EXPONENTIAL ENERGY DECAY FOR THE KADOMTSEV-PETVIASHVILI (KP-II) EQUATION

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Abstract. In this paper we study the exponential decay of the energy of the externally damped Kadomtsev-Petviashvili (KP-II) equation. Our main tool is the classical dissipation-observability method. We use multiplier techniques to establish the main estimates, and obtain exponential decay result.

Contents

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1. Introduction

Consider the initial value problem (IVP) associated with the Kadomtsev-Petviashvili (KP) equation,

$$\begin{cases} (u_t + u_{xxx} + uu_x)_x = \alpha u_{yy}, & (x,y) \in (0,L)\times(0,L),\ t\in\mathbb{R} \\ u(x,y,0) = u_0(x,y), \end{cases} \tag{1.1}$$

where $u = u(x,y,t)$ is a real valued function, $L > 0$ and $\alpha = \pm 1$. This model was derived by Kadomtsev and Petviashvili [8] to describe the propagation of weakly nonlinear long waves on the surface of a fluid, when the wave motion is essentially one-directional with weak transverse effects along y-axis. Equation (1.1) is known as KP-I or KP-II equation depending whether $\alpha = 1$ or $\alpha = -1$. In this paper we consider the KP-II equation, that is (1.1) with $\alpha = -1$.

The KP-II equation is a two dimensional generalization of the Korteweg-de Vries (KdV) equation

$$u_t + u_{xxx} + uu_x = 0, \qquad x,\, t \in \mathbb{R}, \tag{1.2}$$

which arises in modeling the evolution of one dimensional surface gravity waves with small amplitude in a shallow channel of water. The KdV model is a widely studied model which arises in various physical contexts and has a very rich mathematical structure.

It is customary to work with (1.1) for $(x,y)\in\mathbb{R}^2$, without any boundary conditions. In this case the L^2 norm, the energy,

$$E(u(t)) := \int_{\mathbb{R}^2} |u|^2 dxdy$$

is a conserved quantity. However, since $(x,y)\in(0,L)\times(0,L)$, if we impose suitable boundary conditions energy may be dissipated. These conditions will be discussed in Section 3. If in addition to these boundary conditions a weak damping is imposed we obtain an exponential decay of the energy. As a damping term, we take a non-negative function $a(x,y)$ and consider the following damped KP-II equation:

$$\begin{cases} (u_t + u_{xxx} + uu_x + au)_x = \alpha u_{yy}, & (x,y) \in (0,L) \times (0,L),\ t \in \mathbb{R} \\ u(x,y,0) = u_0(x,y). \end{cases} \tag{1.3}$$

If $a > 0$ everywhere, an easy computation shows exponential decay. If, however, we allow a to vanish the exponential decay requires additional work. In the case of the KdV equation this problem was studied in [13]. In this paper we extend these results to the KP-II equation by introducing suitable boundary conditions and using the unique continuation principle proved in [12]. We note that our techniques do not rely on the Holmgren's Uniqueness Theorem but in a unique continuation principle. Therefore it works both for linear and non-linear equations.

KP models are extensively investigated in the recent literature see for example [2], [4], [5], [6] [7], [9], [11], [12], [14], [15], [16] and references therein, for issues such as local and global well-posedness, gain of regularity and unique continuation principles. As we are interested in the case when the initial data has sufficiently high Sobolev regularity, the question of well-posedness follows, for instance, by the semi-group theory. So, we will omit the details of this aspect.

The plan of the paper is as follows: firstly, in Section 2 we give a compact presentation of the dissipation-observability method which is the basis for many decay results. Then in Section 3 we establish, using multiplier techniques, the main technical estimates which allow us to address the exponential dissipation of energy for the KP-II equation.

