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**“The Adequacy of the Traditional Econometric  
Approach to  
Nonlinear Cycles”**

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# The Adequacy of the Traditional Econometric Approach to Nonlinear Cycles

by

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## Abstract

To show that the traditional econometric approach is not able to deal with deterministic chaos, we use an extension of Goodwin's growth cycle model to generate artificial data for output. An EGARCH model is estimated to describe the data generation process. Although using some traditional econometric tests no evidence of misspecification is found the estimated process is qualitatively wrong: it is dynamically stable when the true process is unstable. We present a specific econometric procedure developed to deal with deterministic chaos: the BDS statistics. Also an explanation for

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the little evidence of deterministic chaos in aggregated macroeconomic time series is suggested.

## 1 A Model of Growth and Cycles

In 1991 Goodwin extended his 1967 predator-prey model in order to accomplish growth and cycles. The model generated a Kondratieff growth cycle, which also incorporated Juglar cycles.

He incorporated the Schumpeterian swarm of innovations according to which, after a weak beginning, the path-breaking innovation proves its importance and more and more firms will adopt the innovation. At the end, the rate of adoption will diminish since the majority of the firms have already adopted it.

For convenience Goodwin's system of equations is reproduced here ( $u$  and  $L$  should be understood as deviations from equilibrium):

$$\begin{cases} u' = hL \\ L' = -du + fL - ez \\ z' = b + gz(L - c) \end{cases} \quad (1)$$

where  $u$  represents labour's proportion of national income,  $L$  is the rate of employment and  $z$  is a control parameter (e.g. government budget surplus). For sufficiently high values of  $g$  the system generates deterministic chaos.

To include growth in the above model we can consider the effects of investment in

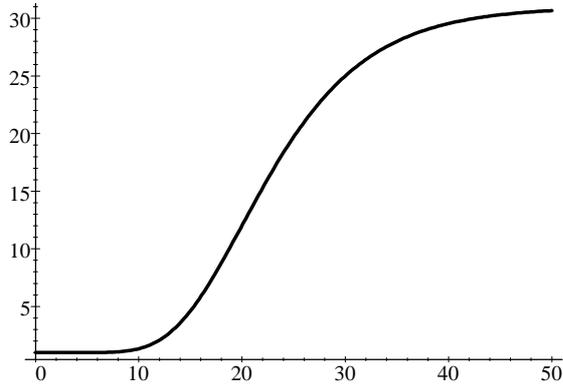


Figure 1: Capital Evolution for 50 years

labour productivity. We admit there is a cyclical component in labour productivity ( $= \gamma \frac{K'}{K}$ , where  $K$  stands for the stock of capital). For investment we admit an historically given fifty years Schumpeterian swarm of innovations. Specifically we will admit that<sup>1</sup>:

$$K' = m e^{n-qt - e^{n-qt}} \quad (2)$$

For example, if we calibrate this equation with the values  $m = 4.5$ ,  $n = 3$ ,  $q = 0.15$  and  $K_0 = 1$  the capital accumulation would be as represented in figure 1.

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<sup>1</sup>This function is known as the Gompertz curve and it is a special case of the generalized logistic. The advantage of this formulation relative to the usual simple logistic is that it is more flexible; namely we are not restricted to a symmetrical curve. If we had used a simple logistic its main implication would have been a greater variation of the series accumulated in early stages. Stone (1990) also used this function to describe the population growth dynamics.

The dynamics of output will be determined by the evolution of employment ( $= L^* + L$ ) and by the evolution of labour productivity ( $= \gamma \frac{K'}{K}$ ):

$$\frac{Y'}{Y} = \frac{L'}{L^* + L} + \gamma \frac{K'}{K} \quad (3)$$

This formulation has one problem. The investment function influences output, but should also be influenced by. To answer, at least partially, to this criticism we will admit that the employment level has its role in the dynamics of the investment, so:

$$K' = me^{n-L-qt-e^{n-L-qt}} \quad (4)$$

In this formulation the employment level enters directly in the investment function, and it can be interpreted as an accelerator mechanism.

