p-q Theory Power Components Calculations

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Abstract — **The "Generalized Theory of the Instantaneous Reactive Power in Three-Phase Circuits", proposed by *Akagi et al.*, and also known as the p-q theory, is an interesting tool to apply to the control of active power filters, or even to analyze three-phase power systems in order to detect problems related to harmonics, reactive power and unbalance.**

In this paper it will be shown that in three phase electrical systems the instantaneous power waveform presents symmetries of 1/6, 1/3, 1/2 or 1 cycle of the power system fundamental frequency, depending on the system being balanced or not, and having or not even harmonics (interharmonics and subharmonics are not considered in this analysis). These symmetries can be exploited to accelerate the calculations for active filters controllers based on the p-q theory. In the case of the conventional reactive power or zero-sequence compensation, it is shown that the theoretical control system dynamic response delay is zero.

Index Terms — **p-q Theory, Active Power Filters, Digital Controller, Sliding Window, Power Quality.**

I. INTRODUCTION

In 1983 *Akagi et al.* [1, 2] proposed a new theory for the control of active filters in three-phase power systems called "Generalized Theory of the Instantaneous Reactive Power in Three-Phase Circuits", also known as "Theory of Instantaneous Real Power and Imaginary Power", or "Theory of Instantaneous Active Power and Reactive Power", or "Theory of Instantaneous Power", or simply as "p-q Theory". The theory was initially developed for three-phase three-wire systems, with a brief mention to systems with neutral wire. Later, *Watanabe et al.* [3] and *Aredes et al.* [4] extended it to three-phase four-wire systems (systems with phases *a*, *b*, *c* and neutral wire).

Since the p-q theory is based on the time domain, it is valid both for steady-state and transient operation, as well as for generic voltage and current waveforms, allowing the control of the active filters in real-time.

Another advantage of this theory is the simplicity of its calculations, since only algebraic operations are required. The only exception is in the separation of some power components in their mean and alternating values. However, as it will be shown in this paper, it is possible to exploit the symmetries of the instantaneous power waveform for each specific power system, achieving a calculation delay that can be as small as 1/6 and never greater than 1 cycle of the power system frequency. It is also shown that calculations for reactive power and zero-sequence compensation do not introduce any delay.

Furthermore, it is possible to associate physical meaning to the p-q theory power components, which eases the understanding of the operation of any three-phase power system, balanced or unbalanced, with or without harmonics.

II. p-q THEORY POWER COMPONENTS

The p-q theory implements a transformation from a stationary reference system in *a-b-c* coordinates, to a system with coordinates *α-β-0*. It corresponds to an algebraic transformation, known as *Clarke* transformation [5], which also produces a stationary reference system, where coordinates *α-β* are orthogonal to each other, and coordinate *0* corresponds to the zero-sequence component. The zero-sequence component calculated here differs from the one obtained by the symmetrical components transformation, or *Fortescue* transformation [6], by a $\sqrt{3}$ factor.

The voltages and currents in *α-β-0* coordinates are calculated as follows:

$$\begin{bmatrix} v_0 \\ v_\alpha \\ v_\beta \end{bmatrix} = T \cdot \begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix} \qquad \begin{bmatrix} i_0 \\ i_\alpha \\ i_\beta \end{bmatrix} = T \cdot \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix}$$

$$\text{where,} \quad T = \sqrt{\frac{2}{3}} \cdot \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 1 & -1/2 & -1/2 \\ 0 & \sqrt{3}/2 & -\sqrt{3}/2 \end{bmatrix} \tag{1}$$

The p-q theory power components are then calculated from voltages and currents in the *α-β-0* coordinates. Each component can be separated in its mean and alternating values (see Fig. 1), which present physical meanings:

A. Instantaneous Zero-Sequence Power (p_0)

$$p_0 = v_0 \cdot i_0 = \bar{p}_0 + \tilde{p}_0 \tag{2}$$

$\bar{p}_0$ – Mean value of the instantaneous zero-sequence power. It corresponds to the energy per time unity that is transferred from the power source to the load through the zero-sequence components of voltage and current.

$\tilde{p}_0$ – Alternating value of the instantaneous zero-sequence power. It means the energy per time unity that is exchanged between the power source and the load through the zero-sequence components of voltage and current.

