

Solving Systems of Inequalities and Equalities by a Nonmonotone Hybrid Tabu Search Method

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Abstract. This paper presents a derivative-free nonmonotone hybrid tabu search to compute a solution of overdetermined systems of inequalities and equalities through the global optimization of an appropriate merit function. The proposed algorithm combines global and local searches aiming to reduce computational effort. Preliminary numerical results show the effectiveness of the combined heuristic.

Keywords: Inequalities and Equalities, Tabu Search, Nonmonotone Reduction

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INTRODUCTION

The primary goal of the paper is to show that a straightforward heuristic used in global optimization can be effective and as efficient as classical methods in solving systems of nonlinear inequalities and equalities. In this paper, we examine the behavior of a derivative-free nonmonotone hybrid tabu search approach when solving nonlinear systems of p inequalities and equalities and n variables, where $p > n$. We assume that the overdetermined system has the form:

$$\begin{cases} f_i(x) \leq 0, & i = 1, \dots, m \\ f_i(x) = 0, & i = m + 1, \dots, p \end{cases} \quad (1)$$

where each $f_i : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}$ and Ω is a closed convex set. The motivation of this work comes mainly from the detection of feasibility in nonlinear optimization problems. This type of systems appear frequently in bound constrained variational inequalities and nonlinear complementarity problems [1]. Classical methods for solving problem (1) use Newton-type methods [2]. In [3], the authors propose a method that combines the use of a modified Newton step and a conventional first-order step and in [4], a solution of (1) is obtained by applying successively a Levenberg-Marquardt algorithm to solve smoothed versions of the problem. The authors use first-order derivatives to construct a smooth function to approximate the objective function of the problem. Smooth reformulation of (1) have also been proposed in [5]. Our strategy however is to solve (1) by reformulating the inequalities into equalities and yielding:

$$\begin{cases} f_i(x) \leq 0 & i = 1, \dots, m \\ f_i(x) = 0, & i = m + 1, \dots, p \end{cases} \Leftrightarrow \begin{cases} F_i(x) \equiv \max\{0, f_i(x)\} = 0, & i = 1, \dots, m \\ F_i(x) \equiv f_i(x) = 0, & i = m + 1, \dots, p \end{cases} \quad (2)$$

Since some functions in the equivalent system (2) are nonsmooth, Newton's method cannot be directly applied to solve it. In this paper, we present a derivative-free hybrid tabu search strategy to obtain a solution of systems of inequalities and equalities by solving the equivalent system of equations alone (2). The most famous techniques to solve nonlinear equations are based on the Newton's method [6]. They are computationally expensive since the Jacobian matrix with analytical first derivatives and the solution of a system of linear equations may be required at each iteration. Quasi-Newton methods are less expensive than the Newton's method since they avoid computing derivatives and solving a full linear system of equations at each iteration [7]. The problem of solving a nonlinear system of equations can be naturally formulated as a global optimization problem. Problem (2) is equivalent to

$$\min_{x \in \Omega \subset \mathbb{R}^n} M(x) \equiv \sum_{i=1}^p F_i(x)^2, \quad (3)$$

in the sense that they have the same solutions. These required solutions are the global minima, and not just the local minima, of the function $M(x)$, known as merit function, in the set Ω . Problem (3) is similar to the usual least squares

problem for which many iterative methods have been proposed. They basically assume that the objective function is twice continuously differentiable. However, the objective M in (3) is only once differentiable even if all $f_i, i = 1, \dots, p$ are twice continuously differentiable. Thus, methods for solving the least squares problem cannot be directly applied to solve (3). Further, local optimization techniques guarantee globality only under certain convexity assumptions. Preventing premature convergence to a local solution, while trying to compute a global optimum, is a very important property of the global solution methods.

The Tabu Search (TS) method is a heuristic for global optimization and aims at exploring all the search space for global optimal solutions. It has the ability to guide the search out of local optima and explore new regions. It is an iterative procedure that maintains a list of the movements most recently made, avoiding in subsequent iterations the execution of movements that lead to solutions already known to have been visited. TS was developed primary for solving combinatorial problems and introduced in continuous optimization in [8]. Like most heuristics for global optimization, TS converges slowly near the global solution. This issue is normally addressed by incorporating a classical local search strategy into the main algorithm. The Directed Tabu Search (DTS) method of Hedar and Fukushima [9] is a variant of the TS that implements a local search in the final stage of the process, therein denoted by the intensification search. Another recent strategy that proved to be very effective in solving systems of equations [10] and inequalities [11] uses a combination of global and local searches. Several conditions to choose which search is the most appropriate at each iteration have already been tested [11, 10]. They rely on the behavior of the merit function. If a sufficient reduction in the merit function is observed, when compared with the merit function value at the previous iteration, a fast downhill progress has been detected and a local search is implemented to refine the search in the vicinity of a global solution. Otherwise, the region does not look to be promising and a global diversifying search is to be carried out looking for a promising region with a global minimum. Numerical experiments have shown that in this combined version, the global search dominates and the iterative process turns out to be expensive.