2. Dissipation - observability method

In this section we put forward a general method to prove energy decay that follows from the energy dissipation law and an observability inequality.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain. Let A be a linear operator, and B a non-linear operator with domain dense in $L^2(\Omega)$. Suppose that u is a solution to the evolution equation in $L^2(\Omega)$*

$$u_t = Au + B(u), \tag{2.1}$$

under suitable boundary conditions. Suppose that the evolution associated to (2.1) *satisfies a semigroup property and that the energy* $E(u(t)) := \int_\Omega u^2 dx$ *is dissipated according to*

$$\frac{d}{dt}E(u(t)) = -Q(u) \leq 0, \tag{2.2}$$

where $\int_\Omega u(Au + B(u)) = -Q(u)$. *Assume that* $\forall\, T > 0$ *there exists* $C > 0$ *such that the following observability inequality*

$$E(u(\cdot, 0)) \leq C \int_0^T Q(u(\cdot, s))ds \tag{2.3}$$

holds. Then the energy E *decays exponentially, i.e, there exists* $\alpha > 0$ *such that* $\forall\, t \geq 0$,

$$E(u(t)) \leq CE(u(0))e^{-\alpha t}. \tag{2.4}$$

Proof. Integrating (2.2) in $(0, T)$ we get,

$$E(u(T)) = E(u(0)) - \int_0^T Q(u(s))\, ds. \tag{2.5}$$

Now, multiplying (2.5) by C, adding to itself and using the observability inequality (2.3), yields

$$\begin{aligned}(1 + C)E(u(T)) &= E(u(0)) - C \int_0^T Q(u(s))\, ds + CE(u(0)) - \int_0^T Q(u(s))\, ds \\ &\leq CE(u(0)) - \int_0^T Q(u(s))\, ds \\ &\leq CE(u(0)).\end{aligned} \tag{2.6}$$

From (2.6) we obtain

$$E(u(T)) \leq \frac{C}{1 + C}E(u(0)). \tag{2.7}$$

Therefore, for some $0 < \alpha < 1$ we have $E(u(T)) \leq \alpha E(u(0))$. Hence, the semigroup property implies the conclusion of the Theorem. □

Lemma 2.2. *Let* u *be as in Theorem 2.1. Then the following estimate holds*

$$E(u_0) \leq \frac{1}{T} \int_0^T E(u(t)) + 2 \int_0^T Q(u). \tag{2.8}$$

Proof. Multiplying (2.1) by $(T-t)u$ and integrating we obtain

$$\int_0^T \int_\Omega (T-t)uu_t dxdt = \int_0^T \int_\Omega (T-t)u(Au+B(u))dxdt. \tag{2.9}$$

After integrating by parts in the t variable, (2.9) yields

$$\frac{1}{2}\int_\Omega \left[\left(\int_0^T u^2\right) - Tu_0^2\right] = -\int_0^T (T-t)Q(u),$$

i.e.,

$$-\frac{T}{2}E(u_0) + \frac{1}{2}\int_0^T E(u)dt + \int_0^T (T-t)Q(u)dt = 0. \tag{2.10}$$

Now the desired estimate (2.8) follows from (2.10). □

Corollary 2.3. *Suppose that*

$$\int_0^T E(u(t)) \leq C(T)\int_0^T Q(u). \tag{2.11}$$

Then the observability inequality (2.3) *holds.*

3. Nonlinear KP-II - exponential energy dissipation

Let $a(x,y) \geq a_0 > 0$ almost everywhere in the complement of a compact non-empty proper subset Θ of $\Omega := (0,L)\times(0,L)$. We assume that for some $\delta > 0$, $\Theta \subset (\delta, L-\delta)\times(\delta, L-\delta)$, so that we can apply the UCP, using an extension technique.

Consider the damped KP-II model

$$\begin{cases} u_t + u_{xxx} + \partial_x^{-1}u_{yy} + uu_x + a(x,y)u = 0, & (x,y)\in(0,L)\times(0,L),\ t\in\mathbb{R},\\ u(L,y,t) = 0 = u(0,y,t), \quad u(x,L,t) = 0 = u(x,0,t), & \\ u_x(L,y,t) = 0, & \\ u(x,y,0) = u_0(x,y), & \end{cases} \tag{3.1}$$

where the operator ∂_x^{-1} is defined as $\partial_x^{-1}f(x,y,t) = g(x,y,t)$ with $g(L,y,t) = 0$ and $g_x(x,y,t) = f(x,y,t)$.

Define the energy as

$$E(u(t)) := \frac{1}{2}\int_0^L\int_0^L u^2(x,y,t)\,dxdy. \tag{3.2}$$

As we prove in the next proposition, the energy is a decreasing function of t. Our main objective is to show that the decay is exponential in time, by using the dissipation-observability method.