The zero-sequence power exists only in three-phase systems with neutral wire. Moreover, the systems must have

both unbalanced voltages and currents, or the same third order harmonics, in both voltage and current, for at least one phase. It is important to notice that $\bar{p}_0$ cannot exist in a power system without the presence of $\tilde{p}_0$ [3]. Since $\tilde{p}_0$ is clearly an undesired power component (it only exchanges energy with the load, and does not transfer any energy to the load), both $\bar{p}_0$ and $\tilde{p}_0$ must be compensated.

B. Instantaneous Real Power (p)

$$p = v_\alpha \cdot i_\alpha + v_\beta \cdot i_\beta = \bar{p} + \tilde{p} \tag{3}$$

$\bar{p}$ – Mean value of the instantaneous real power. It corresponds to the energy per time unity that is transferred from the power source to the load, in a balanced way, through the a-b-c coordinates (it is, indeed, the only desired power component to be supplied by the power source).

$\tilde{p}$ – Alternating value of the instantaneous real power. It is the energy per time unity that is exchanged between the power source and the load, through the a-b-c coordinates. Since $\tilde{p}$ does not involve any energy transference from the power source to load, it must be compensated.

C. Instantaneous Imaginary Power (q)

$$q = v_\beta \cdot i_\alpha - v_\alpha \cdot i_\beta = \bar{q} + \tilde{q} \tag{4}$$

$\bar{q}$ – Mean value of instantaneous imaginary power.

$\tilde{q}$ – Alternating value of instantaneous imaginary power.

The instantaneous imaginary power, q, has to do with power (and corresponding undesirable currents) that is exchanged between the system phases, and which does not imply any transference or exchange of energy between the power source and the load.

Rewriting equation (4) in a-b-c coordinates the following expression is obtained:

$$q = \frac{\left[(v_a - v_b) \cdot i_c + (v_b - v_c) \cdot i_a + (v_c - v_a) \cdot i_b\right]}{\sqrt{3}} \tag{5}$$

This is a well known expression used in conventional reactive power meters, in power systems without harmonics and with balanced sinusoidal voltages. These instruments, of the electrodynamic type, display the mean value of equation (5). The instantaneous imaginary power differs from the conventional reactive power, because in the first case all the harmonics in voltage and current are considered.

In the special case of a balanced sinusoidal voltage supply and a balanced load, with or without harmonics, $\bar{q}$ is equal to the conventional reactive power ($\bar{q} = 3 \cdot V \cdot I_1 \cdot \sin \phi_1$).
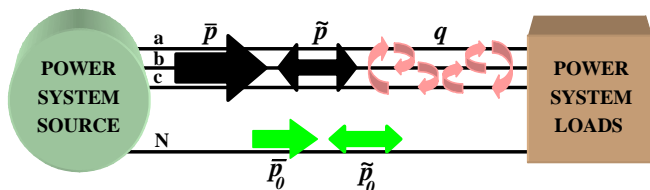

Fig. 1 p-q theory power components

It is also important to note that the three-phase instantaneous power (p_3) can be written in both coordinates systems, a-b-c and α-β-0, assuming the same value:

$$p_3 = v_a \cdot i_a + v_b \cdot i_b + v_c \cdot i_c = p_a + p_b + p_c \tag{6}$$

$$p_3 = v_\alpha \cdot i_\alpha + v_\beta \cdot i_\beta + v_0 \cdot i_0 = p + p_0 \tag{7}$$

Thus, to make the three-phase instantaneous power constant, it is necessary to compensate the alternating power components $\tilde{p}$ and $\tilde{p}_0$. Since, as seen before, it is not possible to compensate only $\tilde{p}_0$, all zero-sequence instantaneous power must be compensated.

Moreover, to minimize the power system currents, the instantaneous imaginary power, q, must also be compensated.

The compensation of the p-q theory undesired power components ($\tilde{p}$, p_0 and q) can be accomplished with the use of an active power filter. The dynamic response of this active filter will depend on the time interval required by its control system to calculate these values.