In general, a monotone decreasing sequence of merit function values is required when global convergence is to be guaranteed in an optimization context. Relaxing the sufficient descent condition on the merit function allows that iterates may be selected even if they do not provide an improvement on the merit function. As a consequence, iterates are accepted more often and a reduction on the number of function evaluations is obtained. We brought this idea to the combined algorithm and use a nonmonotone condition on the merit function to choose between local and global search. Furthermore, our strategy is to use a simple and effective local search. The well-known coordinate directions search procedure is used. During global search, a simplified version of the DTS method is used.

The remaining paper is organized as follows. Firstly, the main steps of the nonmonotone combination of global and local searches are described. Then, some numerical results and remarks are presented.

NONMONOTONE COMBINED GLOBAL AND LOCAL SEARCHES

The goal of a nonmonotone strategy is to relax the condition that is usually used to control the decreasing behavior of merit function values along the iterative process, i.e., the progress towards a minimizer of the merit function [12]. The Algorithm 1 contains the pseudo-code of the herein presented nonmonotone combined global search and local search (nm-CGSLS) method.

Algorithm 1 nm-CGSLS algorithm

Given: $x^0, \eta^* > 0, \eta_0 > \eta^*, 0 < \gamma_2 < 1, 0 < \gamma_1 \leq 1, k_{\max} > 0, s_{\max} \geq 0$

Set $k = 0, s^k = 0, \text{flag} = 1$

While $M(x^k) > \eta^*$ and $k < k_{\max}$ **do**

If $\text{flag} = 1$ **then** compute x^{k+1} by *local search* **else** compute x^{k+1} by *global search* **endif**

If $M(x^{k+1}) \leq \gamma_1 \max_{0 \leq j \leq s^k} M(x^{k-j})$ **then** set $\text{flag} = 1$ **else** set $\text{flag} = 0$ **endif**

Set $k = k + 1, \eta_k = \max\{\eta^*, \gamma_2 \eta_{k-1}\}, s^k = \min\{s^{k-1} + 1, s_{\max}\}$

Endwhile

It is required that the solution x^{k+1} obtained by either the *local search* or the *global search* be an η_k -approximation, where the sequence of η_k values decreases and converges to η^* : $\eta_k = \max\{\eta^*, \gamma_2 \eta_{k-1}\}$ for $0 < \gamma_2 < 1$. This means that as the iterative process proceeds, higher accurate solutions are required. The condition

$$M(x^{k+1}) \leq \gamma_1 \max_{0 \leq j \leq s^k} \{M(x^{k-j})\}, \text{ for } 0 < \gamma_1 \leq 1, \text{ where } s^k = \min\{s^{k-1} + 1, s_{\max}\}, k \geq 1 \text{ and } s^0 = 0, \quad (4)$$

that requires a nonmonotone reduction of the merit function to choose the *local search*, allows separate local intensification of the DTS more often. As a consequence, the initial approximation of the next iteration will be closer to the optimal solution and a reduction on the overall number of function evaluations is observed. The condition in (4) compares the merit function value at the current iterate with the maximum value of the merit function attained within the last $s^k + 1$ iterations. s^k is a nondecreasing integer, bounded by some fixed integer s_{\max} .

Local Search. The local search procedure of the Algorithm 1 is the classical coordinate search method [13]. At each iteration, a pattern search type method uses a pattern of points to search for a minimizer. At least $n + 1$ points are provided by the pattern, where n is the number of variables. One of these points is the current point, x^j , where j represents the iteration counter in this inner iterative process, and one of the others, denoted by trial point, y^i , is generated along a search direction (starting from the current point) with a certain step size $\Delta^j > 0$:

$$y^i = x^j + \Delta^j d^i$$

where d^i is the search direction chosen from a finite set \mathcal{D} of positive spanning directions in \mathbb{R}^n . The most used set contains the $2n$ coordinate directions, defined as the positive and negative unit coordinate vectors $\mathcal{D} = \{e_1, \dots, e_n, -e_1, \dots, -e_n\}$. The most important property is that at least one of the coordinate directions is a descent direction for M , so long as the current point is not a stationary point of M . In this coordinate search method, all $2n$ points $y^i, i = 1 \dots, 2n$ are computed and evaluated. The point with the smallest merit function value is chosen, y^{best} , and compared with the merit function at x^j . If this search fails to generate a trial point that is better than the current point, the iteration is called *unsuccessful*, the step size Δ^j is halved, in order to refine the search hereafter, and $x^{j+1} \leftarrow x^j$. The step size is then compared to a specified stopping tolerance. Since an η_k -approximate solution is required, when the step size falls below η_k , the local search terminates and the current point is the required x^{k+1} . However, if at the end of each iteration, a simple decrease in M is verified, then the iteration is *successful*, Δ^j is not changed and $x^{j+1} \leftarrow y^{best}$.

