3D-Mappings by Means of Monogenic Functions and their Approximation

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Abstract

We consider quasi-conformal 3D-mappings realized by hypercomplex differentiable (monogenic) functions and their polynomial approximation. Main tools are the series development of monogenic functions in terms of hypercomplex variables and the generalization of L. V. Kantorovich's approach for approximating conformal mappings by powers of a small parameter.

1 Introduction

In contrast to the planar case, in $\mathbb{R}^n$, with $n \geq 3$, the set of conformal mappings is only the set of Möbius transformations (due to Liouville's theorem [9]).

The difficulties in characterizing Möbius transformations in $\mathbb{R}^4$ by some differentiability property have been studied in detail in [8]. In the case of $\mathbb{R}^4$ the application of quaternions is natural, a fact that has already been noticed in [15] and [3], for instance. But the theory of generalized holomorphic functions developed on the basis of Clifford algebras (with quaternions as a special case, cf. [3]; for historical reasons they are also called *monogenic functions*, cf. [2]) does *not* cover the set of Möbius transformations in $\mathbb{R}^n$ if $n \geq 3$. Möbius transformations are *not monogenic* and therefore *monogenic functions* are not directly related to conformal mappings in $\mathbb{R}^n$, $n \geq 3$. Here one can only expect that monogenic functions realize quasi-conformal mappings. Obviously, such a situation has originated many questions concerning the extension of theoretical and practical conformal mapping methods in $\mathbb{C}$ to higher dimensions in the setting of Clifford Analysis (see [12] for a special approach). Note that, in this setting, contrary to the case of several complex variables there are no restrictions on the real dimension being even or odd. This implies that the real 3-dimensional Euclidean space, the most important space for concrete applications, can be subject to a treatment similar to the complex one.

Some results based on the application of Bergman's reproducing kernel method (BKM) in the Clifford setting are described in [1]. Our goal is to present in this case study a different from BKM approach. In fact, it is an extension of ideas of L. V. Kantorovich (c.f. [7] and [4]) to the 3-dimensional case.

As usual, we identify each element $x = (x_0, x_1, x_2) \in \mathbb{R}^3$ with the *paravector* (sometimes also called *reduced quaternion*) $z = x_0 + x_1 e_1 + x_2 e_2$.

For $C^1(\Omega, \mathbb{R}^3)$ define the generalized Cauchy-Riemann operator $D = \frac{\partial}{\partial x_0} + e_1 \frac{\partial}{\partial x_1} + e_2 \frac{\partial}{\partial x_2}$.

Solutions of the differential equations $Df = 0$ (resp. $fD = 0$) are called left-monogenic (resp. right-monogenic) functions in the domain Ω. Let us remind that the differential operator D is not only a formal linear combination of the real partial derivatives $\frac{\partial}{\partial x_k}$ but, when applied to a given function $f : \Omega \to \mathbb{H}$, is nothing else than an areolar derivative in the sense of Pompeiu (cf. [14] and [13]). The same is true for the conjugate Cauchy-Riemann operator $\overline{D} = \frac{\partial}{\partial x_0} - e_1 \frac{\partial}{\partial x_1} - e_2 \frac{\partial}{\partial x_2}$.

But if f is a monogenic function in Ω, its areolar derivative Df vanishes and this is equivalent to the fact that the areolar derivative $\frac{1}{2}\overline{D}f$ can be considered as the hypercomplex derivative of the function f. In $\mathbb{C}$ for a complex differentiable function f we have $f' = \frac{df}{dz} = \frac{1}{2}(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}) = \frac{\partial f}{\partial x}$. The same is true in our case, i.e. $\frac{1}{2}\overline{D}f = \frac{\partial f}{\partial x_0}$. Obviously, this formula guarantees that the (hypercomplex) derivative of a monogenic function is again a monogenic function.

In general we have to assume that a monogenic function f has values in $\mathbb{H}$, i.e., it is of the form $f(x) = f_0(x) + f_1(x)e_1 + f_2(x)e_2 + f_3(x)e_3$, where $f_k,\ k = 0, 1, 2, 3$ are real valued functions in Ω. But if we are dealing with mappings from one 3-dimensional domain to another 3-dimensional domain we have to restrict f to be a quaternion-valued function with one identically zero component. This can be done by different choices. Here we consider f also as a paravector, i.e., as being of the form $f(x) = f_0(x) + f_1(x)e_1 + f_2(x)e_2$.

