Microscopic dynamics for the porous medium equation

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Abstract In this work, I present an interacting particle system whose dynamics conserves the total number of particles but with gradient transition rates that vanish for some configurations. As a consequence, the invariant pieces of the system, namely, the hyperplanes with a fixed number of particles can be decomposed into an irreducible set of configurations plus isolated configurations that do not evolve under the dynamics. By taking initial profiles smooth enough and bounded away from zero and one and for parabolic time scales, the macroscopic density profile evolves according to the porous medium equation. Perturbing slightly the microscopic dynamics in order to remove the degeneracy of the rates the same result can be obtained for more general initial profiles.

1 Introduction

The purpose of this work is to present the hydrodynamic limit for an non-ergodic interacting particle system. The non ergodicity translates by saying that each hyperplane with a fixed number of particles (which is a conserved quantity of the system) can be decomposed into a irreducible set of configurations plus isolated configurations that do not evolve under the dynamics. In contrast with erdogic systems it is not possible to pick randomly one configuration η from a certain hyperplane and get to any other configuration in the same hyperplane with jumps that are allowed by the dynamics. This is the main difficulty when establishing the hydrodynamic limit for this class of processes. The process considered here belongs to the class of *kinetically constrained lattice gases* (KCLG) which are used in physical literature to model liquid/glass and more general jamming transitions. In this context, the constraints are devised to mimic the fact that the motion of a particle in a dense medium

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can be inhibited by the geometrical constraints induced by the neighboring particles. Here I present the hydrodynamic limit for a particle system associated to the porous medium equation. The process is of gradient type and is one of the simplest models in the KCLG class. The porous medium equation is given by $\partial_t \rho(t,u) = \partial_u^2 \rho^2(t,u)$ and it can be written in divergence form as $\partial_t \rho(t,u) = \nabla(D(\rho(t,u))\nabla(\rho(t,u)))$ with diffusion coefficient given by $D(\rho(t,u)) = 2\rho(t,u)$ and thus the equation looses the parabolic character as $\rho \to 0$. One of the properties of the solutions is that they can be compactly supported at each fixed time. A second observation is that the solutions of the equation can be continuous on the domain of definition, without being smooth at the boundary. In the next section I will present a Markov process whose macroscopic density behavior $\rho : [0,T] \times \mathbb{T} \to [0,1]$ evolves according to the partial differential equation above, the so called hydrodynamic equation. Here $\mathbb{T}$ denotes the one-dimensional torus.

2 Markov Process

Let η_t be a continuous time Markov process with space state $\{0,1\}^{\mathbb{T}_N}$, where $\mathbb{T}_N$ denotes the one-dimensional discrete torus. For a site x on the microscopic space, $\eta(x)$ denotes the number of particles at that site and $\eta(x) = 1$ will have the physical meaning as the site x being occupied by a particle, while $\eta(x) = 0$ will denote a vacancy at that site. For a configuration η, $c(x,y,\eta)$ denotes the rate at which a particle jumps from x to y. We restrict to the case of nearest-neighbor jumps, so that $c(x,y,\eta) = 0$ if $|x-y| > 1$ and the exclusion rule, a particle at site x jumps to y if the site y is empty otherwise the jump is suppressed. The jump rates are degenerate and of gradient type, in fact we consider $c(x,x+1,\eta) = \eta(x-1) + \eta(x+2)$ and $c(x,x+1,\eta) = c(x+1,x,\eta)$. This Markov process has generator given on local functions $f : \{0,1\}^{\mathbb{T}_N} \to \mathbb{R}$ by

$$(\mathcal{L}_P f)(\eta) = \sum_{\substack{x,y \in \mathbb{T}_N \\ |x-y|=1}} c(x,y,\eta)\eta(x)(1-\eta(y))(f(\eta^{x,y}) - f(\eta)). \tag{1}$$

In order to have a non-trivial temporal evolution of the density profile the process is evolving on the parabolic time scale tN^2. Since the jump rates are symmetric, the Bernoulli product measures $(\nu_\alpha)_\alpha$ in $\{0,1\}^{\mathbb{T}_N}$ are invariant and in fact reversible. This chosen rates define a **gradient system** since the instantaneous current $W_{0,1}(\eta) = c(0,1,\eta)\Big[\eta(0)(1-\eta(1)) - \eta(1)(1-\eta(0))\Big]$ can be rewritten as the gradient of a local function, namely $W_{0,1}(\eta) = h(\eta) - \tau_1 h(\eta)$, with $h(\eta) = \eta(0)\eta(1) + \eta(0)\eta(-1) - \eta(-1)\eta(1)$. The relation between h and the hydrodynamic equation is that $\partial_t \rho(t,u) = \partial_u^2 \tilde{h}(\rho(t,u))$ where $\tilde{h}(\rho) = E_{\nu_\rho}(h(\eta)) = \rho^2$.

