FEM-BASED FORMULATION FOR LINEAR ANALYSIS OF CURVED SHELLS

Joaquim António Oliveira de Barros

Report 12-DEC/E-27

This document is an updated of the first version published in May 2004

Date: November 2012

Pages: 54

Keywords: Ahmad shell, Finite element method, Linear elastic analysis

School of Engineering Department of Civil Engineering University of Minho

TABLE OF CONTENTS

1 INTRODUCTION 3

2 BASIC HYPOTHESIS 5

3 TYPES OF NODAL POINTS 6

4 COORDINATE SYSTEMS 9

4.1 Introduction 9
4.2 Global coordinate system $x_i \left(x_1, x_2, x_3\right)$ 10
4.3 Natural coordinate system $s_i \left(s_1, s_2, s_3\right)$ 10
4.4 Nodal coordinate system $n_i \left(n_1, n_2, n_3\right)$ 11
4.5 Tangential coordinate system $t_i \left(t_1, t_2, t_3\right)$ 13
4.6 Auxiliary coordinate system $a_i \left(a_1, a_2, a_3\right)$ 13

5 GEOMETRY DEFINITION 14

6 DISPLACEMENT FIELD 16

7 STRAIN COMPONENTS 23

8 STRESS COMPONENTS 34

9 CONSTITUTIVE EQUATIONS FOR LINEAR-ELASTIC MATERIALS 37

10 STIFFNESS MATRIX FOR LINEAR-ELASTIC MATERIALS 39

11 LOAD VECTOR 40

11.1 Introduction 40
11.2 Point loads 40
11.3 Gravity load 40
11.4 Generalized forces per unit length 40
11.5 Generalized forces per unit area 43
11.6 Uniform temperature variation 44
11.7 Diferential temperature variation 45
11.8 Prescribed displacements 47

12 RESULTANT STRESSES 48

13 REFERENCES 53

ANNEX I 54

1 INTRODUCTION

A shell is a thin structure with in-plane dimensions much larger than its thickness. There are shells formed by plane components, but curvilinear complex geometries can also be found amongst the most beautiful world-wide shell constructions. In this type of structures, in-plane strains and out-of-plane shear strains can simultaneously occur, inducing membrane forces, bending moments and out-of-plane shear forces. The in-plane forces and the bending moments are the most significant. However, near high stress concentration areas, like point and line loads, and at the supports, the out-of-plane shear forces can be appreciable. These forces can also be significant in shells of moderate thickness.

Using the shell concept, very interesting solutions can be obtained, not only from the structural and architectural point of view, but also from technical and economic reasons. Large areas can be covered with lightweight and elegant shell structures, resulting lower seismic loads and smaller loads transferred to the supports and to the soil.

The major part of the shells is made by reinforced concrete. Recent advances in fibre reinforced concrete (FRC) indicate that fibres can replace part of the conventional reinforcement (steel bars and meshes). In the last years, composite materials are also being used to build shell roofs of large span, but the sensitivity of these materials to fire is yet a big obstacle for their use in this type of structures.

The Ahmad shell finite element (Ahmad et al. 1970) results from the degeneration process of the volume finite element. The degeneration process is represented in Figure 1. The Ahmad shell finite element has been extensively used in the linear and nonlinear analysis of reinforced concrete shell structures (Figueiras 1983, Póvoas 1991). This element has been proved to be appropriated not only for very thin shells of composite materials (Figueiras 1983), but also for reinforced concrete shells of moderate thickness.

The present work has the main purpose of describing the finite element based formulation of Ahamd shell implemented into the FEMIX computer package (Azevedo et al. 2003) in order to simulate the linear and elastic behaviour of curved shells. To be capable of analyzing shells or curved geometry and where the out-of-plane shear cannot be neglected, the Ahmad shell formulation (Zienkiewicz and Taylor1989, Oñate 1995) is adopted. This formulation is rewritten for a simple implementation of the layered concept that is currently used in the material nonlinear analysis of laminate concrete structures (Barros 1995).

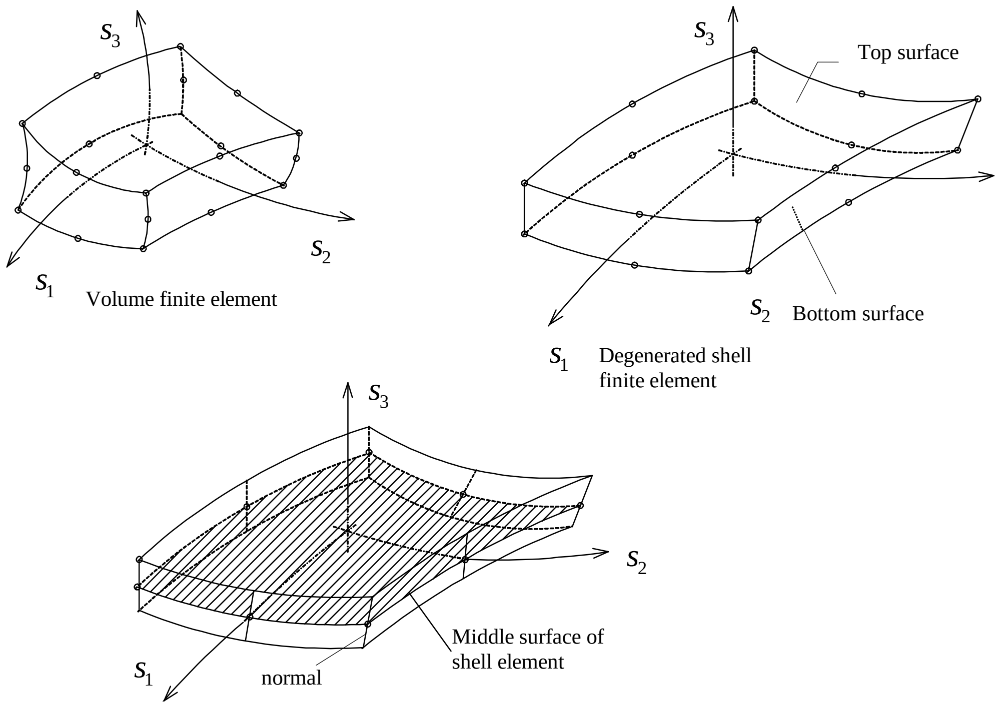


Figure 1 - Degeneration process of a volume finite element into an Ahmad shell finite element.

2 BASIC HYPOTHESIS

The Ahmad shell theory is supported on the following hypothesis (Oñate 1995):

- The deflection orthogonal to the shell middle surface of any point is small when compared to the shell thickness;
- The stress orthogonal to the shell middle surface is negligible;
- Normal to the undeformed shell middle surface remains orthogonal to the deformed shell middle surface.

3 TYPES OF NODAL POINTS

Some thin structures, like the one represented in Figure 2, are composed by components forming kink edges. To analyze this type of structures, using the shell approach, two types of nodes should be distinguished: coplanar and kink nodes. The last ones are in the kink edges, while the coplanar nodes are the remaining nodes.

The process to distinguish a kink from a coplanar node is based on the angle formed amongst the vectors that, at this node, are orthogonal to the middle surface of the shell elements connecting to this node. If this angle is less than one degree, the node is declared coplanar node, otherwise it is kink node.

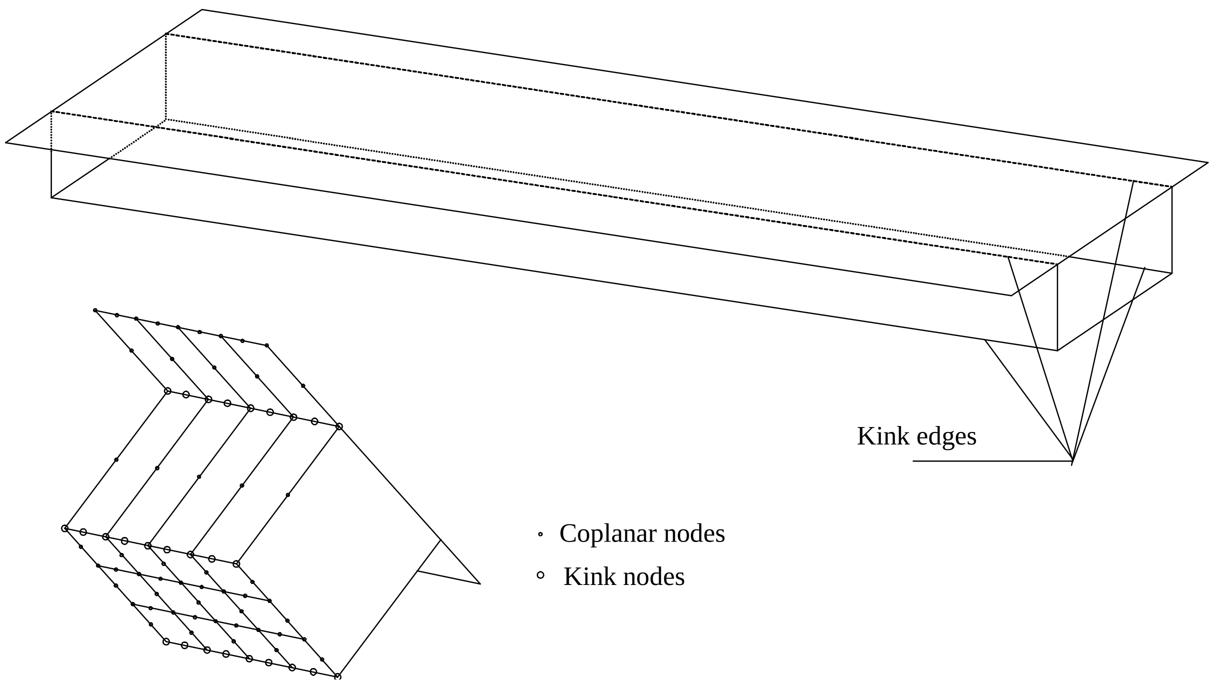
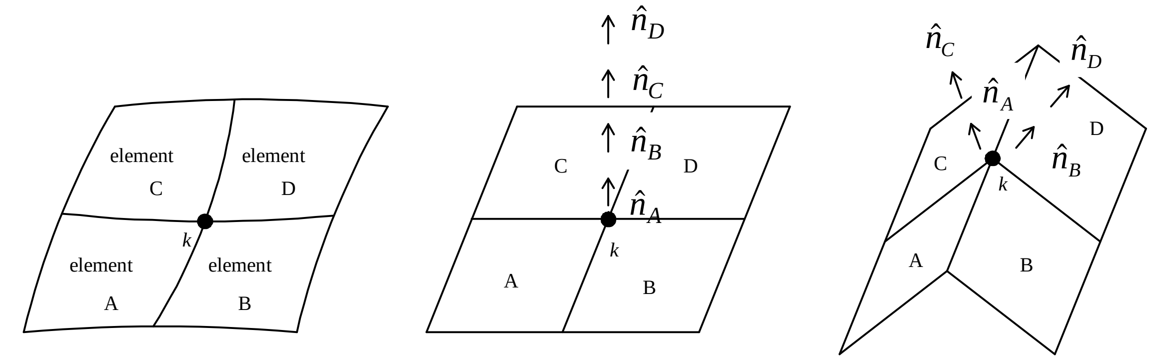


Figure 2 - Structures that have coplanar and kink nodes.