III. EXPLOITING POWER SYMMETRIES

Once the p-q theory power components are calculated it is important to decompose the instantaneous real power (p) in its mean value ($\bar{p}$) and alternating value ($\tilde{p}$), since only the second quantity must be compensated [7, 8].

The dynamic response of an active power filter controller depends mainly on the time required to separate $\bar{p}$ and $\tilde{p}$. With a digital controller it is possible to exploit the instantaneous real power waveform symmetries, so that only samples of a fraction of the period of the measured voltage and current waveforms are required to separate the referred components. In fact, only 1/6, 1/3, 1/2 or 1 cycle is enough, depending on the system being balanced or not, and having or not even harmonics, as shown next. These symmetries can be exploited by a digital control system that makes use of a sliding window with a number of samples corresponding to the symmetry period.

A. Symmetry of 1/6 Cycle

In power systems with balanced voltages and currents, without even harmonics, the real instantaneous power presents symmetry equal to 1/6 of the fundamental period (3.33 ms to a fundamental frequency of 50 Hz). This means that the frequency of the instantaneous real power waveform is 6 times the fundamental frequency.

B. Symmetry of 1/3 Cycle

In power systems with balanced voltages and currents, but with even harmonics in voltages and/or currents, the real instantaneous power presents symmetry equal to 1/3 of fundamental frequency (6.66 ms to a fundamental frequency of 50 Hz). This means that the frequency of instantaneous real power waveform is 3 times the fundamental frequency.

C. Symmetry of 1/2 Cycle

In power systems with unbalanced voltages and/or currents, without even harmonics, the real instantaneous power presents symmetry equal to 1/2 of fundamental frequency (10 ms to a fundamental frequency of 50 Hz). This means that the frequency of instantaneous real power waveform is 2 times the fundamental frequency.

D. Symmetry of 1 Cycle

In power systems with unbalanced voltages and/or currents, and with even harmonics in voltages and/or currents, the real instantaneous power presents symmetry equal to the fundamental frequency (20 ms to a fundamental frequency of 50 Hz).

Fig. 2 shows the results of the instantaneous real power mean value calculations exploiting the four different types of symmetry previously described, and the using of a sliding window approach.. Assuming that the power system does not presents interharmonics or subharmonics (as result of *flicker* problems, for instance), one cycle is enough to calculate the value of $\bar{p}$ for all cases.

For some active filters controllers it is important to ob-
tain the mean value of the instantaneous zero-sequence power ($\bar{p}_0$) [4]. It can be observed that, in the cases where p_0 exists, it presents a symmetry equal or smaller than the symmetry of the instantaneous real power (p). Therefore, the same sliding window time interval used to determine $\bar{p}$ can be applied to obtain $\bar{p}_0$.

