# Musical Composition Without Standard Musical Knowledge

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## Abstract

Musical composition is based on pre-established rules that define relations between the musical notes. In this workshop we would like to give a numerical version of these rules. Musical notes will be regarded as numbers and the usual rules are translated into numerical relations. These new set of rules will be applied in usual composition exercises translated numerically.

## Motivation

When a music student starts learning musical composition he or she usually starts with counterpoint theory. The very first steps of counterpoint are the following: you are given a set of notes and you are supposed to produce a piece of music giving, for each known note, another one, in order to obtain two notes chords. The notes previously given are either the lower notes or the higher notes. More precisely you have to produce the missing notes (the higher if the lower are known, or the lower if the higher are known). There are however some rules that must be obeyed. This is called first species counterpoint.

Basically we can image the final result to be a set of pairs $\{x_i, y_i\}$ where $i = 1, ..., n$, $n$ being the number of notes previously known. The sequence previously known can be either $\{x_i\}$, the lower notes, or the $\{y_i\}$, the higher notes.

It is possible to look at these rules and translate them numerically. We will look at the rules as they are given in [**?**] or [**?**] and try to make it possible for anyone to solve some first species counterpoint exercises.

## Introduction

So what rules are we talking about? The set of rules given here is as shown in [**?**]. The main purpose is to make the participant in the workshop build the possible pairs of notes in order to feel the restrictions musicians feel. But there are other approaches that could have been chosen, namely those given in Tymoczko in [**?**]. In this paper Tymoczko gives a list of admissible progression of chords. In more simple terms Tymoczko list pairs of notes that can appear in sequence. Tymoczko's paper is based on the approach given by Mazzola in [**?**]. Mazzola's approach is very difficult to follow for non-mathematicians or for mathematitians who would like to quickly understand the rules of composition.

Let us start by listing the rules as they appear in [**?**].

We will introduce the rules just as they are stated in order to justify the arithmetic rules shown later. Everything will become clear further ahead so even if the reader does not have any musical knowledge, do not give up.

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Unison, perfect fifths, twelfth and octaves are called perfect consonances. All other chords are called imperfect consonances.

**Rule 1** The commencement must be a perfect concord and the termination also: so that the first bar may be either in fifth, in octave, (or in unison), and the last bar should be simply in octave, or in unison. Fifth is also understood as twelfth and octave as fifteenth.

**Rule 2** The parts should progress always by concords with endeavour to avoid the unison excepting at the first or last bar.

**Rule 3** It is sometimes admissible to let the higher part pass beneath the lower part, always however, taking care that they shall be in concord and not allowing this method to continue too long.

**Rule 4** Several perfect concords of the same denomination should never be permitted to succeed each other, at whatever pitch they may occur (two fifths or two octaves in succession are prohibited).

**Rule 5** Do not pass to a perfect concord by direct movement, except when one of the two proceeds by semitone.

**Rule 6** All movement should be either diatonic or natural in regard to melody; and conjunct movement better suits the style of strict counterpoint then disjunct movement. Accordingly movement of the major and minor second, of the major and minor third of the perfect fourth or the perfect fifth, of the minor sixth and the octave are permitted either in ascending or descending. The movement of superfluous fourth, or tritone of the imperfect fifth and of the major and minor seventh are expressly prohibited either in ascending or descending.

**Rule 7** False relation of the octave, and of the tritone between the parts should be avoided.

**Rule 8** Except at the first bar, and the last, imperfect concords should always be introduced, in the course of composition, in preference to perfect ones.

Let us now write these rules in another way.