(a) The k node is common to the elements A, B, C and D.

(b) The coplanar node k has five degrees of freedom.

(c) The kink nde k has six degrees of freedom.

Figure 3 - Schematic representation of the procedure to distinguish coplanar and kink nodes.

Coplanar nodes have five degrees of freedom (see Figure 4), three displaments in the global coordinate system, $(\delta_1 = u_1)_k$, $(\delta_2 = u_2)_k$ and $(\delta_3 = u_3)_k$, and two rotations in the nodal coordinate system, $(\delta_4 = \theta_1^n)_k$ and $(\delta_5 = \theta_2^n)_k$. The vectors n_1 and n_2 in Figure 4 define the plane containing the k node and it is tangent to the element middle surface $(s_3 = 0)$ at this node. The procedures to define the nodal coordinate system are given in the next section.

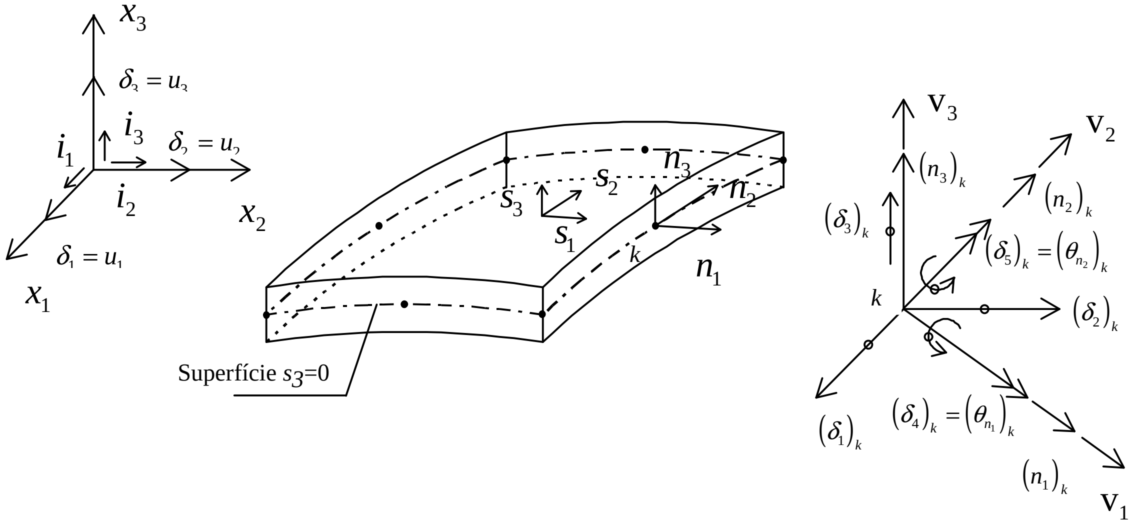


Figure 4 - Degrees of freedom and coordinate systems for coplanar nodes.

Kink nodes have six degrees of freedom, three displacement components and three rotation components, all of them in the global coordinate system, $(\delta_1 = u_1)_k$, $(\delta_2 = u_2)_k$, $(\delta_3 = u_3)_k$, $(\delta_4 = u_4)_k$, $(\delta_5 = u_5)_k$ and $(\delta_6 = u_6)_k$, see Figure 5.

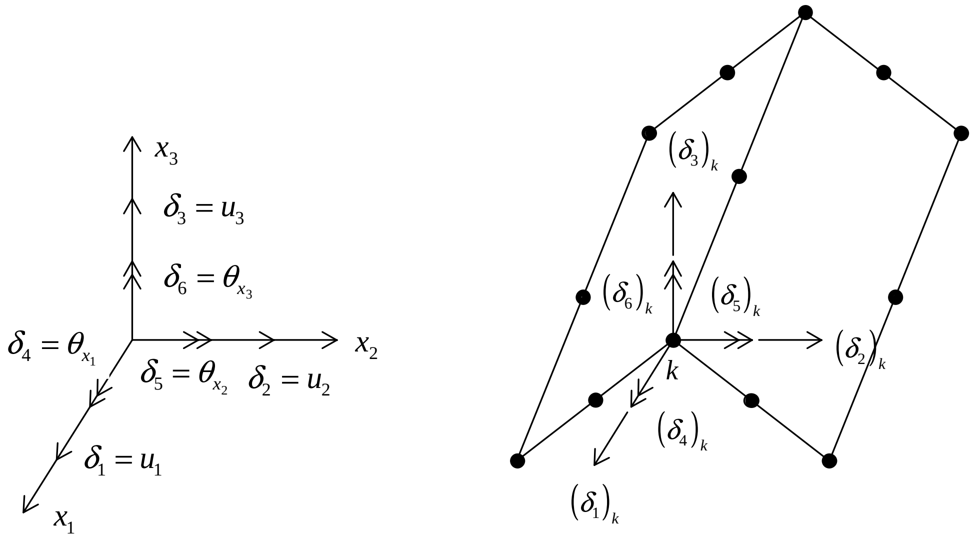


Figure 5 - Degrees of freedom and coordinate system for kink nodes.

4 COORDINATE SYSTEMS

4.1 INTRODUCTION

The four coordinate systems used in the Ahmad shell finite element formulation are represented in Figure 6. Auxiliary coordinate systems (see Figure 7), in general, are used to define support conditions that prescribe some degrees of freedom in directions distinct of the global and nodal coordinate systems.

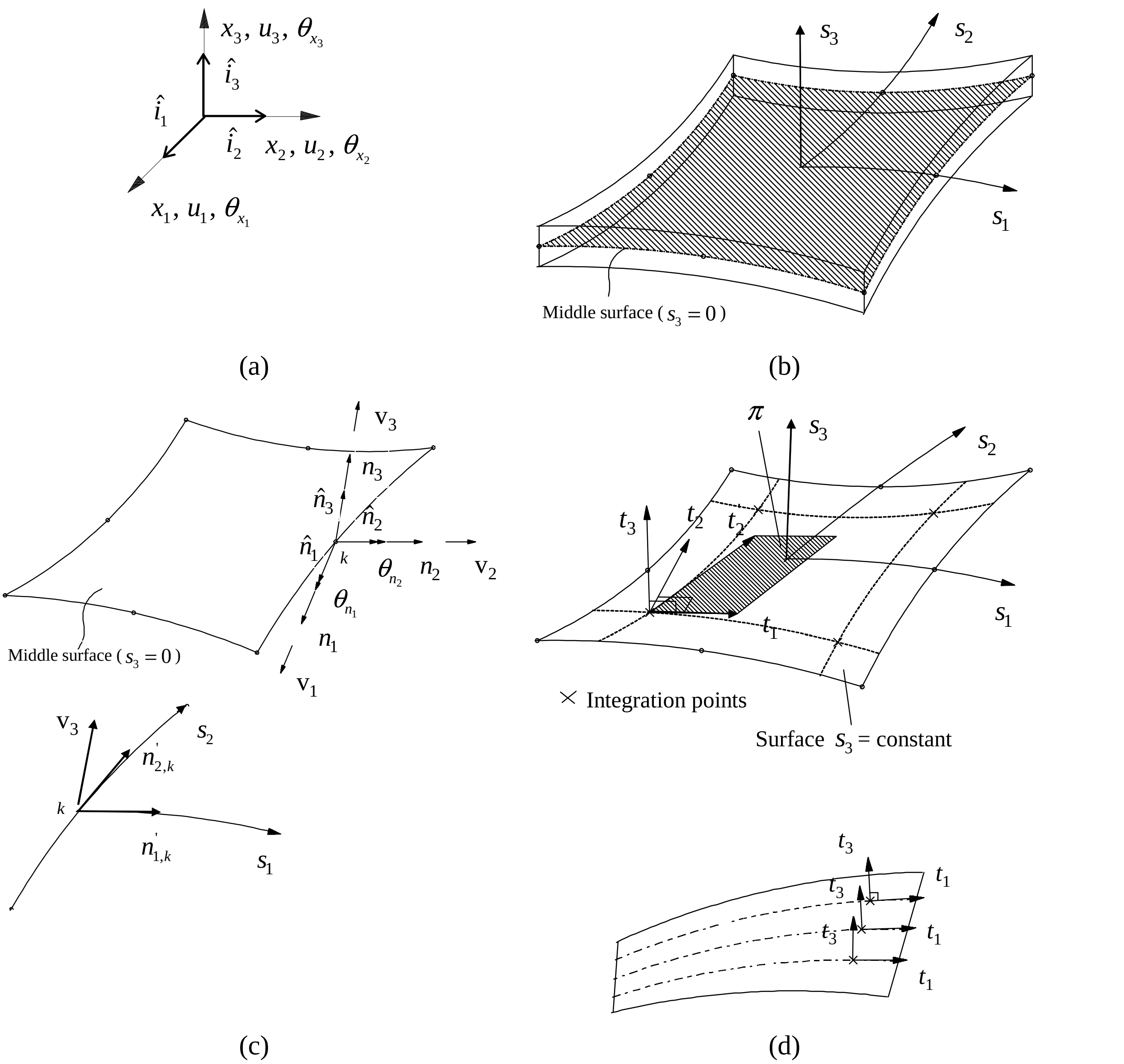


Figure 6 - Coordinate systems for the Ahmad finite element formulation: (a) global, (b) natural, (c) nodal and (d) tangential.

