ANALYSING THE ELASTICITY DIFFERENCE TENSOR OF GENERAL RELATIVITY

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Abstract. The elasticity difference tensor, used in [1] to describe elasticity properties of a continuous medium filling a space-time, is here analysed. Principal directions associated with this tensor are compared with eigendirections of the material metric. Examples concerning spherically symmetric and axially symmetric space-times are then presented.

1. Introduction

In recent years there has been a growing interest in the theory of general relativistic elasticity. Based on the classical Newtonian elasticity theory going back to the 17th century and Hooke's law, some authors began to adapt the theory of elasticity to relativity due to the necessity to study astrophysical problems, such as the deformations of neutron star crusts. One of the first elastic phenomenon considered in the relativistic context was Weber's observation of the elastic response of an aluminium cylinder to gravitational radiation and the detection of gravitational waves [2], [3] and [4]. Neutron stars have attracted attention since it has been argued [5] that the crusts of neutron stars are in elastic states and since it has been established the existence of a solid crust and speculated the possibility of solid cores in neutron stars, [6], [7], [8].

There were many attempts to formulate a relativistic version of elasticity theory. Thereby laws of non relativistic continuum mechanics had to be reformulated in a relativistic way. The study of elastic media in special relativity was first carried out by Noether [9] in 1910 and by Born [10], Herglotz [11] and Nordström [12] in 1911. The discussion of elasticity theory in general relativity started with Synge [13], De Witt [14], Rayner [15], Bennoun [16], [17], Hernandez [18] and Maugin [19] [1]. In 1973 Carter and Quintana [20] developed a relativistic formulation of the concept of a perfectly elastic solid and constructed a quasi-Hookean perfect elasticity theory suitable for applications to high-pressure neutron star matter. Recently, Karlovini and Samuelsson [1] have made an important contribution to this topic, extending the results of Carter and Quintana (see also [21], [22]). Other relevant formulations of elasticity in the framework of general relativity were given by Kijowski and Magli ([23], [24]) who presented a gauge-type theory of relativistic elastic media and a corresponding generalization [25]. The same authors also studied interior solutions of the Einstein field equations in elastic media ([26], [27]).

The recent increasing consideration of relativistic elasticity in the literature shows the win of recognition and importance of this topic, motivating toward a detailed study of quantities used in this context, the elasticity difference tensor defined in [1] being one of them. This tensor occurs contracted with the relativistic Hadamard

[1]Relativistic elasticity has been treated in the mid-20th century until the early seventies by many other authors. For further references, see, for example, [19], and for later references see also [23], [1].

elasticity tensor in the Euler equations for elastic matter. However, one can recognize the geometric role of the elasticity difference tensor, since, in principle, it can be used to understand the influence of the material metric (inheriting elastic properties) on the curvature of the space-time.

Here, in section 2, general results about relativistic elasticity are presented. In section 3, the elasticity difference tensor is analysed and the principal directions associated with this tensor are compared with the eigendirections of the pulled-back material metric. A specific orthonormal tetrad is introduced to write a general form of the elasticity difference tensor, which brings in Ricci rotation coefficients used in the $1+3$ formalism [28] and the linear particle densities.

Finally, in section 4, we apply the results obtained to a static spherically symmetric space-time and an axially symmetric non-rotating space-time. The software *Maple GRTensor* was used to perform some calculations.

2. General results

Let (M, g) be a space-time manifold, i.e. a 4-dimensional, paracompact, Hausdorff, smooth manifold endowed with a Lorentz metric g of signature $(-,+,+,+)$, $U \subseteq M$ being a local chart around a point $p \in M$. We assume that the space-time is time orientable. Suppose that U is filled with a continuous material. The *material space* $(\mathcal{X}, k)$ is a 3-dimensional manifold, each point in $\mathcal{X}$ representing an idealized particle of the material, and k being a Riemannian metric, the *material metric*, measuring distances between particles in the "locally relaxed state" of matter. The space-time configuration of the material is described by a smooth mapping

$$\Psi : U \subseteq M \longrightarrow \mathcal{X},$$

the *configuration function*, which associates to each point p of the space-time the particle $\bar{p} = \Psi(p) \in \mathcal{X}$ of the material at the event p. The operators push-forward Ψ_* and pull-back Ψ^* will be used to take contravariant tensors from M to $\mathcal{X}$ and covariant tensors from $\mathcal{X}$ to M, respectively, in the usual way.

If $\{\xi^A\}$ $(A = 1, 2, 3)$ is a coordinate system in $\mathcal{X}$ and $\{\omega^a\}$ $(a = 0, 1, 2, 3)$ [2] a coordinate system in $U \subseteq M$, then the configuration of the material can be described by the fields $\xi^A = \xi^A(\omega^a)$. The mapping $\Psi_* : T_pM \longrightarrow T_{\Psi(p)}\mathcal{X}$ gives rise to a (3×4) matrix (the relativistic deformation gradient) whose entries are $\xi^A_a = \frac{\partial \xi^A}{\partial \omega^a}$.

It is required that the relativistic deformation gradient has maximal rank and that its Kernel is a one-dimensional timelike subspace of T_pM, $\forall p \in M$. Since (M, g) is time orientable, we can choose a generator u^a of the Kernel such that: $u^0 > 0$, $u^a u_a = -1$, $u^a \xi^B_a = 0$. The vector field u^a is called the matter four-velocity and for each $\bar{p} \in \mathcal{X}$, $\Psi^{-1}(\bar{p})$ is an integral curve of u, the worldline of the particle $\bar{p}$.

The pull-back of the material metric

$$k_{ab} = \Psi^* k_{AB} = \xi^A_a \xi^B_b k_{AB} \tag{1}$$

and the (usual) projection tensor

$$h_{ab} = g_{ab} + u_a u_b \tag{2}$$

[2] Capital Latin indices A,B,... range from 1 to 3 and denote material indices. Small Latin indices a,b,... take the values 0,1,2,3 and denote space-time indices.

are Riemannian metric tensors on the subspaces of T_pM orthogonal to u^a. These tensors are symmetric and satisfy $k_{ab}u^a = 0 = h_{ab}u^a$ and $\mathcal{L}_u k_{ab} = 0$.

The tensor $k^a_b = g^{ac}k_{cb}$ has three positive eigenvalues, here called n_1^2, n_2^2, n_3^2, associated with spacelike eigenvectors. The positive quantity $n = n_1 n_2 n_3 = \sqrt{det(k^a_b)}$ is the particle density of the material. This definition is justified by the continuity equation $\nabla_a(nu^a) = 0$.

The state of strain of the material can be measured by the relativistic strain tensor, according to e.g. [26], [27]:

$$s_{ab} = \frac{1}{2}(h_{ab} - k_{ab}). \tag{3}$$

The material is said to be "locally relaxed" at a particular point of space-time if the material metric and the projection tensor agree at that point, i.e. if the strain tensor vanishes.

When considering elastic matter sources in general relativity, one is confined to a stress-energy tensor taking the form $T_{ab} = -\rho g_{ab} + 2\frac{\partial \rho}{\partial g^{ab}} = \rho u_a u_b + p_{ab}$, where $p_{ab} = 2\frac{\partial \rho}{\partial g^{ab}} - \rho h_{ab}$, the energy density being written as $\rho = n\epsilon$, where ϵ is the energy per particle.

