The Einstein field equations for cylindrically symmetric elastic configurations

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Abstract. In the context of relativistic elasticity it is interesting to study axially symmetric space-times due to their significance in modeling neutron stars and other astrophysical systems of interest. To approach this problem, here, a particular class of these space-times is considered. A cylindrically symmetric elastic space-time configuration is studied, where the material metric is taken to be flat. The components of the energy-momentum tensor for elastic matter are written in terms of the invariants of the strain tensor, here chosen to be the eigenvalues of the pulled-back material metric. The Einstein field equations are presented and a condition confirming the existence of a constitutive function is obtained. This condition leads to special cases, in one of which a new system for the metric functions and an expression for the constitutive function are deduced. The new system depends on a particular function, which builds up the constitutive equation.

1. Introduction - Relativistic elasticity

General relativistic elasticity was formulated in the mid-twentieth century due to the necessity to study astrophysical problems such as deformations of neutron star crusts, which can be modelled by axially symmetric metrics. Relevant contributions to the theory of general relativistic elasticity were given, for example, by Carter and Quintana [1], Magli and Kijowski [2], Beig and Schmidt [3], Karlovini and Samuelsson [4]. The work here presented is based on Magli [5], [6] and Brito, Carot and Vaz [7], [8].

Consider a spacetime (M, g) filled with an elastic material, where M is a four-dimensional Hausdorff, simply connected manifold of class C^2 at least and g is a Lorentz metric with signature $(-,+,+,+)$. Let x^a, $a = 0, 1, 2, 3$, denote the coordinates in M. The material space X is a three-dimensional manifold, whose points represent the particles of the material. The material space is equipped with a Riemannian metric γ, which is called the material metric. This metric measures the distance between particles, calculated in the locally relaxed state of the material. The material coordinates will be represented by y^A, $A = 1, 2, 3$. The configuration of the material is described by the configuration mapping $\Psi : M \longrightarrow X$, which gives rise to the rank three matrix $y^A_a = \dfrac{\partial y^A}{\partial x^a}$, the relativistic deformation gradient. The velocity field of the matter u^a, a future oriented, timelike unit vector field, which spans the one-dimensional Kernel of the relativistic deformation gradient, is defined by the conditions: $u^a y^A_a = 0$, $u^a u_a = -1$, and $u^0 > 0$. The pulled-back material metric $k_{ab} = y^A_a y^B_b \gamma_{AB}$ is orthogonal to the velocity field and can be used to measure the state of strain of the material. Thus, one can define the strain operator as $K^a{}_b = -u^a u_b + k^a{}_b$, and the relativistic strain tensor is then defined by $s_{ab} = \frac{1}{2}(k_{ab} - h_{ab}) = \frac{1}{2}(g_{ab} - K_{ab})$, where $h_{ab} = g_{ab} + u_a u_b$. The material is said to be in an unstrained state if $s_{ab} = 0$. Assuming that the internal energy of an elastic deformation, accumulated in an infinitesimal portion of the material, is invariant with respect to the space-time orientation of the material, the energy depends only on the invariants of the strain tensor. This energy is called the constitutive equation of the material and will be denoted by $v = v(I_1, I_2, I_3)$, where I_1, I_2, I_3 are scalar invariants constructed out of K_{ab}. Here we will use [6]

$$I_1 = \frac{1}{2}\left(\mathrm{Tr}K - 4\right), \quad I_2 = \frac{1}{4}\left[\mathrm{Tr}K^2 - (\mathrm{Tr}K)^2\right] + 3, \quad I_3 = \frac{1}{2}\left(\det K - 1\right). \tag{1}$$

These invariants can be written in terms of the eigenvalues of $K^a{}_b$.
The energy density ρ is defined by

$$\rho = \epsilon v(I_1, I_2, I_3), \tag{2}$$

where ϵ represents the particle number density. The energy density can also be rewritten as $\rho = \epsilon_0 \sqrt{\det K} v(I_1, I_2, I_3)$, where ϵ_0 is the particle number density of the relaxed material.

The energy-momentum tensor for elastic matter can be derived from the Lagrangian $\Lambda = \sqrt{-g}\rho$, which depends on y^A, y^A_a and x^a. The corresponding Euler-Lagrange equations are given by $\dfrac{\partial \Lambda}{\partial y^A} - \partial_a \left(\dfrac{\partial \Lambda}{\partial y^A_a}\right) = 0$. Using Noether's theorem one constructs the canonical energy-momentum tensor

$$T^a{}_b = \frac{1}{\sqrt{-g}} \frac{\partial \Lambda}{\partial y^A_a} y^A_b - \delta^a{}_b \Lambda, \tag{3}$$

which satisfies the conservation law $\nabla_a T^{ab} = 0$. The energy-momentum tensor can be rewritten in terms of the invariants of K_{ab}, see [6] for details, and the resulting expression can then be used to construct the Einstein field equations for elastic matter $G^a{}_b = 8\pi T^a{}_b$.

