1905 and 1940, the term *semigroup* was used with different meanings, for example, *associative magma* Hilton (1908) [11], *left cancellative associative magma* Bell (1930) [1], *left cancellative, commutative, associative magma with identity* Clifford (1938) [3]. From 1940 onwards, the term *semigroup* became fixed with its modern definition.
- Up to 1941, most of the results obtained were analogous of known results for groups and rings. A paper by Clifford, establishing and proving a result that has no precursor in either group or ring theory, was published in 1941 [4]. Quoting Howie [14], this paper can be considered *the beginning of an independent theory of semigroups.*

In Section 2 we give a brief overview of the development of the theory of algebraic semigroups, up to the publication of the major textbooks in the

1960's. In Section 3 we present a short description of the group-theoretic circumstances which led to the initial definition of a *semigroup*.

Section 4 describes the development of the first major theorem in semigroup theory: the Rees Theorem. In this theorem, using the Wedderburn Theorem in Ring Theory as a template, David Rees described the structure of what we now call *completely* 0-*simple semigroups*: semigroups which are unions of groups. Clifford took up this study. This is the subject of Section 5, which also contains the first *independent* theorem of semigroup theory.

Section 6 contains a comment on what we consider to be the most important reference work in semigroup theory to this day.

2 A brief review of the development of the theory

As we will see in Section 3, the term *semigroup* first appeared in order to provide a name for algebraic structures, different from groups, which arose during attempts to extend, to the infinite case, known results on finite groups. This work cannot, however, be regarded as a *semigroup theory*.

The theory of semigroups itself began to appear in the 1920s, with the work of the russian mathematician Anton Suschkewitsch, so much so that he has earned the title of the world's first semigroup theorist. He was the first mathematician to prove some of the results which we now take for granted. For example, he proved the representation theorem for semigroups, the semigroup analogue of Cayley's Theorem for groups: *every semigroup can be embedded in a full transformation monoid* (Suschkewitsch 1926). Suschkewitsch wrote the textbook *The theory of generalised groups* published in 1937 [23]. However, due to the political circumstances under which he lived in Ukraine, his work failed to find a wide audience during his lifetime. In fact, many post-World War II researchers did not realise that they were reproducing results first obtained by Anton Suschkewitsch. For example, the embedding of a finite semigroup in a full transformation monoid was reproduced by Stoll (1944) in [21].

During the decade of the 30s, the study of semigroups was still very influenced by the existing work on both groups and rings: semigroups were approached either by dropping some of the group axioms or by ignoring the addition operation from a ring. As the decade progressed, the theory gradually consolidated and gained ground, culminating with the publication of three highly influential papers: Rees (1940) [20], Clifford (1941) [4] and Dubreil (1941) [9]. The Rees and the Clifford papers are notable for containing substantial theorems - we will look at them in more detail in Sections 4 and 5. The Dubreil paper, primarily concerned with theory building, is extremely creative and proved to be immensely influential.

As a result of the influence of the above-mentioned papers, the theory of semigroups went from strength to strength and there was a significant increase in the number of published papers. The theory did not emerge fully structured, however, and the following comment was made by Nathan Jacobson in the preface of *Lectures in Abstract Algebra* (1951) [15]:

> *Though this notion appears to be useful in many connections, the theory of semi-groups is comparatively new and it certainly cannot be regarded as having reached a definitive stage.*

The 1950's saw the introduction of three broad concepts, still of enormous use and importance in the modern theory: *Green's Relations*, *Regular semigroups* and *inverse semigroups.* In 1951 J. A. Green [10] defined five equivalence relations on a semigroup S, in terms of its principal ideals. These relations, which reduce to the universal relation if S is a group, proved to be an immensely powerful tool in examining semigroups and studying their structure. In the same paper, Green introduced the notion of *regular* semigroup by analogy with that of *von Neumann regularity* in rings; the concept of a regular ring had been introduced by von Neumann in 1936, as an algebraic tool for the study of complemented modular lattices [24]. The study of classes of regular semigroups has proved particularly fruitful over the years. The first of these classes to make an appearance is believed to be the class of *completely regular semigroups*, in Clifford (1941) [4], as we shall see in Section 5. The third concept referred to above is that of an *inverse semigroup.* Inverse semigroups were introduced independently by Wagner, in 1952 [25], who called these semigroups *generalised groups*, and by Preston, in 1954 [18], who called them *inverse semigroups.* These semigroups arose from both the study of systems of partial one-one mappings of a set and the aim of finding an abstract structure corresponding to such a system, like abstract groups correspond to systems of permutations of a set.

In the 1960s the theory expanded so much that more textbooks appeared. First, Lyapin, *Semigroups* (1960), with the english translation appearing in 1963, [17]. In 1961 the first volume of Clifford and Preston *The algebraic theory of semigroups* [5] was published and the second volume followed in 1967, [6].

The subsequent development of semigroup theory clearly reflects the solid foundation provided by these three semigroup textbooks. In 1970, a journal completed devoted to semigroups was founded, *Semigroup Forum*, giving the theory an effective platform for further development.

3 The group structure and the concept of semigroup

The definition of *group*, as we know it today, exists since the beginning of the twentieth century. Following Kleiner [16], the first book to survey groups under the abstract point of view was *Éléments de la Théorie des Groups Abstraits*, by de Séguier (1904). Although most of this book deals with finite groups, various attempts to generalise, to the infinite case, the more general theorems, were made. It was in this context of generalisation that de Séguier realised that there were certain algebraic systems that were groups when they are finite and failed to be, when they are infinite. The will to provide a name for these *non-groups* led de Séguier to the definition of a new concept: that of a *semigroup*. De Séguier's definition, as translated by Dickson (1904) [7], is the following:

Definition 1. *A set G, which has generating set $S \subseteq G$ with respect to a given binary operation, forms a semigroup if the following postulates hold:*

(1) $(ab)c = a(bc)$, *for all* $a, b, c \in G$*;*

(2) *for any* $a \in S$ *and any* $b \in G$*, there is at most one solution,* $x \in G$*, of* $ax = b$*;*

(3) *similarly for* $xa = b$*.*

In his book, de Séguier mentions that left cancellation in a semigroup S follows from the definition. While suggesting that de Séguier's argument must have been based on the decomposition of an element as a product of the generators of S and on the repeated application of axiom (2), in [7] Dickson objects to the fact that de Séguier does not explicitly demand, in his definition, the closure of the binary operation. As a result, he modified the definition accordingly and called the new system 'semi-group', meaning, perhaps ,'half a group' and emphasising the connection with groups. We point out that, in defining a binary operation on a set S to be a mapping from $S \times S$ into S, the closure is ensured and is, therefore, implicit in de Séguier's definition. In a subsequent paper [8], Dickson introduced the following new definition, which removes the reference to the generator set and includes, explicitly, the closure and the cancellation law:

Definition 2. *A set G forms a semi-group under a given binary operation, if the following postulates hold:*

(1') *if $a, b \in G$ then $ab \in G$;*

(2') $(ab)c = a(bc)$, *for all $a, b, c \in G$;*

(3') *for any $a, x, x' \in G$ if $ax = ax'$ then $x = x'$;*

(4') *for any $a, x, x' \in G$ if $xa = x'a$ then $x = x'$.*

This is what it is called today a *cancellative semigroup.* It is known that every cancellative semigroup that is finite is a group. Thus, de Séguier's semigroup was only of interest when the set is infinite and does not form a group. In general, in the infinite case, a cancellative semigroup is not a group: for instance, under addition, $\mathbb{N}$ is a cancellative semigroup which is not a group. An example, using a rather elaborate method, was constructed by Dickson.

In the early decades of the twentieth century the distinction between a semigroup and a group was not clear and quite often researchers referred to systems corresponding to semigroups simply as 'groups'. This is probably due to the fact that the axiomatic definition of a group was still very new and also to some uncertainty in the move to the infinite case. The groups that many of theses researchers were studying were finite - groups of permutations of a finite set - and so only closure need be postulated to ensure that they were groups.

This 'unclear' situation persisted until the beginning of the 'real' theory of algebraic semigroups.

4 The Rees Theorem

Structure theorems are an important part of any algebraic theory. Basically, a structure theorem for a class $\mathscr{A}$ of algebras involves:

(1) buildings blocks belonging to a class $\mathscr{B}$;

(2) a 'recipe' for constructing algebras in $\mathscr{A}$ from algebras in $\mathscr{B}$;

(3) an isomorphism theorem indicating that the building blocks from $\mathscr{B}$ and the construction recipe are essentially unique.

In a good structure theorem, the class $\mathscr{B}$ must be substantially better understood than the class $\mathscr{A}$ and the recipe must be easy to implement.

In this section, we present the first major structure theorem of semigroup theory: the Rees Theorem. Established and proved by David Rees in 1940 [20], this theorem provides the structure of completely (0 -) simple semigroups and is, indeed, a good structure theorem! We begin with the definition of this class of semigroups.