Proposition 3.1. *Suppose u solves (3.1), and let E be given by (3.2). Then*

$$\frac{d}{dt}E(u(t)) = -\frac{1}{2}\int_0^L [u_x^2(0,y,t)+(\partial_x^{-1}u_y)^2(0,y,t)]\,dy - \int_0^L\int_0^L a(x,y)u^2(x,y)\,dxdy \leq 0. \quad (3.3)$$

Proof. We have

$$\begin{aligned}\frac{d}{dt}E(u(t)) &= \frac{1}{2}\frac{d}{dt}\int_0^L\int_0^L u^2(x,y,t)\,dxdy \\ &= \int_0^L\int_0^L uu_t(x,y,t)\,dxdy \\ &= \int_0^L\int_0^L u(-u_{xxx} - \partial_x^{-1}u_{yy} - uu_x - a(x,y)u)\,dxdy.\end{aligned} \quad (3.4)$$

Now observe that

$$\begin{aligned}-\int_0^L\int_0^L uu_{xxx}\,dxdy &= \int_0^L \Big[\int_0^L u_xu_{xx}\,dx - uu_{xx}\Big|_{x=0}^L\Big]dy \\ &= \int_0^L \Big[\frac{1}{2}\int_0^L (u_x)_x^2\,dx\Big]dy \\ &= \frac{1}{2}\int_0^L \Big[u_x^2\Big|_{x=0}^L\Big]dy \\ &= -\frac{1}{2}\int_0^L u_x^2(0,y,t)\,dy.\end{aligned} \quad (3.5)$$

Define v by $u_y = v_x$, then

$$
\begin{aligned}
-\int_0^L\int_0^L u\partial_x^{-1}u_{yy}\,dxdy &= -\int_0^L\int_0^L uv_y\,dydx \\
&= \int_0^L \Big[\int_0^L u_y v\,dy - uv\Big|_{x=0}^{L}\Big]dx \\
&= \int_0^L\int_0^L v_x v\,dxdy \\
&= \frac{1}{2}\int_0^L\int_0^L (v^2)_x\,dxdy \\
&= \frac{1}{2}\int_0^L \Big[v^2(L,y,t) - v^2(0,y,t)\Big]dy \\
&= -\frac{1}{2}\int_0^L v^2(0,y,t)dy,
\end{aligned} \tag{3.6}
$$

where we have used $v = 0$ at (L, y, t).

Also, integrating by parts yields

$$
-\int_0^L\int_0^L u^2u_x dxdy = 0. \tag{3.7}
$$

Now, using (3.5), (3.6) and (3.7) in (3.4) we obtain (3.3). □

Now we state and prove the main result of this work that deals with the exponential decay of energy of the nonlinear KP-II equation.

Theorem 3.2. *Given $M > 0$, let u be a solution of* (3.1) *with data $u_0 \in H^s(\mathbb{R}^2)$, $s \geq 3$, satisfying $\|u_0\|_{L^2(\Omega)} \leq M$, and let $E(u(t))$ be the energy as defined in* (3.2)*. Then the energy $E(u(t))$ decays exponentially.*

Proof. The proof of this Theorem follows from Theorem 2.1, using Corollary 2.3 with

$$
Q(u) = \frac{1}{2}\int_0^L [u_x^2(0,y,t) + (\partial_x^{-1}u_y)^2(0,y,t)]\,dy + \int_0^L\int_0^L a(x,y)u^2(x,y)\,dxdy
$$

and the estimate

$$
\int_0^T\int_0^L\int_0^L |u|^2 \leq \tilde{C}\Big\{\int_0^T\int_0^L\Big[|u_x(0,y,t)|^2 + |\partial_x^{-1}u_y(0,y,t)|^2\Big]dydt + 2\int_0^T\int_0^L\int_0^L a(x,y)|u|^2dxdydt\Big\}, \tag{3.8}
$$

which holds for some $\tilde{C} > 0$ independent of solution u to (3.1) with initial data u_0 satisfying $\|u_0\|_{L^2(\Omega)} \le M$ for any given $M > 0$, as will be shown in Lemma 3.5 below. □

Now we prove the following result which will be used in the proof of (3.8).