Joining equations 1, 3, and 4, the complete model becomes:

$$\left\{ \begin{array}{l} u' = hL \\ L' = -du + fL - ez \\ z' = b + gz(L - c) \\ K' = me^{n-L-qt-e^{n-L-qt}} \\ Y' = \left( \frac{-du+fL-ez}{L^*+L} + \gamma \frac{me^{n-L-qt-e^{n-L-qt}}}{K} \right) Y \end{array} \right. \quad (5)$$

To understand the kind of output dynamics generated by this model we calibrate it with the following parameter values:  $b = 0.001, c = 0.048, d = 0.5, e = 0.8, f = 0.15, g = 85, L^* = 0.9, n = 3, p = 35, q = 0.15, \gamma = 0.3$ . The data are generated for

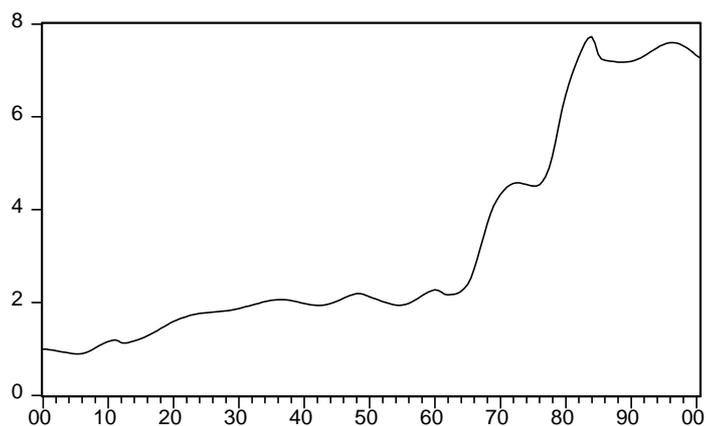


Figure 2: Output Time Evolution

a hundred years. Since the parameters chosen to the investment equation imply a swarm of fifty years we have to introduce a second swarm of fifty years. So in the first fifty years  $m$  takes the value  $m = 4.5$ . Beyond that  $m$  takes the value  $m = 135$ . The evolution of investment is essentially the same in each half century. The initial values were  $u = 0.02, L = 0.04, z = 0, k = 1, y = 1$ . The results can be observed in figure 2. One interesting feature of the time series is that the generated cycles are not identical, even considering identical capital accumulation dynamics for both half centuries. We can observe that a chaotic deterministic system can generate a quite erratic behaviour.

The possibility that an erratic behaviour can be purely deterministic raises an important question: to what extent are the traditional econometric techniques ap-

propriate to deal with this new issue? We will try to sketch the answer in the next point.

## **2 An Econometric Application to Our Artificial Model**

Blatt (1983) alerted to the dangerous consequences of an error in the identification of the stability properties of an economic system. He then asked if the traditional econometric tools were a good instrument to analyze the stability of an economic system. To answer this question he made a simple test.

He generated some economic time series with the help of a nonlinear, locally unstable, macro-model Hicks proposed. With these artificial data he tried to estimate the original model. The results are quite unpleasant: the estimated model did not identify (not even close) the inherent instability of the original model. Basically, a dynamically stable model was estimated and the endogenous cycles were attributed to stochastic shocks, with no statistic evidence of misspecification.

Louçã (1997) made a similar approach. He considered a more general model to generate artificial data for output<sup>2</sup> which was able to simulate growth and cycles endogenously. But, in his treatment of the time series output, he extracted a linear

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<sup>2</sup>The model he used was very similar to the system of equations 5. The main difference is that he represented the investment dynamics with a simple logistic and did not introduce an accelerator component in the investment function.

trend<sup>3</sup> and then modelled the residuals as a linear autoregressive process. The problem with Louçã's approach is that when one tries to apply usual econometric procedures to time series data it is not possible to forget testing the stationarity of the series before extracting a (linear) time trend. Depending on the results of that test extracting a linear time trend may, or may not, be appropriate.

We tested the stationarity of time series represented in figure 2 (the test used was the ADF test and was applied after logarithmizing the series). According to the test result, we could not reject the null hypothesis of nonstationarity for the log of the output ( $LY$ ), while for the growth rate ( $DLY$ ) we reject the null hypothesis accepting the growth rate<sup>4</sup> to be stationary around a constant. So an applied econometrist would not extract a linear trend to stationarize the series. He would rather consider the growth rate of  $Y$ .