Definition 3. *(Howie [13]) A semigroup S is called simple if its only two-sided ideal is itself. A semigroup S with 0 is called 0-simple if its only two-sided ideals are itself and $\{0\}$, and $S^2 \neq \{0\}$.*

A semigroup S without 0 (respectively, with 0) is said to be completely simple (completely 0-simple) if the following conditions hold:

(CS1) *S is simple (0-simple);*

(CS2) *S has a primitive idempotent, i.e., a non-zero idempotent e such that, for all non-zero idempotents $f \in S$, $ef = fe = f \Longrightarrow e = f$.*

Completely (0-)simple semigroups appeared in Rees' work when considering semigroup analogues of certain properties for rings. They also emerged, around the same time and independently, in Clifford's work (1941) [4].

(1) The building blocks

Let G be a group, let I, Λ be non-empty sets and let P be a matrix over the 0-group $G^0 := G \cup \{0\}$ which is regular in the sense that each row and each column contains at least one non-zero entry.

(2) The recipe

In the set $I \times G \times \Lambda \cup \{0\}$, define a multiplication by

$$\forall (i, g, \lambda) \in I \times G \times \Lambda \cup \{0\}, \quad 0\,(i, g, \lambda) = 0 = (i, g, \lambda)\,0 \text{ and}$$

$$(i, g, \lambda)(j, h, \mu) = \begin{cases} (i, gp_{\lambda j}h, \mu) & \text{if } p_{\lambda j} \neq 0 \\ 0 & \text{if } p_{\lambda j} = 0 \end{cases}$$

The set $I \times G \times \mathit{\Lambda} \cup \{0\}$ together with this binary operation is a completely 0-simple semigroup. This semigroup is denoted by $\mathscr{M}(G; I, \Lambda; P)$ and is called the $I \times \Lambda$ *Rees matrix semigroup over* G^0 *with sandwich matrix* P.

(3) The isomorphism theorem

Every completely (0-)simple semigroup is isomorphic to a semigroup constructed as above.

Rees Theorem 1 (1940)**.** *Let S be a Rees matrix semigroup over a 0-group with regular sandwich matrix. Then S is completely 0-simple. Conversely, every completely 0-simple semigroup is isomorphic to such a Rees matrix semigroup.*

A Rees matrix semigroup *without* 0 is defined by dropping, in the above definition, all references to 0, the requirement of P being 'regular' and defining multiplication simply by $(i, g, \lambda)(j, h, \mu) = (i, gp_{\lambda j}h, \mu)$.

The completely simple version of the Rees Theorem is also due to Rees:

Rees Theorem 2 (1940)**.** *Let S be a Rees matrix semigroup without 0. Then S is completely simple. Conversely, every completely simple semigroup is isomorphic to such a Rees matrix semigroup.*

Although the Rees Theorem is the first important structure theorem of semigroup theory, it is yet another result with an analogue in ring theory:

the Wedderburn-Artin Theorem. This theorem states, in particular, that *any ring satisfying the descending chain condition for principal right ideals is simple and isomorphic to some ring of square matrices over a division ring.* We will meet, in the next section, the first 'independent' theorem of semigroup theory.

5 The First Independent Theorem

Prior to the Rees Theorem, two papers gave the structure of certain special classes of simple semigroups. Accordingly to Hollings [12], Suschkewitsch (1928) determined the structure of finite simple semigroups - in particular, he studied right groups and showed that these are unions of groups. A. Clifford (1933) [2] extended this work to infinite right groups. We start with a brief summary of this latter paper.

Given a set with a binary operation, Clifford considered the following axioms:

(I) If $a, b \in G$ then $ab \in G$;

(II) for all $a.b.c \in G$, $a(bc) = (ab)c$;

(III) for each $a \in G$, there exists at least one left identity $e \in G$;

(IV$_L$) for each $a \in G$ and each left identity e of a, there exists at least one left inverse b of a, with respect to e;

(IV$_R$) for each $a \in G$ and each left identity e of a, there exists at least one right inverse b of a, with respect to e;

(V$_L$) for each $a \in G$, there exists at least one left identity e of a and at least one left inverse b, with respect to e;

(V$_R$) for each $a \in G$, there exists at least one left identity e of a and at least one right inverse b, with respect to e.

He showed that the systems (**I**, **II**, **III**, **IV**$_R$), (**I**, **II**, **III**, **V**$_L$) and (**I**, **II**, **III**, **V**$_R$) define the same algebraic structure and that this structure can be alternatively defined as follows: a *semigroup S in which, for all $a, b \in S$, the equation $ax = b$ has a unique solution in S.* Clifford showed further that such a semigroup is a union of groups and named the structure accordingly: a *multiple group*. The main theorem of the paper determines the structure of multiple groups.

With this work, Clifford extended part of the work of Suschkewitsch to infinite right groups and built towards the later Rees Theorem. More importantly, he initiated the study of semigroups which are unions of groups and this gave rise to the first major independent theorem of semigroup theory.

Definition 4. *(Clifford [4]) A semigroup S is said to admit relative inverses if, for every $a \in S$,*

(1) there exists $e \in S$ such that $ae = a = ea$ and

(2) there exists $a' \in S$ such that $aa' = e = a'a$.

Semigroups with relative inverses are now called *completely regular* semigroups.

Theorem 5 (Clifford [4], Theorem 2)**.** *Every completely regular semigroup S determines a semilattice P such that each $\alpha \in P$ corresponds to a subsemigroup S_α of S with the following properties:*

(1) S is the disjoint union of the S_α;

(2) each S_α is a completely simple semigroup;

(3) $S_\alpha S_\beta \subseteq S_{\alpha\beta}$.

Conversely, any semigroup S with this structure is completely regular. We say that S is a semilattice of completely simple semigroups.

The special case of completely regular semigroups in which the idempotents form a semilattice (i.e. the idempotents commute with each other) was considered by Clifford because this particularity makes the S_α groups. These semigroups are called *Clifford semigroups*. In a similar way that he had proved that a completely regular semigroup is a semilattice of completely simple semigroups, Clifford showed that a Clifford semigroup is a semilattice of groups. The following theorem provides a clear recipe for the construction of such a semigroup.

Theorem 6 (Clifford [4], Theorem 3). *Let P be a semilattice. To each $\alpha \in P$, we assign a group S_α in such a way that distinct groups S_α are disjoint. For each pair $\beta < \alpha$ (i.e., $\alpha\beta = \beta$), let $\phi_{\alpha\beta} : S_\alpha \longrightarrow S_\beta$ be a morphism such that $\phi_{\alpha\beta}\phi_{\beta\gamma} = \phi_{\alpha\gamma}$, if $\gamma < \beta < \alpha$, and let $\phi_{\alpha\alpha}$ be the identity automorphism of S_α. We let S be the union of the S_α and define the product of $a_\alpha \in S_\alpha$ by $b_\beta \in S_\beta$ to be*

$$a_\alpha b_\beta = (a_\alpha \phi_{\alpha\gamma})(b_\beta \phi_{\beta\gamma}),$$

where $\gamma = \alpha\beta$. Then S is a Clifford semigroup. Conversely, every Clifford semigroup is isomorphic to a semigroup constructed in this way.

Neither of these two theorems due to Clifford has an analogue in either group or ring theory. Theorem 3 is considered to be the first major structure theorem of an independent semigroup theory. Theorem 4 is certainly the second. Theorem 3 is also important because it provides a construction which has been the basis for many semigroup structure theories ever since.

Clifford's 1941 paper marks the beginning of an independent theory of semigroups. Moreover, the effect of this paper in the subsequent development of the theory was enormous. More than half a century later, referring to this paper, Preston wrote [19]:

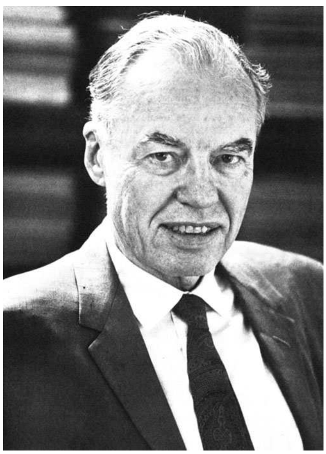
> *[It] was immensely influential. It contained definitive results that have been in continual use since. It introduced new concepts that provided powerful new tools for semigroup theory.*

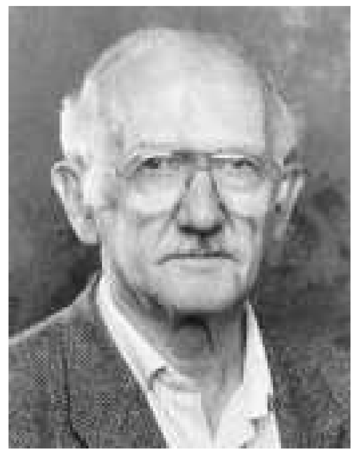
Clifford's papers certainly influenced many researchers not only because of the mathematical content but also because of the manner in which the content was presented, attracting to the domain of semigroups many outstanding mathematicians. Douglas Munn was one of them - as he often said,

> *the clarity of Clifford's papers appealed to me greatly and this consolidated my decision to work in the field.*

6 The most important reference work

Rees Theorem had a tremendous impact in semigroup theory and a remarkable influence in its subsequent development. After 1940, the number of research papers on semigroups appearing in the literature increased to an average annual production of about 30 papers. In response to this developing interest, two mathematicians, one american, *A. H. Clifford* (1908-1992), and the other british, *G. B. Preston* (1925-), wrote the work which remains, to this day and almost half a century later, **the most important reference work** in semigroup literature:





A. H. Clifford and **G. B. Preston**

The algebraic theory of semigroups

Mathematical Surveys No.7 Vol I (1961), Vol II (1967), AMS

These two books are the most influential semigroup textbooks to date. Not only did they collate many of the existing results on semigroups but they also added new ones and, above all, standardised the notation and terminology of the theory. The early work of Suschkewitsch, Clifford and Rees provided the foundation for the first volume and both volumes provided a solid foundation for the subsequent development of the theory. The books by Clifford and Preston were one of the major driving forces for the formation of a coherent theory of semigroups.