3 Decomposition of the space state

By the definition of the dynamics, the number of particles is obviously a preserved quantity, and as a consequence the state space can be decomposed into hyperplanes with a fixed number of particles, namely $\Sigma_{N,k} = \{\eta \in \{0,1\}^{\mathbb{T}_N} : \sum_{x\in\mathbb{T}_N} \eta(x) = k\}$. It is said that $\mathcal{O}$ is an irreducible component of $\Sigma_{N,k}$ if for every η, $\xi \in \mathcal{O}$ it is possible to go from η to ξ by jumps that are allowed by the dynamics. Since the dynamics is defined by the presence of particles in the neighboring positions to the site where the particle jumps, it is natural to have a critical density for which in a regime under that critical density some configurations, that do not evolve under the dynamics, arise. For this process if $k > N/3$, each hyperplane with k particles is not decomposable into smaller ergodic subsets; however, for $k \leq N/3$, each hyperplane is decomposable into an irreducible component (the set of configurations that contain at least one couple of particles at distance at most two) plus many irreducible sets: configurations that do not evolve under the dynamics - to which we call frozen.

4 Hydrodynamic Limit

To investigate the hydrodynamic limit, define the empirical measure by:

$$\pi_t^N(du) = \pi^N(\eta_t, du) = \frac{1}{N}\sum_{x\in\mathbb{T}_N} \eta_t(x)\delta_{\frac{x}{N}}(du). \tag{2}$$

Fix an initial profile $\rho_0 : \mathbb{T} \to [0,1]$ and denote by $(\mu^N)_N$ a sequence of probability measures on $\{0,1\}^{\mathbb{T}_N}$.

Definition 1. A sequence $(\mu^N)_N$ is associated to an initial profile ρ_0, if for every continuous function $H : \mathbb{T} \to \mathbb{R}$ and for every $\delta > 0$

$$\lim_{N\to+\infty} \mu^N\Big[\Big|\frac{1}{N}\sum_{x\in\mathbb{T}_N} H\Big(\frac{x}{N}\Big)\eta(x) - \int_{\mathbb{T}} H(u)\rho_0(u)du\Big| > \delta\Big] = 0. \tag{3}$$

We can translate the definition above by saying that a sequence of measures $(\mu^N)_N$ is associated to a profile ρ_0 if a Law of Large Number (in the weak sense) holds for the empirical measure at time $t = 0$ under the probability μ^N. We can rewrite (3) as

$$\lim_{N\to+\infty} \mu^N\Big[\Big|\int_{\mathbb{T}} H(u)\pi_0^N(du) - \int_{\mathbb{T}} H(u)\rho_0(u)du\Big| > \delta\Big] = 0. \tag{4}$$

The goal in hydrodynamic limit consists in showing that if at time $t = 0$ the empirical measures are associated to some initial profile ρ_0, then at time t they are associated to a profile ρ_t, where ρ_t is the solution of the hydrodynamic equation, then if a Law of Large Numbers holds for the empirical measure at time $t = 0$

then it holds at any time t. The hydrodynamic limit can be derived in two different ways. One is known as the Relative Entropy Method and it was first introduced by Yau [5], when proving the hydrodynamic limit for Ginzburg-Landau models. This method requires the existence of smooth solutions of the hydrodynamic equation. The second one is known as the Entropy Method and it is due to Guo, Papanicolau and Varadhan [2]. In contrast with the first method, this requires the uniqueness of weak solutions of the hydrodynamic equation. Before proceeding we recall the definition of a weak solution of the porous medium equation.

Definition 2. Fix a bounded profile $\rho_0 : \mathbb{T} \to \mathbb{R}$. A bounded function $\rho : [0,T] \times \mathbb{T} \to \mathbb{R}$ is a weak solution of the hydrodynamic equation, if for every function $H : [0,T] \times \mathbb{T} \to \mathbb{R}$ of class $C^{1,2}([0,T] \times \mathbb{T})$

$$\begin{aligned} \int_0^T dt \int_{\mathbb{T}} du \Big\{ \rho(t,u)\partial_t H(t,u) + (\rho(t,u))^2 \partial_u^2 H(t,u) \Big\} \\ + \int_{\mathbb{T}} \rho_0(u) H(0,u) du = \int_{\mathbb{T}} \rho(T,u) H(T,u) du. \end{aligned} \tag{5}$$

4.1 The Relative Entropy Method

Fix $\varepsilon > 0$ and let $\rho_0 : \mathbb{T} \to [0,1]$ be a profile of class $C^{2+\varepsilon}(\mathbb{T})$. By a well known result, the porous medium equation admits a solution denoted by $\rho(t,u)$ of class $C^{1+\varepsilon,2+\varepsilon}(\mathbb{R}_+ \times \mathbb{T})$. In order to apply the method, there is a technical condition that has to be assumed: the existence of a constant $\delta_0 > 0$ such that the profile is bounded away from 0 and 1: $\forall u \in \mathbb{T}$ it holds that $\delta_0 \leq \rho_0(u) \leq 1 - \delta_0$. Let $\nu^N_{\rho_0(\cdot)}$ be the product measure in $\{0,1\}^{\mathbb{T}_N}$ such that $\nu^N_{\rho_0(\cdot)}\{\eta, \eta(x) = 1\} = \rho_0(x/N)$. This means that for a fixed site $x \in \mathbb{T}_N$, $\eta(x)$ has Bernoulli distribution of parameter $\rho(0,x/N)$ and $(\eta(x))_x$ are independent. For two measures μ and ν in $\{0,1\}^{\mathbb{T}_N}$ define the relative entropy of μ with respect to ν as:

$$H(\mu/\nu) = \sup_f \Big\{ \int f d\mu - \log \int e^f d\nu \Big\}.$$

The supremum is taken over all continuous functions.