Choosing an orthonormal tetrad $\{u, x, y, z\}$ in M, with u in the direction of the velocity field of the matter and x, y, z spacelike vectors along the eigendirections of $k^a_b = g^{ac}k_{cb}$, the orthogonality conditions are $-u_a u^a = x_a x^a = y_a y^a = z_a z^a = 1$, all other inner products being zero. For this tetrad, k and g can be written as

$$k_{ab} = n_1^2\, x_a x_b + n_2^2\, y_a y_b + n_3^2\, z_a z_b, \tag{4}$$

and

$$g_{ab} = -u_a u_b + h_{ab} = -u_a u_b + x_a x_b + y_a y_b + z_a z_b. \tag{5}$$

It should be noticed that the eigenvectors x, y, z are automatically orthogonal whenever the corresponding eigenvalues are distinct. However, if the eigenvalues are not all distinct, the eigendirections associated with the same eigenvalue can (and will) be chosen orthogonal.

Consider the spatially projected connection D_a acting on an arbitrary tensor field $t^{b\cdots}_{c\ldots}$ as follows:

$$D_a t^{b\cdots}_{c\ldots} = h^d_a h^b_e \ldots h^f_c \ldots \nabla_d t^{e\cdots}_{f\ldots}. \tag{6}$$

Here ∇ is the connection associated with g and one has $D_a h_{bc} = 0$. Now, consider a differential operator $\tilde{D}_a$ acting on space-time tensors obtained from the Levi-Civita connection $\tilde{D}_A$ of k_{AB} under the following hypothesis [1]:

(i) there exists a torsion-free connection $\tilde{\nabla}$ on M such that

$$\tilde{D}_a t^{b\cdots}_{c\ldots} = h^d_a h^b_e \ldots h^f_c \ldots \tilde{\nabla}_d t^{e\cdots}_{f\ldots}; \tag{7}$$

(ii) for all space-time vector fields V^b and Z^a, Z^a having zero convected derivative

$$\Psi_*(V^b \tilde{D}_b Z^a) = V^B \tilde{D}_B Z^A, \qquad V^B = \Psi_*(V^b), \qquad Z^A = \Psi_*(Z^a).$$

It follows that

$$\tilde{D}_b X^a - D_b X^a = h^m_b h^a_n (\tilde{\nabla}_m X^n - \nabla_m X^n) = S^a{}_{bc} X^c, \tag{8}$$

for any space-time vector field X. The tensor field $S^a{}_{bc}$ is the elasticity difference tensor as introduced by Karlovini and Samuelsson in [1]. Using hypothesis (ii), this third order tensor can be written as

$$S^a{}_{bc} = \frac{1}{2} k^{-am}(D_b k_{mc} + D_c k_{mb} - D_m k_{bc}), \tag{9}$$

where k^{-am} is such that $k^{-am}k_{mb} = h^a_b$. It occurs in the Euler equations $\nabla_b T^{ab} = 0$ for elastic matter contracted with the Hadamard elasticity tensor as given by the same authors.

The covariant derivative of the timelike unit vector field u can be decomposed as follows

$$u_{a;b} = -\dot{u}_a u_b + D_b u_a = -\dot{u}_a u_b + \frac{1}{3}\Theta h_{ab} + \sigma_{ab} + \omega_{ab}, \tag{10}$$

where $\dot{u}_\alpha$ is the acceleration, $\sigma_{\alpha\beta}$, the symmetric tracefree rate of shear tensor field, $\omega_{\alpha\beta}$, the antisymmetric vorticity tensor field and Θ, the expansion scalar field for the congruence associated with u.

3. The Elasticity Difference Tensor

In this section we investigate the elasticity difference tensor. This tensor arises when studying elasticity within the framework of general relativity and is related to the connection of the space-time, as shown in the previous section.

The following two properties of the elasticity difference tensor are straightforward:

(i) it is symmetric in the two covariant indices, i. e.

$$S^a{}_{bc} = S^a{}_{cb}; \tag{11}$$

(ii) it is a completely flowline orthogonal tensor field, i.e.

$$S^a{}_{bc} u_a = 0 = S^a{}_{bc} u^b = S^a{}_{bc} u^c. \tag{12}$$

The following result provides a mathematical construction for the elasticity difference tensor which requires a second metric defined on M and its associated Levi-Civita connection.

It is a well known result that the difference between two connections $\tilde{\nabla}$ and ∇, associated with two different metrics $\tilde{g}$ and g, respectively, defined on U, is the following $(1,2)$ tensor:

$$C^n{}_{ml} = \tilde{\Gamma}^n{}_{ml} - \Gamma^n{}_{ml}, \tag{13}$$

$\tilde{\Gamma}^n{}_{ml}$ and $\Gamma^n{}_{ml}$ being the Christoffel symbols associated with those two metrics. In a local chart, this tensor can be written as ([29], [30])

$$C^n{}_{ml} = \frac{1}{2} \tilde{g}^{np}(\tilde{g}_{pm;l} + \tilde{g}_{pl;m} - \tilde{g}_{ml;p}), \tag{14}$$

where $\tilde{g}^{np}$ is such that $\tilde{g}^{np}\tilde{g}_{pr} = \delta^n_r$ and a semi-colon ; represents the covariant derivative with respect to g. The difference tensor $C^n{}_{ml}$ can be used to write the difference of the Riemann and the Ricci tensors associated with the two metrics in the following form (see e.g. [31]):

$$\tilde{R}^a{}_{bcd} - R^a{}_{bcd} = -C^a{}_{bd;c} + C^a{}_{bc;d} - C^a{}_{lc}C^l{}_{bd} + C^a{}_{ld}C^l{}_{bc} \tag{15}$$

and

$$\tilde{R}_{bd} - R_{bd} = -C^a{}_{bd;a} + C^a{}_{ba;d} - C^a{}_{la}C^l{}_{bd} + C^a{}_{ld}C^l{}_{ba}. \tag{16}$$

The projection of the difference tensor orthogonally to u is defined by the expression

$$h^a_n\, h^m_b\, h^l_c\, C^n{}_{ml}. \tag{17}$$

Assume that ∇ and $\tilde{\nabla}$ are the Levi-Civita connections associated with the two metric tensors $g_{ab} = -u_a u_b + h_{ab}$ and $\tilde{g}_{ab} = -u_a u_b + k_{ab}$. Then, writing (14) explicitly and projecting the resulting expression according to (17), one obtains

$$h^a_n\, h^m_b\, h^l_c\, C^n{}_{ml} = \frac{1}{2} k^{-am}(D_b k_{mc} + D_c k_{mb} - D_m k_{bc}). \tag{18}$$

The expression on the right hand side of (18) is the elasticity difference tensor given in (9).

Under this approach, the elasticity difference tensor equals the projection, orthogonal to u, of the difference between two Levi-Civita connections, one associated with the space-time metric and the other with the metric $\tilde{g}_{ab} = -u_a u_b + k_{ab}$, where k_{ab} is the pull-back of the material metric k_{AB}.

Using (6) and (17) the calculation of the spatial projection of equation (15) yields the following expression for the difference of the Riemann tensors:

$$\begin{aligned} &h^f_m\, h^n_g\, h^p_e\, h^q_h\, [\, h^m_a\, h^b_n\, h^c_p\, h^d_q\, (\tilde{R}^a{}_{bcd} - R^a{}_{bcd})] \\ &= -D_e S^f{}_{gh} + D_h S^f{}_{ge} - S^f{}_{ke} S^k{}_{gh} + S^f{}_{kh} S^k{}_{ge}. \end{aligned} \tag{19}$$

The spatial projection of (16) expressing the difference of the Ricci tensors can be obtained analogously by equating the indices $a = c$ in (19).

Therefore, these expressions, which contain the elasticity difference tensor, give the difference between the projected Riemann and Ricci tensors associated with the metrics referred to above.