As for the theory of semigroups, in the words of C. Hollings [12],

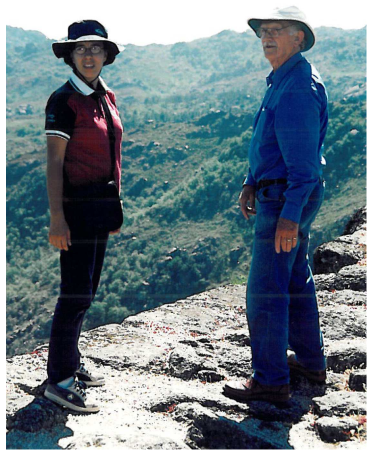
> *The investigations of Suschkewitsch, on finite simple semigroups, of Rees, on completely (0-)simple semigroups, and of Clifford, on unions of groups, served not only as a solid starting-point for the theory of semigroups, providing elegant methods and a framework for subsequent research, but also as a source of further interesting problems. Thanks to the early boost that these early researchers provided, the theory of semigroups continues to go from strength to strength.*

7 A privilege and a joy

While studying for her doctorate, at the University of St. Andrews, in Scotland (1980-1983), the author had the honor and the pleasure of meeting many important mathematicians, responsible for great advances in semigroup theory and whose contributions cannot be ignored by anybody researching on the history of semigroup theory. Douglas Munn (1929-2008) and John Howie (1936-) are outstanding examples. The author also had the immense privilege of meeting A. H. Clifford (1982) and having discussed with him her first progress in her research problem. As if that was not enough, the author lived the unforgettable experience of hearing A. H. Clifford talking about his own achievements, ideas and difficulties of the previous 40 years.

With very best wishes to Paula Marques -
Alfred Clifford
St. Andrews, 31 July 1982 -

Almost twenty years afterwards, the privelege was ‘complete’: while organising the International Conference on Semigroups, which was held at the Universidade do Minho, the author had the pleasure of having Gordon Preston accepting the invitation to participate in the conference. This new contact brought the joy of providing, to the younger generation of semigroup researchers, a privilege similar to the one the author was offered eighteen years before!



Suzana Gonçalves and G. B. Preston
(*Castro Laboreiro*, 1999)

Acknowledgements

The author is pleased to acknowledge her gratitude to Christopher Hollings, for kindly providing access to historical archives.

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Research Seminar
on History and Epistemology of Mathematics, 173–184

Finite Semigroup Theory, an historical perspective

M. Lurdes Teixeira[1]
Centro de Matemática
Universidade do Minho
mlurdes@math.uminho.pt

Abstract

In this brief note it is presented what is necessarily a personal, and certainly by no means complete, point of view on the central results and the most fruitful methods in finite semigroup theory. The aim is to justify the autonomy of this field and to put it into an historical context. The initial motivation for studying finite semigroups was not originated by algebraic problems, by the contrary it comes from outside, from theoretical computer science as a consequence of the connections of semigroups with formal languages and automata. It was with the advent of electronic computers, in the 1950's, that the study of formal models of computers, such as automata and sequential machines, was given the attention of many researchers that develop an algebraic approach for computational problems. As an algebraic field, the theory of finite semigroup grows up using techniques from group theory, ring theory and universal algebra, but their richness is a consequence of the interaction with other areas of Mathematics and with Computer Science.

Keywords: semigroup, automaton, rational language, pseudovariety, implicit operation, equation, category, decidability.

[1]Work supported by the Research Centre of Mathematics of the University of Minho through the FCT Pluriannual Funding Program.

This exposition is not meant to be self-contained, so it is expected that the reader has familiarity with basic algebraic concepts and techniques and with rudiments of automata theory.

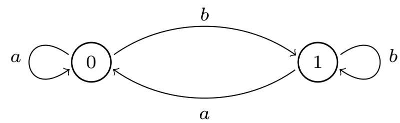
The first steps about finite semigroups had the aim to classify semigroups up to isomorphism, as it is usual in algebraic theories. At the time, the finiteness of the semigroup order was not considered an important property, and finite semigroups are only particular cases of semigroups. A first successful result concerning finite semigroups is due to Anton Kazimirovich Suschkewitsch, which described the structure of the minimal ideal of a finite semigroup [24] in 1928. In 1940 and 1941, David Rees extends this result describing, up to isomorphism, every completely simple and completely 0-simple semigroups as a certain matrix semigroups over groups [16, 17]. The sequel of this work was the James Alexander Green's theory based in the introduction of the well known equivalence relations $\mathscr{R}, \mathscr{L}, \mathscr{H}, \mathscr{D}$ and $\mathscr{J}$, actually known by Green's relations, which are essential for understanding how a semigroup is built up, both locally and globally. From a local point of view, Green's relations yield the notion of a coordinate system for a regular $\mathscr{D}$-class [10], generalizing Suschkewitsch's results. Note that in finite semigroups the relations $\mathscr{J}$ and $\mathscr{D}$ coincide. A missing point was concerned with the product of elements of different $\mathscr{D}$-classes. The gap was filled in by Marcel-Paul Schützenberger with the representation of semigroups by monomial matrices [20, 21]. From the global point of view this representation gives wreath product coordinates to the action of a semigroup on the left or right of a $\mathscr{D}$-class.

There are too many isomorphism classes of finite semigroups and few of them cover almost all finite semigroups, namely the classes of 3-nilpotent semigroups, which are semigroups such that all products of three elements are equal. In fact as the order of the semigroups grows up the percentage of 3-nilpotent semigroups goes to 1. So, the classification of finite semigroups up to isomorphism is hopeless and the applications confirm this. Hence, groups and semigroups are quite different. In a classical algebraic approach the class of groups is richer than the class of semigroups, but semigroups have wider applicability.

The crucial steps in direction to the autonomy of the field of finite semigroup theory appear only in the 1950's, and they were a consequence of the development of electronic computers and the study of simple formal models of computers such as automata and sequential machines. The starting point of the automata theory is connected to a famous theorem due to Stephen Kleene, which characterizes rational languages as the languages recognized by a finite automata [11]. The basic notions and results of automata and rational language theory can be interpreted in a very rich way in algebraic and logical terms. An (finite) automaton can be algebraically interpreted as an action of the free semigroup on an alphabet (which is a finite set) on the finite set of states. Hence, to each automaton can be associated a finite semigroup called the transition semigroup, which is generated by the partial transformations of the states defined by the letters of the alphabet. Moreover, every semigroup is isomorphic to the transition semigroup of a finite automaton. The free semigroup on an alphabet A over the class of all semigroups is denoted by A^+ and it contains all nonempty words on A. Rational languages on A are subsets of A^+ defined by rational expressions, which can be thought of as a generalization of polynomials involving three operations: union (which plays the role of addition), set concatenation (which plays the role of product), and the plus operation (which generalizes the power operation since it gives as a result the union of the iterated concatenations of a set, of any positive order). If it is used the star operation instead the plus operation, which means that one accepts the existence of a empty word and considers iterated concatenations of any non negative order, then one works in the monoid context instead in the semigroup context, which in some instances are not identical.

In 1957, John Myhill proved that a language L on an alphabet A is recognized by a finite automata if it is recognized by a finite semigroup, which means that there is a finite semigroup S and a homomorphism $\varphi : A^+ \to S$ such that $L = \varphi^{-1}(\varphi(L))$ [15]. He introduced the notion of syntactic semigroup of a language, which is the quotient of the free semigroup by the largest congruence that saturates the language (see also the work of Michael Rabin and Dana Scott [18]). The syntactic semigroup of a language is the smaller semigroup that recognizes the language, so a language is recognizable if and only if its syntactic semigroup is finite . Hence, rational languages are precisely subsets of the free semigroup saturated by a finite index congruence.

The connection between automata, rational languages and semigroups was first used to obtain computability results. In the mid-1960's, Kenneth Krohn an John Rhodes studied a decomposition theory of machines inspired in the theory of groups and they achieved a successful theory of decomposition of machines and of semigroups. First, they proved that every finite automaton can be decomposed as an iterated cascade product of elementary components, which are simple permutation automata (in other words, automata whose transition semigroups are simple groups) and the two state flip-flop.