So, we try to model  $DLY$  as an autoregressive process. The estimated results are<sup>5</sup>:

$$\begin{aligned}
 DLY_t = & 0.0009 + 1.9571DLY_{t-1} - 1.6237DLY_{t-2} + \\
 & \quad \quad \quad (2.836) \quad \quad (29.026) \quad \quad \quad (-11.511) \\
 & \quad \quad \quad + 0.9392DLY_{t-3} - 0.3552DLY_{t-4} \\
 & \quad \quad \quad \quad \quad \quad (6.66) \quad \quad \quad (-5.273)
 \end{aligned} \tag{6}$$

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<sup>3</sup>The trend was extracted from logarithmized time series, so it is an exponential trend relative to the original data.

<sup>4</sup> $DLY$  is the first difference of  $LY$ . Since  $LY = \log Y$ ,  $DLY$  will be the growth rate of  $Y$ .

<sup>5</sup>Since we have considered semi-annual data, we have the growth rate of period  $t$  depending on the previous four semesters.

where the values in parenthesis are the  $t$ -statistics. The R-squared (and the adjusted R-squared) is about 96%. The residuals show no evidence of serial correlation. The number of lags chosen was based on the Akaike and Schwarz information criteria and were strengthened by the fact that higher lags were not statistically significant<sup>6</sup>.

It is interesting to note that equation 6 is a simple difference equation with an explicit analytic solution:

$$\begin{aligned}
 DLY_t = & 0.0104 + 0.6627^t (A_1 \cos(1.4073t) + A_2 \sin(1.4073t)) + \\
 & + 0.8994^t (A_3 \cos(0.2534t) + A_4 \sin(0.2534t))
 \end{aligned}
 \tag{7}$$

where  $A_1, A_2, A_3,$  and  $A_4$  are arbitrary constants that can be determined with the help of four initial conditions.

This estimated model fits perfectly in Slutsky- Frisch's paradigm: we have an exogenous trend (determined by the constant 0.01042) and two different growth cycles (one with 4.5 years and the other with 24.8 years) aggregated additively. In figure 3 we can see how this estimated model would work in the absence of stochastic shocks.

To explain the persistence of cycles in this model, Frisch would suggest the addition of a stream of exogenous shocks. This is what we do next. We add a stream of exogenous shocks with mean zero and variance 0.0001444 to equation 6 with the help of a normal random number generator<sup>7</sup>. In figure 4 we compare the original artificial

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<sup>6</sup>For example, if we introduced a fifth lag, its P-value would be 0.39.

<sup>7</sup>The variance was chosen in such a way that the original time series and this new time series

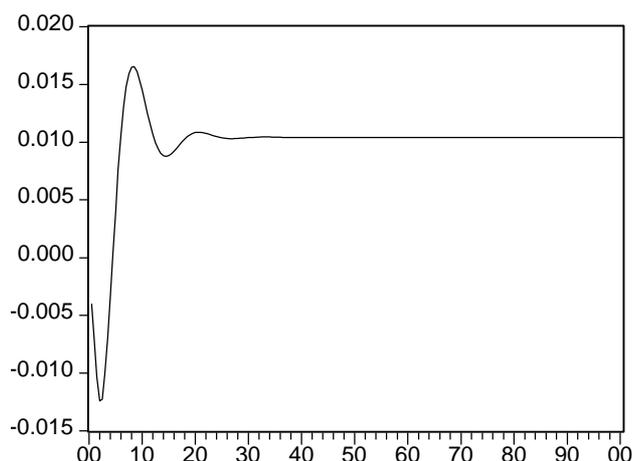


Figure 3: The Estimated Model Dynamics

time series with the time series generated by the estimated model (augmented with the stochastic shocks)<sup>8</sup>.

## 2.1 Problems with Heteroskedasticity

We have so far neglected the possibility of having heteroskedastic disturbances. In traditional time series analysis it was usual to consider homoskedastic processes (associating heteroskedasticity to cross-sectional data). But, at least since Engle (1982), one cannot put aside the possibility of having an Autoregressive Conditional Heteroskedasticity (ARCH) model or one of its extensions, as we shall see.

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have the same variance.

<sup>8</sup>Although not reproduced here, we would have reached the same qualitative results if we had applied the Hodrick-Prescott filter instead and then fitted an autoregressive process to the residuals.