Lemma 3.3. *Let u be a solution of* (3.1). *Then the following estimate holds:*

$$\|u\|^2_{L^2(0,T;H^1(\Omega))} \le \frac{2L}{3}\|u_0\|^2_{L^2(\Omega)} + \frac{4CT}{81}\|u_0\|^3_{L^2(\Omega)}. \tag{3.9}$$

Proof. Multiply equation (3.1) by xu and integrate on $(0,L)\times(0,L)\times(0,T)$. The resulting identity is composed of five terms, that we simplify next:

$$\begin{aligned}\int_0^T\int_0^L\int_0^L xuu_t\,dxdydt &= \frac{1}{2}\int_0^T\int_0^L\int_0^L x\frac{d}{dt}(u^2)\,dtdxdy\\ &= \frac{1}{2}\int_0^L\int_0^L xu^2(x,y,T)\,dxdy - \frac{1}{2}\int_0^L\int_0^L xu_0^2(x,y)\,dxdy;\end{aligned} \tag{3.10}$$

the next term is

$$\begin{aligned}\int_0^T\int_0^L\int_0^L xuu_{xxx} &= \int_0^T\int_0^L\Big[-\int_0^L (xu)_x u_{xx}\,dx + xuu_{xx}\Big|_0^L\Big]dydt\\ &= \int_0^T\int_0^L\Big[-\int_0^L uu_{xx}\,dx - \int_0^L xu_xu_{xx}\,dx\Big]dydt\\ &= \int_0^T\int_0^L\Big[\int_0^L u_x^2\,dx - uu_x\Big|_0^L + \int_0^L (xu_x)_xu_x\,dx - xu_x^2\Big|_0^L\Big]dydt\\ &= \int_0^T\int_0^L\Big[\int_0^L u_x^2\,dx + \int_0^L u_x^2\,dx + \int_0^L xu_{xx}u_x\,dx\Big]dydt\\ &= \int_0^T\int_0^L\Big[2\int_0^L u_x^2\,dx - \int_0^L (xu_{xx})_xu\,dx + xu_{xx}u\Big|_0^L\Big]dydt\\ &= \int_0^T\int_0^L\Big[2\int_0^L u_x^2\,dx - \int_0^L u_{xx}u\,dx - \int_0^L xu_{xxx}u\,dx\Big]dydt\\ &= \int_0^T\int_0^L\Big[2\int_0^L u_x^2\,dx + \int_0^L u_x^2\,dx - u_xu\Big|_0^L - \int_0^L xu_{xxx}u\,dx\Big]dydt,\end{aligned} \tag{3.11}$$

which, from (3.11), yields

$$\int_0^T\int_0^L\int_0^L xuu_{xxx}\,dxdydt = \frac{3}{2}\int_0^T\int_0^L\int_0^L u_x^2\,dxdydt; \tag{3.12}$$

for the next term, as earlier, we set $u_y = v_x$, and obtain

$$\int_0^T\int_0^L\int_0^L xu\partial_x^{-1}u_{yy}\,dxdydt = \frac{1}{2}\int_0^T\int_0^L\int_0^L v^2\,dxdydt. \tag{3.13}$$

Also, integrating by parts we get,

$$\int_0^T\int_0^L\int_0^L xuuu_x\,dxdydt = \frac{1}{3}\int_0^T\int_0^L\int_0^L x(u^3)_x\,dxdydt = -\frac{1}{3}\int_0^T\int_0^L\int_0^L u^3\,dxdydt. \tag{3.14}$$

Finally, the last term is simply

$$\int_0^T\int_0^L\int_0^L xa(x,y)u^2\,dxdydt. \tag{3.15}$$

Now, adding (3.10), (3.12),(3.13), (3.14) and (3.15) we get

$$\begin{aligned}\frac{1}{2}\int_0^L\int_0^L xu^2(x,y,T)\,dxdy - \frac{1}{2}\int_0^L\int_0^L xu_0^2(x,y)\,dxdy + \frac{3}{2}\int_0^T\int_0^L\int_0^L u_x^2\,dxdydt \\ + \int_0^T\int_0^L\int_0^L xa(x,y)u^2\,dxdydt + \frac{1}{2}\int_0^T\int_0^L\int_0^L v^2\,dxdydt - \frac{1}{3}\int_0^T\int_0^L\int_0^L u^3\,dxdydt = 0.\end{aligned} \tag{3.16}$$