Flip-flop automaton

The associative operation wreath product of semigroups models the cascade product of automata and a fundamental notion to obtain a successful algebraic decomposition theory is the division of semigroups. One says that a semigroup S divides a semigroup T if S is homomorphic image of a subsemigroup of T. Thus, for wreath product the prime semigroups are the simple groups and the divisors of the tree element monoid of transformations of the set $\{0, 1\}$ consisting of the constant maps and the identity map. So, the prime semigroups which are not groups are the divisors of the flip-flop transitions monoid, which are aperiodic semigroups (that is semigroups whose subgroups are trivial). Hence in semigroup context the fundamental result, known by Krohn-Rhodes Prime Decomposition Theorem, states that every finite semigroup S divides an iterated wreath product of prime semigroups, where the simple group factors are divisors of S [12]. This is another theorem that led to develop the theory of finite semigroups, which had not previously deserved any specific attention from semigroups's researchers. Krohn and Rhodes proposed as a measure of (group) complexity of a finite semigroup the minimum number of group factors (not necessarily simple groups) in a simplified form of the prime decomposition which is an iterated alternate wreath product of groups and aperiodic semigroups. Note that the wreath product of groups is a group and of aperiodic semigroups is an aperiodic semigroup. The existence of an uniform algorithm to compute the complexity of a semigroup has been a central question in finite semigroup theory. Although various announcements of a solution were made, a complete and correct answer to the problem has not been published yet.

The first reference book on the subject, *Algebraic Theory of machines, languages and semigroups* [6], was edited by Michael Arbib and contains an exposition of the state of the art of the semigroup theory approach to automata an languages theory as it is by the end of 1960's. The book had contributions from twelve authors with special reference to the contributions by Krohn and Rhodes.

One precursor example of a successful attempt to use algebraic proprieties of finite semigroups to effectively characterize natural classes of rational languages is the Schützenberger's theorem, published in 1965, that states that star-free languages are the languages whose syntactic semigroups are aperiodic [22]. The appropriate framework for such applications was provided by Samuel Eilenberg together with Schützenberger and Bret Tilson, in the mid-1970's. They introduced the notion of pseudovariety of semigroups, that is a class of finite semigroups closed for homomorphic images, subsemigroups and finite direct products, and a correspondence that to a pseudovariety V associates the variety of rational languages whose syntactic semigroups are elements of V. Examples of pseudovarieties are G, the class of all finite groups, A, the class of all finite aperiodic semigroups, and N, the class of all finite n-nilpotent semigroups for any natural n. Since pseudovarieties of semigroups constitute a complete lattice for the inclusion ordering, in actual language Eilenberg's fundamental theorem establish that the lattice of varieties of languages is isomorphic to the lattice of pseudovarieties of semigroups. In this context, the Schützenberger's theorem is an instance of this correspondence in the case of the pseudovariety A whose image is the variety of star free languages. Similar examples are due to Janusz Brozozowski and Imre Simon and to Robert McNaughton for the variety of locally testable languages which is the image of the pseudovariety LSl of the local semilattices [8, 14], and to I. Simon for the variety of the piecewise testable languages which is the image of the pseudovariety J of the $\mathscr{J}$-trivial semigroups [23].

The classification of semigroups in pseudovarieties driven by applications was the successful one and the initial direction had to be abandoned. Through Eilenberg's framework a program for the classification of rational languages was provided and many problems in finite semigroup theory motivated by applications in computer science may be formulated in terms of the decidability of pseudovarieties of semigroups. A pseudovariety V is decidable if there is an uniform algorithm to check if a finite semigroup belongs to V or not. In the mid-1970's finite semigroup theory essentially became the study of pseudovarieties and operators on pseudovarieties, most of them reflecting important combinatorial operations on varieties of rational languages. Examples of such operators are the wreath product, join, Mal'cev product and the power. Pseudovarieties that are operators images are often defined by a set of generators. In such cases the study of decidability becomes more difficult, because often it is unknown a procedure to check whether a semigroup is not in the pseudovariety.

At the pseudovariety level the operators semidirect product and wreath product coincide, from which it follows that the semidirect product of pseudovarieties is associative. The Krohn-Rhodes decomposition theorem formulated in terms of pseudovarieties states that every semigroup lies in to a pseudovariety of the form $\mathsf{A} * \mathsf{G} * \mathsf{A} * \cdots * \mathsf{G} * \mathsf{A}$, where $*$ denotes the semidirect product. Hence, the complexity problem became the problem of the existence of a uniform decision procedure for iterated semidirect products of this form.

A second reference book is *Automata, languages and machines, vol. B* [9], written in 1976 by Eilenberg, which contains two chapters with contributions from Tilson. The aim is to present the application of algebraic methods on the study of recognizable sets and sequential functions. The 1970's where rich in the development of Eilenberg's program that encloses two difficult problems. One is related to the computation of semidirect products of pseudovarieties and the other to the syntactic characterization of pseudovarieties. The attempts to solve such problems conduces finite semigroup theory to its actual situation.

Influenced by previous works of several authors, in particular on graph congruences and wreath products, Tilson realize that categories and semigroupoids (*i.e.* categories without the requirement of local identities) viewed as partial algebras generated by graphs, are the essential ingredients to understand semidirect products. Hence, semigroups (respectively, monoids) are particular cases of semigroupoids (respectively, categories) by viewing its elements as edges at a single virtual vertex. The notion of division naturally drives on to the notion of relational morphism introduced by Tilson and Rhodes, as a relation between two semigroups, S and T, such that the image of every element of S is a nonempty subset of T, and for every two elements of S the product of its image sets is contained in the image set of the product of the elements. In this context, Tilson introduces the concept of derived semigroupoid which establishes an intimate link between relational morphisms and wreath product decomposition. The Derived Semigroupoid Theorem [25] clarifies this link and implies that, if V and W are pseudovarieties of semigroups, the semidirect product $\mathsf{V}*\mathsf{W}$ is the pseudovariety of all semigroups S such that there exists a semigroup T in W and a relational morphism $\varphi : S \longrightarrow\!\!\!\!\!\!\circ\!\!\!\!\rightarrow T$ such that the derived semigroupoid associated to φ divides a semigroup on V. Hence, there is an alternative definition of semidirect products of pseudovarieties based on proprieties of its elements, instead defined by a class of generators. Naturally this drives on to the extension of the concept of pseudovariety to categories and semigroupoids.

Free semigroups over a pseudovariety are often infinite semigroups. Moreover, in general pseudovarieties can not be characterized by equations. For example, there is no nontrivial semigroup equation satisfied by all finite groups or by all finite nilpotent semigroups. The free semigroups on a set A over G and over N are equal to A^+. So, classical results of Universal Algebra, like the equational characterization of varieties in the Birkhoff's theory, does not hold for pseudovarieties. The solution proposed by Eilenberg and Schützenberger was the characterization of a pseudovariety by an infinite sequence of equations that ultimately defines it, in the sense that every semigroup in the pseudovariety satisfies all equations of the sequence after some sufficiently large order. In practice this approach is useless and it was necessary to define limits on A^+. In order to define limits, semigroups needed to be endowed with a topological structure and so one consider that all finite sets are endowed with the discrete topology and A^* is endowed with the initial topology for the homomorphisms on semigroups of V.

In 1982, Jan Reiterman proposed the adequate approach to solve the equational characterization problem of pseudovarieties. He proved that pseudovarieties of semigroups are the classes of finite semigroups defined by sets of pseudoidentities [19], defined as formal equalities between implicit operations. The notion of implicit operation was already known and it is due to F. William Lawere [13]. An implicit operation on a class of semigroups is a family of semigroup operations with the same arity, indexed by the class, which commutes with homomorphisms between semigroups in the class. Note that every word on an alphabet with n elements determines a n-ary operation on each semigroup, so each element of the free semigroup determines an implicit operation. Reiterman defined a metric on sets of implicit operations but no algebraic structure.

An independent proof of Reiterman's theorem was done by Bernhard Banaschewski, who introduced pseudoidentities as formal identities between members of a free profinite semigroup [7]. In general, given a pseudovariety V, a semigroup is said to be pro-V if it is compact and residually in V in the sense that distinct elements could be separated by continuous homomorphisms on elements of V. A profinite semigroup is a pro-S semigroup where S is the pseudovariety of all finite semigroups. Of course, semigroups of V are pro-V semigroups and, for every alphabet A, the class of pro-V semigroups has a free object on A. Such free pro-V semigroups capture the algebraic and combinatorial properties of the semigroups of V. The elements of the free profinite semigroup on a set A could be identified with limits of sequences of words on A.

However, it was only under the impetus of Jorge Almeida that profinite methods and the syntactic approach became fundamental tools in semigroup theory. Almeida's early work on this subject explores connections with Universal Algebra in order to improve the knowledge about the lattice of pseudovarieties of semigroups and to compute operators on pseudovarieties. This is the aim of finite semigroup theory nowadays. Much of this work can be founded in his book *Finite semigroups and universal algebra* [1].

In general operators on pseudovarieties do not preserve decidability, so a central question is under what conditions on the arguments the operator image is decidable. The notions of hyperdecidability, introduced by Almeida [2], and of tameness, introduced by Almeida and Benjamin Steinberg [4, 5], came about precisely in trying to find a stronger form of decidability which would be preserved or at least guarantee decidability of the operator image.

In the case of the semidirect product, the profinite approach was extended to pseudovarieties of semigroupoids and one goal was to find basis of pseudoidentities for pseudovarieties of the form $\mathsf{V} * \mathsf{W}$ given pseudoidentity basis for V and for W. In general the problem is still open but an upper bound is provided by the usually designed "basis theorem" of Almeida and Weil. Particular results were known, namely for cases where the upper bound given by the basis theorem coincides with the semidirect product. In [3] some key problems on finite semigroups that have guided the theory of pseudovarieties until the end of the XX century are reviewed and several open problems at the time are discussed.