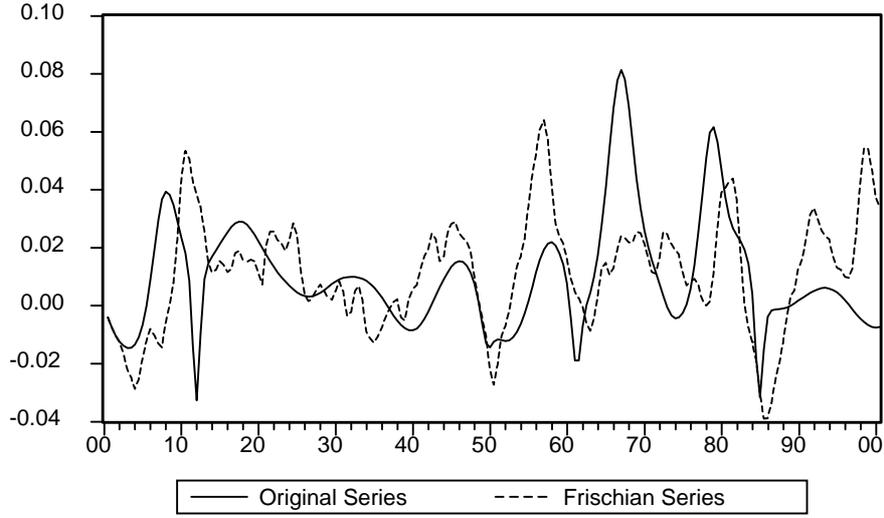


Figure 4: Two Simulated Time Series

Consider a  $p^{th}$  order ARCH process:

$$Y_t = \beta' \mathbf{X}_t + \epsilon_t \quad (8)$$

$$\epsilon_t = u_t \sqrt{\varpi + \sum_{i=1}^q \alpha_i \epsilon_{t-i}^2} \quad (9)$$

where  $u_t$  follows a standard normal. It is easy to derive the conditional and unconditional variances of  $\epsilon_t$ :

$$Var[\epsilon_t | \epsilon_{t-1}, \dots, \epsilon_{t-p}] = \sigma_t^2 = \varpi + \sum_{i=1}^q \alpha_i \epsilon_{t-i}^2 \quad (10)$$

$$Var[\epsilon_t] = \frac{\varpi}{1 - \sum_{i=1}^q \alpha_i} \quad (11)$$

In this situation, although the Ordinary Least Squares (OLS) estimator is BLUE it is not BUE, *i.e.*, it is the best linear unbiased estimator, but there is a more efficient

	obs×R <sup>2</sup>	P-value
1 <sup>st</sup> order	2.979	0.084
2 <sup>nd</sup> order	15.199	0.001
3 <sup>rd</sup> order	15.149	0.002
4 <sup>th</sup> order	16.030	0.003

Table 1: ARCH LM Test

nonlinear estimator. Engle (1982) derived the likelihood function for this model and also presented a Lagrange Multiplier (LM) test for the ARCH process.

In table 5.2 we can see the results of the ARCH LM test. With these results an applied econometrist would have to deal with the conditional heteroskedasticity problem. In our work we begin by considering a Generalized ARCH (GARCH) model proposed by Bollerslev (1986). The advantage of this approach is that it is usually more parsimonious with the number of lags needed. In a GARCH( $p, q$ ) model the conditional variance is given by:

$$\sigma_t^2 = \varpi + \sum_{i=1}^q \alpha_i \epsilon_{t-i}^2 + \sum_{j=1}^p \beta_j \sigma_{t-j}^2 \quad (12)$$

Bollerslev *et al.* (1994) show that this is equivalent to saying that  $\epsilon_t^2$  can be modelled as an ARMA( $[\max(p, q), p]$ ) model. Obviously, if equation 12 is correctly specified the residuals should not exhibit additional ARCH.

After considering several GARCH models of different orders we conclude that the

standardized residuals continued to exhibit ARCH, indicating that equation 12 was misspecified.