IV. COMPARISON WITH AN ANALOG CONTROLLER

Fig. 3 (a) shows the time necessary to obtain the value of $\bar{p}$ when using a 4th order Butterworth low-pass filter, with cut-off frequency of 50 Hz, when the power system operating conditions are the same of those presented in Fig. 2 (a) (balanced system without even harmonics). It can be seen that the digital solution, using a sliding window and exploiting the 1/6 cycle power symmetry, is much faster (3.33 ms) than the Butterworth (that takes about 40 ms to obtain the correct value of $\bar{p}$). Fig. 3(b) illustrates that this same Butterworth filter would not be suitable to get the value of $\bar{p}$ in an unbalanced power system without even harmonics, like the one of Fig. 2(c).
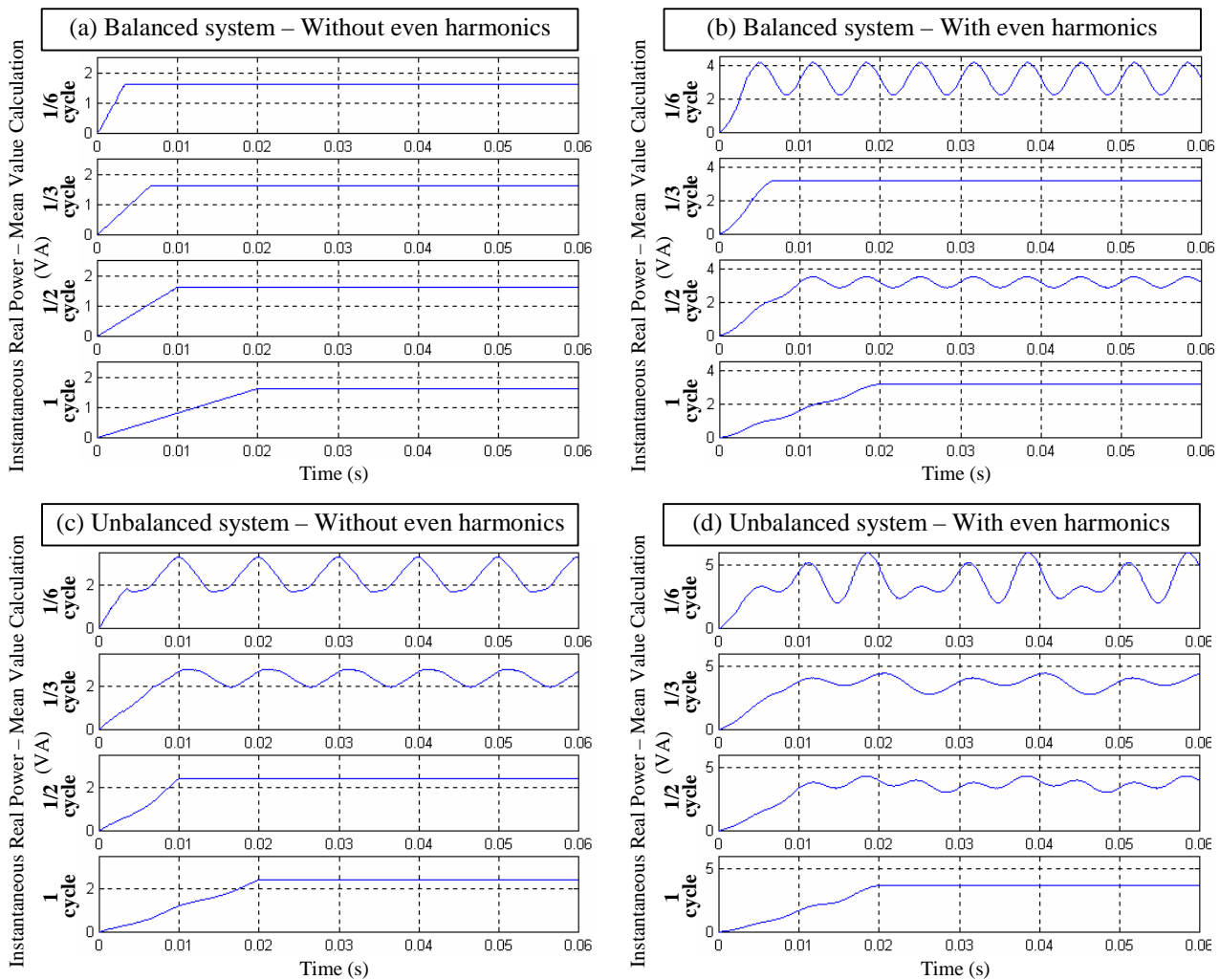

Fig. 2 Calculation of $\bar{p}$ exploiting symmetries with the use of a sliding window for different power system conditions
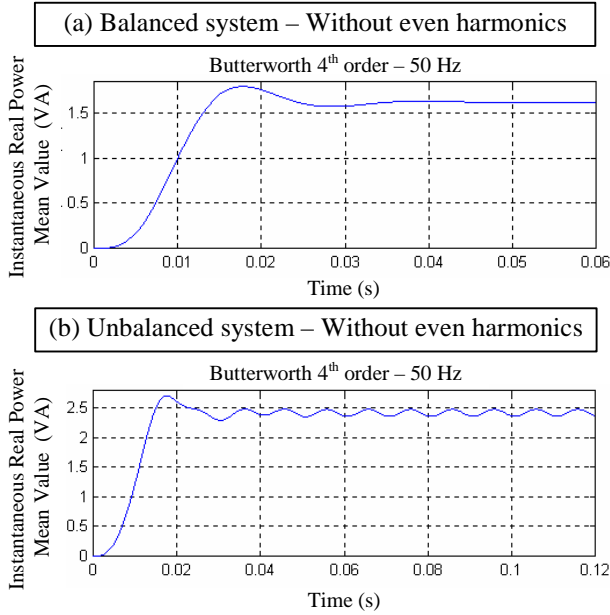
Fig. 3 Obtaining of $\bar{p}$ with a Butterworth filter