The most recent book on finite semigroup theory is *The* q*-theory of finite semigroups*, from Rhodes and Steinberg, published in 2009. The q-theory is a new theory and the aim is to provide a unifying approach to finite semigroup theory via quantization. The authors classify this book as a research manuscript that contains a contemporary exposition of the complete theory of the complexity of finite semigroups, including classical results that profit from a recasting in a modern language. In this approach, relational morphisms were considered instead homomorphisms in the perspective that they should be the key structure. They introduce the product and the division of relational morphisms and develop a new language defining pseudovarieties of relational morphisms aiming the construction of a Reiterman's theory for such pseudovarieties.

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Time and Pedro Nunes' pursuit of mathematical certainty

Ângela Lopes
Escola Secundária
Vila Cova da Lixa
angelafafe@gmail.com

M. Elfrida Ralha[1]
Centro de Matemática
Universidade do Minho
elfrida@math.uminho.pt

Carlos Vilar[1]
Centro de Matemática
Universidade do Minho
cvilar@math.uminho.pt

Abstract

Time has been a familiar topic for as long as one can imagine but, as St. Augustine puts it, one is not capable of easily and briefly explaining it. Besides, measuring time poses problems which may, in particular, interfere with specific standards of mathematical certitude.

The purposes of regularity and convenience in Calendars embody an ancient and complex mathematical problem. One may even think that timetables, in which human activities are scheduled and recorded, rule life with natural perfection. However, no matter how natural may seem some cycles of one's life and no matter how perfect calendars may seem to be, the mathematical modeling of such cycles - under social, cultural, religious and/or other constraints - is an artificial erroneous exercise whose ideal has often appeared as an impossible task.

[1]Work supported by the Research Centre of Mathematics of the University of Minho through the FCT Pluriannual Funding Program.

In this article we will analyse some historical and some mathematical data about calendars, reporting especially on Portuguese literature. In particular, we will focus our attention on the Gregorian Calendar and we will refer to some events and works in the life of the Portuguese mathematician Pedro Nunes. It is our belief that Nunes' criteria for mathematical rigor are well reflected in his approach to the problem of measuring time.

Keywords: Calendars, Portuguese literature, Mathematical Certitude, Pedro Nunes.

Dedicated, with affection, to our friend Maria Fernanda Estrada.

> ... For what is time?
> Who can easily and briefly explain it? Who can even comprehend it in thought or put the answer into words? Yet is it not true that in conversation we refer to nothing more familiarly or knowingly than time? And surely we understand it when we speak of it; we understand it also when we hear another speak of it.
> What, then, is time?
> If no one asks me, I know what it is. If I wish to explain it to him who asks me, I do not know.
>
> St. Augustine, 4^{th} Century, *Confessions*, Book XI

1 Introduction

Time has naturally been a human concern for as long as one can imagine. Day and night, cyclic seasons and other periodic events came to be studied by looking at the sky (the Sun and the Moon, in particular) which offered the answers to planning the "right" moments for the most important daily tasks: to sow, to harvest, to give birth, and so on.

Marshack's research, as pointed out by [Richards2000], came to suggest that men have been recording the phases of the Moon for more than 30 000 years. The birth of civilization pushed people even further to conceive calendars as physical tools based on astronomical events defining basic measures for time.

Astrologers, physicists and many other experts have attempted, more or less successfully, to develop tools for measuring time aiming to reach a necessary equilibrium between Religious demands and Mathematical solutions. In this context, Mathematics was once more to prove itself a powerful instrument for organizing the practical knowledge of adapting the day in accordance with the religious aspirations of different people on different occasions and in various geographies.

The divisions of time, for instance at the beginning of Portugal's own history (12th century), were essentially the same as they are nowadays, in spite of the Gregorian reform having taken place in the 16th century. The traditional cycles that served as the inspiration for measuring time (making calendars) are related to the Moon ($\sim 29,5305$ days) and to the Sun ($\sim 365,2422$ days) and those are not nowadays, neither they were in the past, easy numbers to work with. In fact, on this quest for finding suitable numbers for measuring time Alphonso X wrote in one of his famous books on Astronomy that *if God had consulted me, I could have offered Him some wise advice*[2].

The Hebrews, for example, choose the Moon and made their month close to the average lunar cycle. From them we have inherited the regular 7-day week possibly because each phase of the Moon has, more or less, this length and later on also used for the biblical account of Creation. The Egyptians, acknowledging the connection between the floods of the Nile and the rise of Sirius, were the first to establish an average cycle for changes in the weather (seasons); from there we have inherited the year of 365 or 366 days. Christians did not give up on an even more difficult task: the synchronization of both the lunar and the solar cycles, establishing a lunisolar calendar.

2 The Julian Calendar legacy

When each 4 years we have one additional day in the annual cycle we are, actually, using a rule already stated by Julius Caesar (by 46 b.C.) for a leap/bissextile year. Acknowledging an error when using 365 days in a year, the Julian calendar proposed that common length and managed

[2] Alphonso X (1221-1284) - "El Sabio" (the Wise/Learned Man) and King of Castile and Leon - upon whose order a highly influential set of astronomical tables and treaties was prepared (in Toledo, around 1260), was the grandfather of the Portuguese King D. Dinis (founder of the Portuguese University). Following Ptolemaic methods, Alphonsine's tables chartered the movement of heavenly bodies and proposed a new length for the tropical year (using a sexagesimal system): 365 days, 5 hours, 49 minutes and 16 seconds.

by having an *ante diem bis sextus Calendas Martii* (a second sixth day before the beginning/*calendas* of March).

With this intercalation method, one year in the Julian calendar lasted, on average, 365 days and 6 hours (one quarter of the day). But let´s check with some easy calculations since this correction amounts to taking the year almost 10 minutes longer than the tropical year itself:

$$365,25 - 365,2422... = 0,0078...$$

Does it seem small? Well, the difference is approximately $3,12$ days for each 400 years (1 day in just over 128 years) and this "small" difference was soon to be noticed by our ancestors.

3 Easter: an extra complexity

Setting the date of Easter (a movable feast registered according to the Gospels) brought new complexity to the organization of the Julian Calendar. By neglecting such movable feasts one has a "simple" arithmetic solar calendar but the addition of these Christian ecclesiastical demands transformed it into an arithmetic lunisolar calendar.

By 325, the first Ecumenical Council, gathered at the Imperial Palace of Nicaea, discussed the dissociation of two dates: the celebration of the Resurrection of Jesus could not match nor precede the Jewish Passover. This Council, hosted by Emperor Constantine himself, gathered 380 representatives of the Christian churches and decided to accommodate the date for the resurrection of Christ in the existing Julian calendar. In addition, the final decision defined Easter by means of the vernal equinox.

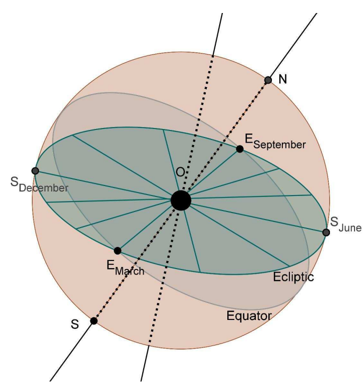

Figure 1: The Celestial Sphere, with the equinoctial and the solsticial points.

From then on, Easter has been scheduled on the Sunday immediately after the first full moon that falls on or next after the date of the vernal equinox.

On the precision of using the equinoxes to set the length of the year one may show (Cf. Figure 1) the Ecliptic (plane), the Celestial Equator (above the Earth's equator) and the Earth's Polar Axis NS (in which N represents the North Celestial Pole and S represents the South Celestial Pole). The intersections shown between the Equator and the Ecliptic are the equinoctial points: the vernal point E_{March} and the autumnal point $E_{September}$. By extension, the term equinox may denote an equinocial point. Due to the obliquity of the Ecliptic (approximately 23.44^o) there are two points S_{June} and $S_{December}$ which are the northernmost and southernmost extremes of the Ecliptic, when the solstices occur.

The bishops of Rome and Alexandria also decided that, each year, they were due to inform, in advance, the correct date for celebrating Easter. However, this procedure proved to be a complicated task and it did not take long until Rome and Alexandria were implementing different practices. Under the circumstances, direct astronomical observation had been long before rejected as a method, and the theoretical mathematical models prevailed.

For instance, in the sixth century, a mathematical model devised by Dionysius Exiguus[3] became internationally accepted. Marking the vernal equinox invariably on the 21^{st} of March for calculating the Paschal full moon as the fourteenth day of a specific lunation beneath the cycle of 19 years of the Golden Number, Dionysius used his mathematical knowledge to build, from those foundations, an Easter table for a cycle of 95 years[4].

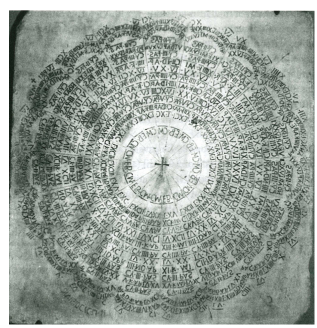

Figure 2: Dionysius Exiguus' Easter Table.

[3]Dionysius Exiguus (Dennis, the "little" 470 – 544), who wrote treaties on elementary mathematics, is best known as the inventor of the *Anno Domini* Christian era.