In this case a monogenic function is monogenic from both sides and its components satisfy the Riesz system

$$\begin{aligned} \frac{\partial f_0}{\partial x_0} - \frac{\partial f_1}{\partial x_1} - \frac{\partial f_2}{\partial x_2} &= 0 \\ \frac{\partial f_0}{\partial x_1} + \frac{\partial f_1}{\partial x_0} &= 0 \\ \frac{\partial f_0}{\partial x_2} + \frac{\partial f_2}{\partial x_0} &= 0 \\ \frac{\partial f_1}{\partial x_2} - \frac{\partial f_2}{\partial x_1} &= 0. \end{aligned} \tag{1}$$

2 Monogenic functions as mapping functions

2.1 Paravector-valued monogenic functions of a paravector in $\mathbb{R}^3$

The description of the series development of monogenic functions will be made here in terms of two hypercomplex monogenic variables $z_k = x_k - x_0 e_k, k = 1, 2$. Indeed, using the general approach for Clifford algebra valued monogenic functions ([11], [5]) restricted to our case of $n = 2$, a second hypercomplex structure of $\mathbb{R}^{2+1}$ different from that given by the set of paravectors $\mathcal{A}$ consists in the following isomorphism:

$$\mathbb{R}^{2+1} \cong \mathcal{H}^2 = \{\vec{z} : z_k = x_k - x_0 e_k; x_0, x_k \in \mathbb{R}\}.$$

where $k = 1, 2$. $\mathcal{C}\ell_{0,2}$-valued functions of the form $f(z) = f_0(z) + f_1(z)e_1 + f_2(z)e_2 + f_{12}(z)e_1e_2$ are considered as mappings

$$f : \Omega \subset \mathbb{R}^3 \cong \mathcal{H}^2 \quad \longmapsto \quad \mathcal{C}\ell_{0,2}.$$

Following [11], we apply

Definition 1. *Let $V_{+,\cdot}$ be a commutative or non-commutative ring, $a_k \in V$ $(k = 1, \dots, n)$, then the symmetric "$\times$"-product is defined by*

$$a_1 \times a_2 \times \cdots \times a_n = \frac{1}{n!} \sum_{\pi(i_1,\dots,i_n)} a_{i_1} a_{i_2} \cdots a_{i_n} \tag{2}$$

where the sum runs over **all** *permutations of all $(i_1, \dots, i_n)$,*

together with the

Convention:
If the factor a_j occurs μ_j-times in (2), we briefly write

$$\underbrace{a_1 \times \cdots \times a_1}_{\mu_1} \times \cdots \times \underbrace{a_n \times \cdots \times a_n}_{\mu_n} \\ = {a_1}^{\mu_1} \times {a_2}^{\mu_2} \times \cdots \times {a_n}^{\mu_n} = \vec{a}^{\,\mu} \tag{3}$$

where $\mu = (\mu_1, \ldots, \mu_n,)$ and set parentheses if the powers are understood in the ordinary way (see [13]).

Since the symmetric products of μ_1 factors z_1 and μ_2 factors z_2 are monogenic functions of homogeneous degree $\mu_1 + \mu_2$ which form a basis for the Taylor series of a monogenic function in $\mathbb{R}^3$ (see [13]) they are called *generalized powers*.

Given a paravector-valued function f in $\mathbb{R}^n$, its power series development ordered by powers of the same homogeneous degree is given by ([11]):

Theorem 1. *Let $f = f(z) = f_0 + f_1 e_1 + f_2 e_2$ be a paravector-valued monogenic function of the paravector $z = x_0 + x_1 e_1 + x_2 e_2$. The Taylor series of $f(z)$ in terms of z_k in a neighborhood of the origin is given by*

$$f(z) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \; {z_1}^{n-k} \times {z_2}^{k} \;\; \alpha_{(n-k,\,k)} \tag{4}$$

with

$$\alpha_{(n-k,\,k)} = \frac{1}{n!} \frac{\partial^n f(0)}{\partial x_1^{n-k} \partial x_2^{k}} \quad , k = 0, \ldots, n, \tag{5}$$

where

$$\alpha_{(n-k,\,k)} = \alpha^0_{(n-k,\,k)} + \alpha^1_{(n-k,\,k)} \, e_1 + \alpha^2_{(n-k,\,k)} \, e_2, \quad n = 0, 1, 2 \ldots; \, k = 0, \ldots, n, \tag{6}$$