Nelson (1991) proposed an Exponential GARCH (EGARCH) model. Equation 12 is replaced by:

$$\ln(\sigma_t^2) = \varpi + \sum_{i=1}^q \left( \alpha_i \left| \frac{\epsilon_{t-i}}{\sigma_{t-i}} \right| + \gamma_i \frac{\epsilon_{t-i}}{\sigma_{t-i}} \right) + \sum_{j=1}^p \beta_j \ln(\sigma_{t-j}^2) \quad (13)$$

When we re-estimate equation 6, admitting that the conditional heteroskedasticity follows an EGARCH(2, 4)<sup>9</sup>:

$$\begin{aligned} DLY_t = & 0.00003 + 3.0965 DLY_{t-1} - 3.6082 DLY_{t-2} \\ & \quad (19.830) \quad (281.625) \quad (-135.434) \\ & + 1.8744 DLY_{t-3} - 0.3679 DLY_{t-4} \\ & \quad (84.615) \quad (-55.842) \end{aligned} \quad (14)$$

$$\begin{aligned} \ln(\sigma_t^2) = & -7.2006 + 2.1347 \left| \frac{\epsilon_{t-1}}{\sigma_{t-1}} \right| - 0.0650 \frac{\epsilon_{t-1}}{\sigma_{t-1}} \\ & \quad (-14.975) \quad (17.355) \quad (-0.816) \\ & + 1.8102 \left| \frac{\epsilon_{t-2}}{\sigma_{t-2}} \right| - 0.0242 \frac{\epsilon_{t-2}}{\sigma_{t-2}} \\ & \quad (9.183) \quad (-0.264) \\ & + 0.0583 \ln(\sigma_{t-1}^2) + 1.3331 \ln(\sigma_{t-2}^2) \\ & \quad (0.930) \quad (36.398) \\ & - 0.1151 \ln(\sigma_{t-3}^2) - 0.5251 \ln(\sigma_{t-4}^2) \\ & \quad (-2.611) \quad (-16.551) \end{aligned} \quad (15)$$

where the values in parenthesis are the  $z$ -statistics. As it can be seen, the results of equation 14 do not differ substantially from the results obtained in equation 6. It is easy to verify that the stability properties do not change. When the ARCH LM test is applied to the standard residuals, the results are conclusive. As we can see in

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<sup>9</sup>The order of the ARCH process was chosen, basically, with the help of the Akaike and Schwartz information criterion, and with significance tests.

	obs $\times$ R <sup>2</sup>	P-value
1 <sup>st</sup> order	0.009	0.924
2 <sup>nd</sup> order	1.693	0.429
3 <sup>rd</sup> order	1.936	0.586
4 <sup>th</sup> order	2.096	0.718

Table 2: ARCH LM Test to EGARCH standard residuals

table 5.3, no economist would reject the null hypothesis of conditional homoskedastic residuals. Even the Jarque-Bera normality test tends to accept the good specification of the model (the Jarque-Bera statistic has a value of 2.4 with a  $P$ -value of 0.3). So, we would even accept the normality of the standard residuals.

Although we did not perform a battery of tests, so that we cannot be sure the estimated model would pass in all specification tests, we can see a tendency to accept this wrong model. We say wrong because equation 14 represents a linear stable model, when we know the true model is a nonlinear unstable one. Even equation 15 tells us that, although the conditional variance of the residuals will vary with time, it will stabilize, unless it is fed with exogenous shocks. The intrinsic instability of the model is not captured by any of the components of the EGARCH estimates.

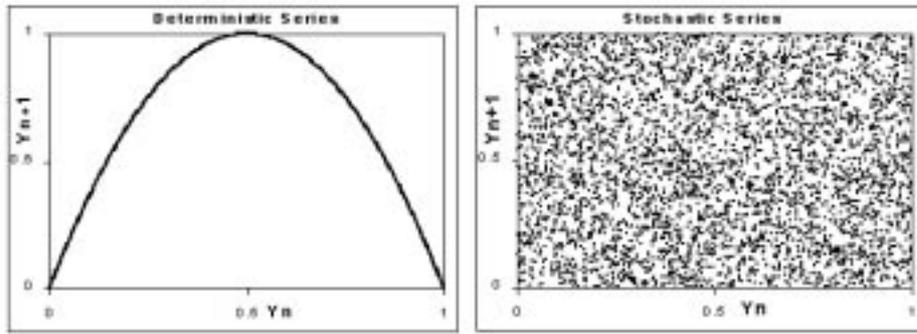


Figure 5: Dimension of Stochastic Processes vs Deterministic Processes

## 2.2 The BDS Statistic

### 2.2.1 Dimension of Stochastic Processes

Although nonlinear deterministic models can generate random processes, one important difference between deterministic and stochastic processes is that while deterministic processes have finite dimension, stochastic processes have infinite dimension.