Now we obtain the tetrad components of the elasticity difference tensor. From now the following notation is used for the orthonormal tetrad: $e^a_\mu = (e^a_0, e^a_1, e^a_2, e^a_3) = (u^a, x^a, y^a, z^a)$. Tetrad indices will be represented by Greek letters from the second half or the first half of the alphabet according to their variation as follows: $\mu, \nu, \rho ... = 0-3$ and $\alpha, \beta, \gamma ... = 1-3$. The Einstein summation convention and the notation for the symmetric part of tensors will be applied to coordinate indices only, unless otherwise stated. The operation of raising and lowering tetrad indices will be performed with $\eta_{\mu\nu} = \eta^{\mu\nu} = diag(-1, 1, 1, 1)$ and one has $g_{ab} = \displaystyle\sum_{\mu,\nu=0}^{3} e_{\mu a} e_{\nu b} \eta^{\mu\nu}$.

The tetrad components of the elasticity difference tensor can be obtained using the standard relationship

$$S^\alpha{}_{\beta\gamma} = S^a{}_{bc} e^\alpha_a e^b_\beta e^c_\gamma \tag{20}$$

the result being

$$\begin{aligned} S^\alpha{}_{\beta\gamma} = \frac{1}{2n^2_\alpha} [&\left(n^2_\alpha - n^2_\gamma\right) \gamma^\alpha{}_{\gamma\beta} + \left(n^2_\alpha - n^2_\beta\right) \gamma^\alpha{}_{\beta\gamma} + \left(n^2_\gamma - n^2_\beta\right) \gamma_{\beta\gamma}{}^\alpha + D_n(n^2_\alpha) e^n_\beta \delta^\alpha_\gamma \\ &+ D_p(n^2_\alpha) e^p_\gamma \delta^\alpha_\beta - D_l(n^2_\beta) e^{l\alpha} \delta_{\beta\gamma}]. \end{aligned} \tag{21}$$

Here the following notation was used for the Ricci rotation coefficients: $\gamma_{\mu\nu\rho} = e_{\mu a;b} e^a_\nu e^b_\rho$.

An alternative form for (21) is:

$$S^{\alpha}{}_{\beta\gamma} = \frac{1}{2}[(1-\epsilon_{\gamma\alpha})\gamma^{\alpha}{}_{\gamma\beta} + (1-\epsilon_{\beta\alpha})\gamma^{\alpha}{}_{\beta\gamma} + (\epsilon_{\gamma\alpha}-\epsilon_{\beta\alpha})\gamma_{\beta\gamma}{}^{\alpha} + m_{\beta\alpha}\delta^{\alpha}_{\gamma} + m_{\gamma\alpha}\delta^{\alpha}_{\beta} - m^{\alpha}{}_{\beta}\delta_{\beta\gamma}\epsilon_{\beta\alpha}], \tag{22}$$

where $\epsilon_{\gamma\alpha} = \left(\dfrac{n_{\gamma}^2}{n_{\alpha}^2}\right)$ and $m^{\alpha}{}_{\beta} = D_a(\ln n_{\beta}^2)e^{a\alpha}$.

For the elasticity difference tensor it is possible to define two independent traces. Here we give their expressions in the orthonormal tetrad already chosen:

$$S^{\alpha}{}_{\alpha\gamma} = \frac{1}{2}m_{\gamma\alpha} = \frac{1}{n_{\alpha}}D_a(n_{\alpha})e^a_{\gamma} \tag{23}$$

and

$$S^{\alpha}{}_{\beta\beta} = m_{\beta\beta} - \frac{1}{2}m^{\alpha}_{\beta}\epsilon_{\beta\alpha} = \frac{2}{n_{\alpha}}D_a(n_{\alpha})e^a_{\alpha} - \frac{n_{\beta}}{n_{\alpha}^2}D_a(n_{\beta})e^{a\alpha}. \tag{24}$$

The Ricci rotation coefficients, when related to the quantities used in the decomposition (10), can be split into the set [32]:

$$\gamma_{0\alpha 0} = \dot{u}_{\alpha} \tag{25}$$

$$\gamma_{0\alpha\beta} = \frac{1}{3}\Theta\delta_{\alpha\beta} + \sigma_{\alpha\beta} - \epsilon_{\alpha\beta\gamma}\omega^{\gamma} \tag{26}$$

$$\gamma_{\alpha\beta 0} = -\epsilon_{\alpha\beta\gamma}\Omega^{\gamma} \tag{27}$$

$$\gamma_{\alpha\beta\gamma} = -A_{\alpha}\delta_{\beta\gamma} + A_{\beta}\delta_{\alpha\gamma} - \frac{1}{2}(\epsilon_{\gamma\delta\alpha}N^{\delta}{}_{\beta} - \epsilon_{\gamma\delta\beta}N^{\delta}{}_{\alpha} + \epsilon_{\alpha\beta\delta}N^{\delta}{}_{\gamma}). \tag{28}$$

The quantities A and N appear in the decomposition of the spatial commutation functions $\Gamma^{\alpha}{}_{\beta\gamma} = \gamma^{\alpha}{}_{\gamma\beta} - \gamma^{\alpha}{}_{\beta\gamma}$ (see [33]), where N is a symmetric object.

The elasticity difference tensor can be expressed using three second order symmetric tensors, designated as $\underset{\alpha}{M}_{bc}$, $\alpha = 1, 2, 3$, as follows:

$$S^a{}_{bc} = \underset{1}{M}_{bc}x^a + \underset{2}{M}_{bc}y^a + \underset{3}{M}_{bc}z^a = \sum_{\alpha=1}^{3}\underset{\alpha}{M}_{bc}e^a_{\alpha}. \tag{29}$$

Here we study these three tensors $\underset{\alpha}{M}_{bc}$ in order to understand to what extent the principal directions of the pulled back material metric remain privileged directions of the elasticity difference tensor through the tensors $\underset{\alpha}{M}_{bc}$, following the eigenvalue-eigenvector approach for these second order tensors.

First, we obtain a general expression for $\underset{\alpha}{M}_{bc}$, $\alpha = 1, 2, 3$, which depends explicitly on the orthonormal tetrad vectors, the Ricci rotation coefficients and the linear particle densities n_{α}. In fact, contracting $S^a{}_{bc}$ in (9) with each one of the spatial tetrad vectors and using then the relationships (4), (6), after some appropriate simplifications the final result becomes:

$$
\begin{aligned}
\underset{\alpha}{M}_{bc} = {} & u^m(e_{\alpha m;(b}u_{c)} + u_{(b}e_{\alpha c);m}) + e_{\alpha(b;c)} - e_{\alpha}^{m} e_{\alpha(c}e_{\alpha b);m} \\
& + \gamma_{0\alpha\alpha}\ u_{(b}e_{\alpha c)} - \ \gamma_{0\alpha 0}\ u_b u_c \\
& + \frac{1}{n_\alpha}[2n_{\alpha,(b}e_{\alpha c)} + 2n_{\alpha,m}u^m u_{(b}e_{\alpha c)} + n_{\alpha,m}e_{\alpha}^{m}e_{\alpha b}e_{\alpha c}] \\
& + \frac{1}{n_\alpha^2}\{-e_{\alpha}^{m}(e_{\beta b}e_{\beta c}n_\beta n_{\beta,m} + e_{\gamma b}e_{\gamma c}n_\gamma n_{\gamma,m}) \\
& + n_\gamma^2[(\gamma_{0\gamma\alpha} - \gamma_{\alpha\gamma 0})u_{(b}e_{\gamma c)} + e_{\alpha}^{m}(e_{\gamma m;(b}e_{\gamma c)} - e_{\gamma(b}e_{\gamma c);m})] \\
& + n_\beta^2[(\gamma_{0\beta\alpha} - \gamma_{\alpha\beta 0})u_{(b}e_{\beta c)} + e_{\alpha}^{m}(e_{\beta m;(b}e_{\beta c)} - e_{\beta(b}e_{\beta c);m})]\}.
\end{aligned}
\tag{30}
$$

Here $\gamma \neq \beta \neq \alpha$ and a comma represents a partial derivative.[3] It should be noticed that this expression also contains the non-spatial Ricci rotation coefficients given in (25), (26) and (27).