Or,

$$\begin{aligned}\frac{3}{2}\int_0^T \|u\|^2_{H^1(\Omega)}\,dt = {} & \frac{1}{2}\int_0^L\int_0^L xu_0^2(x,y)\,dxdy - \frac{1}{2}\int_0^L\int_0^L xu^2(x,y,T)\,dxdy \\ & - \int_0^T\int_0^L\int_0^L xa(x,y)u^2\,dxdydt - \frac{1}{2}\int_0^T\int_0^L\int_0^L v^2\,dxdydt \\ & + \frac{1}{3}\int_0^T\int_0^L\int_0^L u^3\,dxdydt,\end{aligned} \tag{3.17}$$

which yields,

$$\begin{aligned}\int_0^T \|u\|^2_{H^1(\Omega)}\,dt = {} & \frac{1}{3}\int_0^L\int_0^L xu_0^2(x,y)\,dxdy - \frac{1}{3}\int_0^L\int_0^L xu^2(x,y,T)\,dxdy \\ & - \frac{2}{3}\int_0^T\int_0^L\int_0^L xa(x,y)u^2\,dxdydt - \frac{1}{3}\int_0^T\int_0^L\int_0^L v^2\,dxdydt \\ & + \frac{2}{9}\int_0^T\int_0^L\int_0^L u^3\,dxdydt.\end{aligned} \tag{3.18}$$

Since $a(x,y)\geq 0$, we obtain from (3.18) that

$$\int_0^T \|u\|^2_{H^1(\Omega)}\,dt \leq \frac{L}{3}\|u_0\|^2_{L^2(\Omega)} + \frac{2}{9}\int_0^T\int_0^L\int_0^L u^3\,dxdydt. \tag{3.19}$$

We have the following Gagliardo-Nirenberg type inequality (see [3] or [10])

$$\|u\|_{L^3(\Omega)} \leq C\|u\|^{\frac{1}{3}}_{\dot{H}^1(\Omega)}\|u\|^{\frac{2}{3}}_{L^2(\Omega)} \leq C\|u\|^{\frac{1}{3}}_{H^1(\Omega)}\|u\|^{\frac{2}{3}}_{L^2(\Omega)}. \tag{3.20}$$

Now using (3.20) and (3.3) the last term in (3.19) can be controlled by

$$\begin{aligned}\int_0^T\int_0^L\int_0^L u^3\,dxdydt &\leq \int_0^T \|u\|_{H^1(\Omega)}\|u\|^2_{L^2(\Omega)}dt\\ &\leq C\|u_0\|^2_{L^2(\Omega)}\int_0^T \|u\|_{H^1(\Omega)}dt\\ &\leq C\|u_0\|^2_{L^2(\Omega)}\sqrt{T}\|u\|_{L^2(0,T;H^1(\Omega))}.\end{aligned} \tag{3.21}$$

Substituting (3.21) in (3.19) yields

$$\begin{aligned}\|u\|^2_{L^2(0,T;H^1(\Omega))} &\leq \frac{L}{3}\|u_0\|^2_{L^2(\Omega)} + C\frac{2}{9}\sqrt{T}\|u_0\|^2_{L^2(\Omega)}\|u\|_{L^2(0,T;H^1(\Omega))}\\ &\leq \frac{L}{3}\|u_0\|^2_{L^2(\Omega)} + \frac{\left(\frac{2}{9}C\sqrt{T}\|u_0\|^2_{L^2(\Omega)}\right)^2}{2} + \frac{\left(\|u\|_{L^2(0,T;H^1(\Omega))}\right)^2}{2}\\ &\leq \frac{L}{3}\|u_0\|^2_{L^2(\Omega)} + \frac{2CT}{81}\|u_0\|^4_{L^2(\Omega)} + \frac{1}{2}\|u\|^2_{L^2(0,T;H^1(\Omega))}.\end{aligned} \tag{3.22}$$

Therefore, from (3.22) we get

$$\|u\|^2_{L^2(0,T;H^1(\Omega))} \leq \frac{2L}{3}\|u_0\|^2_{L^2(\Omega)} + \frac{4CT}{81}\|u_0\|^4_{L^2(\Omega)}, \tag{3.23}$$

which yields the required result. □

In addition to the Lemma 3.3, unique continuation principle (UCP) for the KP-II equation (1.1) will also be used in the proof the estimate (3.8). The UCP for the KP-II equation (1.1) was established in [12] whose precise statement is given in the following theorem.