An active power filter with a control system that presents a fast response, like the one based on the p-q theory, has basically two advantages:

− It presents a good dynamic response, producing the correct values of compensating voltage or current in a short time after variations of the power system operating conditions. If this response is good enough, the active filter can be used in harsh electrical environments, where the loads or the voltage system suffer intense and numerous variations. An active filter with such a controller has also the capacity of performing well in power systems with some types of subharmonics, like *flicker*. It may happen because *flicker* usually has a period (not necessarily constant) of some or even many power system cycles, and the proposed controller can respond in a time interval always inferior or equal to one cycle.

− If the active filter has a fast control system, its energy storage element will suffer less to compensate the power system parameters variation.

Fig. 4 shows the voltage at the capacitor used in the DC side of the inverter of a shunt active power filter, like the one presented in Fig. 5, when a load change occurs.
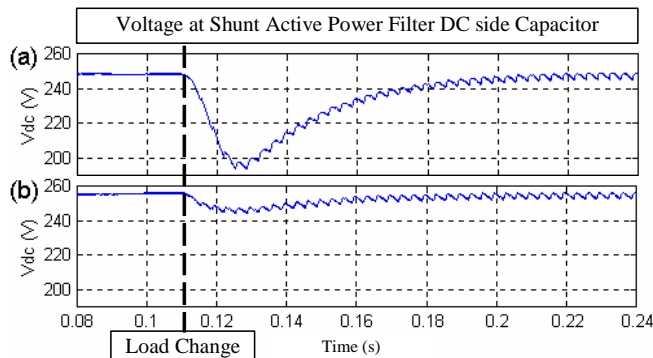

Fig. 4 Active filter capacitor voltage when load changes:

(a) When $\bar{p}$ is obtained with a Butterworth filter;

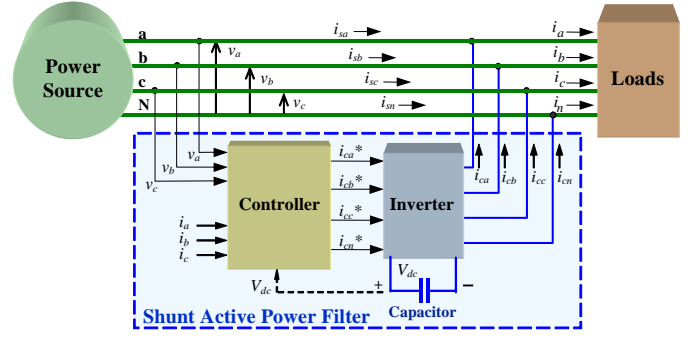(b) When $\bar{p}$ is calculated with a sliding window

Fig. 5 Block diagram of a shunt active power filter

When there is a load change, the shunt active filter can act incorrectly, delivering (as shown in this case) or accumulating energy. If it delivers energy, the capacitor voltage falls. This behaviour must be considered, and is one of the factors that influence the sizing of the capacitor. Fig. 4(a) refers to the behaviour of the capacitor voltage when a certain new load is added to the power system, and the control system uses the already described Butterworth filter to obtain the value of $\bar{p}$. Fig 4 (b) refers to the same situation, but with a digital control system that calculates $\bar{p}$ with a 1/6 cycle symmetry sliding window. The second solution presents advantages: the active filter can compensate greater load variations, or, under the same operating conditions, the capacitor can be downsized.

V. SIMULATED EXAMPLE OF COMPENSATION

Fig. 6 shows an example of compensation performed by a shunt active power filter with control system based on the p-q theory. The simulation was made with *Matlab/Simulink*

The voltages of this power system are balanced and sinusoidal. The loads are unbalanced and present 2nd order harmonic currents, so a sliding window with symmetry of 1 cycle was chosen to dynamically calculate the value of $\bar{p}$. Initially there are only two single-phase loads: two half-wave rectifiers with a resistive load. Then a three-phase full-wave rectifier with RL load is turned-on (Fig. 6 (a)).