The expressions obtained for $\underset{\alpha}{M}_{bc}$ still satisfy the conditions $\underset{\alpha}{M}_{bc}u^b = 0$, as a consequence of the orthonormality conditions for the tetrad together with (29).

The eigenvalue-eigenvector problem for $\underset{\alpha}{M}_{bc}$ is quite difficult to solve in general. However, one can investigate the conditions for the tetrad vectors to be eigenvectors of those tensors, the results being summarized in the two following theorems.

On what follows, intrinsic derivatives of arbitrary scalar fields Φ, as derivatives along tetrad vectors, will be represented by Δ_{e_α} and defined as:

$$\Delta_{e_\alpha}\Phi = \Phi_{,m}e_{\alpha}^{m},$$

where a comma stands again for a partial derivative.

Theorem 1. *The tetrad vector e_α is an eigenvector for $\underset{\alpha}{M}$ iff n_α remains invariant along the two spatial tetrad vectors e_β, such that $\beta \neq \alpha$, i.e. $\Delta_{e_\beta}(\ln n_\alpha) = 0$ whenever $\beta \neq \alpha$.*
The corresponding eigenvalue is $\lambda = \Delta_{e_\alpha}(\ln n_\alpha)$.

Proof: In order to solve this eigenvector-eigenvalue equation the following algebraic conditions are used

$$\underset{\alpha}{M}{}_b^c e_{\alpha}^{b} e_{\alpha c} = \lambda, \tag{31}$$

$$\underset{\alpha}{M}{}_b^c e_{\alpha}^{b} e_{\beta c} = 0 \tag{32}$$

and

$$\underset{\alpha}{M}{}_b^c e_{\alpha}^{b} e_{\gamma c} = 0, \tag{33}$$

where $\gamma \neq \beta \neq \alpha$. Considering the orthogonality conditions satisfied by the tetrad vectors and the anti-symmetry of the Ricci rotation coefficients on the first pair of indices, expressions (32) and (33) yield $\Delta_{e_\beta}(\ln n_\alpha) = 0 = \Delta_{e_\gamma}(\ln n_\alpha)$. Therefore $\Delta_{e_\beta}n_\alpha = 0 = \Delta_{e_\gamma}n_\alpha$. On the other hand, from (31) one obtains, after some calculations, the eigenvalue $\lambda = \Delta_{e_\alpha}(\ln n_\alpha)$. □

[3]To read (30) properly one must see that each value of $\alpha = 1, 2, 3$ fixes exactly one pair of values for (β, γ). For example, $\alpha = 1$ fixes (β, γ) as either $(2, 3)$ or $(3, 2)$, yielding the same result for both choices.

For each value of α, the eigenvalue λ in Theorem 1 vanishes iff n_α remains constant along e_α. However this condition is equivalent to $n_\alpha = c$, with c as a constant. In this case, $k_{ab} = c^2\, e_{\alpha a}\, e_{\alpha b} + \sum\limits_{\beta\neq\alpha} n_\beta^2\, e_{\beta a}\, e_{\beta b}$.

Theorem 2. *e_β is an eigenvector of $\underset{\alpha}{M}$ (with $\alpha \neq \beta$) iff the following conditions are satisfied:*

(i) $\Delta_{e_\beta}(\ln n_\alpha) = 0$, *i.e. n_α remains invariant along the direction of e_β;*
(ii) $\gamma_{\alpha\gamma\beta}[n_\alpha^2 - n_\gamma^2] + \gamma_{\alpha\beta\gamma}[n_\alpha^2 - n_\beta^2] + \gamma_{\beta\gamma\alpha}[n_\gamma^2 - n_\beta^2] = 0$, *where $\gamma \neq \beta \neq \alpha$ for one pair (β, γ).*

The corresponding eigenvalue is $\lambda = -\frac{n_\beta}{n_\alpha^2}\Delta_{e_\alpha} n_\beta + \gamma_{\alpha\beta\beta}(-\frac{n_\beta^2}{n_\alpha^2} + 1)$.

Proof: Contracting $\underset{\alpha}{M}{}_b^c e_\beta^b = \lambda e_\beta^c$ with $e_{\alpha c}$ one obtains $\Delta_{e_\beta}(\ln n_\alpha) = 0$. This condition is satisfied whenever $\Delta_{e_\beta} n_\alpha = 0$. The second condition is a consequence of $\underset{\alpha}{M}{}_b^c e_\beta^b e_{\gamma c} = 0$.
Contracting $\underset{\alpha}{M}{}_b^c e_\beta^b = \lambda e_\beta^c$ with $e_{\beta c}$ yields the eigenvalue λ.
The simplifications performed are based on the orthogonality conditions of the tetrad vectors and on the properties of the rotation coefficients. □

Notice that the two conditions (i) and (ii) in Theorem 2 are satisfied simultaneously if $n_\alpha = n_\beta = n_\gamma = c$, with c a constant, in which case $\lambda = 0$ and $k_{ab} = c^2\, x_a x_b + c^2\, y_a y_b + c^2\, z_a z_b$.

The previous theorems show that strong conditions have to be imposed both on n_α ($\alpha = 1, 2, 3$) and the metric if one requires that the spatial tetrad vectors are principal directions of $\underset{\alpha}{M}$, for $\alpha = 1, 2, 3$.

However, the conditions for e_α to be an eigenvector of $\underset{\alpha}{M}$ are less restrictive then the conditions for e_β to be an eigenvector of the same tensor, for all values of $\beta \neq \alpha$: in the first case the conditions to be satisfied contain only intrinsic derivatives of the quantities n_α; in the second case, besides conditions on the intrinsic derivatives on the n_α, one also has conditions containing the Ricci rotation coefficients.
Furthermore, for e_α to be an eigenvector of $\underset{\alpha}{M}$ only conditions on n_α have to be satisfied: n_α is to remain constant along the directions of e_β for all values of $\beta \neq \alpha$. In this case the eigenvalue corresponding to e_α depends on n_α only. Moreover, the conditions for the vectors e_β, for all $\beta \neq \alpha$, to be eigenvectors of $\underset{\alpha}{M}$ depend explicitly on the three quantities n_1, n_2 and n_3.

Finally we use the previous theorems to establish the conditions for each vector e_α, with $\alpha = 1, 2, 3$, to be an eigenvector of the three tensors $\underset{1}{M}$, $\underset{2}{M}$, $\underset{3}{M}$ simultaneously. One can show that those conditions are:

(i) $\Delta_{e_\beta}(\ln n_\alpha) = 0$,
(ii) $\Delta_{e_\alpha}(\ln n_\beta) = 0$,
(iii) $\gamma_{\alpha\beta\gamma}\ [n_\alpha^2 - n_\beta^2] + \gamma_{\alpha\gamma\beta}\ [n_\gamma^2 - n_\alpha^2] + \gamma_{\beta\gamma\alpha}[n_\beta^2 - n_\gamma^2] = 0$,

for all values of β and γ such that $\beta \neq \gamma \neq \alpha$.
Here conditions (i), (ii) and (iii) must be satisfied for all values of $\beta \neq \alpha$.

Ruling out the solution $n_1 = n_2 = n_3 =$ constant, which is not physically interesting, it is not easy to solve these last equations. However one can say that, in general, the principal directions of the pulled back material metric k are not principal directions of the three tensors $\underset{1}{M}$, $\underset{2}{M}$ and $\underset{3}{M}$.