We can see in figure 5 the phase portrait of a chaotic deterministic series based on the logistic equation ( $Y_{n+1} = 4Y_n(1 - Y_n)$ ) and of a stochastic series. It is easy to see that while the deterministic process is one-dimensional, the stochastic series fills out the entire area. Thus the stochastic process is at least two dimensional. If we plot a three-dimensional phase portrait we will conclude that the stochastic process fills out the entire cube and so on for higher dimensions. So a stochastic process approaches an infinite dimension.

### 2.2.2 The Correlation Dimension and the BDS Statistic

Based on the above notion of dimension Brock *et al.* (1987)<sup>10</sup> propose a statistical procedure to test departures from independently and identically distributed (i.i.d.) observations.

Consider  $T$  observations of a time series  $(x_1, x_2, \dots, x_T)$  after removing all non-stationary components. Define the  $m$ -histories of  $x_t$  process as the vectors  $(x_1, \dots, x_m)$ ,  $(x_2, \dots, x_{m+1})$ , ...,  $(x_{T-m+1}, \dots, x_T)$ . Now define the correlation integral as the fraction of the distinct pairs of  $m$ -histories lying within a distance  $\varepsilon$  in the sup norm<sup>11</sup>:

$$C_{\varepsilon, m, T} = \frac{1}{(T - m + 1)(T - m)} \sum_{\substack{i=1 \\ i \neq j}}^T \sum_{j=1}^T H(\varepsilon - \text{sup norm}(\mathbf{x}_i, \mathbf{x}_j)) \quad (16)$$

where  $\mathbf{x}_i = (x_i, \dots, x_{i+m-1})$  and  $H(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$ .

Under some assumptions  $C_{\varepsilon, m, T}$  converges to a limit  $C_{\varepsilon, m}$ . The true correlation dimension is given by  $\frac{d \ln(C_{\varepsilon, m})}{d \ln \varepsilon}$ . It is possible to show that the correlation dimension has the Hausdorff dimension as its upper bound. If  $\frac{d \ln(C_{\varepsilon, m, T})}{d \ln \varepsilon}$  increases without bound with  $m$  then one conclude that data is stochastic, if  $\frac{d \ln(C_{\varepsilon, m, T})}{d \ln \varepsilon}$  tends to a constant

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<sup>10</sup>Brock, W. Dechert, W. and Sheinkman, J. (1987), "A Test for Independence Based on the Correlation Dimension", University of Wisconsin, Madison, University of Houston, and University of Chicago, cit. in Brock *et al.* (1991).

<sup>11</sup>Brock (1986) showed that the correlation dimension was independent of the choice of the norm, so it is not restrictive to consider the sup norm.

then data is consistent with deterministic chaotic behaviour,

Brock *et al.* (1986) employed the correlation dimension to obtain a statistical test of nonlinearity: they proved that under the null ( $x_t$  i.i.d.)  $\ln(C_{\varepsilon,m}) = m \ln(C_{\varepsilon,1})$ , which is the basis for the BDS statistic:

$$BDS = \frac{C_{\varepsilon,m,T} - (C_{\varepsilon,1,T})^m}{\sigma_{\varepsilon,m,T}} \quad (17)$$

where  $\sigma_{\varepsilon,m,T}$  is the standard deviation consistently estimated<sup>12</sup>. Under the null  $BDS$  has a limiting standard normal distribution. The asymptotic distribution behaves reasonably well if the sample size is not less than 500, but it behaves poorly for smaller sample dimensions.

To implement the BDS test, Monte Carlo Simulations of Brock *et al.* (1991) suggested that  $\varepsilon$  should vary between 0.5 and 2 standard deviations of the data, and  $m$  between 2 and 5.

### 2.2.3 BDS Applied to our Model

We now apply the test to the residuals of our model. Since an ARCH model and its extensions typically assume i.i.d. standard residuals, Bollerslev *et al.* (1994) suggest the use of the BDS test as a specification test applied to the standardized residuals of a model. We have already seen that the Jarque-Bera test applied to the standardized residuals of our EGARCH(2,4) did not reject the normality of those residuals. We

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<sup>12</sup>See Brock et al. (1991) for details on how to estimate  $\sigma_{\varepsilon,m,T}$ .

