# Fractional Viscoelastic Models on Time Scales

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*Abstract:* The main goal of this work is the generalisation of fractional viscoelastic models on Time Scales, applying the Riemann-Liouville derivative to the Maxwell's viscoelastic constitutive law. We show that, in time scale, the solution found is also expressed through the Miller-Ross function.

*Key–Words:* Time scale; Riemann-Liouville fractional derivatives; Viscoelastic models; Miller-Ross function

## 1 Introduction

Time scale analysis is a method that unifies discrete and continuous analysis, see e.g. [1], [2] and [3]. This method can be applicable to several physical phenomena, in particular for those that can be described through fractional differential equations. In the literature, the mechanical behaviour of viscoelastic materials is described as being a combination of an ideal solid (elastic) and purely viscous (fluid) components, that are represented by a spring and a dashpot, respectively. The so called Blair's model is an intuitive generalisation of the classical viscoelastic models for which the integer derivatives are replaced by non-integer derivatives. In recent years, several authors extended theoretical results in the contexts of mathematical analysis to time scale, see e.g.[3], [5] [4], [6], [9], [10], [11], [12]. These results can, for instance, be applied in the field of material mechanics in order to improve conventional constitutive laws. In this paper, as an example of a potential application, we apply the Laplace transform and Miller-Ross function, on time scales, to viscoelastic constitutive laws [15], in order to explore the potentialities of this unified analysis.

This paper is organised as follows: In Section 2 are summarised importants results regarding Time Scale Calculus; In section 3, using the Riemann-Liouville derivative on time scale, the viscoelastic constitutive models. Also, in this section, the relaxation modulus is written with respect to Miller-Ross function, in time scale, applying Laplace transforms. The conclusions are given in Section 4.

## 2 Fundamentals of the Fractional Calculus in Time Scale

Some preliminarily definitions and theorems on time scales can be found in [3], [4], [6], [7], [8], [12], [9], [10], [11], and [13].

**Definition 1** *The Taylor monomials or generalized polynomials, $h_k : \mathbb{T}^2 \to \mathbb{R}$, $k \in \mathbb{N}_0$, are defined recursively by*

$$\begin{cases} h_0(t,\tau) = 1, & k = 0 \\ h_{k+1}(t,\tau) = \displaystyle\int_s^t h_k(s,\tau)\Delta s, & k \in \mathbb{N}, \end{cases}$$

*for all $t, \tau \in \mathbb{T}$.*

The time scale $\Delta$-Riemann-Liouville type fractional integral of the function $f \in L^1([a,b] \cap \mathbb{T})$ of order $\alpha \in \mathbb{R}_+$ is defined by $(a, b \in \mathbb{T})$: $I_a^0 f = f$ and

$$I_a^\alpha f(t) = \int_a^t h_{\alpha-1}(t,\sigma(s)) f(s)\Delta s \in L^1([a,b] \cap \mathbb{T}). \tag{1}$$

Let $\alpha, \beta > 0$. Then,

$$I_0^\alpha (I_0^\beta f)(t) = I_0^{\alpha+\beta} f(t). \tag{2}$$

**Definition 2** *In the fractional calculus the $\Delta$-Riemann-Liouville fractional derivative of order $\alpha > 1$, $m-1 < \alpha < m \in \mathbb{N}$ is defined as follows, respectively, $\forall t \in [a,b] \cap \mathbb{T}$,*

$$\begin{aligned} D_{RL}^\alpha f(t) &= \Delta^m I_0^{m-\alpha} f(t) \\ &= \Delta^m \int_a^t h_{m-\alpha-1}(t,\sigma(\tau)) f(\tau)\Delta\tau \end{aligned} \tag{3}$$

*where $f \in C_{rd}^m([a,b] \cap \mathbb{T})$ and $\Delta^m g(t) = f^{\Delta^m}(t)$ is the usual derivative of $g(t)$.*

**Proposition 3** *(Taylor's Formula) Let $f \in C_{rd}^m([a,b] \cap \mathbb{T})$, $m \in \mathbb{N}, \mathbb{T}^k = \mathbb{T}$, $a, b \in \mathbb{T}$. Then*

$$f(t) = \sum_{k=0}^{m-1} h_k(t,a) f^{\Delta^k}(a) + \underbrace{\int_a^t h_{m-1}(t, \sigma(s)) f^{\Delta^m}(s)\Delta s}_{\left(I_a^m f^{\Delta^m}\right)(t)}. \quad (4)$$