Being obtained by the partial derivatives of a function $f : \Omega \rightarrow \mathcal{A}$ it is evident that the coefficients $\alpha_{(n-k,\,k)}$ should have this form (6) of a paravector if explicitly written with their real and imaginary parts. As we now prove, the last equation of the Riesz system (1) implies a special relationship between adjacent coefficients of the same homogeneous degree.

Theorem 2. *A homogeneous monogenic polynomial of degree n given by*

$$f(z_1, z_2) := \sum_{k=0}^{n} \binom{n}{k} \; {z_1}^{n-k} \times {z_2}^{k} \alpha_{(n-k,\,k)} \tag{7}$$

with arbitrary paravector-valued coefficients of the form (6) is paravector-valued if and only if

$$\alpha^2_{(n-k,\,k)} = \alpha^1_{(n-k-1,\,k+1)} \;\; n = 0, 1, 2 \ldots; \, k = 0, \ldots, n. \tag{8}$$

Proof: To see that (8) is necessary we start from the last equation of (1), i.e.

$$\frac{\partial f_2}{\partial x_1} = \frac{\partial f_1}{\partial x_2}. \tag{9}$$

Since monogenic functions are infinitely hypercomplex differentiable and therefore also real differentiable of arbitrary order we can differentiate both sides of (9) $(n-k-1)$-times by x_1 and k-times by x_2 to end up with

$$\frac{\partial^n f_2}{\partial x_1^{n-k} \partial x_2^{k}} = \frac{\partial^n f_1}{\partial x_1^{n-k-1} \partial x_2^{k+1}}. \tag{10}$$

Formula (10), together with (5), (6), leads to the assertion (8). The fact that (8) is also a sufficient condition can be shown by induction over k. □

Note, that due to (8) an $L-$monogenic paravector-valued function of a paravector of arbitrary dimension is also $R-$monogenic, i.e. bi-monogenic. This was proved in [10] by arguments relying on hypercomplex differentiability.

For the purpose of polynomial approximation by means of the Taylor series development (4) of a monogenic function we still have to discuss some other aspects inspired by the corresponding complex approach.

As it is usual in the case of conformal transformations in the complex plane, the domain $\Omega \subset \mathbb{R}^3$ that we are going to transform should contain the origin. More concretely, we suppose, that the domain into which we are mapping should be a ball $\mathcal{B} \subset \mathbb{R}^3$ with the center at the origin. Then follows immediately from (4) that $\alpha_{0,\,0} = 0$ and each monogenic approximation polynomial begins with an expression of the form

$$u = z_1\alpha_{(1,\,0)} + z_2\alpha_{(0,\,1)}. \tag{11}$$

Clearly, the invariance of the origin guarantees in our settings that the interior of Ω will be mapped to the interior of the ball. On the other hand, (11) represents the first order (linear) approximation of the mapping function that we are looking for and, as such, is naturally related to its hypercomplex derivative in the origin.

What is the corresponding situation in $\mathbb{C}$? In the complex case we know that the Riemann mapping theorem still allows to prescribe, for example, the direction in which the real axis should be mapped. Such behavior is simply related to a property of the complex derivative. For instance, often the positivity or a special value of the argument of the derivative in the origin is demanded. Or take for example the requirement that $f'(0) = 1,$ where for the moment $f : \Omega \to \mathcal{B} \subset \mathbb{C}$ (cf. [7]). This leads to the class $\mathcal{S}$ of univalent functions and means that the first (linear) approximation of the mapping function f is given by $w = f(z) = z, z \in \mathbb{C}$. In other words, in the first step of approximation nothing else than the identity function is used. Moreover, this means also that in the first step the unit ball in the image plane $\{w : |w| \leq 1\}$, has as its pre-image the unit ball $\{z : |z| \leq 1\}$. But step by step the approximation by polynomials of higher degree changes the situation. In some sense we could say that in $\mathbb{C}$, from the geometric viewpoint the conditions $f(0) = 0$ as well as $f'(0) = 1$, are *normalizing the first step* in the approximation process of a domain Ω to a circle: the simplest polynomial of degree 1, namely the identity $w = f(z) = z$, is used and therefore in this step a $w-$circle is obtained from the corresponding $z-$circle. This works independently from the considered domain Ω and in so far we do not only have the $w-$circle as the canonical "target" domain, but also the $z-$circle as the canonical "starting" domain.