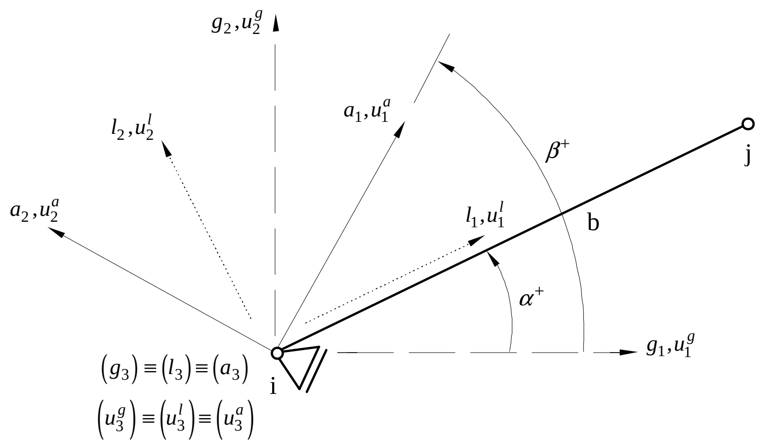


Figure 7 – Auxiliary coordinate system.

4.2 GLOBAL COORDINATE SYSTEM $x_i \ (x_1, \ x_2, \ x_3)$

The global coordinate system is used to define the geometry of the structure (Figure 6a). The nodal coordinates, the nodal displacements (except the rotations of coplanar nodes and the degrees of freedom in auxiliary directions), the stiffness matrix of the structure (apart the terms corresponding to the rotations of the coplanar nodes and the terms corresponding to degrees of freedom in auxiliary directions) and the load vector (except the moments of the coplanar nodes and the terms corresponding to degrees of freedom in auxiliary directions) are all referred to the global coordinate system.

4.3 NATURAL COORDINATE SYSTEM $s_i \ (s_1, \ s_2, \ s_3)$

The finite element shape functions are defined in the natural coordinate system (Figure 6b). The term natural is adopted since s_i axis ranges between -1 and +1, independently of the finite element geometry. The s_1 and s_2 axis are in the element middle surface, while s_3 axis is orthogonal to this surface.

4.4 NODAL COORDINATE SYSTEM $n_i \left(n_1, n_2, n_3 \right)$

With its origin in the element middle surface, this referential is defined for each *k* node of the element, performing the following procedures (see Figure 6c).

Evaluation of the unit vector $\hat{n}_3$

In the first step, vectors tangent to s_1 ($\underline{n}_1^{'}$) and s_2 ($\underline{n}_2^{'}$) at *k* node are determined:

$$\underline{n}_{1,k}^{'} = \left(\frac{\partial x_1}{\partial s_1} \; \frac{\partial x_2}{\partial s_1} \; \frac{\partial x_3}{\partial s_1} \right)_k^T \tag{1}$$

$$\underline{n}_{2,k}^{'} = \left(\frac{\partial x_1}{\partial s_2} \; \frac{\partial x_2}{\partial s_2} \; \frac{\partial x_3}{\partial s_2} \right)_k^T \tag{2}$$

The cross product of vector $\underline{n}_{1,k}^{'}$ by vector $\underline{n}_{2,k}^{'}$ gives the vector orthogonal to the element middle surface at node *k*, $\underline{n}_{3,k}$, and after the calculation of its Euclidean norm the unit vector $\hat{n}_{3,k}$ is obtained,

$$\hat{n}_{3,k} = \frac{1}{\left\| \underline{n}_{1,k}^{'} \times \underline{n}_{2,k}^{'} \right\|} \left(\underline{n}_{1,k}^{'} \times \underline{n}_{2,k}^{'} \right) \tag{3}$$

If more than one element has this *k* node, the final $\hat{n}_{3,k}$ is the vectorial addition of the $\hat{n}_{3,k}^{(e)}$ of the elements adjacent to *k* node.

If $\underline{n}_{3,k}$ is parallel to x_3 global axis, the nodal coordinate system is parallel to the global coordinate system ($\underline{n}_{1,k} = \hat{i}_1$, $\underline{n}_{2,k} = \hat{i}_2$ and $\underline{n}_{3,k} = \hat{i}_3$, where $\hat{i}_1$, $\hat{i}_2$ and $\hat{i}_3$ are the unit vectors of x_1, x_2 and x_3, respectively). If $\underline{n}_{3,k}$ has opposite direction to x_3, $\underline{n}_{1,k} = \hat{i}_1$, $\underline{n}_{2,k} =- \hat{i}_2$ and $\underline{n}_{3,k} =- \hat{i}_3$.

If $\underline{n}_{3,k}$ is not in these special conditions, $\underline{n}_{1,k}$ is assumed to be orthogonal to the plane defined by $\hat{i}_3$ and $\hat{n}_{3,k}$, i.e. $\underline{n}_{1,k}$ is parallel to (x_1, x_2) plane,

$$\hat{n}_{1,k} = \frac{1}{\left\| \hat{i}_3 \times \hat{n}_{3,k} \right\|} \left(\hat{i}_3 \times \hat{n}_{3,k} \right) = -n_{22,k}\,\hat{i}_1 + n_{13,k}\,\hat{i}_2 - 0\,\hat{i}_3 \qquad (4)$$

Finally, the $\hat{n}_{2,k}$ results from the cross product of vector $\hat{n}_{3,k}$ by vector $\hat{n}_{1,k}$

$$\hat{n}_{2,k} = \left(\hat{n}_{3,k} \times \hat{n}_{1,k} \right) \qquad (5)$$

The rotations of a coplanar node, θ_1^n and θ_2^n, are defined in n_1 and n_2 directions. The $\underline{n}_3$ vector defines the direction orthogonal to the element middle surface at this node. In the present formulation, the $\underline{n}_3$ is always orthogonal to the element middle surface, but in the majority part of the formulations this vector is not necessary orthogonal to the element middle surface, since it is obtained from the coordinates of the corresponding nodes at bottom and top element surfaces, such is illustrated in Figure 8 (Figueiras 1983, Póvoas 1991). These formulations, however, cannot analyze structures with kink edges (see Figure 2)

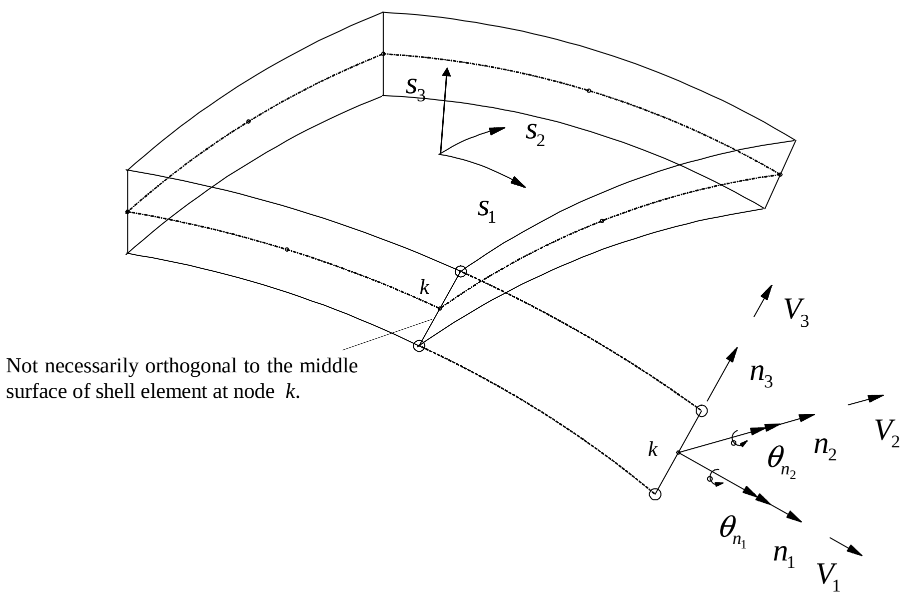


Figure 8 - The $\hat{n}_3$ vector at k node is defined from the coordinates of the corresponding nodes at top and bottom element surfaces.

4.5 TANGENTIAL COORDINATE SYSTEM t_i (t_1, t_2, t_3)

This referential is defined in each finite element integration point. The strain, stress and resultant stress components are determined in this coordinate system. The $\underline{t}_1$ and t_2 axes form a plane that is tangent to the surface of s_3 =constant, at the integration point. The procedures for evaluating t_i axes are equal to the one adopted for n_i, but the t_i axis are evaluated in the integration points, while n_i are defined in the element nodal points. Figure 6 shows a graphical representation of the tangential coordinate system.

4.6 AUXILIARY COORDINATE SYSTEM a_i (a_1, a_2, a_3)

To prescribe degrees of freedom in directions do not coinciding with global and nodal coordinate systems, auxiliary coordinate systems having axis coincident with the direction of these prescribed displacements, should be used (see Figure 6e).

5 GEOMETRY DEFINITION

The global coordinates of a generic point of a shell element is obtained from following expression:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \sum_{k=1}^{n} N_k \begin{bmatrix} \overline{x}_1 \\ \overline{x}_2 \\ \overline{x}_3 \end{bmatrix}_k + \sum_{k=1}^{n} N_k \frac{h_k}{2} s_3 \begin{bmatrix} n_{31} \\ n_{32} \\ n_{33} \end{bmatrix}_k \tag{6}$$

where N_k is the shape function of node k of the element of n nodes, $\underline{\overline{x}}_k$ is the vector of the global coordinates of node k, n_{3i} are the components of the $\hat{n}_{3,k}$ in the global coordinate system and h_k is the element thickness at k node (see Figure 9).