*for all $t \in [a,b] \cap \mathbb{T}$*

**Definition 4** *Assume that the function $f : \mathbb{T} \to \mathbb{C}$ is regulated. The Laplace transform of $f$, denoted by $\mathcal{L}\{f\}$, is defined by*

$$\mathcal{L}\{f\}(z) \coloneqq \int_0^\infty f(t) e_{\ominus z}(\sigma(t), 0)\Delta t, \quad (5)$$

*for all $z \in \mathbb{C}$ where $\ominus z$ is the operation from [3].*

**Proposition 5** *If $f : \mathbb{T} \to \mathbb{C}$ is such that $f^\Delta$ is regulated, then*

$$\mathcal{L}\{f^\Delta\}(z) := z\mathcal{L}\{f\}(z) - f(0) \quad (6)$$

*for all $z \in D\{f\}$such that $\lim_{t\to\infty} \{f(t)e_{\ominus s}(t,0)\} = 0$.*

**Proposition 6** *If $f : \mathbb{T} \to \mathbb{C}$ be a generalized exponential, hyperbolic, trigonometric, or polynomial function, and let $g : \mathbb{T} \to \mathbb{C}$ be regulated. The, subject to a certain limit condition,*

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f\} \cdot \mathcal{L}\{g\} \ \text{ on } \ D\{f * g\}, \quad (7)$$

*where $*$ is the convolution's operator.*

**Proposition 7** *If $f : \mathbb{T} \to \mathbb{C}$ is regulated and $F(t) = \int_0^t f(u)\Delta u$ for $t \in \mathbb{T}$, then*

$$\mathcal{L}\{F\}(z) = \frac{\mathcal{L}\{f\}(z)}{z} \quad (8)$$

*for all $z \in D\{f\}$such that $\lim_{t\to\infty} \{F(t)e_{\ominus s}(t,0)\} = 0$.*

**Proposition 8** *Let $\alpha > 0$. Then,*

$$\mathcal{L}\{h_\alpha(t,0)\} = \mathcal{L}\left\{\int_0^t h_{\alpha-1}(t,0)\Delta u\right\} = \frac{1}{z^{\alpha+1}}. \quad (9)$$

**Proposition 9** *Let $\alpha > 0$, $\mathbb{T}$ be a time scale, and $f : \mathbb{T} \to \mathbb{R}$. The fractional integral of $f$ of order $\alpha$ on the time scale*

$$\mathcal{L}\{I_0^\alpha f(t)\} = \frac{F(z)}{z^\alpha}, \quad (10)$$

*where $F(z) = \mathcal{L}\{f(t)\}(z)$.*

**Definition 10** *Let $\alpha > 0$, $\beta > 0$ and $a \in \mathbb{R}^+$. The Miller-Ross function is defined through*

$$R_{\alpha,\beta}(t; \pm a) := \sum_{k=0}^\infty (\pm a)^k h_{\alpha k+\beta-1}(t,0), \quad (11)$$

*and its Laplace transform is given by*

$$\mathcal{L}\{R_{\alpha,\beta}(t;\pm a)\} = \int_0^\infty R_{\alpha,\beta}(t;\pm a) e_{\ominus s}^\sigma(\sigma(t),0)\Delta t.$$

**Proposition 11** *Let $\alpha > 0$, $\beta > 0$ and $a \in \mathbb{R}^+$. The Laplace transformations for several Miller-Ross functions are summarised below:*

$$\mathcal{L}\{R_{\alpha,\beta}(t;\pm a)\} = \frac{z^{\alpha-\beta}}{z^\alpha \mp a}. \quad (12)$$

## 3 Viscoelasticity models on time scale

In this section we consider an application on the fractional calculus in time scale in the theory of viscoelasticity.

Viscoelastic materials when undergoing deformation exhibits a mechanical behavior that can be considered as a combination of purely elastic and viscous phenomena. In fact, constitutive laws that describe viscoelastic behaviour are a combination of springs (elastic) and dashpot (viscous) components. The spring (elastic) component, also called Hooke's element is described by the formula

$$\sigma(t) = Ee(t),$$

where $\sigma$ is the stress, $E$ the Young Modulus (elastic constant) and $e$ the applied strain. The dashpot (viscous) component, also called Newton's element, is described by the relation

$$\sigma(t) = \eta \frac{e(t)}{dt},$$

where $\eta$ is the viscosity of the material.