In the 3D-case (and, in general, for any real dimension $n > 2$) the situation is different, due to the nature of the used function class. Whereas in $\mathbb{C}$ the identity $f(z) = z$ with $f'(z) \equiv 1$ is a holomorphic function, it is not the case that $f(z) = z = x_0 + x_1e_1 + x_2e_2 \in \mathcal{A}$ is a monogenic function. Indeed, for a linear monogenic function f with hypercomplex derivative $\frac{1}{2}\overline{D}f(\mathbf{0}) = 1$ according to formula (11)we must have that

$$\frac{1}{2}\overline{D}f(\mathbf{0}) = -e_1\alpha_{(1,\,0)} - e_2\alpha_{(0,\,1)} = 1. \tag{12}$$

Applying (6) and (8) the formula (12) is equivalent to

$$\frac{1}{2}\overline{D}f(\mathbf{0}) = -e_1(\alpha^0_{(1,\,0)} + \alpha^1_{(1,\,0)}\,e_1) - e_2(\alpha^0_{(0,\,1)} + \alpha^2_{(0,\,1)}\,e_2) = 1. \tag{13}$$

This condition for a linear monogenic function with hypercomplex derivative equal to 1 is equivalent to

$$\alpha^0_{(1,\,0)} = \alpha^0_{(0,\,1)} = 0$$

as well as

$$\alpha^1_{(1,\,0)} + \alpha^2_{(0,\,1)} = 1. \tag{14}$$

Thus, together with (8) in the form $\alpha^2_{(1,\,0)} = \alpha^1_{(0,\,1)} = c$, the linear approximation is obtained as

$$\begin{aligned} w = f(z) &= z_1(\alpha^1_{(1,\,0)}\,e_1 + c\,e_2) + z_2(c\,e_1 + \alpha^2_{(0,\,1)}\,e_2) \\ &= \alpha^1_{(1,\,0)}z_1\,e_1 + \alpha^2_{(0,\,1)}z_2\,e_2 + c\,(z_2\,e_1 + z_1\,e_2) \\ &= x_0 + \alpha^1_{(1,\,0)}x_1\,e_1 + \alpha^2_{(0,\,1)}x_2\,e_2 + c\,(x_2\,e_1 + x_1\,e_2) \end{aligned} \tag{15}$$

Whereas in the complex case the demand for a function with derivative $f'(0) = 1$ immediately leads to a well defined linear approximation, we see that in dimension three one condition concerning the hypercomplex derivative is not enough for this purpose. In fact, three (real) parameters are still left. To overcome this problem we impose one more initial condition to the mapping function, namely that

$$\tilde{D}f(\mathbf{0}) := (\frac{\partial}{\partial x_0} + e_1\frac{\partial}{\partial x_1} - e_2\frac{\partial}{\partial x_2})f(\mathbf{0}) = 1. \tag{16}$$

Direct calculation carried out on (15) together with (12) results in the following relationship between the three until now not fixed real parameters: $-\alpha^1_{(1,0)} + \alpha^2_{(0,1)} + 2ce_1e_2 = 0$; hence $c = 0$ and (14) leads to

$$\alpha^1_{(1,0)} = \alpha^2_{(0,1)} = \frac{1}{2}. \tag{17}$$

Using these values in (15) we finally obtain for the initial ("first step") approximation the linear polynomial

$$w = f(z) \quad = \quad \frac{1}{2}(z_1e_1 + z_2e_2) = x_0 + \frac{1}{2}(x_1\, e_1 + x_2\, e_2). \tag{18}$$

From the geometrical point of view the result seems not to be a surprise. Indeed, formula (18) means nothing else than that, in the first step of approximation, the interior of the unit $w-$ball

$$\mathcal{B} = \{w : |w| \leq 1\}$$

is obtained from the interior of a unit *oblate ellipsoid* or *oblate spheroid* given by

$$\mathcal{O} = \{(x_0, x_1, x_2) : {x_0}^2 + \frac{1}{4}{x_1}^2 + \frac{1}{4}{x_2}^2 = 1\}. \tag{19}$$

From the analytical point of view the additional condition (16) which led to this situation is also not very surprising. Besides others, we mention only two arguments:

(1) Due to the real dimension three, the use of three $\mathcal{H}^2$-linear hypercomplex differential operators is necessary for describing the three real partial derivatives in terms of hypercomplex differential expressions.