We will consider the numerical notation for the usual twelve musical notes:

| | | | |
|---|---|---|---|
| C | 0 | C♯ (or D♭) | 1 |
| D | 2 | D♯(or E♭) | 3 |
| E | 4 | F (or E ♯) | 5 |
| F ♯ | 6 | G | 7 |
| G ♯ | 8 | A | 9 |
| A♯ | 10 | B | 11 |

**Table 1** : *Correspondence between musical notes and numbers*

This numerical notation introduces some ambiguities when we translate numbers into notes. In Table 1 we give some examples showing that each number can correspond to different notes. But, for instance in a piano, all the notes corresponding to the same number will correspond to the same key. This is a modulo 12 numerical representation. If one has two musical notes whose numerical representation is respectively $n$ and $n+12$, they will be played on a piano keyboard exactly one octave appart. Counting all keys black and white they are 12 keys appart. In each exercise we will use 7-note musical scales. For instance if we are using the musical scale C D E F G H A B (C major) then $x$ and $y$ can be one of the following numbers: 0,2,4,5,7,9 or 11 (mod 12). In rigor we should talk about Gregorian modes but this choice of C major keeps things simple.

Essentially there is:

1. a set of rules that for each sequence of note numbers $\{x_i\}$ and $\{y_i\}$ .
2. a set of that gives pairs $\{x_i, y_i\}$ that is admissible.

3. a set of rules that establish the sequence of pairs $\{x_s, y_s\}$, $\{x_{s+1}, y_{s+}\}$ that is admissible

We will demand that $\{x_i\} \leq \{y_i\}$. This will mean that we will not consider the exception situation in rule 3.

## First Set of Rules

The first set of rules can be translated as follows. Each sequence $\{x_i\}$ and $\{y_i\}$ must have consecutive elements that are one of the following:

Let $\{x_s\}$ be an element of the sequence with $s$ neither the first nor the last element, then $\{x_{s+1}\}$ is one of this notes

| | |
|---|---|
| $x_s \pm 1$ | Minor second (ascending or descending) |
| $x_s \pm 2$ | Major second (ascending or descending) |
| $x_s \pm 3$ | Minor third (ascending or descending) |
| $x_s \pm 4$ | Major third (ascending or descending) |
| $x_s \pm 5$ | Perfect fourth (ascending or descending) |
| $x_s \pm 7$. | Perfect fifth (ascending or descending) |
| $x_s \pm 10$. | Minor sixth (ascending or descending) |
| $x_s \pm 12$. | Octave (ascending or descending) |

(We will establish the rules for $\{x_i\}$, for simplicity, but they are the same for both sequences).

This rule translates Cherubini Rule 6. You should use mainly very small steps (minor or major seconds), i.e., consecutive notes on the scale.

## Second Set of Rules

The second set of rules states that the only pairs (musical intervals) allowed are:

| Name of the interval | Numerical correspondence |
|---|---|
| Unison | $\{x, x\}$, $(x = y)$. |
| Minor third | $\{x, x+3\}$, $(y = x+3)$. |
| Major third | $\{x, x+4\}$, $(y = x+4)$. |
| Perfect fifth | $\{x, x+7\}$, $(y = x+7)$. |
| Minor sixth | $\{x, x+8\}$, $(y = x+8)$. |
| Major sixth | $\{x, x+9\}$, $(y = x+9)$. |
| Octave | $\{x, x+12\}$, $(y = x+12)$. |
| Minor tenth | $\{x, x+15\}$, $(y = x+15)$. |
| Major tenth | $\{x, x+16\}$, $(y = x+16)$. |
| Perfect twelfth | $\{x, x+19\}$, $(y = x+19)$. |

This set of rules correspond to Cherubini Rule 1.

## Third Set of Rules

The third set of rules establishes that:

| | |
|---|---|
| The first chord | must be$(x, x+7)$, $(x, x+12)$, $(x, x+24)$ or $(x, x)$. |
| The final chord | must be$(x, x+12)$ or $(x, x+24)$. |
| The penultimate chord | must be$(x, x+9)$ if the lower notes are given. |

The penultimate chord must be $(y - 3, y)$ if the higher notes are given.
The notes in each sequence cannot “cross”, i.e., $y_i > x_i$ always.
Equal perfect consonances cannot be used consecutively.
Unison can only be used in the beginning.
Use preferably imperfect consonances.
Repetitions of the same consonances are forbidden.
The motion from any consonant to a perfect consonance has to be oblique or contrary.