In linear viscoelasticity, Maxwell's model correspond to a serial combination of spring and dashpots is used to model the relaxation effect (constant strain). Voigt-Kelvin model, expressed as parallel combination of spring and dashpot, is used to model the creep effect (constant stress). Also, is normal to perform combinations of these two models in order to obtain generalized models that best describe the observed physical phenomena.

Following the work detailed in [15] the main idea is to substitute the integer derivative (first derivative term) of the Newton's model by a non integer $\alpha$-derivative ($0 \le \alpha \le 1$). With this intuitive generalization we obtain the so called Blair's model, expressed through:

$$\sigma(t) = E\tau^{\alpha} D^{\alpha}_{RL} e(t),$$

where the order of differential-integration is given by $\alpha$ and $\tau$ is the relaxation time

As usually applied in the viscoelastic field, in order to obtain a more representative model, we can use various combinations of Blair's elements.

$$\sigma(t) = E_1 \tau_1^{\alpha} D^{\alpha}_{RL} e(t), \tag{13}$$

$$\sigma(t) = E_2 \tau_2^{\beta} D^{\beta}_{RL} e(t). \tag{14}$$

Let us assume without loss of generality that $\alpha > \beta$ and apply the operator $D^{\alpha-\beta}_{RL}$ to the previous equation

$$D^{\alpha-\beta}_{RL} \sigma(t) = E_2 \tau_2^{\beta} D^{\alpha-\beta}_{RL} \left[ D^{\beta}_{RL} e_2(t) \right] \tag{15}$$

where, from (2), (3) and (4), we have

$$\begin{aligned} D^{\alpha-\beta}_{RL} \left[ D^{\beta}_{RL} e_2(t) \right] &= \Delta^m I^{m-\alpha+\beta} (D^{\beta}_{RL} e_2(t)) \\ &= \Delta^m I_0^{m-\alpha+\beta} (\Delta^m I_0^{m-\beta} e_2(t)) \\ &= \Delta^m I_0^{-\alpha+\beta} I_0^{m} (\Delta^m I_0^{m-\beta} e_2(t)) \\ &= \Delta^m I^{-\alpha+\beta} \Big\{ I_0^{m-\beta} e_2(t) \\ &\quad - \sum_{k=0}^{m-1} h_k(t,0) I_0^m (\Delta^m I_0^{m-\beta} e_2(0)) \Big\}. \end{aligned}$$

If we assume that the strain is zero for $t \le 0$, then the last term in previous expression disappears

$$\begin{aligned} D^{\alpha-\beta}_{RL} \left[ D^{\beta}_{RL} e_2(t) \right] &= \\ &= \Delta^m I^{-\alpha+\beta} I_0^{m-\beta} e_2(t) = \Delta^m I^{m-\alpha} e_2(t) \\ &= D^{\alpha}_{RL} e_2(t) \end{aligned}$$

and equation (15) becomes

$$D^{\alpha-\beta}_{RL} \sigma(t) = E_2 \tau^{\beta} D^{\alpha}_{RL} e_2(t). \tag{16}$$

Combining equation (13) with (16) we obtain the following model:

$$\frac{1}{E_1 \tau_1^{\alpha}} \sigma(t) + \frac{1}{E_2 \tau_2^{\beta}} D^{\alpha-\beta}_{RL} \sigma(t) = D^{\alpha}_{RL} e(t), \tag{17}$$

where $e(t) = e(t)_1 + e_2(t)$. Now, we apply Laplace transform, on both side of the equation, to be able to transform functions from the time domain to the Laplace domain. In Laplace domain, equation (17) reads

$$\frac{1}{E_1 \tau_1^{\alpha}} \mathcal{L}\{\sigma(t)\} + \frac{1}{E_2 \tau_2^{\beta}} \mathcal{L}\{D^{\alpha-\beta}_{RL} \sigma(t)\} = \mathcal{L}\{D^{\alpha}_{RL} e(t)\}. \tag{18}$$

where, from (3), can be expressed through

$$\begin{aligned} &\frac{1}{E_1 \tau_1^{\alpha}} \mathcal{L}\{\sigma(t)\} + \frac{1}{E_2 \tau_2^{\beta}} \mathcal{L}\{\Delta^m I^{m-\alpha+\beta} \sigma(t)\} \\ &= \mathcal{L}\{\Delta^m I^{m-\alpha} e(t)\}. \end{aligned}$$