(2) The hypercomplex derivative of a monogenic function given by

$$\frac{1}{2}\overline{D}f = -e_1\frac{\partial f}{\partial x_1} - e_2\frac{\partial f}{\partial x_2} = \frac{\partial f}{\partial x_0}$$

reflects several essential qualitative properties of a monogenic function, but from the quantitative point of view does not allow to describe the influence of the partial derivatives with respect to x_1 and x_2 separately. As we saw, the use of the operator $\tilde{D}$ defined by (16) solved this problem and is equivalent with imposing a symmetric behavior of the first approximation with respect to x_1 and x_2. Needless to note that by (16) the number of initial conditions has been increased by one, exactly the same as the real dimension of the considered Euclidean space increased by one compared with the complex case.

Summarizing we notice that the general form of the series that we shall use for approximating a mapping of $\Omega \subset \mathbb{R}^3$ into a ball $\mathcal{B}_\varrho \subset \mathbb{R}^3$ is

$$f(z_1, z_2) = \frac{1}{2}(z_1e_1 + z_2e_2) + \sum_{n=2}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\; {z_1}^{n-k} \times {z_2}^k\alpha_{(n-k,\,k)} \tag{20}$$

together with the compatibility condition (8), i.e.,

$$\alpha^2_{(n-k,\,k)} = \alpha^1_{(n-k-1,\,k+1)} \;\; n = 0, 1, 2 \ldots;\; k = 0, \ldots, n.$$

2.2 The approximate solution of a special 3D-mapping problem by monogenic polynomials involving a small parameter

We suppose further that the boundary of Ω can be embedded with a sufficiently small real parameter λ in a family of surfaces parameterized by s and t of the form

$$z = z(s, t, \lambda).$$

Suppose also that the family of surfaces includes the origin for all λ.

This idea follows Kantorovich's method in the complex plane ([7], Ch. V, §5), where an analogous family of curves $z = z(t, \lambda)$ is considered. The corresponding problem (mapping into a circle) together with the usual standardization of the mapping function leads to a series analogous to (20):

$$\varphi(z, \lambda) = z + \alpha_2(\lambda)z^2 + \alpha_3(\lambda)z^3 + \dots \tag{21}$$

already written with indeterminate coefficients $\alpha_n(\lambda), n = 2, 3, \dots$. The determination of those $\alpha_n(\lambda)$ by resolution of a non-linear system of algebraic equations depending on relationships between the boundaries of the considered domains is the core of the method.

From the previous subsection it is now clear that we generalize Kantorovich's method by considering the series

$$\varphi(z_1, z_2, \lambda) = \frac{1}{2}(z_1e_1 + z_2e_2) + \sum_{n=2}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\ {z_1}^{n-k} \times {z_2}^{k}\alpha_{(n-k,\,k)}(\lambda) \tag{22}$$

where the indeterminate coefficients are paravectors satisfying the compatibility property

$$\alpha^2_{(n-k,\,k)}(\lambda) = \alpha^1_{(n-k-1,\,k+1)}(\lambda)\ \ n = 0, 1, 2 \dots;\ k = 0, \dots, n.$$

As a concrete example, our case study is concerned with the mapping of the interior of the oblate ellipsoid $\mathcal{E}_\lambda,\ (0 \le \lambda < 1)$, defined by

$$x_0 = (1 + \lambda)\cos s,\ x_1 = 2(1 - \lambda)\sin s \cos t,\ x_2 = 2(1 - \lambda)\sin s \sin t$$

with $0 \le s \le \pi$ and $0 \le t < 2\pi$, into the interior of a ball $\mathcal{B}$.