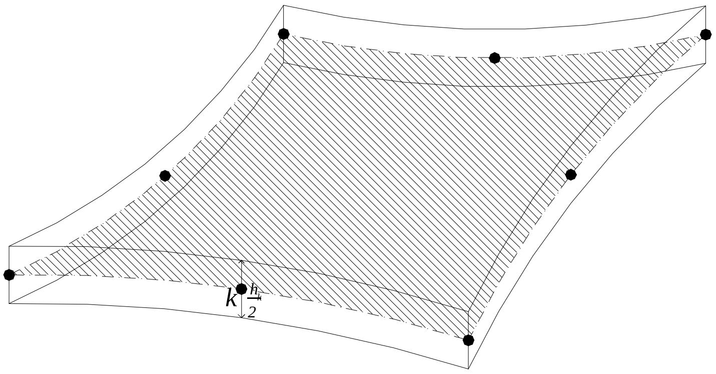


Figure 9 - Thickness at k node.

Giving another format to (6) results,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} N_1 & 0 & 0 & N_1 \frac{h_1}{2} s_3 & 0 & 0 & \dots & N_n & 0 & 0 & N_n \frac{h_n}{2} s_3 & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_1 \frac{h_1}{2} s_3 & 0 & \dots & 0 & N_n & 0 & 0 & N_n \frac{h_n}{2} s_3 & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_1 \frac{h_1}{2} s_3 & \dots & 0 & 0 & N_n & 0 & 0 & N_n \frac{h_n}{2} s_3 \end{bmatrix} \begin{bmatrix} \overline{x}_{1,1} \\ \overline{x}_{2,1} \\ \overline{x}_{3,1} \\ n_{31,1} \\ n_{32,1} \\ n_{33,1} \\ \dots \\ \overline{x}_{1,n} \\ \overline{x}_{2,n} \\ \overline{x}_{3,n} \\ n_{31,n} \\ n_{32,n} \\ n_{33,n} \end{bmatrix} \tag{7}$$

or,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \underline{N}_{x,1} \\ \underline{N}_{x,2} \\ \underline{N}_{x,3} \end{bmatrix} \underline{X} \qquad (8)$$
$$= \underline{N}_x \, \underline{X}$$

where $\underline{N}_{x,1}$, $\underline{N}_{x,2}$ and $\underline{N}_{x,3}$ have the dimension of (1,6*n*) and $\underline{X}^{(e)}$ has the dimension of (6*n*,1), where *n* is the number of element nodes.

6 DISPLACEMENT FIELD

It was already referred that the number of the degrees of freedom of a given node is dependent on the type of node. In case of a coplanar node, the degrees of freedom are five, while a kink node has six degrees of freedom. In both types of nodes the first three degrees of freedom are displacements in the global coordinate system (u_i *c/ i=1,2,3*). In coplanar node, the fourth and the fifth degrees of freedom are rotations of the orthogonal to the element middle surface in turn of the $\underline{n}_1$ and $\underline{n}_2$ axes of the nodal coordinate system, θ_1^n and θ_2^n, respectively. In kink node, the fourth to sixth degrees of freedom are rotations in turn of the global coordinate system (θ_1^g, θ_2^g, θ_3^g).

Figure 10 represents the displacements occurred in a given nodal point. Starting for assuming that this is a coplanar node, it has five degrees of freedom.

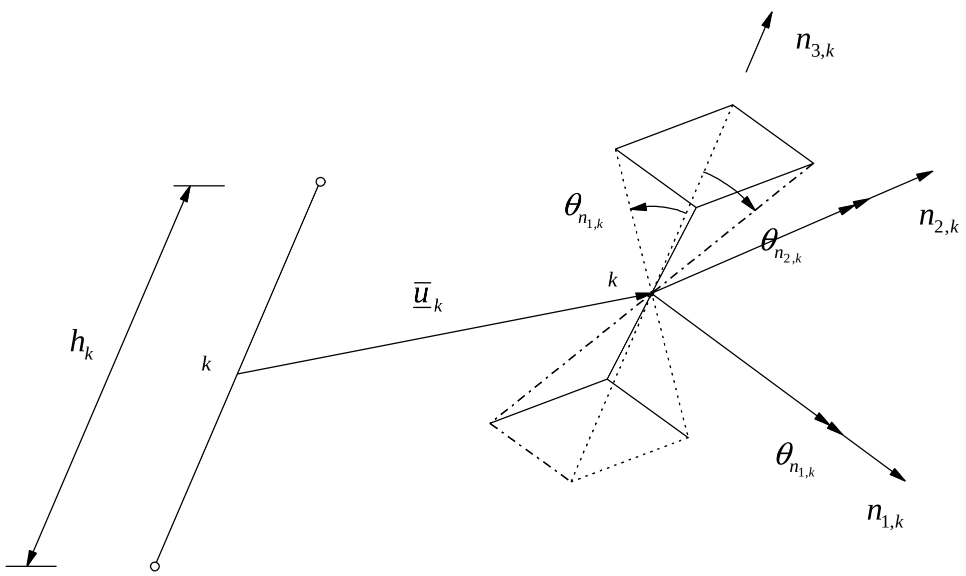


Figure 9 - Displacements of a given nodal point.

The displacements of a point of the interior of the element are the result of the displacements ocurred in the points at the element middle surface, plus the displacements due to the rotations.

The contribution of the displacements of the nodal points at the element middle surface is evaluated from the following expression:

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \sum_{k=1}^{n} N_k \begin{bmatrix} \overline{u}_{1,k} \\ \overline{u}_{2,k} \\ \overline{u}_{3,k} \end{bmatrix} \quad (9)$$

where $\left[\overline{u}_{1,k} \quad \overline{u}_{2,k} \quad \overline{u}_{3,k}\right]^T$ are the displacement components in global coordinate system of *k* node at element middle surface.

The rotations contribute for the displacement field according to the schematic representation of Figure 11.

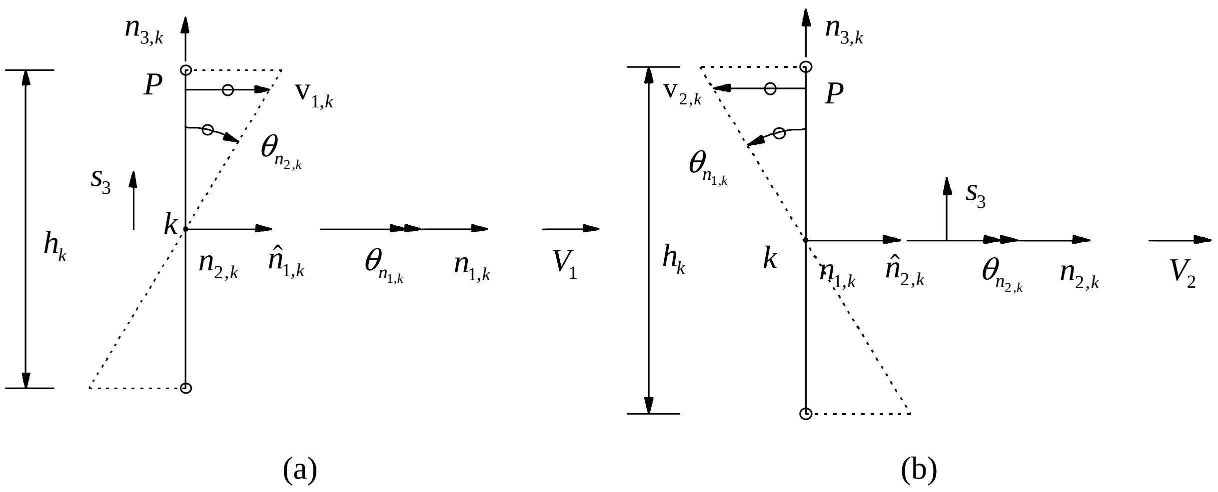


Figure 11 - Contribution of the rotations for the displacement field.

Assuming that a rotation occurs in turn of $n_{2,k}$ axis (Figure 11a), i.e., in the plane $n_{1,k}n_{3,k}$, the displacement of a point *P* at a level s_3 is obtained from the following equations (the rotations are infinitesimal):

$$\mathrm{v}_{1,k} = \frac{h_k}{2} s_3 \, \theta_{2,k}^{n} \quad (10)$$

$$\begin{aligned} \underline{\mathrm{v}}_{n_1,k} &= \mathrm{v}_{1,k}\,.\hat{n}_{1,k} \\ &= \frac{h_k}{2}\, s_3\, \theta^n_{2,k}\,.\hat{n}_{1,k} \end{aligned} \tag{11}$$

In the same way, due to a rotation in turn of $n_{1,k}$ axis (Figure 11b), i.e., in the plane $n_{2,k}n_{3,k}$, the displacement of a point *P* at a level s_3 is obtained from the following equations:

$$\mathrm{v}_{2,k} = -\frac{h_k}{2}\, s_3\, \theta^n_{1,k} \tag{12}$$

$$\begin{aligned} \underline{\mathrm{v}}_{n_2,k} &= \mathrm{v}_{2,k}\,.\hat{n}_{2,k} \\ &= -\frac{h_k}{2}\, s_3\, \theta^n_{1,k}\,.\hat{n}_{2,k} \end{aligned} \tag{13}$$

Adding the contribution from the rotations in turn of $n_{2,k}$ and $n_{1,k}$ axis results (adding (11) and (13)),

$$\underline{\mathrm{v}}_k = \underline{\mathrm{U}}^g_k = \frac{h_k}{2}\, s_3 \begin{bmatrix} -\hat{n}_{2,k} & \hat{n}_{1,k} \end{bmatrix} \begin{bmatrix} \theta^n_{1,k} \\ \theta^n_{2,k} \end{bmatrix} \tag{14}$$

where $\underline{\mathrm{U}}^g_k$ is the displacement vector of *k* node and $\hat{n}_{1,k}$ and $\hat{n}_{2,k}$ are the unit vectors of n_1 and n_2 axis of the nodal coordinate system, at *k* node. Therefore, the contribution of the rotations of all nodes of the element for the displacements in any point of the shell element is obtained from the following expression:

$$\underline{U}^g_k = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \sum_{k=1}^{n} N_k\, \frac{h_k}{2}\, s_3 \begin{bmatrix} -\hat{n}_{2,k} & \hat{n}_{1,k} \end{bmatrix} \begin{bmatrix} \theta^n_{1,k} \\ \theta^n_{2,k} \end{bmatrix} \tag{15}$$

The displacement components of a generic *k* node in the global coordinate system can be transferred to the nodal coordinate,

$$\underline{U}^n_k = T^{gn}_k\, \underline{U}^g_k \tag{16a}$$

where

$$\underline{U}_k^n = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}_k^n \qquad (16b)$$

are the displacement components in the nodal coordinate system,

$$\underline{U}_k^g = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}_k^g \qquad (16c)$$

are the displacement components in the global coordinate system, and

$$\underline{T}_k^{gn} = \begin{bmatrix} \hat{n}_1^T \\ \hat{n}_2^T \\ \hat{n}_3^T \end{bmatrix}_k = \begin{bmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \\ n_{31} & n_{32} & n_{33} \end{bmatrix}_k \qquad (16d)$$

is the transformation matrix that converts displacements from global coordinate system to nodal coordinate system. The lines of T_k^{gn} are formed by the unit vectors $\hat{n}_1$, $\hat{n}_2$ and $\hat{n}_3$ of the n_1, n_2 and n_3 nodal coordinate system.