4. Examples

In this section examples concerning a static spherically symmetric metric and an axially symmetric, non-rotating metric are presented and the analysis developed in the last section is applied. The main problem when dealing with examples lies in the difficulties of finding an orthonormal tetrad for the space-time metric such that the corresponding spacelike tetrad vectors are precisely the principal directions of the pulled back material metric. However, in the examples presented, this difficulty was overcome.

4.1. **The static spherically symmetric case.** In this subsection we analyse the elasticity difference tensor and corresponding eigendirections for a static spherically symmetric metric, due to its significance on modelling neutron stars. The metric regarded here can be thought of as the interior metric of a non-rotating star composed of an elastic material.
For a static spherically symmetric spacetime the line-element can be written as

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \tag{34}$$

where the coordinates $\omega^a = \{t, r, \theta, \phi\}$ are, respectively, the time coordinate, the radial coordinate, the axial coordinate and the azimuthal coordinate. Choosing the basis one-forms $u_a = (-e^{\nu(r)}, 0, 0, 0)$, $x_a = (0, e^{\lambda(r)}, 0, 0)$, $y_a = (0, 0, r, 0)$ and $z_a = (0, 0, 0, r\sin\theta)$ for the orthonormal tetrad, the metric is given by $g_{ab} = -u_a u_b + x_a x_b + y_a y_b + z_a z_b$ and $h_{ab} = x_a x_b + y_a y_b + z_a z_b$ defines the corresponding projection tensor. Using this tetrad, the pulled-back material metric becomes

$$k_{ab} = n_1^2 x_a x_b + n_2^2 y_a y_b + n_2^2 z_a z_b, \tag{35}$$

where we have chosen $n_3 = n_2$ since for this material distribution k has only two different eigenvalues.

Let $\xi^A = \{\tilde{r}, \tilde{\theta}, \tilde{\phi}\}$ be the coordinate system in the material space $\mathcal{X}$. Since the space-time is static and spherically symmetric, $\tilde{r}$ can only depend on r and one can take $\tilde{\theta} = \theta$ and $\tilde{\phi} = \phi$ so that the configuration of the material is entirely described by the material radius $\tilde{r}(r)$. Moreover, the only non-zero components of the deformation gradient are $\frac{d\xi^1}{d\omega^1} = \frac{d\tilde{r}}{dr}$, $\frac{d\xi^2}{d\omega^2} = 1$ and $\frac{d\xi^3}{d\omega^3} = 1$.
In $\mathcal{X}$ the material metric is $k_{AB} = \tilde{x}_A\tilde{x}_B + \tilde{y}_A\tilde{y}_B + \tilde{z}_A\tilde{z}_B$, with $\tilde{x}_A = e^{\tilde{\lambda}}d\tilde{r}_A$, $\tilde{y}_A = \tilde{r}d\tilde{\theta}_A$, $\tilde{z}_A = \tilde{r}sin\tilde{\theta}d\tilde{\phi}_A$ and $\tilde{\lambda} = \lambda(\tilde{r})$. The pull-back of the material metric becomes

$$k^a{}_b = g^{ac}k_{cb} = g^{ac}(\xi^C_c \xi^B_b k_{CB}) = \left(\frac{d\tilde{r}}{dr}\right)^2 e^{2\tilde{\lambda}-2\lambda}\delta^a_1\delta^1_b + \frac{\tilde{r}^2}{r^2}\delta^a_2\delta^2_b + \frac{\tilde{r}^2}{r^2}\delta^a_3\delta^3_b. \tag{36}$$

Comparing (35) and (36) it is simple to obtain the following values for the linear particle densities (all positive), which are found to depend on r alone:

$$n_1 = n_1(r) = \frac{d\tilde{r}}{dr}e^{\tilde{\lambda}-\lambda} \tag{37}$$

$$n_2 = n_2(r) = n_3(r) = \frac{\tilde{r}}{r}. \tag{38}$$

The non-zero components of the strain tensor (3), when written as functions of the quantities n_α, are

$$
\begin{aligned}
s_{rr} &= \frac{1}{2}e^{2\lambda}(1-n_1^2)\\
s_{\theta\theta} &= \frac{1}{2}r^2(1-n_2^2)\\
s_{\phi\phi} &= \frac{1}{2}r^2 sin^2\theta(1-n_2^2).
\end{aligned}
$$

Using the expressions obtained for the n_α one can find that the condition for this tensor to vanish identically is $\tilde{r}=r$.

Calculating the quantities given in (10) one obtains

$$
\begin{aligned}
&\Theta = 0\\
&\dot{u}_a = \left(0, e^{2\nu}\frac{d\nu}{dr}, 0, 0\right)\\
\sigma_{ab}:\ &\sigma_{12} = \frac{1}{2}e^{4\nu}\frac{d\nu}{dr} = \sigma_{21}\\
\omega_{ab}:\ &\omega_{12} = e^{2\nu}\frac{d\nu}{dr} + \frac{1}{2}e^{4\nu}\frac{d\nu}{dr}\\
&\omega_{21} = -\omega_{12},
\end{aligned}
$$

where the remaining components of σ_{ab} and ω_{ab} vanish.

The non-zero components of the elasticity difference tensor S^a_{bc} are:

$$
\begin{aligned}
S^r{}_{rr} &= \frac{1}{n_1}\frac{dn_1}{dr}\\
S^\theta{}_{\theta r} &= \frac{1}{n_2}\frac{dn_2}{dr}\\
S^\phi{}_{\phi r} &= \frac{1}{n_2}\frac{dn_2}{dr}\\
S^r{}_{\theta\theta} &= re^{-2\lambda} - re^{-2\lambda}\frac{n_2^2}{n_1^2} - e^{-2\lambda}r^2\frac{n_2}{n_1^2}\frac{dn_2}{dr}\\
S^r{}_{\phi\phi} &= e^{-2\lambda}rsin^2 - e^{-2\lambda}rsin^2\theta\frac{n_2^2}{n_1^2} - e^{-2\lambda}r^2sin^2\theta\frac{n_2}{n_1^2}\frac{dn_2}{dr}.
\end{aligned}
$$

Since $S^a{}_{bc} = S^a{}_{cb}$, there are only seven non-zero components for this tensor on the coordinate system chosen above.
Again, using (37) and (38) one obtains:

(i) the components $S^\theta{}_{\theta r}$ and $S^\phi{}_{\phi r}$ are zero whenever the function $\tilde{r}$ is of the form $\tilde{r} = c_1 r$, where c_1 is a constant;
(ii) $S^r{}_{rr}$ is zero whenever $\tilde{r} = c_2 + c_3\int e^{\lambda-\tilde{\lambda}}dr$;
(iii) the components $S^r{}_{\theta\theta}$ and $S^r{}_{\phi\phi}$ are zero whenever $\tilde{r} = c_4 e^{\int\frac{e^{-2\tilde{\lambda}+2\lambda}}{r}dr}$.

The second order symmetric tensors $\underset{\alpha}{M}$, for $\alpha = 1, 2, 3$, have the following non-zero components:

$$
\begin{aligned}
\underset{1}{M}_{rr} &= \frac{e^{\lambda}}{n_1}\frac{dn_1}{dr} \\
\underset{1}{M}_{\theta\theta} &= e^{-\lambda} r - e^{-\lambda} r \frac{n_2^2}{n_1^2} - e^{-\lambda} r^2 \frac{n_2}{n_1^2}\frac{dn_2}{dr} \\
\underset{1}{M}_{\phi\phi} &= e^{-\lambda} r sin^2\theta - e^{-\lambda} r sin^2\theta \frac{n_2^2}{n_1^2} - e^{-\lambda} r^2 sin^2\theta \frac{n_2}{n_1^2}\frac{dn_2}{dr} \\
\underset{2}{M}_{r\theta} &= \underset{2}{M}_{\theta r} = \frac{r}{n_2}\frac{dn_2}{dr} \\
\underset{3}{M}_{r\phi} &= \underset{3}{M}_{\phi r} = \frac{r sin\theta}{n_2}\frac{dn_2}{dr}.
\end{aligned}
$$

The eigenvalues and eigenvectors of these tensors are presented in tables 1, 2 and 3. The eigenvectors are then compared with the eigendirections of the material metric.