Theorem 3.4 ([12]). *Let $u\in C(\mathbb{R};H^s(\mathbb{R}^2))$ be a solution to the IVP associated with the KP-II equation with $s>0$ large enough. If there exists a non trivial time interval $I=[-T,T]$ such that for some $\beta>0$*

$$\operatorname{supp} u(t) \subseteq [-\beta,\beta]\times[-\beta,\beta], \quad \forall\, t\in I,$$

then $u \equiv 0$.

In the following lemma we prove the main estimate (3.8) used in the proof of Theorem 3.2.

Lemma 3.5. *given* $M > 0$, *there exists a constant* $\tilde{C} > 0$ *such that the following estimate holds*

$$\int_0^T\int_0^L\int_0^L |u|^2 \leq \tilde{C}\Big\{ \int_0^T\int_0^L [|u_x(0,y,t)|^2+|\partial_x^{-1}u_y(0,y,t)|^2]dydt+2\int_0^T\int_0^L\int_0^L a(x,y)|u|^2\,dxdydt\Big\}, \tag{3.24}$$

for all solutions u *of* (3.1) *with initial data* u_0 *satisfying* $\|u_0\|_{L^2(\Omega)} \leq M$.

Proof. We prove it by contradiction with use of the Lemma 3.3 and the unique continuation principle stated in Theorem 3.4.

Suppose that (3.24) is false. Then there exists a sequence of solution u_n of (3.1) such that

$$\lim_{n\to\infty} \frac{\|u_n\|^2_{L^2(0,T;L^2(\Omega))}}{\int_0^T\int_0^L [|\partial_x u_n(0,y,t)|^2 + |\partial_x^{-1}\partial_y u_n(0,y,t)|^2]dydt + 2\int_0^T\int_0^L\int_0^L a(x,y)|u_n|^2\,dxdydt} = +\infty. \tag{3.25}$$

Let

$$\lambda_n = \|u_n\|_{L^2(0,T;L^2(\Omega))} \qquad \text{and} \quad v_n(x,y,t) = \frac{1}{\lambda_n}u_n(x,y,t).$$

From Lemma 3.3, we have that λ_n is a bounded sequence for $\|u_n(0)\|_{L^2(\Omega)} \leq M$. Therefore, extracting a subsequence if necessary, we can assume that $\lambda_n \to \lambda \geq 0$.

We notice that v_n solves

$$\begin{cases} (v_n)_t + (v_n)_{xxx} + \partial_x^{-1}(v_n)_{yy} + \lambda_n(v_n)(v_n)_x + a(x,y)(v_n) = 0, & (x,y)\in\Omega,\ t\in\mathbb{R}, \\ (v_n)(L,y,t) = 0 = (v_n)(0,y,t), \quad (v_n)(x,L,t) = 0 = (v_n)(x,0,t), \\ (v_n)_x(L,y,t) = 0, \end{cases} \tag{3.26}$$

with initial data $\frac{1}{\lambda_n}u_n(x,y,0)$. Moreover,

$$\|v_n\|_{L^2(0,T;L^2(\Omega))} = 1 \tag{3.27}$$

and from (3.25)

$$\int_0^T\int_0^L [|\partial_x v_n(0,y,t)|^2 + |\partial_x^{-1}\partial_y u_n(0,y,t)|^2] dydt + 2\int_0^T\int_0^L\int_0^L a(x,y)|v_n|^2\, dxdydt \to 0, \quad (3.28)$$

as $n \to \infty$.

In view of Lemma 2.2, $v_n(x,y,0)$ is bounded in $L^2(\Omega)$. Thus, combining with an analogue of (2.8), we have

$$\|v_n(\cdot,\cdot,t)\|_{L^2(\Omega)} \leq M, \quad \forall\, 0 \leq t \leq T. \quad (3.29)$$

Now, from Lemma 3.3,

$$\|v_n\|^2_{L^2(0,T;H^1(\Omega))} \leq C, \quad (3.30)$$

for all $n \in \mathbb{N}$.