[4]Dionysius' Easter table was later continued by Isidore of Seville (for another 95 years) and Bede, the Venerable, completed them until 1064.

Easter tables such as Dionysius' were, invariably, limited and, gradually, new algorithms for modeling time were being devised: with new techniques (using fingers, wheels, etc.) priests could, on their own, calculate the important dates for planning the celebrations and still be sure that they were tuned to the Pope's rules.

An example of these tools may be found in the Portuguese book *Regra geral para aprender a tirar pela mão as festas mudáveis*. In addition, in his work, Trancoso ([Trancoso1570]) explained that the computation of Easter, in a certain year Y, should follow some stages, each one with several alternative (and equivalent) methods. In an updated language Trancoso's rules may be read as follows:

Let Y be

$$Y = a \times 1000 + b \times 500 + c \times 100 + d \times 20 + e$$

with $a \in N_0, b \in \{0,1\}, c, d \in \{n \in N_0 : n \leq 4\}, e \in \{n \in N_0 : n \leq 19\}$.

One would have

1. To calculate the figure of the Golden Number valid for Y: $gn(Y)$, by definition,

   $$\text{mod}_{19}(\text{mod}_{19}(\text{mod}_{19}(\text{mod}_{19}(\text{mod}_{19}(12a) + 6b) + 5c) + 1d) + e) + 1$$

2. To identify the Dominical Letter(s) valid for Y, over a "hand" matrix W with general term $w_{i,j}$:

   $$W = \begin{bmatrix} dc & b & a & g \\ fe & d & c & b \\ ag & f & e & d \\ cb & a & g & f \\ ed & c & b & a \\ gf & e & d & c \\ ba & g & f & e \end{bmatrix} \begin{array}{l} dl(Y) = w_{i,j} \\ i = 1 + \text{mod}_7(a \times 12 + b \times 4 + c \times 5 + \lfloor \frac{d}{4} \rfloor) \\ j = 1 + \text{mod}_4(d) \end{array}$$

3. To find out which of the 29 conjunctures for the hand (Cf. Figure 3) has the calculated figure for the Golden Number.

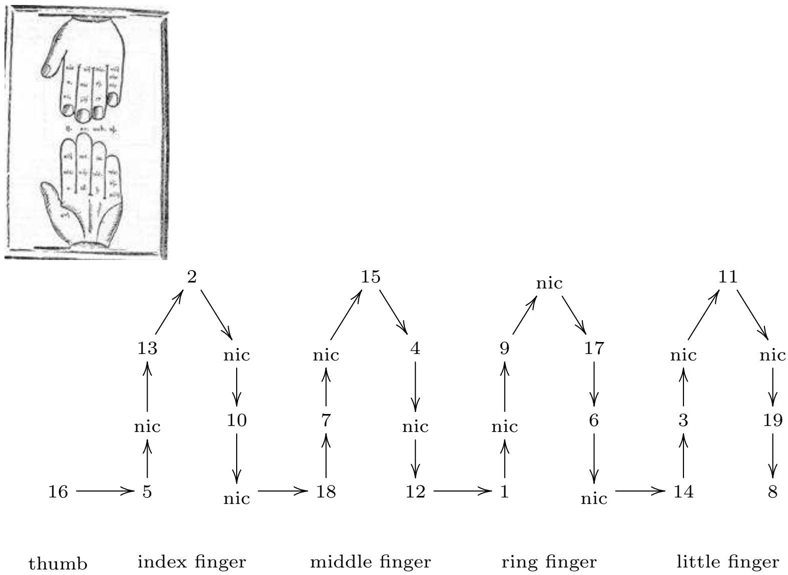

Figure 3: Using Hands for evaluating the Golden Number, in [Trancoso1570] and in a diagram.

By analysing the criteria behind the numerical sequence and the allocation of *nichel*[5] one may find:

$$t_i \in \{0, 1, 2, \ldots, 19\}$$
$$t_1 = 16 \qquad \text{With } b_n = t_n + 8.$$
$$\begin{cases} t_{n+1} = b_n - 19, & \text{if } b_n \notin \{1, 2, \ldots, 19\} \\ t_{n+1} = 0 \wedge t_{n+2} = b_n, & \text{otherwise.} \end{cases}$$

It means that in each iteration 8 years are added, leaving one empty space whenever that sum does not lead to a new Golden Number cycle. Indeed, if a new 19 years cycle starts, after 8 year epact, the age of the moon on the 1st of January, decreases one unit and every full moon comes one day

[5] *Nichel* is a variant of *nichil*, which is, in turn, the scholastic form of *nihil*, for "nothing", in classical Latin. In the present example it means that the junct works as an empty space and one must remember that, at that time, a numeral for "zero" was not in use.

later; otherwise, epact decreases two units and full moons are two days delayed.

Finally, to find out which conjuncture in the hand matches Easter there is yet another "hand" in Trancoso.

In this case the conjunctures were signified by the letters d, e, f, g, a, b and c.

Where should one then mark the dominical letter of Y (second dominical letter if the year is bissextile) after the conjuncture that has the Golden Number of Y?

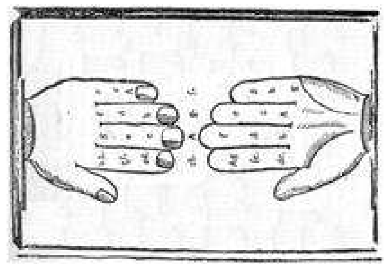

Figure 4: Using hands to evaluate the Dominical Letter, in [Trancoso1570].

Movable feasts depended on those two variables - Golden Number and Dominical Letter. They also depended on a constant (for Easter the earliest day is 22^{nd} of March). Therefore, one should start counting on from that date as many days as the conjunctures until the marked one is reached...

Easter date would then be found.

4 Measuring Time: tackling a historical problem

Understanding the rules implemented in calendars may be a tricky task and it does not seem to have been by chance that some of the first books printed in Portugal also dealt with calendar algorithms and/or calendar tables. We have referred to Trancoso's text but, before that, we may find, for example, the *Almanach Perpetuum* (Cf. [Zacuto1496]) aiming at being, as expressed in the title itself, a perpetual calendar[6].

[6]Printed in Leiria it associated the work of three Jews: Abraham Zacuto (an astronomer who came to Portugal expelled from Spain), José Vizinho (the main cosmographer of the Kingdom of Portugal) and Abraão Samuel Dortas (the editor). The book consisted of numerous tables for Dominical Letters and tables to help the search for the dates of the movable feasts (such as Easter). Previously, in Portugal, it had been printed yet another treaty with some calendar details: we are referring to the *Breviarium Bracarense de* 1494 (Cf. [Breviarium1494]), commissioned by the Archdiocese of

Authors such as Zacuto or Trancoso were, even, presenting their tables together with instructions for some correction factors. Nevertheless, in spite of numerous efforts, in spite of distinguishing texts/tables for practical consultation of dates from texts/treatises for the explanation of mathematical models behind such tables, the truth is that neither the treatises nor the tables could ever have answered accurately to the problem of measuring time.

In a different literary domain one also finds Duarte Pacheco Pereira[7] who, in 1506, wrote a memorable work on the Places of the Earth - *Esmeraldo de Situ Orbis* - where he wrote that

> *Em todo o outro tempo do anno sobe o Sol noventa graaos entrando na dita linha, salvo nos dias 11 de Março e 14 de Setembro em que faz dous equinocios*[8].

The conventional Roman date for the nominal equinox, the 21st of March, was, according to Duarte Pacheco Pereira, falling 10 days apart from the real vernal equinox. That was, for sure, a serious problem whose solution was, apparently, only involving a simple adjustment of the calendar but the mathematical contours of the problem of measuring accurately the time remained unclear.

Pedro Nunes (1502 – 1578), for example, showed that he was much aware of the relevance of mathematical proofs and, on the quest for measuring time, we believe that, he clearly applied his principles of mathematical accuracy. In fact, whereas in *De Crepusculis* Nunes measured some time events, namely the duration of the twilights, acknowledging, in particular, important peculiarities related to geographic as well as seasonal factors and achieving his goal of mathematical certitude. On the other hand, in his analysis of a proposed reform for the Julian calendar, Pedro

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Braga.

[7]Duarte Pacheco Pereira (1460 – 1533) is well known as the expert mind behind the famous Treaty of Tordesilhas, which split the world in two parts (to be divided by the Portuguese and the Spanish empires) but he was, undoubtedly, a very competent ship's captain who knew profoundly the art of navigation. Pacheco Pereira is also credited as the author of the following phrase *A experiência é a mãe de todas as coisas* ("Experience is the mother of everything", where the word experience is, perhaps, better understood as a synonym for practice), which might be seen as Pacheco Pereira's attitude towards knowledge.

[8]"In every other day of the year the Sun rises 90^o... except for the 11th of March and the 14th of September where it makes two Equinoxes".

Nunes commented on the difficulties of the original problem and criticized the solution, certainly also aware of the lack of mathematical rigor in the process of making calendars.

5 *De Crepusculis*: Pedro Nunes and the minimum twilight

Pedro Nunes, in his *De Crepusculis* (1942)[9], established a profound study where one notices clearly his opinion on how important mathematics was to practical knowledge as well as to the theoretical one.