Table 1 - Eigenvectors and eigenvalues for $\underset{1}{M}$

Eigenvectors	Eigenvalues
x	$\mu_1 = \frac{e^{-\lambda}}{n_1}\frac{dn_1}{dr}$
y	$\mu_2 = \frac{e^{-\lambda}}{r} - \frac{e^{-\lambda}}{r}\frac{n_2^2}{n_1^2} - e^{-\lambda}\frac{n_2}{n_1^2}\frac{dn_2}{dr}$
z	$\mu_3 = \frac{e^{-\lambda}}{r} - \frac{e^{-\lambda}}{r}\frac{n_2^2}{n_1^2} - e^{-\lambda}\frac{n_2}{n_1^2}\frac{dn_2}{dr}$

Notice that, in the present example, $\underset{1}{M}$ maintains the eigenvectors of k, namely x, y and z, the two last ones being associated with the same eigenvalue. Therefore the canonical form for $\underset{1}{M}$ is $\underset{1}{M}_{bc} = \mu_1 x_b x_c + \mu_2 (y_b y_c + z_b z_c)$, where μ_1 and μ_2 are the eigenvalues corresponding to x and y ($\equiv z$), respectively.

Table 2 - Eigenvectors and eigenvalues for $\underset{2}{M}$

Eigenvectors	Eigenvalues
$x + y$	$\mu_4 = \frac{e^{-\lambda}}{n_2}\frac{dn_2}{dr}$
$x - y$	$\mu_5 = -\frac{e^{-\lambda}}{n_2}\frac{dn_2}{dr}$
z	$\mu_6 = 0$

In this case, only z is simultaneously an eigenvector of k and $\underset{2}{M}$. The corresponding eigenvalue is now equal to zero. The other two eigenvectors are $x + y$ and $x - y$ so that the canonical form for $\underset{2}{M}$ can be expressed as $\underset{2}{M}_{bc} = 2\mu_4 (x_b y_c + y_b x_c)$, where $\mu_4 = e^{-\lambda}\left(\frac{1}{\tilde{r}}\frac{d\tilde{r}}{dr} - \frac{1}{r}\right)$.

Table 3 - Eigenvectors and eigenvalues for $\underset{3}{M}$

Eigenvectors	Eigenvalues
$x + z$	$\mu_7 = \frac{e^{-\lambda}}{n_2}\frac{dn_2}{dr}$
$x - z$	$\mu_8 = -\frac{e^{-\lambda}}{n_2}\frac{dn_2}{dr}$
y	$\mu_9 = 0$

Comparing $\underset{2}{M}$ and $\underset{3}{M}$, it is easy to see that the role of z and y is interchanged. The eigenvalues of $\underset{2}{M}$ are equal to the eigenvalues of $\underset{3}{M}$ and the canonical form of this tensor field can be written as $\underset{3}{M}_{bc} = 2\mu_7(x_b z_c + z_b x_c)$, where $\mu_7 = e^{-\lambda}\left(\frac{1}{\tilde{r}}\frac{d\tilde{r}}{dr} - \frac{1}{r}\right)$.

It should be noticed that for n_2 =constant the tensors $\underset{2}{M}$ and $\underset{3}{M}$ vanish. Therefore this is not an interesting case to analyse.

The condition for x, y and z to remain eigenvectors for $\underset{2}{M}$ and $\underset{3}{M}$ is that $\tilde{r} = cr$, in which case $\underset{2}{M}$ and $\underset{3}{M}$ vanish identically.

The tetrad components of the elasticity difference tensor can be calculated from (22), yielding:

$$
\begin{aligned}
S^1{}_{11} &= e^{-\lambda}\frac{1}{n_1}\frac{dn_1}{dr} \\
S^2{}_{21} &= e^{-\lambda}\frac{1}{n_2}\frac{dn_2}{dr} \\
S^3{}_{31} &= e^{-\lambda}\frac{1}{n_2}\frac{dn_2}{dr} \\
S^1{}_{22} &= e^{-\lambda}\frac{1}{r} - e^{-\lambda}\frac{1}{r}\frac{n_2^2}{n_1^2} - e^{-\lambda}\frac{n_2}{n_1^2}\frac{dn_2}{dr} \\
S^1{}_{33} &= e^{-\lambda}\frac{1}{r} - e^{-\lambda}\frac{1}{r}\frac{n_2^2}{n_1^2} - e^{-\lambda}\frac{n_2}{n_1^2}\frac{dn_2}{dr}.
\end{aligned}
$$

The expressions for the Ricci rotation coefficients are

$$
\begin{aligned}
\gamma_{122} &= \frac{e^{-\lambda}}{r} \\
\gamma_{133} &= \frac{e^{-\lambda}}{r} \\
\gamma_{233} &= \frac{\cos\theta}{r\sin\theta}.
\end{aligned}
$$

4.2. **The axially symmetric non-rotating case.** First, consider an elastic, axially symmetric, uniformly rotating body in interaction with its gravitational field. The exterior of the body may be described by the following metric, [27],

$$ds^2 = -e^{2\nu}dt^2 + e^{2\mu}dr^2 + e^{2\mu}dz^2 + e^{2\psi}(d\phi - \omega dt)^2, \tag{39}$$

where ν, ψ, ω, μ are scalar fields depending on r and z.

Assume that the material metric is flat. Introducing in $\mathcal{X}$ cylindrical coordinates $\xi^A = \{R, \zeta, \Phi\}$, then the material metric takes the form:

$$ds^2 = dR^2 + d\zeta^2 + R^2 d\Phi^2, \tag{40}$$

where $\Phi(t, r, z, \phi) = \phi - \Omega(r, z)t$ and the parameters R, ζ depend on r and z.

The space-time metric for the limiting case of an axially symmetric non-rotating elastic system can be written as

$$ds^2 = -e^{2\nu}dt^2 + e^{2\mu}dr^2 + e^{2\mu}dz^2 + e^{2\psi}d\phi^2. \tag{41}$$

This metric is obtained from (39), when $\omega = 0$ and the angular velocity Ω vanishes.

Imposing $R = R(r)$, $\zeta = z$ and $g_{ab} = g_{ab}(r)$, one obtains a further reduction to cylindrical symmetry. This reduction is considered in [27].

We will work with the space-time metric presented in (41), where ν, μ, ψ depend on r only, so that $g_{ab} = -u_a u_b + x_a x_b + y_a y_b + z_a z_b$, with $u_a = (-e^{\nu(r)}, 0, 0, 0)$, $x_a = (0, e^{\mu}, 0, 0)$, $y_a = (0, 0, e^{\mu(r)}, 0)$ and $z_a = (0, 0, 0, e^{\psi(r)})$. The space-time coordinates are $\omega^a = \{t, r, z, \phi\}$.