Therefore, the rotations in the global coordinate system can be transformed to the nodal coordinate system,

$$\begin{bmatrix} \theta_{1,k}^n \\ \theta_{2,k}^n \end{bmatrix} = \begin{bmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \end{bmatrix}_k \begin{bmatrix} \theta_{1,k}^g \\ \theta_{2,k}^g \\ \theta_{3,k}^g \end{bmatrix} = \begin{bmatrix} \hat{n}_{1,k}^T \\ \hat{n}_{2,k}^T \end{bmatrix} \begin{bmatrix} \theta_{1,k}^g \\ \theta_{2,k}^g \\ \theta_{3,k}^g \end{bmatrix} \qquad (17)$$

Introducing (17) into (15) results:

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \sum_{k=1}^{n} N_k \frac{h_k}{2} s_3 \left[-\hat{n}_{2,k} \quad \hat{n}_{1,k} \right] \begin{bmatrix} \hat{n}_{1,k}^T \\ \hat{n}_{2,k}^T \end{bmatrix} \begin{bmatrix} \theta_{1,k}^g \\ \theta_{2,k}^g \\ \theta_{3,k}^g \end{bmatrix} \tag{18}$$

Therefore, for coplanar nodes the contribution of the rotations for the displacement field is obtained from (15), which can be rewritten in the following format:

$$\begin{aligned} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} &= \sum_{k=1}^{n} N_k \frac{h_k}{2} s_3 \begin{bmatrix} -n_{21} & n_{11} & 0 \\ -n_{22} & n_{12} & 0 \\ -n_{23} & n_{13} & 0 \end{bmatrix}_k \begin{bmatrix} \theta_{1,k}^n \\ \theta_{2,k}^n \\ \theta_{3,k}^n \end{bmatrix} \\ &= \sum_{k=1}^{n} N_k \frac{h_k}{2} s_3 \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}_k \begin{bmatrix} \theta_{1,k}^n \\ \theta_{2,k}^n \\ \theta_{3,k}^n \end{bmatrix} \end{aligned} \tag{19}$$

where $\theta_{3,k}^n$ is the drill rotation, a degree of freedom not considered, and

$$\begin{matrix} P_{11} = -n_{21} & P_{12} = n_{11} & P_{13} = 0 \\ P_{21} = -n_{22} & P_{22} = n_{12} & P_{23} = 0 \\ P_{31} = -n_{23} & P_{32} = n_{13} & P_{33} = 0 \end{matrix} \tag{20}$$

In the case of a kink node, multiplying the two matrices of (18) that include the unit vectors of the nodal coordinate system yields,

$$\begin{aligned} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} &= \sum_{k=1}^{n} N_k \frac{h_k}{2} s_3 \begin{bmatrix} -n_{21} & n_{11} \\ -n_{22} & n_{12} \\ -n_{23} & n_{13} \end{bmatrix}_k \begin{bmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \end{bmatrix}_k \begin{bmatrix} \theta_{1,k}^g \\ \theta_{2,k}^g \\ \theta_{3,k}^g \end{bmatrix} \\ &= \sum_{k=1}^{n} N_k \frac{h_k}{2} s_3 \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}_k \begin{bmatrix} \theta_{1,k}^g \\ \theta_{2,k}^g \\ \theta_{3,k}^g \end{bmatrix} \end{aligned} \tag{21}$$

where

$$\begin{array}{lll} P_{11}=0 & P_{12}=-n_{21}n_{12}+n_{11}n_{22} & P_{13}=-n_{21}n_{13}+n_{11}n_{23} \\ P_{21}=-n_{22}n_{11}+n_{12}n_{21} & P_{22}=0 & P_{23}=-n_{22}n_{13}+n_{12}n_{23} \\ P_{31}=-n_{23}n_{11}+n_{13}n_{21} & P_{32}=-n_{23}n_{12}+n_{13}n_{22} & P_{33}=-n_{23}n_{13}+n_{13}n_{23} \end{array} \tag{22}$$

Adding the contribution of the displacements of the nodal points at the element middle surface (Equation (9)) to the contribution of the rotations (Equations (19) and (21)) results,

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}^g = \sum_{k=1}^{n} N_k \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \end{bmatrix}_k^g + \sum_{k=1}^{n} N_k \frac{h_k}{2} s_3 \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}_k \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}_k \tag{23}$$

where

$$\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}_k = \begin{bmatrix} \theta_1^n \\ \theta_2^n \\ \theta_3^n \end{bmatrix}_k \tag{24}$$

if k is a coplanar node, and

$$\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}_k = \begin{bmatrix} \theta_1^g \\ \theta_2^g \\ \theta_3^g \end{bmatrix}_k \tag{25}$$

if k is a kink node. Expanding (23),

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}^g = \begin{bmatrix} N_1 & 0 & 0 & N_1 \frac{h_1}{2} s_3 P_{11,1} & N_1 \frac{h_1}{2} s_3 P_{12,1} & N_1 \frac{h_1}{2} s_3 P_{13,1} & \dots & N_n & 0 & 0 & N_n \frac{h_n}{2} s_3 P_{11,n} & N_n \frac{h_n}{2} s_3 P_{12,n} & N_n \frac{h_n}{2} s_3 P_{13,n} \\ 0 & N_1 & 0 & N_1 \frac{h_1}{2} s_3 P_{21,1} & N_1 \frac{h_1}{2} s_3 P_{22,1} & N_1 \frac{h_1}{2} s_3 P_{23,1} & \dots & 0 & N_n & 0 & N_n \frac{h_n}{2} s_3 P_{21,n} & N_n \frac{h_n}{2} s_3 P_{22,n} & N_n \frac{h_n}{2} s_3 P_{23,n} \\ 0 & 0 & N_1 & N_1 \frac{h_1}{2} s_3 P_{31,1} & N_1 \frac{h_1}{2} s_3 P_{32,1} & N_1 \frac{h_1}{2} s_3 P_{33,1} & \dots & 0 & 0 & N_n & N_n \frac{h_n}{2} s_3 P_{31,n} & N_n \frac{h_n}{2} s_3 P_{32,n} & N_n \frac{h_n}{2} s_3 P_{33,n} \end{bmatrix} \begin{bmatrix} \bar{u}_{1,1} \\ \bar{u}_{2,1} \\ \bar{u}_{3,1} \\ \theta_{1,1} \\ \theta_{2,1} \\ \theta_{3,1} \\ \dots \\ \bar{u}_{1,n} \\ \bar{u}_{2,n} \\ \bar{u}_{3,n} \\ \theta_{1,n} \\ \theta_{2,n} \\ \theta_{3,n} \end{bmatrix} \tag{26}$$

or, in a more compact form,

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}^g = \begin{bmatrix} \underline{N}_{u,1} \\ \underline{N}_{u,2} \\ \underline{N}_{u,3} \end{bmatrix} \underline{U} = \underline{N}_u \, \underline{U} \tag{27}$$

where $\underline{N}_{u,1}$, $\underline{N}_{u,2}$ and $\underline{N}_{u,3}$ have the dimension (1,6*n*) and $\underline{U}$ is the vector with the element nodal displacements, of a dimension (6*n*,1).

7 STRAIN COMPONENTS

The strains and the corresponding stresses are defined in the tangential coordinate system. Since shell is a thin laminar structure, the stress orthogonal to the shell middle surface is assumed null ($\sigma_{t_3} = 0$). Taking the Von Karmán hypotheses (Fung 1965), the Cauchy strain vector in the tangential coordinate system is defined from the following expression:

$$\underline{\varepsilon}^t = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{13} \end{bmatrix}^t = \begin{bmatrix} \dfrac{\partial w_1}{\partial t_1} \\ \dfrac{\partial w_2}{\partial t_2} \\ \dfrac{\partial w_1}{\partial t_2} + \dfrac{\partial w_2}{\partial t_1} \\ \dfrac{\partial w_2}{\partial t_3} + \dfrac{\partial w_3}{\partial t_2} \\ \dfrac{\partial w_1}{\partial t_3} + \dfrac{\partial w_3}{\partial t_1} \end{bmatrix} \tag{28}$$

where w_1, w_2 and w_3 are the displacements in t_1, t_2 and t_3 axis. These strain components can be extracted from the matrix:

$$\underline{\overline{\varepsilon}}^t = \begin{bmatrix} \dfrac{\partial w_1}{\partial t_1} & \dfrac{\partial w_1}{\partial t_2} & \dfrac{\partial w_1}{\partial t_3} \\ \dfrac{\partial w_2}{\partial t_1} & \dfrac{\partial w_2}{\partial t_2} & \dfrac{\partial w_2}{\partial t_3} \\ \dfrac{\partial w_3}{\partial t_1} & \dfrac{\partial w_3}{\partial t_2} & \dfrac{\partial w_3}{\partial t_3} \end{bmatrix} \tag{29}$$

The matrix $\underline{\overline{\varepsilon}}^t$ can be obtained from $\underline{\overline{\varepsilon}}^g$, which includes the derivatives of the displacements in the global coordinate system (u_1, u_2, u_3) in relation to the axis of this system,

$$\underline{\bar{\varepsilon}}^{g} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix} \qquad (30)$$

using for this purpose the following expression:

$$\underline{\bar{\varepsilon}}^{t} = \underline{T}^{gt} \underline{\bar{\varepsilon}}^{g} \left[\underline{T}^{gt} \right]^{T} \qquad (31)$$

where $\underline{T}^{gt}$ is the transformation matrix from global to tangential coordinate systems with the rows being composed by the unit vectors of the axes of the tangential coordinate system,

$$\underline{T}^{gt} = \begin{bmatrix} \hat{t}_1 \\ \hat{t}_2 \\ \hat{t}_3 \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{bmatrix} \qquad (32)$$