In this case, reflecting upon the application of mathematics to the concrete/physical reality of watching the skies and understanding that reality, Pedro Nunes proposed the mathematical certitude for solving a famous problem of measuring time: the problem of the "minimum crepuscule"[10].

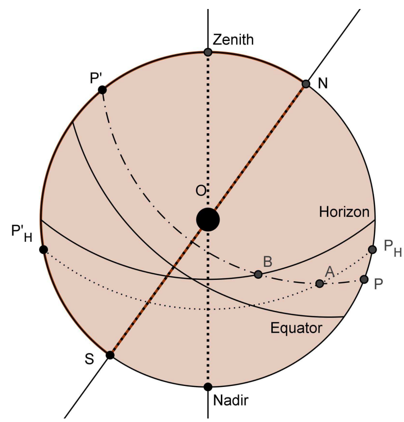

Figure 5: Celestial Sphere, showing the Celestial Equator and the plane of the Horizon of a certain location L.

According to Sacrobosco, (Cf. [Sacrobosco1478]), crepuscule/twilight is: *a diffused light, in-between day and night, so that there are, daily, one early morning twilight, which occurs at dawn just before sunrise, and an evening one, which occurs immediately when the sun sets.*

[9]We are, in what relates to *De Crepusculis*, mostly reporting to the facts presented in the study conducted by Carlos Vilar, in [Vilar2006].

[10]It is possible to find some misleading information on the solution presented by Pedro Nunes and on citing Bernoulli and d'Alembert as the mathematicians who, in fact, solved such problem (see, for example, [Dorrie1965]) but the truth is that Jacob Bernoulli, himself, had already dismissed such opinion in a "Solutio Problematis de Minimo Crepusculo" (Jacob Bernoullium communicata in litteris, Basileae, die 20 Julii, 1692, datis) and, later on, in a letter,1693, published in Journal des Sçavans [JourSavants1693] where he credited Pedro Nunes as having first solved this problem.

It was believed, since very ancient times, that the crepuscule, at a certain place L, started when the Sun, in its daily (apparent) movement around the Earth through a parallel, $PABP'$, to the Equator lays at an angular distance of 18^o below the horizon of L[11].

On Figure 5 we are representing the Earth's Polar Axis NS (in which N is the North Celestial Pole and S is the South Celestial Pole) and the Zenith-Nadir Axis.

Point A is an intersection between the parallel $P_HAP'_H$ to Horizon (18^o below it) and $PABP'$ parallel to Equator, described by the Sun in its daily movement. Point B is an intersection between that same parallel $PABP'$ and Horizon.

Therefore, according to our figure, the morning crepuscule starts in A and ends in B. Those points mark the instances between which the Sun describes on $PABP'$ the arch AB, said to be the *crepuscular arch.* The amplitude of the crepuscular arch divided by 15^o gives us the length of the crepuscule, stated in hours, minutes and seconds.

Pedro Nunes, in his treatise composed of 6 appendices and 19 propositions[12], admitted the Ptolemaic view of the universe, as stated by the geocentric theory, and also admitted that the *depressão do Sol*[13], in the limits of the crepuscules is 18^o. The geocentrism was not important to his conclusions (even by today's standards) but on the *depressão do Sol* Pedro Nunes taught, in his 1st Proposition, that this value was not constant, it depended instead on gases emanating from the Earth and existing on the spherical surface of the Earth itself: it varied, therefore, from place to place on the same day and it also varied from day to day in the same place.

> **(De Crepusculis' Prop. I)** *Demonstrar que o arco da distância do Sol ao horizonte, no princípio do crepúsculo matutino, ou no fim do vespertino, não pode ser sempre o mesmo e que varia necessariamente com as mudanças do tempo*[14].

[11] It seems important to refer that the magnitude of 18° (set by Ptolemy), for the *depressão do Sol*, both at the morning crepuscule and the evening one, although very common, at the time, was, nevertheless, not unique. For example: 18^o for Ptolemy, 17.5^o for Strabo, 19^o for Alhacen and Vitellius.

[12] For a summary see [Vilar2006], pp. 78 and 133.

[13] The *depressão do Sol* is defined by Pedro Nunes as being, at the beginning of the morning crepuscule, or at the end of the evening one, the meridian distance of the Sun to the horizon of the place.

[14] "To prove that the arch of the distance from de Sun to the horizon, at the beginning

In Prop. XVIII he completes his study by evaluating "the maximum height of the gases that make the air dense and thick, being able to reflect the light of the Sun, provoking the crepuscules":

> **(De Crepusculis' Prop. XVIII)** *Avaliar a altura máxima dos vapores*[15].

Moreover, Pedro Nunes shows, in his Prop. XVI, how to determine the *depressão do Sol*

> **(De Crepusculis' Prop. XVI)** *Dada a duração do crepúsculo, deduzir a distância do Sol ao horizonte*[16].

Pedro Nunes, himself, made the necessary observations and found such value, in Lisbon on the 1^{st} of October 1541, to be $16^o 2'$.

His Prop. XVII is, really, both a geometrical and a detailed mathematical hymn to the problem of the minimum crepuscule. With this important proposition, Pedro Nunes solved, in fact, two problems on the minimum crepuscule: finding the dates where they occur and calculating/measuring its length.

> **(De Crepusculis' Prop. XVII)** *Explicar a causa do aumento e da diminuição dos crepúsculos*[17].

The equation for evaluating the length of the minimum crepuscule is (Cf. Figure 5)

$$\sin\left(\frac{CrM}{2}\right) = \frac{\sin\left(\frac{DpS}{2}\right)}{\sin\big(AE\big)},$$

with CrM being the minimum crepuscular arch, DpS being the *depressão do Sol* when the crepuscule starts and AE being the maximum height of the Equator above Horizon.

The equation for finding the declination of the point on the Ecliptic, occupied by the Sun, when the minimum crepuscule occurs is

$$\sin\big(D\big) = \sin\big(AO\big)\sin\phi,$$

of the morning crepuscule, or at the end of the evening one, can not be always the same and that it varies with the changes of the time".

[15] To evaluate the maximum height of the gases.

[16] "From the duration of the crepuscule, to deduce the distance from the Sun to the horizon".

[17] "To explain the causes for the increase and the decrease of the crepuscules".

where D is the declination of the point on the Ecliptic, occupied by the Sun, when the minimum crepuscule occurs, AO is the ortiva amplitude and ϕ is the latitude of the place.

Pedro Nunes could not have known the techniques of differential calculus that, more than one century later, the two Bernoulli brothers came to use for their solution on the minimum crepuscule but it seems fair to remark that in spite of not having the calculus tools, Pedro Nunes managed to include in his solution, by means of geometric methods, the evaluation of the length of this crepuscule which the Bernoulli(s)' solution did not consider. Furthermore, Pedro Nunes applied his study by/to evaluating, in particular, the crepuscular arch in Lisbon in the 25^{th} of February and on the 26th of September, namely[18]: $20^o34'40''$.

Pedro Nunes' quest for mathematical rigor was long validated even at his time and Nunes' international recognition as a mathematician was soon to be called upon for dealing with yet another problem on measuring time: the reform of the Julian calendar.

6 The Gregorian Calendar: Pedro Nunes' comments

Given the discrepancy between the calendar year and the true solar year, as shown in section 4 of the present article, the vernal equinox had gradually moved away from the established date of the 21^{st} of March. Other Ecumenical Councils, after Nicaea, had also attempted to tackle the problem and some Popes had shown interest in solving the problem but it was not until the 16th-century that Pope Gregory XIII $(1502 - 1585)$[19] assigned to himself the task of reforming the Julian calendar.

Gregory XIII, Pope from 1572 until 1585, was born Ugo Boncompagni and acknowledged the errors with the date of Easter, as defined in Nicaea. Known as having led a simple life, but not free of some scandals, Pope Gregory XIII embraced the difficult task of reforming the Catholic Church.

[18] These values are mathematically correct even for today's standards (where a simple change of referential would be needed: a conversion from Julian to Gregorian calendar).

[19] According to [CatholicEncyc1913], Gregory XIII's *election was greeted with joy by the Roman people, as well as by foreign rulers. Emperor Maximilian II, the kings of France, Spain, Portugal, Hungary, Poland, the Italian and other princes sent their representatives to Rome to tender their obedience to the newly-elected pontiff.*

However, it was the reformation of the Julian calendar, introduced to Portugal and other Catholic countries in 1578, that was to gain him a lasting fame.

A reform commission, in which one of the leading members was the Jesuit priest/astronomer Christopher Clavius, was assigned to analyse several proposals for the reform of the calendar and other member, Antonio Lilio, an Italian scientist, would see his brother Luigi's project stand as a basis for the implemented solution.

The structural ideas of Lilio's proposal for the reform were written by Chacón in [Compendium1577]: the *Compendium novae rationis restituendi Kalendarium* and further advised that assessment was required from the best experts (universities and academies) throughout the Catholic territories.

Pedro Nunes was naturally aware of the equinocial shift long before. Nunes, by then already an emeritus professor of Mathematics at the University of Coimbra, was asked to make the Portuguese appraisal of that *Compendium*. Nunes' condition was, however, that of an elderly man, worried by some family scandals and seriously ill.