Remark 1. *We notice that, due to (19), we have $\mathcal{E}_0 = \mathcal{O}$, which means that we are studying a small perturbation of the* canonical oblate spheroid $\mathcal{O}$ *which is mapped into the unit sphere by the linear monogenic function $w = \frac{1}{2}(z_1e_1 + z_2e_2) = x_0 + \frac{1}{2}(x_1\ e_1 + x_2\ e_2)$ (c.f. 18). This can also be seen from the hypercomplex equation of $\mathcal{E}_\lambda$, which, in terms of $w = \frac{1}{2}(z_1e_1 + z_2e_2)$, is given by*

$$(1 + \lambda^2)w\,\overline{w} - \lambda(w\,w + \overline{w}\,\overline{w}) = (1 - \lambda^2)^2$$

and where the choice of $\lambda = 0$ leads immediately to $w\,\overline{w} = 1$.

The numerical efficiency of Kantorovich's methods relies also on simplifications in the series (21) by making use of symmetry properties of the considered domain Ω. For instance, the fact that, in some cases, one or both of the coordinate axes (or other symmetry axes) can be considered as invariant under the mapping immediately implies a substantial reduction of the indeterminate coefficients $\alpha_n(\lambda),\ n = 2, 3, \dots,$ and therefore reduces the numerical costs.

Carrying out similar calculations and simplifications in the case of $\partial\Omega = \mathcal{E}_\lambda$, we arrived to the following result, which we present here without the straightforward but rather cumbersome proof.

Theorem 3. *Let the series $\varphi(z_1, z_2, \lambda)$ be given by formula (22) with the compatibility condition*

$$\alpha^2_{(n-k,\,k)}(\lambda) \quad = \quad \alpha^1_{(n-k-1,\,k+1)}(\lambda)\ \ n = 2, 3, \dots;\ k = 0, \dots, n, \tag{23}$$

being fulfilled.

(i) The hyperplane $x_0 = 0$ is invariant (i.e., $x_0 = 0$ implies that $\varphi(z_1, z_2, \lambda)$ admits only pure imaginary values), if

$$\alpha^0_{(n-k,k)}(\lambda) := 0, \quad \textit{for } k = 0, \dots, n, \quad n=2,3,\dots. \tag{24}$$

(ii) The hyperplane $x_0 = 0$ and the real axis $x_1 = x_2 = 0$ are invariant, if

$${z_1}^{n-k} \times {z_2}^{k} = 0, \quad \textit{for every even } n, \textit{ and } k = 0, \dots, n.$$

and

$$\begin{cases} \alpha^2_{(n-k,k)} = 0, & \textit{for every odd } n \textit{ and even } k, \\ \alpha^1_{(n-k,k)} = 0, & \textit{for every odd } n \textit{ and odd } k, k = 0, 1, \dots, n. \end{cases}$$

These invariance properties immediately lead to the following

Corollary 1. *The mapping of domains $\Omega \subset \mathbb{R}^3$, which are axially symmetric with respect to the real axis and admit a planar symmetry with respect to the imaginary hyperplane $x_0 = 0$, into a ball $\mathcal{B} \subset \mathbb{R}^3$ centered in the origin can be realized by a monogenic series of the form*

$$\begin{aligned} \varphi(z_1, z_2) \quad = \quad & \frac{1}{2}(z_1 e_1 + z_2 e_2) + (\alpha^1_{(3,0)}\ z_1^3 e_1 + 3\alpha^2_{(2,1)}\ z_1^2 \times z_2^1 e_2 + \\ & +3\alpha^1_{(1,2)}\ z_1^1 \times z_2^2 e_1 + \alpha^2_{(0,3)}\ z_2^3 e_2) + (\alpha^1_{(5,0)}\ z_1^5\ e_1 + \\ & +5\alpha^2_{(4,1)}\ z_1^4 \times z_2\ e_2 + 10\alpha^1_{(3,2)}\ z_1^3 \times z_2^2\ e_1 + 10\alpha^2_{(2,3)}\ z_1^2 \times z_2^3\ e_2 + \\ & +5\alpha^1_{(1,4)}\ z_1 \times z_2^4\ e_1 + \alpha^2_{(0,5)}\ z_2^5\ e_2) + \cdots \end{aligned} \tag{25}$$

with real coefficients $\alpha^l_{(n-k,k)}; n = 3, 5, 7, \dots; k = 0, 1, \dots, n; l = 1$ for even k and $l = 2$ for odd k.