This set of rules relates to Cherubini Rules 1, 2, 4, 5 and 8.

What is the meaning of the last rule? Motions in music are very important and they will be explained in detail in the next section.

## Musical Motion

The last rule in the previous section mentions motions. In order to explain what this is let us look at some musical examples.

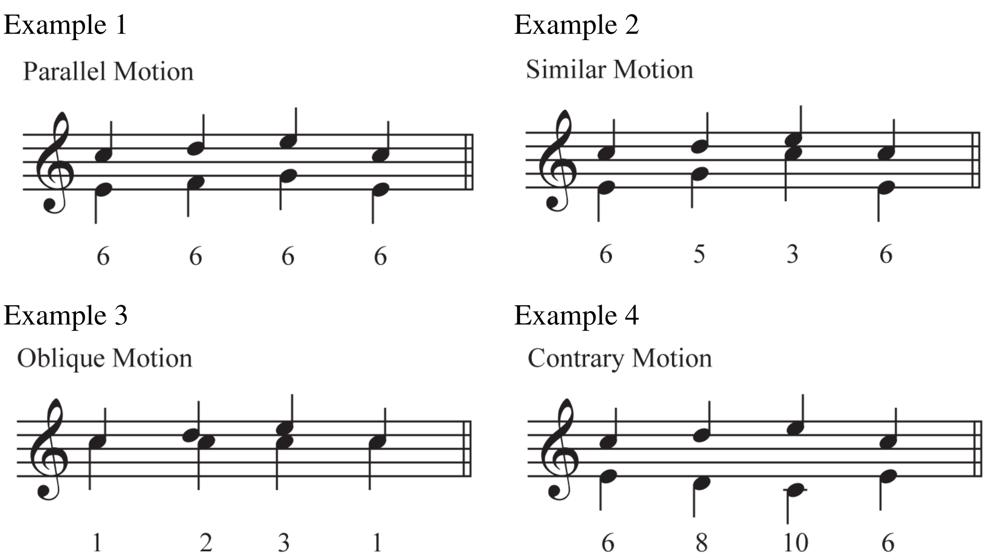


**Figure 1** : *Examples of motions in music: Examples 1 and 2 refer to direct motion*

The previous figures show the four types of musical motion. Example 1 and Example 2 show parallel motion and similar motion respectively. These are the direct movements mentioned in Rule 5 from Cherubini. Parallel motion corresponds to a sequence of pairs of type $(x, y)$, $(x+a, y+a)$, where $a$ is constant. Similar motion would correspond to pairs of type $(x, y)$, $(x + a, y + b)$ with the constants $a$ and $b$ both positive or both negative. Rule 5 states that if $(x, y)$ is a perfect chord then the previous chord cannot be of type $(x+a, y+a)$, with $a$ a constant positive or negative or of type $(x+a, y+b)$ with $a$ $b$ constants both positive or both negative. Parallel motion should always be avoided even in imperfect consonances, similar motion is acceptable with imperfect consonances.

In the Example 2 there is an error, because the interval of perfect fifth in the second pair of notes could not be preceded by two notes that where both lower. The perfect fifth results from similar motion. Example 3 and Example 4 show oblique motion and contrary motion. In example 4 the octave is reached by contrary motion.

It is easy to translate oblique and contrary motion numerically.

Tast rule in the previous section establishes that not all motions can be used to attain all intervals. One has to be careful. In his book *A Geometry of Music* [**?**], Tymoczko introduces a lattice that allows these motions to be seen as paths in a plane for two note chords or paths in space for three note chords.

Tymoczko builds a lattice like the one shown in Table 2.