Fig. 6 (b) shows that, by the action of the shunt active filter the source currents become sinusoidal and balanced.

The active filter controller compensates instantaneously the neutral wire current, and this current is kept always equal to zero in the source, as seen in Fig. 6 (c).

The controller also compensates the instantaneous imaginary power immediately, maintaining a null value for q at the source (Fig. 6 (e)).

However, the instantaneous real power (p) must be separated in its mean and alternating value, and the sliding window takes 1 cycle (20 ms) to perform this task with this power system operating conditions. It can be seen in Fig. 6 (f) that, after the load change, while the calculation of the new correct value of $\bar{p}$ is not finished, the source delivers less energy to the loads than it should do. So, during this time interval, the capacitor of the active filter must supply energy to the load, as already explained.
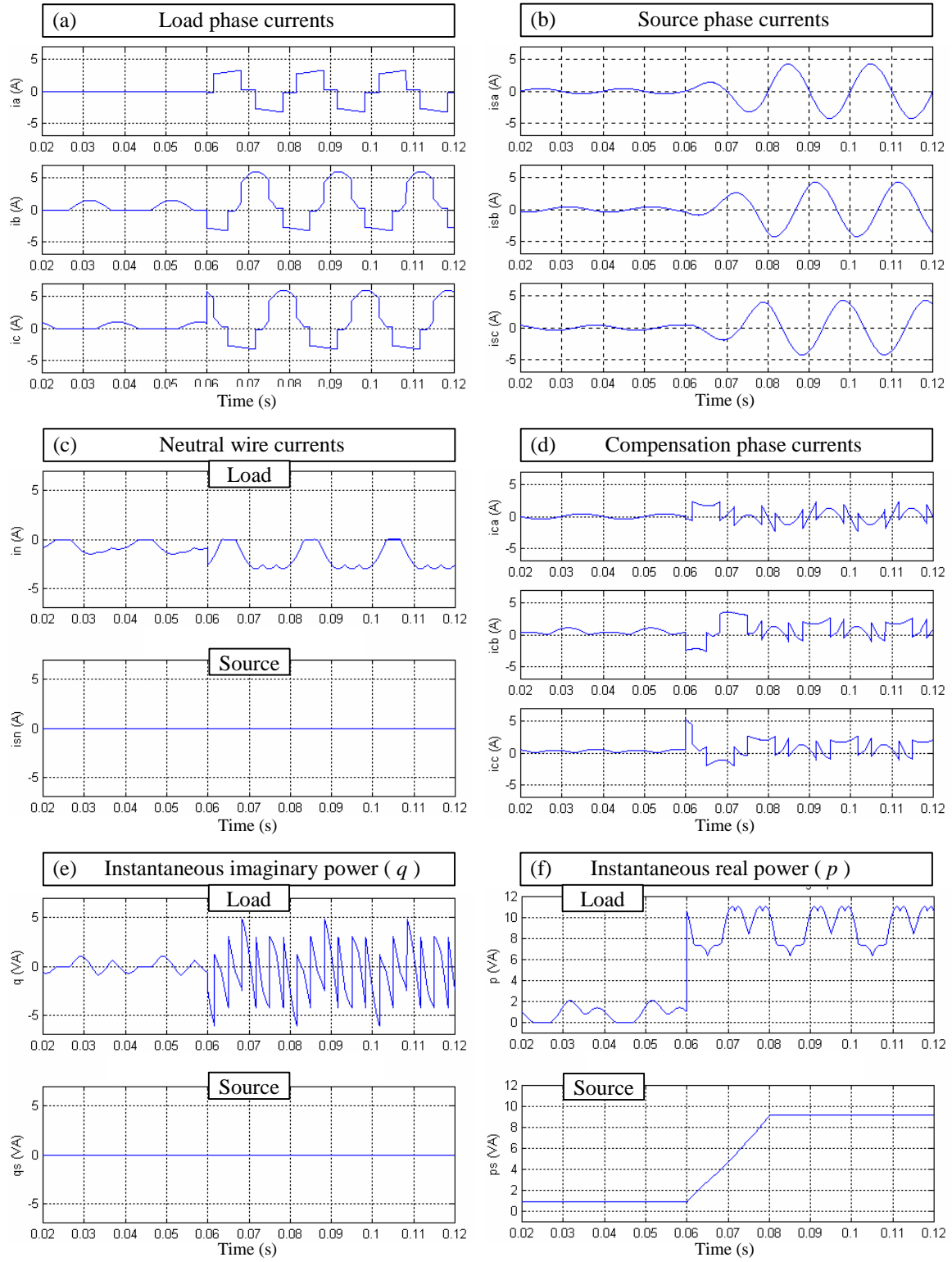
Fig. 6 Operation Example of a shunt active filter with control system based on the p-q theory and with sliding window