According to (6),

$$\begin{aligned} &\frac{1}{E_1 \tau_1^{\alpha}} \mathcal{L}\{\sigma(t)\} + \frac{1}{E_2 \tau_2^{\beta}} s^m \mathcal{L}\{I^{m-\alpha+\beta} \sigma(t)\} \\ &- \frac{1}{E_2 \tau_2^{\beta}} \left( s^{m-1} I^{m-\alpha+\beta} \sigma(0) + \cdots + I^{m-\alpha+\beta} \sigma(0) \right) \\ &= s^m \mathcal{L}\{I^{m-\alpha} e(t)\} - s^{m-1} I^{m-\alpha} e(0) - \cdots - I^{m-\alpha} e(0). \end{aligned}$$

where, from (10) and taking into account that all quantities are zero for $t \le 0$,

$$\frac{1}{E_1 \tau_1^{\alpha}} \mathcal{L}\{\sigma(t)\} + \frac{1}{E_2 \tau_2^{\beta}} s^m \frac{\mathcal{L}\{\sigma(t)\}}{s^{m-\alpha+\beta}} = s^m \frac{\mathcal{L}\{e(t)\}}{s^{m-\alpha}}.$$

Hence

$$\mathcal{L}\{\sigma(t)\} = \frac{E_2 \tau_2^{\beta} s^{\alpha} \mathcal{L}\{e(t)\}}{\frac{E_2 \tau_2^{\beta}}{E_1 \tau_1^{\alpha}} + s^{\alpha-\beta}}.$$

If the strain function is following

$$e(t) = \varepsilon_0 H(t)$$

where $\varepsilon_0$ is constant and $H(t)$ is called Heaviside or unit step function defined as:

$$H(t) = \begin{cases} 0 & , \quad t < 0 \\ 1 & , \quad t \ge 0 \end{cases},$$

then, from (9), we obtain

$$\mathcal{L}\{\sigma(t)\} = \frac{E_2 \tau_2^{\beta} s^{\alpha-1} \varepsilon_0}{\frac{E_2 \tau_2^{\beta}}{E_1 \tau_1^{\alpha}} + s^{\alpha-\beta}}.$$

Taking the inverse transform, using the convolution theorem (7) and the relation (12), we derive the relaxation modulus

$$\begin{aligned}
\frac{\sigma(t)}{\varepsilon_0} &= E_2\tau_2^\beta \mathcal{L}^{-1}\left\{\frac{s^{\alpha-\beta+\beta-1}}{\frac{E_2\tau_2^\beta}{E_1\tau_1^\alpha}+s^{\alpha-\beta}}\right\} \\
&= E_2\tau_2^\beta \mathcal{L}^{-1}\left\{\frac{s^{\alpha-\beta-1+m}}{\frac{E_2\tau_2^\beta}{E_1\tau_1^\alpha}+s^{\alpha-\beta}}\right\} * \mathcal{L}^{-1}\left\{s^{\beta-m}\right\} \\
&= E_2\tau_2^\beta R_{\alpha-\beta,1-m}\left(t;-a\right) * \mathcal{L}^{-1}\left\{\frac{1}{s^{m-\beta}}\right\} \\
&= E_2\tau_2^\beta R_{\alpha-\beta,1-m}\left(t;-a\right) * h_{m-\beta-1}(t,0) \\
&= E_2\tau_2^\beta \int_0^t R_{\alpha-\beta,1-m}\left(s;-a\right) h_{m-\beta-1}(t-\sigma(s),0)\Delta s \\
&= E_2\tau_2^\beta \sum_{k=0}^{\infty}(-a)^k h_{(\alpha-\beta)k-\beta}(t,0) \\
&= E_2\tau_2^\beta R_{\alpha-\beta,1-\beta}\left(t;-a\right),
\end{aligned}$$

where $a = \frac{E_2\tau_2^\beta}{E_1\tau_1^\alpha}$ and

$$R_{\alpha-\beta,1-m}\left(x;-a\right) = \sum_{k=0}^{\infty}(-a)^k h_{(\alpha-\beta)k+1-m-1}(x,0).$$

For a continuous case, $\mathbb{T} = \mathbb{R}$, the relaxation modulus is given by

$$\frac{\sigma(t)}{\varepsilon_0} = E_2\tau_2^\beta t^{-\beta} E_{\alpha-\beta,\beta-1}(-at^{\alpha-\beta})$$

where the Mittag-Leffler function, $E_{\mu,\rho}(at^\mu)$, is defined through

$$E_{\mu,\rho}(at^\mu) = \sum_{k=0}^{\infty}(-a)^k \frac{(t^\mu)^K}{\Gamma(\mu k+\rho)}.$$