Joaquim de Carvalho, in [Carvalho1952], gathered some facts and testimonies into a well-established narration aiming to obtain Nunes' statement over the project under discussion in the *Compendium* and, mainly from his studies, we know that Monsignor Roberto Fontana, by then one of the representatives of the Pope in Lisbon, is credited as having, after Pedro Nunes' death, maintained talks with D. Henrique[20] on Nunes' opinion about the proposed reform of the calendar. We are, on the subject, told that D. Henrique, himself, searched for some related documents having arranged the moving of Pedro Nunes' writings from Coimbra to Lisbon[21]. Fontana was, in the meantime, reporting to the Vatican the status of the ongoing task in Portugal and registering that delays were caused by the illness of the mathematician.

In spite of the absence of Nunes' writings on the reform of the calendar, one knows of the comments by Friar Luís de Souto Maior, a Dominican priest at the College of St. Thomas and a relative of Pedro Nunes. Leon Bourdon, in [Bourdon1953], publicized Nunes's opinion on the subject

[20] Cardinal D. Henrique was uncle of the deceased Portuguese king D. Sebastião and succeeded him as Regent of the kingdom.

[21] We know, nevertheless, that no document that alludes to reforming the calendar was found in two boxes containing Nunes' manuscripts.

through a trustworthy testimony of Friar Luís who wrote a report on his last visits to Pedro Nunes and where he recalled that

> *O Doutor Pero Nunes cosmographo moor, estando na cama muito doente pouquos dias antes que morresse, me disse por vezes que S. A. lhe mandara que visse hum certo tratado enviado pello Santo Padre de celebration Pasche para que scrivesse o seu parecer acerqua disto e que por elle estar tam doente nam podia fazer isto como desejava; mas que elle nam era de parecer que se fizesse nenh ua mudança no kalendario acerqua deste ponto e que era melhor proceder desta maneira que procede a igreja catholica tantos annos ha que nam fazer esta novidade, porque de nenh ua maneira se podem evitar todos os enconvenientes nem as regras que o autor do sobredito tratado daa sam muito certas, antes sam incertas e falsas ou fallivees, como elle determinava mostrar se nam morrera tam depressa. Em fee disto asiney aqui de minha mão. Oje dia de S. Catarina martyr 1578*[22].

The content of that letter explicitly assured Nunes' opposition to those methods exposed in the Compendium set to amend the Julian Calendar.

Alternative solutions to analysing the *Compendium*, after Nunes' death, were also sought by D. Henrique who appointed Tomás da Orta and Manuel Mendes Vizinho, doctors and cosmographers, to accomplish the task.

Orta compared the actual astronomical movements with the dates of Easter, as calculated by the Church, and settled that between 1576 and 1604 there would be five years in which the given difference would be no less than 28 days. He expressed, as a consequence, his objection on the proposed abolition of ten days to fix the equinoctial shift stated in the

[22] "Dr. Pero Nunes, chancel cosmographer, being very ill in bed a few days before he died, told me that sometimes Y. H. appointed him to see a certain treaty sent by the Holy Father de celebration Pasche [On the Easter Celebration] to write his opinion about it and that because he was so sick he could not do it as he wished, but he was not of the opinion to make any change to the Calendar about this point and it was better to proceed in this way the Catholic church has been making for many years to not make this news, because there is no way to avoid all the hassle, nor are the rules that the author of the aforesaid treaty gives very certain but they are false or uncertain and fallible as he was determined to show if he had not died so soon. In this faith I signed here in my hand. Today Day St. Catherine martyr 1578".

Compendium. He was also uncomfortable with a computation founded on the average motions of the Sun and the Moon.

Manuel Mendes Vizinho registered an inventory of the disadvantages and weaknesses of Lilio's proposed method: *complex, arduous, uncertain, based on the misfits of fictional motion cycles: the epacts.* He also recalled that, some 60 years before, the Portuguese Master Diogo[23] submitted a relevant draft to Pope Leo X's request on the same subject.

Orta and Vizinho were aware of Nunes' point of view but they both conducted their own analysis and finally also sustained Pedro Nunes' objections.

Those were not easy times for Portugal which was, by then, ruled by a Castilian Monarch Philippe. Moreover the Portuguese experts' opinions on the proposed reform for the calendar were unanimously against it. We are also acquainted with the fact that many other European experts shared and expressed Nunes' view on the quality of Lilio's proposal. But, in spite of all this, in his Papal Bull of February 24^{th}, 1582 - *Inter Gravissimas* - Pope Gregory XIII finally decided that the calculation of the date of Easter was, from then on, to depend on a cycle of epacts and that, in particular, 10 days were to be suppressed from the calendar; the choice fell over that same October of 1582.

Portugal was, nevertheless, one of the few countries, in the world, to immediately adopt the papal decision.

7 Final Remarks

Legal implementation of the Gregorian reform for the calendar went ahead but it was definitely a very controversial decision from its very beginning. Divergent opinions immediately arose in a struggle where Clavius became prime advocate in favour of the changes.

At the same time, new strategies facilitating the transition from the Julian to the Gregorian model were required, while adapting and reinventing algorithms to everyone's computations. For instance, it seemed to us rather interesting to compare[24] Trancoso's approach (summarized

[23] One suspects that Master Diogo (Mendes Vizinho) is José Vizinho, whom we refereed to previously, accredited for having worked with Abraham Zacuto on his *Almanach Perpetuum.*

[24] For an extensive comparative study of Portuguese works on Julian and Gregorian Calendars see [Lopes2007].

in section 3) to the one given by Sequeira - in a text (Cf. [Sequeira1612]) with an odd title *Thesouro de Prudentes* - where he obviously aims at explaining how the changes are slight.

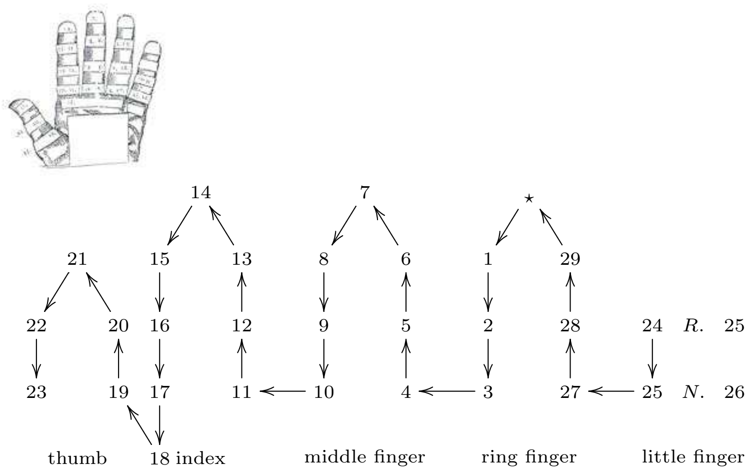

Figure 6: Using Hands, with Epact figures, to compute Easter, in [Sequeira1612] and in a diagram.

Epact was calculated using the thumb as in Trancoso, similarly after translation of two conjunctures, a new iteration that should be emended due to non bissextile secular years. To find out which conjuncture in the hand matched Easter a hand for Epacts was used for an immediate result; instead Golden Number and hand for Letters suffered only the proper adjustments[25].

Yet, such rules, claimed to be simpler, did not please neither convince the majority of people. Many seemed suspicious of the tricky mathematical structure behind the Gregorian reform.

[25]Nevertheless, Sequeira explains the purposes of R for "Rubros" and N for "Nibros" in the Epacts hand inwards a strategy to control an exception: to avoid epact 24 incompatibilities with the boundaries for Easter date settled in Niceae. In that case - R - theoretical Easter full moon must come one day earlier (replacing epact 24 to 25); if epact 25 already occurs in the same 19 year Golden number cycle - N - epact 25 should be also substituted by 26.

Pedro Nunes was largely acquainted with the importance of mathematics to practical as well as theoretical knowledge; with both classical and current literature; and with the benefits of using mathematics (and its certitude) as a tool to explaining real/concrete/physical problems. Acquainted with all of these things, Pedro Nunes knew exactly the difficulties behind an erroneous system for measuring time to which he referred by saying, in particular, that *the rules of the Compendium are uncertain and false or prone to failure.*

The same system, the erroneous one is, nevertheless still used at present, and the advice left by Nunes himself may offer us, in the twenty-first-century, some kind of comfort because, with respect to time measurement, *there is no way to avoid all the inconvenience.*

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Estas Actas são o resultado de um Colóquio sobre História e Epistemologia da Matemática, realizado em 20 de Maio de 2010, como forma de prestarmos homenagem à nossa amiga Maria Fernanda Estrada, por ocasião do seu 80º aniversário.
Aos artigos, resultantes das comunicações apresentadas no Colóquio, juntaram-se as contribuições de Helmut Malonek e de João Filipe Queiró que, gentilmente, se associaram a este projecto.

A organização do Colóquio esteve a cargo de:

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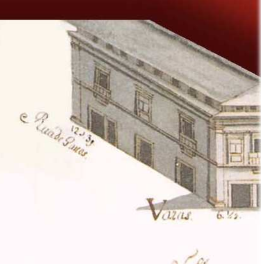

FCT Fundação para a Ciência e a Tecnologia

MINISTÉRIO DA CIÊNCIA, TECNOLOGIA E ENSINO SUPERIOR Portugal

Universidade do Minho

ISBN: 978-989-95122-4-5