The derivatives of $\underline{\bar{\varepsilon}}^{g}$ cannot be directly evaluated, since shape functions, depending on the natural coordinates (s_i), are used to define the displacement field (see expression (23)). To overcome this obstacle the following expression is used:

$$\underline{\bar{\varepsilon}}^{g} = \underline{\bar{\varepsilon}}^{s} \, \underline{J}^{-1} \qquad (33)$$

where $\underline{\bar{\varepsilon}}^{s}$ is the matrix with the derivatives of the displacements in the global coordinate system (u_1, u_2, u_3) in relation to the axis of the natural system s_i,

$$\underline{\varepsilon}^{s} = \begin{bmatrix} \frac{\partial u_1}{\partial s_1} & \frac{\partial u_1}{\partial s_2} & \frac{\partial u_1}{\partial s_3} \\ \frac{\partial u_2}{\partial s_1} & \frac{\partial u_2}{\partial s_2} & \frac{\partial u_2}{\partial s_3} \\ \frac{\partial u_3}{\partial s_1} & \frac{\partial u_3}{\partial s_2} & \frac{\partial u_3}{\partial s_3} \end{bmatrix} \tag{34}$$

and $\underline{J}$ is the Jacobian matrix,

$$\underline{J} = \begin{bmatrix} \frac{\partial x_1}{\partial s_1} & \frac{\partial x_1}{\partial s_2} & \frac{\partial x_1}{\partial s_3} \\ \frac{\partial x_2}{\partial s_1} & \frac{\partial x_2}{\partial s_2} & \frac{\partial x_2}{\partial s_3} \\ \frac{\partial x_3}{\partial s_1} & \frac{\partial x_3}{\partial s_2} & \frac{\partial x_3}{\partial s_3} \end{bmatrix} \tag{35}$$

To evaluate the Jacobian matrix a new format will be given to expression (7),

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \overline{x}_{1,1} & \cdots & \overline{x}_{1,n} \\ \overline{x}_{2,1} & \cdots & \overline{x}_{2,n} \\ \overline{x}_{3,1} & \cdots & \overline{x}_{3,n} \end{bmatrix} \begin{bmatrix} N_1 \\ \vdots \\ N_n \end{bmatrix} + s_3 \begin{bmatrix} n_{31,1} & \cdots & n_{31,n} \\ n_{32,1} & \cdots & n_{32,n} \\ n_{33,1} & \cdots & n_{33,n} \end{bmatrix} \begin{bmatrix} \frac{h_1}{2} & & 0 \\ & \ddots & \\ 0 & & \frac{h_n}{2} \end{bmatrix} \begin{bmatrix} N_1 \\ \vdots \\ N_n \end{bmatrix} \tag{36}$$

or, in a more condensed format,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underline{\overline{X}}^{T} \, \underline{N}_V + s_3 \, \underline{L}_{n3}^{T} \, \underline{H} \, \underline{N}_V \tag{37}$$

where

$$\underline{\bar{X}}^T = \begin{bmatrix} \bar{x}_{1,1} & \cdots & \bar{x}_{1,n} \\ \bar{x}_{2,1} & \cdots & \bar{x}_{2,n} \\ \bar{x}_{3,1} & \cdots & \bar{x}_{3,n} \end{bmatrix} \tag{38}$$

is the matrix (3×*n*) with the global coordinates of the nodes of the finite element,

$$\underline{N}_V = \begin{bmatrix} N_1 \\ \vdots \\ N_n \end{bmatrix} \tag{39}$$

is the vector of the shape functions of the nodes of the element, with a dimension of (*n*×1),

$$\underline{L}_{n3}^T = \begin{bmatrix} n_{31,1} & \cdots & n_{31,n} \\ n_{32,1} & \cdots & n_{32,n} \\ n_{33,1} & \cdots & n_{33,n} \end{bmatrix} \tag{40}$$

is a (3×n) matrix with the ith column being composed by the components of the n_3 axis of node ith, and

$$\underline{H} = \begin{bmatrix} \frac{h_1}{2} & & 0 \\ & \ddots & \\ 0 & & \frac{h_n}{2} \end{bmatrix} \tag{41}$$

is a (*n*×*n*) diagonal matrix with the half of the thickness of node i^{th} $(h_i/2)$ at the H_{ii} coefficient. To obtain the Jacobian matrix, the (37) expression is differentiated in respect to s_i natural coordinate axis resulting,

$$\frac{\partial x_1}{\partial s_1} = \frac{\partial N_1}{\partial s_1}\bar{x}_{1,1} + \cdots + \frac{\partial N_n}{\partial s_1}\bar{x}_{1,n} + s_3\frac{\partial N_1}{\partial s_1}\frac{h_1}{2}n_{31,1} + \cdots + s_3\frac{\partial N_n}{\partial s_1}\frac{h_n}{2}n_{31,n} \tag{42}$$

$$\frac{\partial x_1}{\partial s_2} = \frac{\partial N_1}{\partial s_2}\bar{x}_{1,1} + \cdots + \frac{\partial N_n}{\partial s_2}\bar{x}_{1,n} + s_3\frac{\partial N_1}{\partial s_2}\frac{h_1}{2}n_{31,1} + \cdots + s_3\frac{\partial N_n}{\partial s_2}\frac{h_n}{2}n_{31,n}$$

$$\frac{\partial x_1}{\partial s_3} = + N_1\frac{h_1}{2}n_{31,1} + \cdots + N_n\frac{h_n}{2}n_{31,n}$$

$$\frac{\partial x_2}{\partial s_1} = \frac{\partial N_1}{\partial s_1}\bar{x}_{2,1} + \cdots + \frac{\partial N_n}{\partial s_1}\bar{x}_{2,n} + s_3\frac{\partial N_1}{\partial s_1}\frac{h_1}{2}n_{32,1} + \cdots + s_3\frac{\partial N_n}{\partial s_1}\frac{h_n}{2}n_{32,n}$$

$$\frac{\partial x_2}{\partial s_2} = \frac{\partial N_1}{\partial s_2}\bar{x}_{2,1} + \cdots + \frac{\partial N_n}{\partial s_2}\bar{x}_{2,n} + s_3\frac{\partial N_1}{\partial s_2}\frac{h_1}{2}n_{32,1} + \cdots + s_3\frac{\partial N_n}{\partial s_2}\frac{h_n}{2}n_{32,n}$$

$$\frac{\partial x_2}{\partial s_3} = + N_1\frac{h_1}{2}n_{32,1} + \cdots + N_n\frac{h_n}{2}n_{32,n}$$

$$\frac{\partial x_3}{\partial s_1} = \frac{\partial N_1}{\partial s_1}\bar{x}_{3,1} + \cdots + \frac{\partial N_n}{\partial s_1}\bar{x}_{3,n} + s_3\frac{\partial N_1}{\partial s_1}\frac{h_1}{2}n_{33,1} + \cdots + s_3\frac{\partial N_n}{\partial s_1}\frac{h_n}{2}n_{33,n}$$

$$\frac{\partial x_3}{\partial s_2} = \frac{\partial N_1}{\partial s_2}\bar{x}_{3,1} + \cdots + \frac{\partial N_n}{\partial s_2}\bar{x}_{3,n} + s_3\frac{\partial N_1}{\partial s_2}\frac{h_1}{2}n_{33,1} + \cdots + s_3\frac{\partial N_n}{\partial s_2}\frac{h_n}{2}n_{33,n}$$

$$\frac{\partial x_3}{\partial s_3} = + N_1\frac{h_1}{2}n_{33,1} + \cdots + N_n\frac{h_n}{2}n_{33,n}$$

If these terms are organized according to (35) expression yields,

$$\underline{J} = \begin{bmatrix} \frac{\partial x_1}{\partial s_1} & \frac{\partial x_1}{\partial s_2} & \frac{\partial x_1}{\partial s_3} \\ \frac{\partial x_2}{\partial s_1} & \frac{\partial x_2}{\partial s_2} & \frac{\partial x_2}{\partial s_3} \\ \frac{\partial x_3}{\partial s_1} & \frac{\partial x_3}{\partial s_2} & \frac{\partial x_3}{\partial s_3} \end{bmatrix} = \begin{bmatrix} \bar{x}_{1,1} & \cdots & \bar{x}_{1,n} \\ \bar{x}_{2,1} & \cdots & \bar{x}_{2,n} \\ \bar{x}_{3,1} & \cdots & \bar{x}_{3,n} \end{bmatrix} \begin{bmatrix} \frac{\partial N_1}{\partial s_1} & \frac{\partial N_1}{\partial s_2} & 0 \\ \vdots & \vdots & \vdots \\ \frac{\partial N_n}{\partial s_1} & \frac{\partial N_n}{\partial s_2} & 0 \end{bmatrix} + \begin{bmatrix} n_{31,1} & \cdots & n_{31,n} \\ n_{32,1} & \cdots & n_{32,n} \\ n_{33,1} & \cdots & n_{33,n} \end{bmatrix} \begin{bmatrix} s_3\frac{\partial N_1}{\partial s_1}\frac{h_1}{2} & s_3\frac{\partial N_1}{\partial s_2}\frac{h_1}{2} & N_1\frac{h_1}{2} \\ \vdots & \vdots & \vdots \\ s_3\frac{\partial N_n}{\partial s_1}\frac{h_n}{2} & s_3\frac{\partial N_n}{\partial s_2}\frac{h_n}{2} & N_n\frac{h_n}{2} \end{bmatrix} \tag{43}$$