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 0 | | 1 1 | | 2 2 | | 3 3 | | 4 4 | | 5 5 | | 6 6 |
| | 0 1 | | 1 2 | | 2 3 | | 3 4 | | 4 5 | | 5 6 | |
| 11 1 | | 0 2 | | 1 3 | | 2 4 | | 3 5 | | 4 6 | | 5 7 |
| | 11 2 | | 0 3 | | 1 4 | | 2 5 | | 3 6 | | 4 7 | |
| 10 2 | | 11 3 | | 0 4 | | 1 5 | | 2 6 | | 3 7 | | 4 8 |
| | 10 3 | | 11 4 | | 0 5 | | 1 6 | | 2 7 | | 3 8 | |
| 9 3 | | 10 4 | | 11 5 | | 0 6 | | 1 7 | | 2 8 | | 3 9 |
| | 9 4 | | 10 5 | | 11 6 | | 0 7 | | 1 8 | | 2 9 | |
| 8 4 | | 9 5 | | 10 6 | | 11 7 | | 0 8 | | 1 9 | | 2 10 |
| | 8 5 | | 9 6 | | 10 7 | | 11 8 | | 0 9 | | 1 10 | |
| 7 5 | | 8 6 | | 9 7 | | 10 8 | | 11 9 | | 0 10 | | 1 11 |
| | 7 6 | | 8 7 | | 9 8 | | 10 9 | | 11 10 | | 0 11 | |
| 6 6 | | 7 7 | | 8 8 | | 9 9 | | 10 10 | | 11 11 | | 0 0 |

**Table 2** : *Tymoczko's lattice*

The numbers correspond to the musical notes. It is easy to see that each line actually shows pairs of notes that correspond to the same type of musical interval. Is is also easy to see that horizontal paths in this lattice correspond to parallel motion and vertical paths to contrary motion. The geometrical structure of this lattice correspond to a Möbius strip. If you twist the right end you can actually glue the right end to the left end.

It is easier to understand rule 7 from Cherubini if we consider this lattice.

Tritones are intervals type $(x, x+6)$. These intervals are all in the horizontal line in the middle of the lattice. "Tritone between the parts should be avoided" means that we cannot choose a path that as a pair belonging to this path. For example, if we have the sequence $(7, 11)$, $(5, 0)$, we may listen to the forbidden pair $(11, 5)$ which is part of the path $(7, 11)$, $(11, 5)$,$(5, 0)$.

**Figure 2** : *Forbidden sequence*

If we use only seven notes the previous lattice is simplified. As an example we show the lattice for the scale with notes corresponding to 0, 2, 4, 5, 7, 9 and 11 (Table 3).

Not all of these intervals are allowed. In the lattice the order is not relevant. We will have to be careful because $(0, 7)$, $(7, 0)$ and $(7, 12)$ correspond to $(0, 7)$ in the lattice but are different situations in our exercises.

The three dimensional structure that will represent three note chords given in [**?**] is built through numerical representation of three note chords in a three orthogonal with a suitable basis. The new basis is a 45 degree rotation of the canonical basis just like happens in the two dimensional lattice. One will choose

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 0 | | | | 2 2 | | | | 4 4 | | 5 5 | | |
| | | | | | | | | | 4 5 | | | |
| | | 0 2 | | | | 2 4 | | | | | | 5 7 |
| | 11 2 | | | | | | 2 5 | | | | 4 7 | |
| | | | | 0 4 | | | | | | | | |
| | | | 11 4 | | 0 5 | | | | 2 7 | | | |
| | | | | 11 5 | | | | | | | | |
| | 9 4 | | | | | | 0 7 | | | | 2 9 | |
| | | 9 5 | | | | 11 7 | | | | | | |
| | | | | | | | | | 0 9 | | | |
| 7 5 | | | | 9 7 | | | | 11 9 | | | | |
| | | | | | | | | | | | 0 11 | |
| | | 7 7 | | | | 9 9 | | | | 11 11 | | 0 0 |

**Table 3** : *Tymoczko's lattice restricted to chords using only seven notes instead of twelve*

the vertices and the edges to avoid repetitions and one obtains a prism with a triangular basis. One edge will have the chords (listed from bottom upwards) $(0,0,0)$, $(1,1,1)$, $(2,2,2)$, $(3,3,3)$, $(4,4,4)$ another the chords $(4,4,4)$, $(5,5,5)$, $(6,6,6)$, $(7,7,7)$, $(8,8,8)$ and another the chords $(8,8,8)$, $(9,9,9)$, $(10,10,10)$, $(11,11,11)$, $(0,0,0)$. And one sees a gluing process similar to the two dimensional case. Inside this prism we will have a structure similar to the one shown in Figure 3 with a model made with Zometool. We can see four cubes the bottom with only one vertex touching the plane.