In $\mathcal{X}$ the material metric k_{AB} is defined by $k_{AB} = \tilde{x}_A \tilde{x}_B + \tilde{y}_A \tilde{y}_B + \tilde{z}_A \tilde{z}_B$, where $\tilde{x}_A = dR_A$, $\tilde{y}_A = dz_A$ and $\tilde{z}_A = Rd\phi_A$. The relativistic deformation gradient has the following non-zero components: $\frac{d\xi^1}{d\omega^2} = \frac{dR}{dr}$, $\frac{d\xi^2}{d\omega^1} = 1$ and $\frac{d\xi^3}{d\omega^3} = 1$. The pull-back of the material metric yields

$$k^a{}_b = g^{ac} k_{cb} = g^{ac} \left(\xi^C_c \xi^B_b k_{CB} \right) = e^{-2\mu} \delta^a_1 \delta^1_b + \left(\frac{dR}{dr} \right)^2 e^{-2\mu} \delta^a_2 \delta^2_b + R^2 e^{-2\psi} \delta^a_3 \delta^3_b. \quad (42)$$

Therefore, the corresponding line-element can be expressed as

$$ds^2 = dr^2 + \left(\frac{dR}{dr} \right) dz^2 + R^2 d\phi^2. \quad (43)$$

On the other hand, the pulled back material metric is given by

$$k_{ab} = n_1^2 x_a x_b + n_2^2 y_a y_b + n_3^2 z_a z_b. \quad (44)$$

Comparing (42) with (44) one concludes that the linear particle densities (all positive) are expressed as

$$n_1 = n_1(r) = e^{-\mu} \quad (45)$$

$$n_2 = n_2(r) = e^{-\mu} \frac{dR}{dr} \quad (46)$$

$$n_3 = n_3(r) = Re^{-\psi}. \quad (47)$$

The components of the strain tensor (3) are:

$$\begin{aligned} s_{rr} &= \frac{1}{2} e^{2\mu} (1 - n_1^2) \\ s_{zz} &= \frac{1}{2} e^{2\mu} (1 - n_2^2) \\ s_{\phi\phi} &= \frac{1}{2} e^{2\psi} (1 - n_3^2). \end{aligned}$$

This tensor vanishes if the condition $R(r) = e^{\psi}$ holds.

In this case the quantities in (10) are given by the following expressions:

$$\begin{aligned} \Theta &= 0 \\ \dot{u}_a &= \left(0, e^{2\nu} \frac{d\nu}{dr}, 0, 0 \right) \\ \sigma_{ab}: \quad \sigma_{12} &= \frac{1}{2} e^{4\nu} \frac{d\nu}{dr} = \sigma_{21} \\ \omega_{ab}: \quad \omega_{12} &= e^{2\nu} \frac{d\nu}{dr} + \frac{1}{2} e^{4\nu} \frac{d\nu}{dr} \\ \omega_{21} &= -\omega_{12}, \end{aligned}$$

where the remaining components of σ_{ab} and ω_{ab} vanish.

The non-zero components of the elasticity difference tensor are listed below:

$$
\begin{aligned}
S^{r}{}_{rr} &= \frac{1}{n_1}\frac{dn_1}{dr} \\
S^{z}{}_{zr} &= \frac{1}{n_2}\frac{dn_2}{dr} \\
S^{\phi}{}_{\phi r} &= \frac{1}{n_3}\frac{dn_3}{dr} \\
S^{r}{}_{zz} &= \frac{d\mu}{dr} - \frac{n_2^2}{n_1^2}\frac{d\mu}{dr} - \frac{n_2}{n_1^2}\frac{dn_2}{dr} \\
S^{r}{}_{\phi\phi} &= e^{-2\psi-2\mu}\left(\frac{d\psi}{dr} - \frac{n_3^2}{n_1^2}\frac{d\psi}{dr} - \frac{n_3}{n_1^2}\frac{dn_3}{dr}\right).
\end{aligned}
$$

One can see that only seven components of the elasticity difference tensor are non-zero. However, using the expressions (45), (46) and (47) to obtain the conditions for those components to vanish, leads to the following results:

(i) $S^{r}{}_{rr}$ is zero whenever $\mu(r) = c$, where c is a constant;
(ii) $S^{z}{}_{zr}$ is zero whenever $R(r) = c_1 + c_2 \int e^{\mu(r)} dr$;
(iii) $S^{\phi}{}_{\phi r}$ is zero whenever $R(r) = c_3 e^{\psi(r)}$;
(iv) $S^{r}{}_{zz}$ is zero whenever $R(r) = \pm \int \sqrt{2\mu(r) + c_4}\, dr + c_5$;
(v) $S^{r}{}_{\phi\phi}$ is zero whenever $R(r) = \pm\sqrt{2\int \frac{e^{2\psi}}{e^{2\mu}}\frac{d\psi}{dr} dr + c_6}$.

In this case, the second-order tensors $\underset{1}{M}$, $\underset{2}{M}$ and $\underset{3}{M}$ have the following non-zero components:

$$
\begin{aligned}
\underset{1}{M}_{rr} &= e^{\mu}\frac{1}{n_1}\frac{dn_1}{dr} \\
\underset{1}{M}_{zz} &= e^{\mu}\left(\frac{d\mu}{dr} - \frac{n_2^2}{n_1^2\frac{d\mu}{dr}} - \frac{n_2}{n_1^2}\frac{dn_2}{dr}\right) \\
\underset{1}{M}_{\phi\phi} &= e^{2\psi-\mu}\left(\frac{d\psi}{dr} - \frac{n_3^2}{n_1^2}\frac{d\psi}{dr} - \frac{n_3}{n_1^2}\frac{dn_3}{dr}\right) \\
\underset{2}{M}_{rz} &= \underset{2}{M}_{zr} = e^{\mu}\frac{1}{n_2}\frac{dn_2}{dr} \\
\underset{3}{M}_{r\phi} &= \underset{3}{M}_{\phi r} = e^{\psi}\frac{1}{n_3}\frac{dn_3}{dr}.
\end{aligned}
$$

The next three tables contain the eigenvalues and eigenvectors for these tensors. Their eigenvectors are then compared with the eigenvectors of the pulled-back material metric.

Table 1 - Eigenvectors and eigenvalues for $\underset{1}{M}$

Eigenvectors	Eigenvalues
x	$\lambda_1 = e^{-\mu}\frac{1}{n_1}\frac{dn_1}{dr}$
y	$\lambda_2 = e^{-\mu}\left(\frac{d\mu}{dr} - \frac{n_2^2}{n_1^2}\frac{d\mu}{dr} - \frac{n_2}{n_1^2}\frac{dn_2}{dr}\right)$
z	$\lambda_3 = e^{-\mu}\left(\frac{d\psi}{dr} - \frac{n_3^2}{n_1^2}\frac{d\mu}{dr} - \frac{n_3}{n_1^2}\frac{dn_3}{dr}\right)$

One can see that x, y and z are eigenvectors for both k and $\underset{1}{M}$, however the eigenvalues are not the same. The canonical form for $\underset{1}{M}$ can be written as $\underset{1}{M}_{bc} = \lambda_1 x_b x_c + \lambda_2 y_b y_c + \lambda_3 z_b z_c$.

Table 2 - Eigenvectors and eigenvalues for $\underset{2}{M}$

Eigenvectors	Eigenvalues
$x+y$	$\lambda_4 = e^{-\mu}\left(\frac{1}{n_2}\frac{dn_2}{dr}\right)$
$x-y$	$\lambda_5 = -e^{-\mu}\left(\frac{1}{n_2}\frac{dn_2}{dr}\right)$
z	$\lambda_6 = 0$

$\underset{2}{M}$ inherits only one eigenvector z from k, which corresponds to a zero eigenvalue. The other two eigenvectors of $\underset{2}{M}$ are linear combinations of x and y, namely $x+y$ and $x-y$, the corresponding eigenvalues being symmetric. The canonical form for $\underset{2}{M}$ can be written as $\underset{2}{M}_{bc} = 2\lambda_4(x_b y_c + y_b x_c)$, where $\lambda_4 = \left(\frac{\frac{d^2R}{dr^2}}{\frac{dR}{dr}} - \frac{d\mu}{dr}\right)e^{-\mu}$.