The Jacobian matrix can be decomposed into the following two submatrices,

$$\underline{J} = \begin{bmatrix} \frac{\partial x_1}{\partial s_1} & \frac{\partial x_1}{\partial s_2} & \frac{\partial x_1}{\partial s_3} \\ \frac{\partial x_2}{\partial s_1} & \frac{\partial x_2}{\partial s_2} & \frac{\partial x_2}{\partial s_3} \\ \frac{\partial x_3}{\partial s_1} & \frac{\partial x_3}{\partial s_2} & \frac{\partial x_3}{\partial s_3} \end{bmatrix} = \begin{bmatrix} \underline{J}_1 & \underline{J}_2 \end{bmatrix} \tag{44}$$

where

$$\underline{J}_1 = \begin{bmatrix} \frac{\partial x_1}{\partial s_1} & \frac{\partial x_1}{\partial s_2} \\ \frac{\partial x_2}{\partial s_1} & \frac{\partial x_2}{\partial s_2} \\ \frac{\partial x_3}{\partial s_1} & \frac{\partial x_3}{\partial s_2} \end{bmatrix} \tag{45a}$$

and

$$\underline{J}_2 = \begin{bmatrix} \frac{\partial x_1}{\partial s_3} \\ \frac{\partial x_2}{\partial s_3} \\ \frac{\partial x_3}{\partial s_3} \end{bmatrix} \tag{45a}$$

From (42) expression, these Jacobian submatrices can assume the following format,

$$\underline{J}_1 = \overline{\underline{X}}^T \frac{\partial \underline{N}_V}{\partial \underline{\overline{s}}} + s_3 \, \underline{L}_{n3}^T \, \underline{H} \frac{\partial \underline{N}_V}{\partial \underline{\overline{s}}} \tag{46}$$

where $\underline{\overline{s}}$ represents the s_1 and s_2 natural axis, and

$$\underline{J}_2 = \underline{L}_{n3}^T \, \underline{H} \, \underline{N}_V \tag{47}$$

Therefore, each coefficient of the Jacobian matrix can be obtained from the following two equations,

$$J_{ij} = \frac{\partial x_i}{\partial s_j} = \sum_{k=1}^{n} \left(\frac{\partial N_k}{\partial s_j} \overline{x}_{i,k} + s_3 \frac{\partial N_k}{\partial s_j} \frac{h_k}{2} n_{3i,k} \right) \quad \text{for } 1 \leq i \leq 3 \ \wedge \ 1 \leq j \leq 2 \tag{48a}$$

$$J_{i3} = \frac{\partial x_i}{\partial s_3} = \sum_{k=1}^{n} N_k \frac{h_k}{2} n_{3i,k} \quad \text{for } 1 \leq i \leq 3 \tag{48b}$$

To evaluate the terms of $\underline{\overline{\varepsilon}}^s$ (expression (34)) the $\partial u_i / \partial s_j$ should be evaluated. Applying chain rule,

$$\begin{aligned}
\frac{\partial u_1}{\partial s_1} &= \frac{\partial u_1}{\partial x_1}\frac{\partial x_1}{\partial s_1} + \frac{\partial u_1}{\partial x_2}\frac{\partial x_2}{\partial s_1} + \frac{\partial u_1}{\partial x_3}\frac{\partial x_3}{\partial s_1} \\
\frac{\partial u_1}{\partial s_2} &= \frac{\partial u_1}{\partial x_1}\frac{\partial x_1}{\partial s_2} + \frac{\partial u_1}{\partial x_2}\frac{\partial x_2}{\partial s_2} + \frac{\partial u_1}{\partial x_3}\frac{\partial x_3}{\partial s_2} \\
\frac{\partial u_1}{\partial s_3} &= \frac{\partial u_1}{\partial x_1}\frac{\partial x_1}{\partial s_3} + \frac{\partial u_1}{\partial x_2}\frac{\partial x_2}{\partial s_3} + \frac{\partial u_1}{\partial x_3}\frac{\partial x_3}{\partial s_3} \\
\frac{\partial u_2}{\partial s_1} &= \frac{\partial u_2}{\partial x_1}\frac{\partial x_1}{\partial s_1} + \frac{\partial u_2}{\partial x_2}\frac{\partial x_2}{\partial s_1} + \frac{\partial u_2}{\partial x_3}\frac{\partial x_3}{\partial s_1} \\
\frac{\partial u_2}{\partial s_2} &= \frac{\partial u_2}{\partial x_1}\frac{\partial x_1}{\partial s_2} + \frac{\partial u_2}{\partial x_2}\frac{\partial x_2}{\partial s_2} + \frac{\partial u_2}{\partial x_3}\frac{\partial x_3}{\partial s_2} \\
\frac{\partial u_2}{\partial s_3} &= \frac{\partial u_2}{\partial x_1}\frac{\partial x_1}{\partial s_3} + \frac{\partial u_2}{\partial x_2}\frac{\partial x_2}{\partial s_3} + \frac{\partial u_2}{\partial x_3}\frac{\partial x_3}{\partial s_3} \\
\frac{\partial u_3}{\partial s_1} &= \frac{\partial u_3}{\partial x_1}\frac{\partial x_1}{\partial s_1} + \frac{\partial u_3}{\partial x_2}\frac{\partial x_2}{\partial s_1} + \frac{\partial u_3}{\partial x_3}\frac{\partial x_3}{\partial s_1} \\
\frac{\partial u_3}{\partial s_2} &= \frac{\partial u_3}{\partial x_1}\frac{\partial x_1}{\partial s_2} + \frac{\partial u_3}{\partial x_2}\frac{\partial x_2}{\partial s_2} + \frac{\partial u_3}{\partial x_3}\frac{\partial x_3}{\partial s_2} \\
\frac{\partial u_3}{\partial s_3} &= \frac{\partial u_3}{\partial x_1}\frac{\partial x_1}{\partial s_3} + \frac{\partial u_3}{\partial x_2}\frac{\partial x_2}{\partial s_3} + \frac{\partial u_3}{\partial x_3}\frac{\partial x_3}{\partial s_3}
\end{aligned} \tag{49}$$

or, in a matrix format,

$$\begin{bmatrix} \frac{\partial u_1}{\partial s_1} & \frac{\partial u_1}{\partial s_2} & \frac{\partial u_1}{\partial s_3} \\ \frac{\partial u_2}{\partial s_1} & \frac{\partial u_2}{\partial s_2} & \frac{\partial u_2}{\partial s_3} \\ \frac{\partial u_3}{\partial s_1} & \frac{\partial u_3}{\partial s_2} & \frac{\partial u_3}{\partial s_3} \end{bmatrix} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix} \begin{bmatrix} \frac{\partial x_1}{\partial s_1} & \frac{\partial x_1}{\partial s_2} & \frac{\partial x_1}{\partial s_3} \\ \frac{\partial x_2}{\partial s_1} & \frac{\partial x_2}{\partial s_2} & \frac{\partial x_2}{\partial s_3} \\ \frac{\partial x_3}{\partial s_1} & \frac{\partial x_3}{\partial s_2} & \frac{\partial x_3}{\partial s_3} \end{bmatrix} \tag{50}$$

which in a more compact format has the following representation:

$$\frac{\partial \underline{u}}{\partial \underline{s}} = \frac{\partial \underline{u}}{\partial \underline{x}} \underline{J} \tag{51}$$

or

$$\underline{\varepsilon}^{g} = \underline{\varepsilon}^{s} \underline{J}^{-1} \tag{52}$$

Introducing (52) in to (31) results,

$$\underline{\varepsilon}^{t} = \underline{T}^{gt} \underline{\varepsilon}^{s} \underline{J}^{-1} \left[\underline{T}^{gt} \right]^{T} \tag{53}$$

The $\underline{\varepsilon}^{s}$ can be obtained from the following expression,

$$\underline{\varepsilon}^{s} = \begin{bmatrix} \frac{\partial u_1}{\partial s_1} & \frac{\partial u_1}{\partial s_2} & \frac{\partial u_1}{\partial s_3} \\ \frac{\partial u_2}{\partial s_1} & \frac{\partial u_2}{\partial s_2} & \frac{\partial u_2}{\partial s_3} \\ \frac{\partial u_3}{\partial s_1} & \frac{\partial u_3}{\partial s_2} & \frac{\partial u_3}{\partial s_3} \end{bmatrix} = \underline{D}_u \underline{U}_M \tag{54}$$

where $\underline{D}_u$ is a matrix of dimension (3×18*n*),

$$\underline{D}_u = \begin{bmatrix} \frac{\partial \underline{N}_{u,1}}{\partial s_1} & \frac{\partial \underline{N}_{u,1}}{\partial s_2} & \frac{\partial \underline{N}_{u,1}}{\partial s_3} \\ \frac{\partial \underline{N}_{u,2}}{\partial s_1} & \frac{\partial \underline{N}_{u,2}}{\partial s_2} & \frac{\partial \underline{N}_{u,2}}{\partial s_3} \\ \frac{\partial \underline{N}_{u,3}}{\partial s_1} & \frac{\partial \underline{N}_{u,3}}{\partial s_2} & \frac{\partial \underline{N}_{u,3}}{\partial s_3} \end{bmatrix} \tag{55}$$

with each $\partial \underline{N}_{u,i} / \partial s_j$ having a dimension of (1×6*n*), and

$$\underline{U}_M = \begin{bmatrix} \underline{U} & 0 & 0 \\ 0 & \underline{U} & 0 \\ 0 & 0 & \underline{U} \end{bmatrix} \tag{56}$$

is a matrix of (18*n*×3) dimension, where $\underline{U}$ is a vector of (6*n*×1) of the degrees of freedom of the nodes of the element. Inserting (54) into (53) results,

$$\underline{\overline{\varepsilon}}^t = \underline{T}^{gt}\, \underline{D}_u\, \underline{U}_M\, \underline{J}^{-1} \left[\underline{T}^{gt} \right]^T \tag{57}$$