If the vertex touching the plane is $(0,4,8)$, then the opposite vertex is $(1,5,9)$. The other vertices connected to $(0,4,8)$ are $(0,4,9)$, $(1,4,8)$, $(0,5,8)$. The vertices connected to $(1,5,9)$ are $(1,5,8)$, $(0,5,9)$, $(1,4,9)$. Vertices $(0,4,9)$ and $(1,4,9)$, $(1,4,8)$ and $(1,5,8)$, $(0,5,8)$, $(1,5,8)$ are connected.

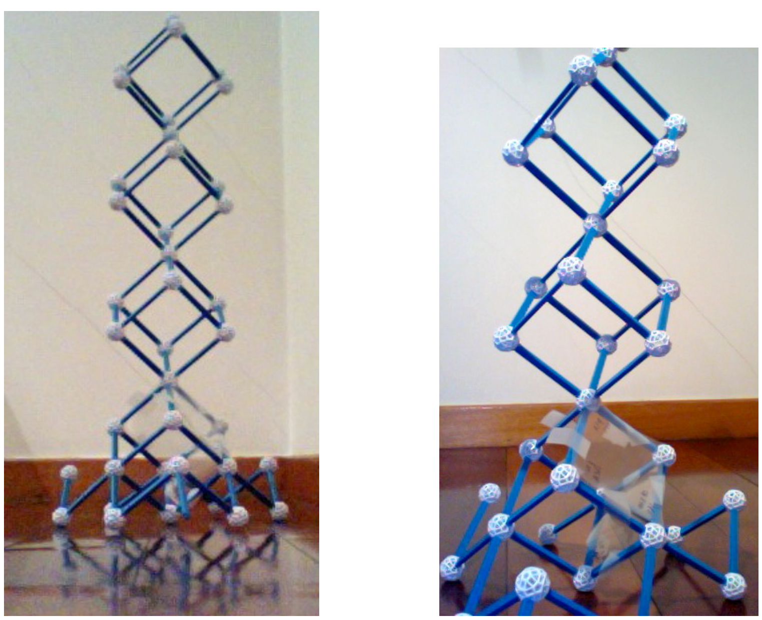

**Figure 3** : *Part of the three-dimensional lattice*

The right side of Figure 3 shows a transparent side of the cube where it is possible to write the chords corresponding to each vertex.

Paths perpendicular to the plane will correspond to a transposition $(x, y, z) \rightarrow (x + a, y + a, z + a)$. We can look for other paths with interesting musical interpretation.

Let us look at each horizontal slice of this structure. A triangle that will look like the following:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | 8 8 8 | | | | |
| | | | | 7 8 9 | | | | |
| | | | 9 9 6 | | 7 7 10 | | | |
| | | | | 6 8 10 | | | | |
| | | | 10 9 5 | | 6 7 11 | | | |
| | | 10 10 4 | | 5 8 11 | | 6 6 0 | | |
| | | | 11 9 4 | | 5 7 0 | | | |
| | | 11 10 3 | | 4 8 0 | | 5 6 1 | | |
| | 11 11 2 | | 0 9 3 | | 4 7 1 | | 5 5 2 | |
| | | 0 10 2 | | 3 8 1 | | 4 6 2 | | |
| | 0 11 1 | | 1 9 2 | | 3 7 2 | | 4 5 3 | |
| 0 0 0 | | 1 10 1 | | 2 8 2 | | 3 6 3 | | 4 4 4 |

**Table 4** : *Horizontal slice*

This structure has some interesting arithmetic properties. For instance the sum of the numbers in each chord in each horizontal line is always the same. In this slice (the base of the structure) it is $0$. In the two dimensional case vertical lines had the same characteristic. It is easy to identify transpositions by a semitone in one or two chords ascending and descending. The structure should correspond to an equilateral triangle and if one rotates it by 120 degrees each chord will be transformed into a new chord a major third higher. For example the chord $(0, 11, 1)$ will be transformed in the chord $(4, 5, 3)$. Further explanations of this structure would be needed to deeply understand its possibilities. For example it is possible to use abstract representations of cross sections of this structure and represent paths where reaching the boundaries of the triangle would swap the order of the notes in the chord.