For discrete time scales,

$$\mathbb{T} = \mathbb{T}_h^q = \left\{q^k + \sum_{i=0}^{k-1} q^i h : \ k \geq 2,\, k \in \mathbb{N}\right\} \cup \left\{\frac{h}{1-q}\right\},$$

which include, for example, $\mathbb{T} = \mathbb{Z}$, $\mathbb{T} = h\mathbb{Z}$ and $\mathbb{T} = \{q^k : k \in \mathbb{Z}\} \cup \{0\}$, the relaxation modulus is expressed by

$$\frac{\sigma(t)}{\varepsilon_0} = E_2\tau_2^\beta R_{\alpha-\beta,1-\beta}\left(t;-a\right)$$

where the Miller-Ross function, also called modified

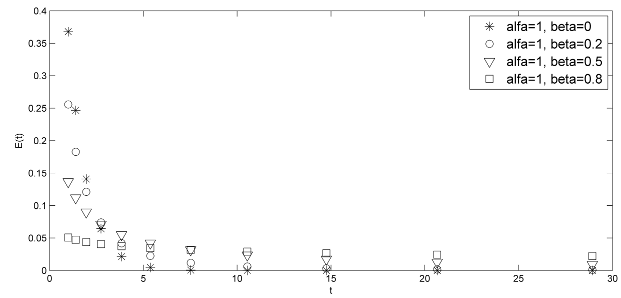


Figure 1: Relaxation modulus in time scale $\mathbb{T}_1^{1.4}$ with $\alpha = 1$.

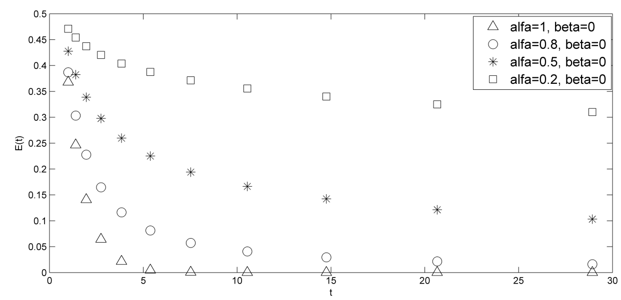


Figure 2: Relaxation modulus in time scale $\mathbb{T}_1^{1.4}$ with $\beta = 0$.

Mittag-Leffler function, reads

$$\begin{aligned}
R_{\alpha-\beta,1-\beta}(t;-\tau^{-(\alpha-\beta)}) &= \sum_{k=0}^{\infty}\left(-\tau^{-\alpha}\right)^k h_{(\alpha-\beta)k-\beta}(t,0) \\
&= \sum_{k=0}^{\infty}\frac{(t-s)_{(\tilde{q},h)}^{((\alpha-\beta)k-\beta)}}{\Gamma_{\tilde{q}}((\alpha-\beta)k-\beta+1)} \\
&= \sum_{k=0}^{\infty}\frac{([t]-[0])_{\tilde{q}}^{((\alpha-\beta)k+\beta)}}{\Gamma_{\tilde{q}}((\alpha-\beta)k-\beta+1)},
\end{aligned}$$

where (see [14])

$$([t]-[s])_{\tilde{q}}^{(\rho)} = [t]^\rho \frac{([s]/[t],\tilde{q})_\infty}{(\tilde{q}^\rho[s]/[t],\tilde{q}))_\infty}, \ t \neq 0$$

with $[x] = x + h\tilde{q}/(1-\tilde{q})$ and $(p,\tilde{q})_\infty = \prod_{j=0}^{\infty}(1-p\tilde{q}^j)$, $x \in \mathbb{R}\backslash\{0,-1,-2,\cdots\}$. Further, the q-Gamma function is defined, for $0 < \tilde{q} < 1$, as

$$\Gamma_{\tilde{q}}(x) = \frac{(\tilde{q},\tilde{q})_\infty(1-\tilde{q})^{1-x}}{(\tilde{q}^x,\tilde{q})_\infty}.$$

With this work it was possible to describe the fractional viscoelastic model, in time scale, in respect to the Miller-Ross function. This unified approach allows the application of Laplace transforms (direct and/or inverse) in discrete time scale with variable time-step. This approach has particular interest for the description of physical phenomena similar to the ones plotted in figures 1 and 2, where it is possible to observe a higher data resolution, for short times, in order to better capture the pronounced decay of the Relaxation modulus for that time range.

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