2. Cylindrically symmetric elastic configuration

Consider a static cylindrically symmetric space-time (M, g), whose metric g is given by the line-element

$$ds^2 = -e^{2\nu} dt^2 + e^{2\mu} dr^2 + e^{2\mu} dz^2 + e^{2\psi} d\phi^2. \tag{4}$$

The space-time coordinates are $x^a = (t, r, z, \phi)$ and ν, μ and ψ depend only on r. The associated material space X is assumed to be such that the configuration mapping ψ preserves the Killing vectors (KVs), so that if $\vec{\xi_A}$ are KVs in M, where $\vec{\xi_1} = \partial_t$, $\vec{\xi_2} = \partial_z$, $\vec{\xi_3} = \partial_\phi$, then $\psi_*(\vec{\xi_A}) = \vec{\eta_A}$ are also KVs in X. Therefore, the material metric γ is also cylindrically symmetric and it can be shown that the coordinates $y^A = (R, \zeta, \Phi)$ in X are defined by $R = R(r)$, $\zeta = z$ and $\Phi = \phi$. The material metric γ can be represented by the line-element $d\Sigma^2 = dR^2 + dz^2 + R^2 d\phi^2$, where $R = R(r)$. Here, we shall assume for simplicity that $R(r) = r$, so that γ takes the form

$$d\Sigma^2 = dr^2 + dz^2 + r^2 d\phi^2. \tag{5}$$

The velocity field of the matter turns out to be $u^a = \left(e^{-\nu(r)}, 0, 0, 0\right)$, and one can then construct the relevant tensors mentioned above, thus:

$$k^a{}_b = e^{-2\mu} \delta^a{}_1 \delta^1{}_b + e^{-2\mu} \delta^a{}_2 \delta^2{}_b + r^2 e^{-2\psi} \delta^a{}_3 \delta^3{}_b. \tag{6}$$

and also

$$K^a{}_b = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{-2\mu} & 0 & 0 \\ 0 & 0 & e^{-2\mu} & 0 \\ 0 & 0 & 0 & r^2 e^{-2\psi} \end{pmatrix}. \tag{7}$$

This operator has one eigenvalue equal to 1 and the other eigenvalues are

$$\eta = e^{-2\mu}, \quad \tau = r^2 e^{-2\psi}, \tag{8}$$

where η has algebraic multiplicity two. The invariants (1) can be expressed as

$$I_1 = \frac{1}{2}\left(2\eta + \tau - 3\right), \; I_2 = -\frac{1}{2}\left(\eta^2 + 2\eta\tau + 2\eta + \tau\right) + 3, \; I_3 = \frac{1}{2}\left(\eta^2\tau - 1\right). \tag{9}$$

The non-zero components of the energy-momentum tensor are then

$$\begin{aligned} T^0{}_0 &= -\rho, \;\; T^1{}_1 = -\rho + \frac{\partial\rho}{\partial I_3}\,\eta^2\tau - \frac{\partial\rho}{\partial I_2}(1+\eta+\tau)\eta + \frac{\partial\rho}{\partial I_1}\eta \\ T^2{}_2 &= T^1{}_1, \;\; T^3{}_3 = -\rho + \frac{\partial\rho}{\partial I_3}\,\eta^2\tau - \frac{\partial\rho}{\partial I_2}(1+2\eta)\tau + \frac{\partial\rho}{\partial I_1}\tau. \end{aligned} \tag{10}$$

Using the fact that

$$\frac{\partial\rho}{\partial\eta} = \frac{\partial\rho}{\partial I_1} - (1+\eta+\tau)\frac{\partial\rho}{\partial I_2} + \eta\tau\frac{\partial\rho}{\partial I_3}, \quad \frac{\partial\rho}{\partial\tau} = \frac{1}{2}\frac{\partial\rho}{\partial I_1} - \left(\eta + \frac{1}{2}\right)\frac{\partial\rho}{\partial I_2} + \frac{1}{2}\eta^2\frac{\partial\rho}{\partial I_3},$$

the components of the energy-momentum tensor can be expressed simply as

$$T^0{}_0 = -\epsilon v, \;\; T^1{}_1 = T^2{}_2 = \epsilon\eta\frac{\partial v}{\partial\eta}, \;\; T^3{}_3 = 2\epsilon\tau\,\frac{\partial v}{\partial\tau}. \tag{11}$$