Implying further in (25) the compatibility conditions (23) and writing as abbreviation for the inner coinciding coefficients

$$\beta_{31} := \alpha^2_{(2,1)} = \alpha^1_{(1,2)},$$

$$\beta_{51} := \alpha^2_{(4,1)} = \alpha^1_{(3,2)},$$

$$\beta_{53} := \alpha^2_{(2,3)} = \alpha^1_{(1,4)},$$

and so on for β_{nm} in all the following polynomials of higher homogeneous degree $n = 7, 9, \dots; m = 1, 3, \dots, n-2$, the formula (25) reduces to

$$\begin{aligned} \varphi(z_1, z_2) \quad = \quad & \frac{1}{2}(z_1 e_1 + z_2 e_2) + (\alpha_{(3,0)}\ z_1^3 e_1 + 3\beta_{(3,1)}\ z_1^2 \times z_2^1 e_2 + \\ & +3\beta_{(3,1)}\ z_1^1 \times z_2^2 e_1 + \alpha_{(0,3)}\ z_2^3 e_2) + (\alpha_{(5,0)}\ z_1^5\ e_1 + \\ & +5\beta_{(5,1)}\ z_1^4 \times z_2\ e_2 + 10\beta_{(5,1)}\ z_1^3 \times z_2^2\ e_1 + 10\beta_{(5,3)}\ z_1^2 \times z_2^3\ e_2 + \\ & +5\beta_{(5,3)}\ z_1 \times z_2^4\ e_1 + \alpha_{(0,5)}\ z_2^5\ e_2) + \\ & + \cdots. \end{aligned} \tag{26}$$

Here also the upper indices on the outer term coefficients in every homogeneous degree are omitted since they are no longer relevant.

Remark 2. *It is obvious that, under the mentioned geometric conditions of Theorem 3, the total degree of freedom d in the choice of real coefficients in (26) corresponding to the homogeneous degree $n = 3, 5, \dots$ is $d = \frac{1}{2}(n+3)$. Similar calculations allow to estimate and compare the number of numerical procedures which are needed with and without additional information about symmetries of the considered domain Ω.*

3 A numerical experiment

We now apply Corollary 1 to the example mentioned before, i.e., to the mapping of the interior of the oblate ellipsoid $\mathcal{E}_\lambda,\ (0 \leq \lambda < 1)$, defined by

$$x_0 = (1+\lambda)\cos s,\ x_1 = 2(1-\lambda)\sin s \cos t,\ x_2 = 2(1-\lambda)\sin s \sin t \tag{27}$$

with $0 \leq s \leq \pi$ and $0 \leq t < 2\pi$, into the interior of a ball $\mathcal{B}$.

In this case the coefficients in (25) depend on the (sufficiently small) parameter λ, i.e. $a^l_{(n-k,k)} = a^l_{(n-k,k)}(\lambda)$. Kantorovich's method makes use of the approximation of the mapping function f through its approximation on the boundary surfaces, i.e., we define the coefficients $a^l_{(n-k,k)}(\lambda)$ of $f = \varphi(z_1, z_2, \lambda)$ up to a certain degree n by considering the mapping of $\mathcal{E}_\lambda$ to $\partial\mathbf{B}_\varrho$ with some radius ϱ.

Since we ask for the transformation into a ball, we have to look for the value of $|\varphi(z_1, z_2, \lambda))|^2$ on the surface $z(s,t,\lambda)$ as described in the beginning of subsection 2.2.