$m$	2	3	4	5
$\varepsilon = 0.5\sigma$	13.85 (0.00)	18.01 (0.00)	25.44 (0.00)	41.47 (0.00)
$\varepsilon = \sigma$	7.16 (0.00)	7.14 (0.00)	7.37 (0.00)	7.80 (0.00)

Table 3: BDS Test to the EGARCH(2,4) residuals

now apply the BDS test to the same residuals. Two difficulties need to be faced with. First, the small dimension of the sample. Second, the asymptotic distribution of the test, which is strongly affected by the fitting of the EGARCH model, and has yet not been derived. To overcome both problems, we follow a procedure suggested by Brock *et al.* (1991), also applied by Louçã (1997): after estimating the BDS statistic we shuffle randomly the time series sample and then re-estimate the statistic. This procedure is repeated 100 times. If the process is purely random the dimension of the process will be unchanged and so will the estimated statistic. If the process is purely deterministic, then shuffling will destroy the correlation structure of the process. In table 4.5 we can see the results achieved. In parenthesis we have the proportion of the statistic values (obtained after reshuffling) that are higher (in absolute value) than the statistic applied to the original series. As we can see, the results of the statistic point, correctly, to a misspecification of the model.

#### 2.2.4 Some Problems

The above results suggest that it is easy to determine whether a time series follows a chaotic process or not. We must take this conclusion very carefully. First, the rejection of the null hypothesis does not tell us anything about the alternative. For example, the data generator process may be a stochastic nonlinear model and not a chaotic deterministic model. Second, there is no practical distinction between a high dimensional chaotic model and a pure stochastic model, so this test is only appropriate to detect low dimensional chaos.

An interesting problem, particularly when we are analyzing macroeconomic time series, is the problem with aggregate data. One of the flaws Schumpeter found in Keynes' work was the use of aggregate functions (consumption, investment, etc.). He argued aggregation could mask innovative processes which are specific to some industries. Goodwin (1991) agreed to this idea and defended the use of large multidimensional systems even though, unfortunately, for simplicity sake, he presented an aggregated model. To illustrate this problem we can see in table 6 the BDS test applied to five different series<sup>13</sup> and to their average ( $f_t = \frac{a_t+b_t+c_t+d_t+e_t}{5}$ ). Since the

sample has 2000 observations, we can use the standard normal distribution to find the

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<sup>13</sup>The series were generated according to the formula:  $x_t = 4x_{t-1}(1 - x_{t-1})$ . The initial values for series  $a_t, b_t, c_t, d_t$ , and  $e_t$  were, respectively, 0.1, 0.2, 0.3, 0.4, and 0.49. 3000 observations were generated, being the first 1000 thrown away.

$m$	2		3		4		5	
	$\frac{\varepsilon}{\sigma} = 0.5$	$\varepsilon = \sigma$						
$a_t$	709	287	939	273	1239	262	1690	265
$b_t$	691	286	915	265	1191	252	1613	244
$c_t$	733	285	969	270	1265	259	1711	252
$d_t$	692	287	925	269	1218	260	1657	254
$e_t$	690	288	912	271	1190	259	1598	252
$f_t$	0.07	-1.08	1.02	-1.36	2.18	-1.25	1.57	-1.18

Table 4: BDS Test to Deterministic Chaotic Time Series

critical values. The results speak for themselves. While for any of the series obtained from a logistic chaotic equation there is overwhelming evidence of nonlinearities, for the average of five chaotic series that evidence has almost completely disappeared: it is impossible to reject the null hypothesis at a 5% significance level except for  $(\frac{\varepsilon}{\sigma}, m) = (0.5, 4)$ .

### 3 Conclusion

We have showed that the traditional econometric techniques are not able to deal with the possibility of deterministic chaos. Using the traditional econometric approach one will tend to accept that the source of the erratic movements is exogenous and that

the system is dynamically stable, even though the model is known to be inherently unstable.

Another problem is that specific econometric techniques, designed to deal with the possibility of deterministic chaos, are not as powerful as one might wish: we saw that aggregation can hide evidence of nonlinearities, a problem that can arise in many macroeconomic time-series

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