Table 3 - Eigenvectors and eigenvalues for $\underset{3}{M}$

Eigenvectors	Eigenvalues
$x+z$	$\lambda_7 = e^{-\mu}\left(\frac{1}{n_3}\frac{dn_3}{dr}\right)$
$x-z$	$\lambda_8 = -e^{-\mu}\left(\frac{1}{n_3}\frac{dn_3}{dr}\right)$
y	$\lambda_9 = 0$

$\underset{3}{M}$ and k have the eigenvector y in common, the corresponding eigenvalue being equal to zero. The other two eigenvectors of $\underset{3}{M}$ are linear combinations of x and z, namely $x+z$ and $x-z$. These two eigenvectors are associated with symmetric eigenvalues. The canonical form for $\underset{3}{M}$ can be written as $\underset{3}{M}_{bc} = 2\lambda_7(x_b z_c + z_b x_c)$, where $\lambda_7 = \left(\frac{1}{R}\frac{dR}{dr} - \frac{d\psi}{dr}\right)e^{-\mu}$.

Imposing that x and y are eigenvectors of $\underset{2}{M}$, one obtains $R(r) = c_1 + \int e^{\mu} dr c_2$, in which case $\underset{2}{M} = 0$. Analogously, for x and z to be eigenvectors for $\underset{3}{M}$ one must require that $R(r) = c_3 e^{\psi}$, in which case $\underset{3}{M} = 0$.

The tetrad components of the elasticity difference tensor are obtained from (22), yielding:

$$
\begin{aligned}
S^1{}_{11} &= e^{-\mu}\frac{1}{n_1}\frac{dn_1}{dr}\\
S^2{}_{21} &= e^{-\mu}\frac{1}{n_2}\frac{dn_2}{dr}\\
S^3{}_{31} &= e^{-\mu}\frac{1}{n_3}\frac{dn_3}{dr}\\
S^1{}_{22} &= e^{-\mu}\frac{d\mu}{dr} - e^{-\mu}\frac{n_2^2}{n_1^2}\frac{d\mu}{dr} - e^{-\mu}\frac{n_2}{n_1^2}\frac{dn_2}{dr}\\
S^1{}_{33} &= e^{-\mu}\frac{d\psi}{dr} - e^{-\mu}\frac{n_3^2}{n_1^2}\frac{d\psi}{dr} - e^{-\mu}\frac{n_3}{n_1^2}\frac{dn_3}{dr}.
\end{aligned}
$$

For the expressions of the Ricci coefficients one obtains:

$$\gamma_{122} = \frac{\frac{d\mu}{dr}}{e^{\mu}}$$

$$\gamma_{133} = \frac{\frac{d\psi}{dr}}{e^{\mu}}.$$

5. Concluding remarks

In the present paper we have presented a detailed study of the elasticity difference tensor. This tensor is decomposed along the spatial eigendirections of the pulled back material metric, here called x, y and z, yielding three second order symmetric tensors named as $\underset{1}{M}$, $\underset{2}{M}$ and $\underset{3}{M}$, respectively. The eigendirections of these tensors are compared with the eigendirections of the pulled back material metric to find conditions for them to coincide. These conditions are presented in the theorems of section 3, showing that only in very restrictive cases those eigendirections coincide. Two classes of static space-times are then considered, one being spherically symmetric and another axially symmetric. In both cases canonical forms for the tensors $\underset{1}{M}$, $\underset{2}{M}$ and $\underset{3}{M}$ are obtained and the following conclusions are drawn:

- the eigenvectors of the pulled back material metric are also eigenvectors of $\underset{1}{M}$;

- the eigenvectors of $\underset{2}{M}$ are $x+y$, $x-y$ and z;

- the eigenvectors of $\underset{3}{M}$ are $x+z$, $x-z$ and y.

For both examples the deformation gradient depends only on the radial coordinate r, the eigenvector associated with this coordinate being x. Therefore a similar pattern is found when the spatial eigenvectors of the pulled back material metric are compared with the eigenvectors of those tensors arising from the decomposition of the elasticity difference tensor referred above.

In the near future we intend to analyse other classes of space-time metrics, in particular non-static cases.

6. Acknowledgements

The authors would like to thank L. Samuelsson for many valuable discussions on this work. We also thank the reviewers for their valuable comments and suggestions.

References

[1] M. Karlovini and L. Samuelsson, *Class. Quantum Grav.*, 20, 3613-3648, (2003).
[2] J. Weber, *Phys. Rev.*, 117, 306-313, (1960).
[3] J. Weber, *General relativity and gravitational waves*, Interscience, New York, (1961).
[4] J. Weber, *Phys. Rev. Lett.*, 22, 1320, (1969).
[5] D. Pines, *Inside Neutron Stars*, Proceedings of the twelfth international conference on low temperature physics, Kyoto, Japan, edited by E. Kanda, Academic Press of Japan, (1971).
[6] S. Shapiro and S. Teukoplsky, *Black Holes, White Dwarfs and Neutron stars*, New York: Wiley, (1983).
[7] P. McDermott, H. Van Horn and C. Hansen *Astrophys. J.*, 325, 725, (1988).
[8] P. Haensel, *Solid interiors of neutron stars and gravitational radiation*, Astrophysical Sources of Gravitational Radiation (Proc. Les Houches School), edited by J. Marck and J. Lasota, (1995).

[9] F. Noether, *Ann. Phys.*, 336, 919-944, (1910).
[10] M. Born, *Physik Zeit.*, 12, 569-575, (1911).
[11] G. Herglotz, *Ann. Phys.*, 341, 493-533, (1911).
[12] G. Nordström, *Physik Zeit.*, 12, 854-857, (1911).
[13] J. L. Synge, *Math. Z.*, 72, 82, (1959).
[14] B. S. De Witt, The quantization of geometry, in *Gravitation: an introduction to current research* (ed. L. Witten), Wiley, (1962).
[15] C. B. Rayner, *Proc. R. Soc. Lond.*, A 272, 44, (1963).
[16] J. F. Bennoun, *Compt. Rennd.*, 259, 3705, (1964).
[17] J. F. Bennoun, *Ann. Inst. Henri Poincar*, A 3, 41, (1965).
[18] W. C. Hernandez, *Phys. Rev.*, D 1, 1013, (1970).
[19] G. Maugin, *Ann. Inst. H. Poincar*, A 15, 275-302, (1971).
[20] B. Carter and H. Quintana, *Proc. R. Soc.*, A 331, 57, (1972).
[21] M. Karlovini et al., *Class. Quantum Grav.*, 21, 1559-1581, (2004).
[22] M. Karlovini and L. Samuelsson, *Class. Quantum Grav.*, 21, 4531-4548, (2004).
[23] J. Kijowski and G. Magli, *Journal of Geometry and Physics*, 9, 207-223, (1992).
[24] J. Kijowski and G. Magli, *Rep. Math. Phys.*, 39, 99-112, (1997).
[25] J. Kijowski and G. Magli, *Class. Quantum Grav.*, 15, 3891-3916, (1998).
[26] G. Magli, *Gen. Rel. Grav.*, 25, 441-460, (1993).
[27] G. Magli, *Gen. Rel. Grav.*, 25, 1277-1293, (1993).
[28] H. Elst, *Extensions and applications of 1+3 decomposition methods in general relativistic cosmological modelling*, Ph.D.Thesis, University of London, (1996).
[29] R. Wald, *General Relativity*, The University of Chicago Press, Chicago and London, (1984).
[30] C. Misner, K. Thorne and J. Wheeler, *Gravitation*, W. H. Freeman and Company, San Francisco, (1970).
[31] N. Rosen, *Annals of Physics*, 22, 1-11, (1963).
[32] G. Ellis and H. Elst, *Cosmological Models*, Cargse Lectures, (1998).
[33] G. Ellis and M. McCallum, *Commun. Math. Phys.*, 12, 108-141, (1969).

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