3. Einstein field equations

The Einstein field equations $G^a{}_b = 8\pi T^a{}_b$ for the cylindrically symmetric elastic configuration presented in the previous section can be written as

$$\frac{\mu'' + \psi'' + \psi'^2}{e^{2\mu}} = -\epsilon v\; 8\pi, \tag{12}$$

$$\frac{\mu'\nu' + \mu'\psi' + \nu'\psi'}{e^{2\mu}} = \epsilon\eta\frac{\partial v}{\partial\eta}\; 8\pi, \tag{13}$$

$$\frac{\nu'^2 + \nu'' + \psi'' + \psi'^2 + \nu'\psi' - \mu'\nu' - \mu'\psi'}{e^{2\mu}} = \epsilon\eta\frac{\partial v}{\partial\eta}\; 8\pi, \tag{14}$$

$$\frac{\nu'^2 + \nu'' + \mu''}{e^{2\mu}} = 2\epsilon\tau\,\frac{\partial v}{\partial\tau}\; 8\pi, \tag{15}$$

where primes denote derivatives with respect to r. Dividing (13) and (15) through by (12) leads to

$$\frac{\partial \ln v}{\partial E} = -\frac{\mu'\nu' + \mu'\psi' + \nu'\psi'}{\mu'' + \psi'' + \psi'^2} \quad \text{and} \quad \frac{\partial \ln v}{\partial T} = -\frac{1}{2}\,\frac{\nu'^2 + \nu'' + \mu''}{\mu'' + \psi'' + \psi'^2}, \tag{16}$$

respectively, where $E = \ln\eta = -2\mu$ and $T = \ln\tau = 2\ln r - 2\psi$. And since $T^2{}_2 = T^1{}_1$, it follows that

$$2\mu'\nu' + 2\mu'\psi' - \nu'^2 - \nu'' - \psi'' - \psi'^2 = 0. \tag{17}$$

The integrability condition $\dfrac{\partial^2 \ln v}{\partial T\partial E} = \dfrac{\partial^2 \ln v}{\partial E\partial T}$ must be satisfied in order for a constitutive equation to exist. Therefrom one obtains

$$\frac{\partial}{\partial T}\left[-\frac{1}{\mu'}\left(\frac{1}{r} - \psi'\right)\right]\frac{\partial \ln v}{\partial T} = 0. \tag{18}$$

Here, we shall consider the particular case in which both factors vanish; i.e.:

$$\frac{\partial}{\partial T}\left[-\frac{1}{\mu'}\left(\frac{1}{r}-\psi'\right)\right]=0 \quad \text{and} \quad \frac{\partial \ln v}{\partial T}=0, \tag{19}$$

which implies

$$\psi=\ln(r)+k_0\mu+k_1 \quad \text{and} \quad T^3_3=0, \tag{20}$$

where k_0 and k_1 are constants. In order to avoid singularities at the axis of symmetry, one must have $e^{2\psi}=r^2L(r)$, where $L(r)\neq 0$ for $r=0$, (see [9]). Then, taking into account (17) and (20) and setting $k_0=1$, allows to write the metric functions as follows

$$\psi(r)=\ln(r)+\frac{1}{2}\ln(L), \quad \mu(r)=\frac{1}{2}\ln(L), \quad \nu(r)=-\frac{1}{4}\ln(L)\ +\ \text{constant}, \tag{21}$$

where the function $L=L(r)$ must satisfy the condition

$$6L'^2Lr-8L'''L^2r^2-8L''L^2r+16L''L'Lr^2-9L'^3r^2+8L'L^2=0, \tag{22}$$

as a consequence of the Einstein field equations (12)-(15). An example of a function $L(r)$ satisfying (22) is given by $L(r)=\exp\left(\frac{7-12r}{8r-7}\right)$. It can be easily shown that the Dominant Energy Condition is satisfied for $r>0.931855$. Since in the present case, where $T^3_3=0$, one has $\frac{\partial \ln v}{\partial r}=-2\mu'\frac{\partial \ln v}{\partial E}$, then, applying (16) one may also write the constitutive function v in terms of L as

$$v(r)=c\ \exp\left(\int\frac{L'^2}{-3L'^2r+4LL'+4L''Lr}dr\right), \tag{23}$$

where c is a constant.

Currently, the generic cases in which only one factor in (18) vanishes are under study, as well as the problem of the matching of these elastic spacetimes to vacuum or cosmological constant ones.

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