Estimates (3.27) and (3.30) yield

$$(v_n)_t = -(v_n)_{xxx} - \partial_x^{-1}(v_n)_{yy} - \lambda_n v_n (v_n)_x - a(x,y)v_n, \quad (3.31)$$

is bounded in $L^2(0,T;H^{-2}(\Omega))$. Note that the non-linear term, is bounded in $L^2(0,T;L^p(\Omega))$ for all $1 \leq p < 2$, which is bounded in $L^2(0,T;H^{-2}(\Omega))$.

Since the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, by Rellich's theorem, it follows that $\{v_n\}$ is relatively compact in $L^2(0,T;L^2(\Omega))$. By extracting a subsequence we may deduce that

$$v_{nj} \rightharpoonup v, \qquad \text{weakly in} \qquad L^2(0,T;H^1(\Omega)) \cap H^1(0,T;H^{-2}(\Omega)) \quad (3.32)$$

and

$$v_{nj} \to v, \qquad \text{strongly in} \qquad L^2(0,T;L^2(\Omega)). \quad (3.33)$$

Since $\|v_{nj}\|_{L^2(0,T;L^2(\Omega))} = 1$, it follows that

$$\|v\|_{L^2(0,T;L^2(\Omega))} = 1. \quad (3.34)$$

By weak lower semicontinuity of convex functionals we have (see (3.28))

$$0 = \liminf_{j\to\infty} \Big\{ \int_0^T\int_0^L [|\partial_x v_{nj}(0,y,t)|^2 + |\partial_x^{-1}\partial_y v_{nj}(0,y,t)|^2] dydt + 2\int_0^T\int_0^L\int_0^L a(x,y)|v_{nj}|^2\, dxdydt \Big\}$$
$$\geq \int_0^T\int_0^L [|\partial_x v(0,y,t)|^2 + |\partial_x^{-1}\partial_y v(0,y,t)|^2] dydt + 2\int_0^T\int_0^L\int_0^L a(x,y)|v|^2\, dxdydt. \tag{3.35}$$

From this we conclude that $a(x,y)v \equiv 0$ in $\Omega \times (0,T)$. Since $a(x,y) > 0$ in Θ^c, so in particular $v \equiv 0$ in $\Theta^c \times (0,T)$. We will show that $v \equiv 0$ in $\Omega \times (0,T)$.

Note that the limit v satisfies

$$v_t + v_{xxx} + \partial_x^{-1} v_{yy} + \lambda v v_x = 0, \tag{3.36}$$

where $\lambda \geq 0$ is the limit of λ_n as $n \to \infty$.

In any case, whether $\lambda = 0$ or $\lambda > 0$, we will use the UCP discussed earlier to conclude that $v \equiv 0$ in $\Omega \times (0,T)$. To be able to apply the UCP, we must show that v is sufficiently regular. Let $Z := (\delta, L-\delta) \times (\delta, L-\delta)$ and define a function,

$$w(x,y,t) = \begin{cases} v(x,y,t), & (x,y,t) \in Z \times (0,T), \\ 0, & (x,y,t) \in \{\mathbb{R}^2 - Z\} \times (0,T). \end{cases} \tag{3.37}$$

Because $\Theta \subset (\delta, L-\delta) \times (\delta, L-\delta)$, this extension is as smooth as v. Furthermore w satisfies

$$\begin{cases} w_t + w_{xxx} + \partial_x^{-1} w_{yy} + \lambda w w_x = 0, & (x,y,t) \in \mathbb{R}^2 \times (0,T) \\ w(x,y,0) = \phi(x,y), \end{cases} \tag{3.38}$$

where

$$\phi(x,y) = \begin{cases} v(x,y,0), & (x,y) \in Z, \\ 0, & (x,y) \in \mathbb{R}^2 - Z. \end{cases} \tag{3.39}$$

Note that, ϕ is a compactly supported function in $H^s(\mathbb{R}^2)$, $s \geq 3$. So, by the regularization property of the KP-II equation (see [9]), the IVP (3.38) has a smooth solution w. Therefore, by the unique continuation property (Theorem 3.4), we conclude that $w \equiv 0$ in $\Omega \times (0,T)$. Consequently we conclude that $v \equiv 0$ in $\Omega \times (0,T)$ which contradicts (3.34). Hence (3.24) must be true. □

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