Theorem 1. *(G.L.T. [1]) Let $\rho_0 : \mathbb{T} \to [0,1]$ be a initial profile of class $C^{2+\varepsilon}(\mathbb{T})$ that satisfies:*

$$\exists \delta_0 > 0 : \forall u \in \mathbb{T}, \quad \delta_0 \leq \rho_0(u) \leq 1 - \delta_0. \tag{6}$$

Let $(\mu^N)_N$ be a sequence of probability measures on $\{0,1\}^{\mathbb{T}_N}$ such that:

$$\lim_{N \to +\infty} \frac{H(\mu^N / \nu^N_{\rho_0(\cdot)})}{N} = 0. \tag{7}$$

Then, for each $t \geq 0$

$$\pi^N_{tN^2}(du) \xrightarrow[N\to+\infty]{} \rho(t,u)du \tag{8}$$

in probability, where $\rho(t,u)$ *is a smooth solution of the porous medium equation.*

We remark that in the last result, there was made two assumptions on the initial profile in order to obtain the result (1) the bound condition

$$\exists \delta_0 > 0 : \forall u \in \mathbb{T}, \quad \delta_0 \leq \rho_0(u) \leq 1 - \delta_0 \tag{9}$$

and (2) the smoothness of class $C^{2+\varepsilon}(\mathbb{T})$. This is too restrictive, since one could want to analyze profiles that are (for example) indicator functions over a certain set. On the other hand, the Entropy Method relies on the full irreducibility of the Markov process when restricted to a hyperplane. We have seen that the process defined above when restricted to a hyperplane with a low density of particles it is not fully irreducible - the frozen states arise. To overcome this problem, the idea is to perturb slightly the dynamics in such a way that the frozen states disappear but the macroscopic density profile still evolves according to the porous medium equation.

4.2 The Entropy Method

In this section we present the hydrodynamic limit for a slightly different dynamics, in which each hyperplane is a unique ergodic piece and whose macroscopic density profile still evolves according to the porous medium equation. For $\theta > 0$, consider a Markov process with generator given by

$$\mathcal{L}_\theta = \mathcal{L}_P + N^{\theta-2}\mathcal{L}_S$$

where $\mathcal{L}_P$ was defined above and $\mathcal{L}_S$ is the generator of the Symmetric Simple Exclusion process:

$$(\mathcal{L}_S f)(\eta) = \sum_{\substack{x,y\in\mathbb{T}_N \\ |x-y|=1}} \frac{1}{2}\eta(x)(1-\eta(y))(f(\eta^{x,y}) - f(\eta)),$$

The Bernoulli product measures $(\nu_\alpha)_\alpha$ are invariant for $\mathcal{L}_\theta$ since they are invariant measures for $\mathcal{L}_P$ and $\mathcal{L}_S$. Its also easy to show that the Markov processes with generators $\mathcal{L}_P$ and $\mathcal{L}_\theta$, have the same hydrodynamic equation as long as $\theta < 2$. This restriction on θ comes from the fact that we want to perturb sightly the dynamics microscopically in order to destroy the frozen configurations, but we do not want to see the effect of this perturbation macroscopically, and for that the Symmetric Simple Exclusion Process has to be speeded up on a time scale less than the parabolic one. Nevertheless, while the former process has each hyperplane (on the low density regime) decomposed in many ergodic or irreducible pieces, which is a consequence of the existence of the frozen states, the latter has each hyperplanes

$\{\Sigma_{N,k} : k = 0,..,N\}$ as a unique ergodic piece. Then the Entropy Method can be applied to the process with generator $\mathcal{L}_\theta$.

For that, denote by $\mathbb{P}_\mu$ the probability measure on $D([0,T],\{0,1\}^{\mathbb{T}_N})$, induced by the Markov process with generator $\mathcal{L}_\theta$, speeded up by N^2 and with initial measure μ.

Theorem 2. *(G.L.T. [1])*

Let $\rho_0 : \mathbb{T} \to [0,1]$ and $(\mu^N)_N$ be a sequence of probability measures on $\{0,1\}^{\mathbb{T}_N}$ ***associated to the profile*** *ρ_0. Then, for every $0 \le t \le T$, for every continuous function $H : \mathbb{T} \to \mathbb{R}$ and for every $\delta > 0$,*

$$\lim_{N\to+\infty} \mathbb{P}_{\mu^N}\Big[\Big|\frac{1}{N}\sum_{x\in\mathbb{T}_N} H\Big(\frac{x}{N}\Big)\eta_t(x) - \int_{\mathbb{T}} H(u)\rho(t,u)du\Big| > \delta\Big] = 0, \tag{10}$$

where $\rho(t,u)$ is the unique weak solution of the porous medium equation.

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