Global Search. The global search in the Algorithm 1 is based on the DTS heuristic of Hedar and Fukushima [9]. The DTS method is composed of three main search procedures: exploration, diversification and intensification. The main loop of the DTS method consists of the exploration and diversification search procedures. The exploration search aims to explore the search space Ω and uses a direct search method to be able to stabilize the search, in particular in the vicinity of a local minimum. Cycling is prevented by the standard Tabu List, as well as by other four TS memory elements: the multi-ranked Tabu List, the Tabu Region, the Semi-Tabu Region and the Visited Region List (VRL). The reader is referred to [9] for details. The diversification procedure aims to generate new trial points outside the previously visited regions. The VRL works as a diversification tool and is used to direct the search towards regions that have not been visited in the search space. When one of the best obtained trial solutions is sufficiently close to a global minimum, or its value has not been changed for a specified number of iterations, the DTS algorithm leaves the exploration and diversification search procedures and enters the intensification procedure. Our implementation of the DTS heuristic also uses the classical coordinate search method in this intensification phase.

NUMERICAL RESULTS AND REMARKS

In this section, we report some preliminary numerical results to test the performance of the nm-CGSLs algorithm. The results of these experiments were obtained in a personal computer with an AMD Turion 2.20 GHz processor and 3 GB of memory. Due to the stochastic nature of the algorithm, each problem was run 30 times and the best of the 30 solutions is shown. We tested the nm-CGSLs algorithm with $s_{\max} = 5$ and $\gamma_1 = 0.995$. The values set to the other parameters of the algorithm are: $\eta_0 = 0.1$, $\gamma_2 = 0.1$, $\Delta^0 = 0.01$ and $10n$ maximum iterations in DTS and coordinate search. If a solution is found with a merit function value less or equal to $\eta^* = 10^{-6}$ the algorithm stops. However, if this last condition fails to be verified, we choose to stop the algorithm when the number of iterations exceeds $15n$. We tested the algorithm using five problems and compare our results with those from the literature: Ex. 6.2 and Ex. 6.6 from [5], Ex. 4.4 and Ex. 4.5 from [3] and Ex. 1, a nonlinear complementarity problem, from [1]. The results obtained with different initial points are depicted in Table 1, where ‘ k ’ is the number of iterations, ‘ n_{feval} ’ is the number of function evaluations (each F -vector evaluation counts for one), ‘ $M(x^*)$ ’ is the merit value at the found solution and ‘-’ means that data is not available. The results show that the nm-CGSLs algorithm is able to reach a solution of the problem. We observed that the overall computational requirements are acceptable and reduced and any randomly generated initial approximation is as effective as any other approximation in the vicinity of a minimizer, when converging to a solution. A nonmonotone combined global search and local search algorithm for solving overdetermined nonlinear systems of inequalities and equalities has been presented. The nonmonotone perspective of the algorithm is directly

TABLE 1. Comparative results.

Problem	x^0	nm-CGSLs			results in [5], [3] and [1]	
		k	$nfeval$	$M(x^*)$	k	$nfeval$
Ex. 6.2 ($p = 6, m = p, n = 2$) in [5]	(0,0)	2	11	0	7	9
	(-1,-1)	2	11	0	-	-
	random	2	11	0	-	-
Ex. 6.6 ($p = 3, m = 1, n = 2$) in [5]	(0,0)	7	96	2.0e-7	4	4
	(-1,1)	8	72	1.2e-8	-	-
	random	8	495	3.3e-8	-	-
Ex. 4.4 ($p = 3, m = p, n = 2$) in [3]	(0.5,-6.0)	2	16	0	5	-
	(1,-1)	2	20	0	-	-
	random	2	21	0	-	-
Ex. 4.5 ($p = 10, m = p, n = 7$) in [3]	(3,0,2,-1.5,1.5,5,0)	2	104	0	8	-
	random	7	874	0	-	-
	random	7	874	0	-	-
Ex. 1 ($p = 3, m = 2, n = 2$) in [1]	(1,1)	8	82	3.5e-7	4	9
	(2,2)	8	89	8.8e-7	12	33
	random	8	97	7.0e-7	-	-

related with the condition that chooses the implementation of a local search in detriment of an exploratory global search. Future developments will be focused on further extending the numerical comparisons, and solving the problem (1) by attacking an equivalent inequality constrained global optimization problem.

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