## The Workshop

The purpose of this workshop is to solve some counterpoint exercises. The participants are given the sequence, for example, $(4, 0, 2, 0, -3, 9, 7, 4, 5, 4)$ corresponding to $(x_i)$ and using the arithmetic rules should be able to produce an admissible $(y_i)$ sequence. According to the remark following Table 1 $-3$ corresponds to the same musical note as $9$, which is **A**.

In order to better understand this structure it will be suggested that the participants use Zometool and build a part. Starting with one chord build a cube and label one vertex with the data of the initial chord $(x, y, z)$. The opposite vertex will be labeled $(x+1, y+1, z+1)$. Label the intermediate vertices according to the example above. Each consecutive vertex can differ only in one coordinate. The new coordinate differs from the previous one by $\pm 1$.

One could proceed studying the musical composition building paths in the structure.

The next pictures show examples of labeled structures.

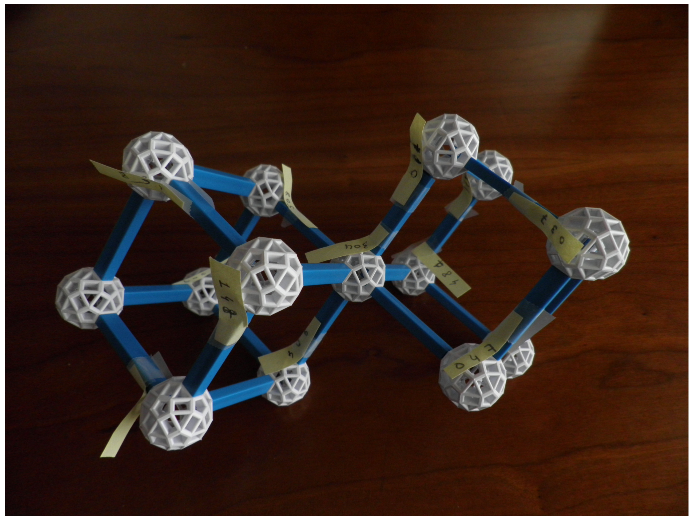

**Figure 4** : *Labeled example 1 corresponding to the center of the chord space*

Figure 5 shows the path from the beginning of the third movement from the piano sonata K576 from Mozart. The path is the red thread.

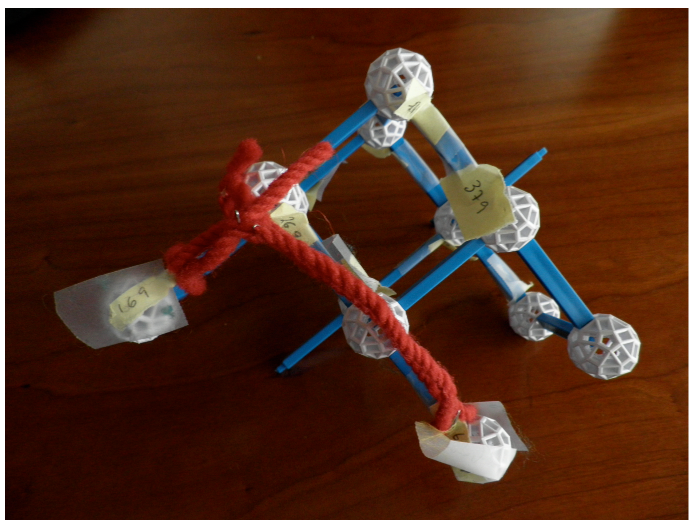

**Figure 5** : *Labeled example 2. Beginning of the third movement from the piano sonata K576 from Mozart*

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