VI. EXPERIMENTAL RESULTS

A shunt active filter control system based on the p-q theory, where $\bar{p}$ is calculated using a sliding window exploiting 1/6 cycle symmetry, was implemented with an *Intel* 80296SA microcontroller. The number of points per cycle is 300. So, the sliding window has 50 points (300/6).

Fig. 7 shows three measured waveforms for phase a: the compensation current reference ($i_{ca}*$) – in the upper window, phase to neutral voltage (v_a), and source current (i_{sa}). Initially only a three-phase balanced RL load is switched-on. Later a three-phase rectifier with RL load is also connected to the power system. It can be seen that $i_{ca}*$ is modified immediately after the load changes. This happens be-

cause the instantaneous imaginary power variation can be compensated without any delay. It is also possible to notice that after a short time interval, necessary to calculate the new value for $\overline{p}$, the compensation current reference waveform reaches steady state again.

The waveforms in Fig. 7 were obtained with the active filter inverter turned-off.

VII. CONCLUSIONS

The paper briefly described the calculations and physical meaning of the p-q theory power components.

In three-phase electrical systems the instantaneous real power presents symmetries of 1/6, 1/3, 1/2 or 1 cycle of the power system fundamental frequency, depending on the system being balanced or not, and having or not even harmonics. Since the instantaneous zero-sequence power presents an equal or smaller symmetry period, the instantaneous three-phase power, which is the sum of p and p_0 (eq. 7), presents the some kind of symmetries.

This paper suggests the utilization of a sliding window, in a digital control system, to calculate the mean value of the instantaneous real power, exploiting the symmetries described above.

The importance of a fast response for the active filter control system is commented. Besides the improvement of the dynamic response of the filter, it can also contribute to a reduction in its cost, since the active filter capacitor can be smaller.

Experimental results prove the good dynamic response of the active filter controller based on the p-q theory, and using a sliding window to calculate $\overline{p}$.

This paper did not intend to discuss practical problems related with the implementation of digital control systems, like the delays imposed by analog-to-digital conversion times and processing times. However, it is certain that these kinds of problems are becoming each time less important, since nowadays there are already available fast ADCs (with conversion times of 80 ns, for instance) and microcontrollers/DSPs operating at frequencies as high as 150 MHz.
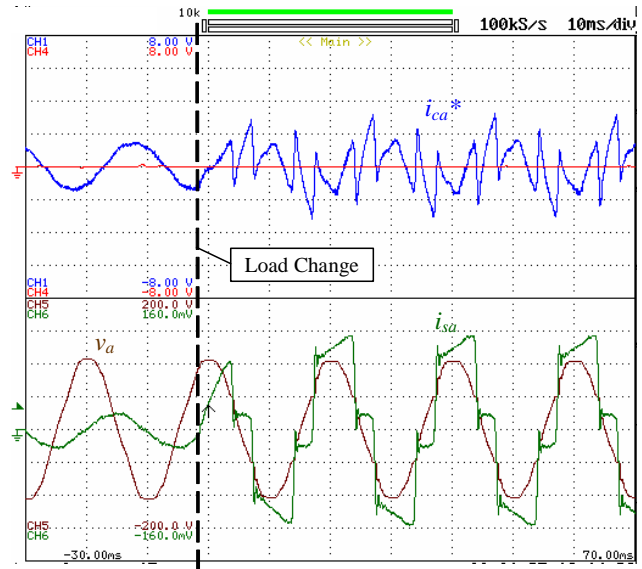

Fig. 7 Experimental results with load changing

IX. REFERENCES

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