It is evident that the multiplication of $\varphi(z_1, z_2, \lambda)$ by its conjugate, together with (27), leads to an expression of the general form:

$$|\varphi(z(s,t,\lambda))|^2 = c_0(\lambda) + \sum_{i,j,k,l} c_{(i,j,k,l)}(\lambda)\ \sin^i s \cos^j s \sin^k t \cos^l t.$$

Due to the fact that on the sphere (i.e. on the boundary of $\mathbf{B}$), the value of $|\varphi(z_1, z_2, \lambda))|^2$ should be constant and equal to ϱ^2, we see that this development of $|\varphi(z(s,t,\lambda))|^2$ results in a nonlinear system of algebraic equations, with $a^l_{(n-k,k)}(\lambda)$ as unknowns, given by

$$c_{(i,j,k,l)}(\lambda) = 0 \tag{28}$$

and, consequently, we have

$$c_0(\lambda) = \varrho^2. \tag{29}$$

We illustrate now the considered approximation by a simple polynomial of degree 3, i.e.

$$\begin{aligned}\varphi_3(z_1, z_2, \lambda) \quad &= \quad \frac{1}{2}(z_1 e_1 + z_2 e_2) + (\alpha_{(3,0)}\ z_1^3 e_1 + \\ &\quad +3\beta_{(3,1)}\ z_1^2 \times z_2^1 e_2 + 3\beta_{(3,1)}\ z_1^1 \times z_2^2 e_1 + \alpha_{(0,3)}\ z_2^3 e_2).\end{aligned}$$

Solving the corresponding system (28) by using the powerful Maple-Quatpackage from [5], we get

$$\begin{cases} \alpha_{(3,0)}(\lambda) = \frac{(59\lambda^4 - 180\lambda^3 + 290\lambda^2 - 180\lambda + 59)\lambda}{12(\lambda^3 - 3\lambda^2 + 3\lambda - 1)(\lambda - 3)(1+\lambda)^3} \\ \beta_{(3,1)}(\lambda) = -\frac{(7 - 2\lambda + 7\lambda^2)\lambda}{12(1+\lambda)(\lambda^4 - 6\lambda^3 + 12\lambda^2 - 10\lambda + 3)} \\ \alpha_{(0,3)}(\lambda) = \frac{(1+\lambda)\lambda}{4(\lambda^4 - 6\lambda^3 + 12\lambda^2 - 10\lambda + 3)} \end{cases}$$

and the radius ϱ of the ball $\mathbf{B}_\varrho$ is obtained as

$$\varrho = \sqrt{\frac{9\lambda^4 - 12\lambda^3 + 22\lambda^2 - 12\lambda + 9}{(\lambda - 3)^2}}.$$

To illustrate the capacity of the package we show in the following figures the approximations of $\mathbf{B}_\varrho$ for different choices of λ

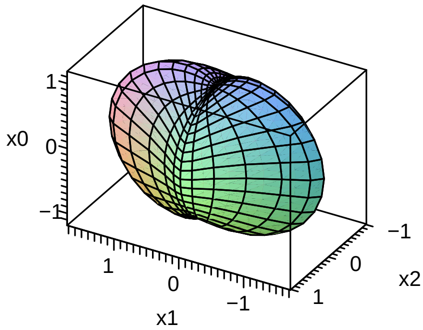

Figure 1: Image in the case of $\lambda = 0.1$

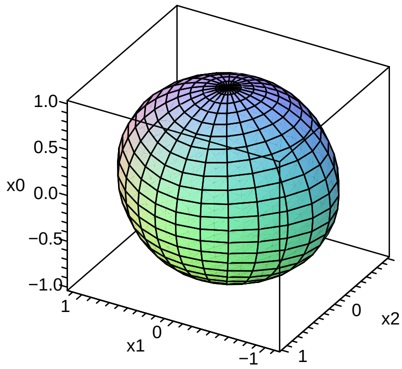

Figure 2: Image in the case of $\lambda = 0.01$

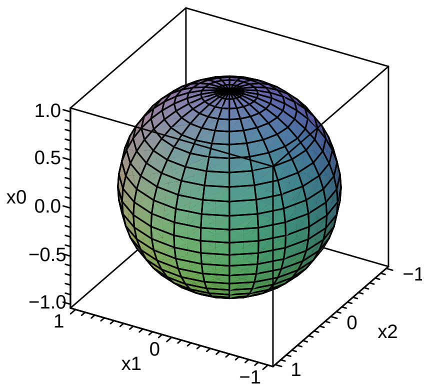

Figure 3: Image in the case of $\lambda = 0.001$

Summing up, we would like to stress that the presented results are not more than a attempt to call attention to a systematic study of monogenic functions as maps between 3D- (or nD -) domains by methods that are almost analogous to the complex case.

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