Evaluating $\partial \underline{N}_{u,i} / \partial s_j$ of $\underline{D}_u$ yields,

$$
\begin{aligned}
\frac{\partial N_{u,1}}{\partial s_1} &= \Big[\frac{\partial N_1}{\partial s_1}\; 0\; 0\; \frac{\partial N_1}{\partial s_1}\frac{h_1}{2}s_3P_{11,1}\; \frac{\partial N_1}{\partial s_1}\frac{h_1}{2}s_3P_{12,1}\; \frac{\partial N_1}{\partial s_1}\frac{h_1}{2}s_3P_{13,1} \cdots \\
&\qquad \cdots \frac{\partial N_n}{\partial s_1}\; 0\; 0\; \frac{\partial N_n}{\partial s_1}\frac{h_n}{2}s_3P_{11,n}\; \frac{\partial N_n}{\partial s_1}\frac{h_n}{2}s_3P_{12,n}\; \frac{\partial N_n}{\partial s_1}\frac{h_n}{2}s_3P_{13,n} \Big] \\
\frac{\partial N_{u,1}}{\partial s_2} &= \Big[\frac{\partial N_1}{\partial s_2}\; 0\; 0\; \frac{\partial N_1}{\partial s_2}\frac{h_1}{2}s_3P_{11,1}\; \frac{\partial N_1}{\partial s_2}\frac{h_1}{2}s_3P_{12,1}\; \frac{\partial N_1}{\partial s_2}\frac{h_1}{2}s_3P_{13,1} \cdots \\
&\qquad \cdots \frac{\partial N_n}{\partial s_2}\; 0\; 0\; \frac{\partial N_n}{\partial s_2}\frac{h_n}{2}s_3P_{11,n}\; \frac{\partial N_n}{\partial s_2}\frac{h_n}{2}s_3P_{12,n}\; \frac{\partial N_n}{\partial s_2}\frac{h_n}{2}s_3P_{13,n} \Big] \\
\frac{\partial N_{u,1}}{\partial s_3} &= \Big[0\; 0\; 0\; N_1\frac{h_1}{2}P_{11,1}\; N_1\frac{h_1}{2}P_{12,1}\; N_1\frac{h_1}{2}P_{13,1} \cdots \\
&\qquad \cdots 0\; 0\; 0\; N_n\frac{h_n}{2}P_{11,n}\; N_n\frac{h_n}{2}P_{12,n}\; N_n\frac{h_n}{2}P_{13,n} \Big] \\
\frac{\partial N_{u,2}}{\partial s_1} &= \Big[0\; \frac{\partial N_1}{\partial s_1}\; 0\; \frac{\partial N_1}{\partial s_1}\frac{h_1}{2}s_3P_{21,1}\; \frac{\partial N_1}{\partial s_1}\frac{h_1}{2}s_3P_{22,1}\; \frac{\partial N_1}{\partial s_1}\frac{h_1}{2}s_3P_{23,1} \cdots \\
&\qquad \cdots 0\; \frac{\partial N_n}{\partial s_1}\; 0\; \frac{\partial N_n}{\partial s_1}\frac{h_n}{2}s_3P_{21,n}\; \frac{\partial N_n}{\partial s_1}\frac{h_n}{2}s_3P_{22,n}\; \frac{\partial N_n}{\partial s_1}\frac{h_n}{2}s_3P_{23,n} \Big] \\
\frac{\partial N_{u,2}}{\partial s_2} &= \Big[0\; \frac{\partial N_1}{\partial s_2}\; 0\; \frac{\partial N_1}{\partial s_2}\frac{h_1}{2}s_3P_{21,1}\; \frac{\partial N_1}{\partial s_2}\frac{h_1}{2}s_3P_{22,1}\; \frac{\partial N_1}{\partial s_2}\frac{h_1}{2}s_3P_{23,1} \cdots \\
&\qquad \cdots 0\; \frac{\partial N_n}{\partial s_2}\; 0\; \frac{\partial N_n}{\partial s_2}\frac{h_n}{2}s_3P_{21,n}\; \frac{\partial N_n}{\partial s_2}\frac{h_n}{2}s_3P_{22,n}\; \frac{\partial N_n}{\partial s_2}\frac{h_n}{2}s_3P_{23,n} \Big] \\
\frac{\partial N_{u,2}}{\partial s_3} &= \Big[0\; 0\; 0\; N_1\frac{h_1}{2}P_{21,1}\; N_1\frac{h_1}{2}P_{22,1}\; N_1\frac{h_1}{2}P_{23,1} \cdots \\
&\qquad \cdots 0\; 0\; 0\; N_n\frac{h_n}{2}P_{21,n}\; N_n\frac{h_n}{2}P_{22,n}\; N_n\frac{h_n}{2}P_{23,n} \Big] \\
\frac{\partial N_{u,3}}{\partial s_1} &= \Big[0\; 0\; \frac{\partial N_1}{\partial s_1}\; \frac{\partial N_1}{\partial s_1}\frac{h_1}{2}s_3P_{31,1}\; \frac{\partial N_1}{\partial s_1}\frac{h_1}{2}s_3P_{32,1}\; \frac{\partial N_1}{\partial s_1}\frac{h_1}{2}s_3P_{33,1} \cdots \\
&\qquad \cdots 0\; 0\; \frac{\partial N_n}{\partial s_1}\; \frac{\partial N_n}{\partial s_1}\frac{h_n}{2}s_3P_{31,n}\; \frac{\partial N_n}{\partial s_1}\frac{h_n}{2}s_3P_{32,n}\; \frac{\partial N_n}{\partial s_1}\frac{h_n}{2}s_3P_{33,n} \Big] \\
\frac{\partial N_{u,3}}{\partial s_2} &= \Big[0\; 0\; \frac{\partial N_1}{\partial s_2}\; \frac{\partial N_1}{\partial s_2}\frac{h_1}{2}s_3P_{31,1}\; \frac{\partial N_1}{\partial s_2}\frac{h_1}{2}s_3P_{32,1}\; \frac{\partial N_1}{\partial s_2}\frac{h_1}{2}s_3P_{33,1} \cdots \\
&\qquad \cdots 0\; 0\; \frac{\partial N_n}{\partial s_2}\; \frac{\partial N_n}{\partial s_2}\frac{h_n}{2}s_3P_{31,n}\; \frac{\partial N_n}{\partial s_2}\frac{h_n}{2}s_3P_{32,n}\; \frac{\partial N_n}{\partial s_2}\frac{h_n}{2}s_3P_{33,n} \Big] \\
\frac{\partial N_{u,3}}{\partial s_3} &= \Big[0\; 0\; 0\; N_1\frac{h_1}{2}P_{31,1}\; N_1\frac{h_1}{2}P_{32,1}\; N_1\frac{h_1}{2}P_{33,1} \cdots \\
&\qquad \cdots 0\; 0\; 0\; N_n\frac{h_n}{2}P_{31,n}\; N_n\frac{h_n}{2}P_{32,n}\; N_n\frac{h_n}{2}P_{33,n} \Big]
\end{aligned}
\tag{58}
$$

Each $\partial \underline{N}_{u,i} / \partial s_j$ coefficient can be obtained from the following three equations,

$$\underline{D}_{u,i1} = \frac{\partial \underline{N}_{u,i}}{\partial s_1} = \left[\cdots \frac{\partial N_k}{\partial s_1} \; 0 \; 0 \; \frac{\partial N_k}{\partial s_1}\frac{h_k}{2} s_3 P_{i1,k} \; \frac{\partial N_k}{\partial s_1}\frac{h_k}{2} s_3 P_{i2,k} \; \frac{\partial N_k}{\partial s_1}\frac{h_k}{2} s_3 P_{i3,k} \cdots \right] \tag{59a}$$

$$\text{for } 1 \leq i \leq 3 \;\wedge\; j = 1$$

$$\underline{D}_{u,i2} = \frac{\partial \underline{N}_{u,i}}{\partial s_2} = \left[\cdots 0 \; \frac{\partial N_k}{\partial s_2} \; 0 \; \frac{\partial N_k}{\partial s_2}\frac{h_k}{2} s_3 P_{i1,k} \; \frac{\partial N_k}{\partial s_2}\frac{h_k}{2} s_3 P_{i2,k} \; \frac{\partial N_k}{\partial s_2}\frac{h_k}{2} s_3 P_{i3,k} \cdots \right] \tag{59b}$$

$$\text{for } 1 \leq i \leq 3 \;\wedge\; j = 2$$

$$\underline{D}_{u,i3} = \frac{\partial \underline{N}_{u,i}}{\partial s_3} = \left[\cdots 0 \; 0 \; 0 \; N_k \frac{h_k}{2} P_{i1,k} \; N_k \frac{h_k}{2} P_{i2,k} \; N_k \frac{h_k}{2} P_{i3,k} \cdots \right] \tag{59c}$$

$$\text{for } 1 \leq i \leq 3 \;\wedge\; j = 3$$

Inserting (59) into (57) and executing the matrix operations, the $\underline{\overline{\varepsilon}}^t$ is calculated, from which the Cauchy strain vector in the tangential coordinate system, indicated in (28), can be obtained (see Annex 1),

$$\underline{\varepsilon}^t = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{13} \end{bmatrix}^t = \begin{bmatrix} \underline{B}_{mb} \\ -- \\ \underline{B}_s \end{bmatrix}^t \underline{U} = \underline{B}^t \, \underline{U} \tag{60}$$

where $\underline{B}_{mb}^t$ is the in-plane strain matrix of a dimension (3×6*n*), and $\underline{B}_s^t$ is the out-of-plane shear strain matrix of a dimension of (2×6*n*). Both submatrices are defined in the tangential coordinate system.

8 STRESS COMPONENTS

The stresses are calculated in the tangential coordinate system of each integration point. For shell element, null value is assumed for the stress orthogonal to its middle surface ($\sigma_{t_3} = 0$). The stress vector, $\underline{\sigma}_t$, in correspondence to the strain vector $\underline{\varepsilon}_t$ has the following components,

$$\underline{\sigma}^t = \left\{ \sigma_1^t ,\ \sigma_2^t ,\ \tau_{12}^t ,\ \tau_{23}^t ,\ \tau_{13}^t \right\}^T \tag{61}$$

the first three ones are in-plane stress components (in plane $t_1 t_2$) that originate the membrane forces and the bending moments,

$$\underline{\sigma}_{mb}^t = \left\{ \sigma_1^t ,\ \sigma_2^t ,\ \tau_{12}^t \right\}^T \tag{62a}$$

and the last two ones are the out-of-plane shear components (orthogonal to $t_1 t_2$ plane) that originate the out-of-plane shear forces,

$$\underline{\sigma}_s^t = \left\{ \tau_{23}^t ,\ \tau_{13}^t \right\}^T \tag{62b}$$

Therefore, the stress vector (61) can be transformed in the following format:

$$\underline{\sigma}^t = \left\{ \underline{\sigma}_{mb}^t ,\ \underline{\sigma}_s^t \right\}^T \tag{62c}$$

The stress components in a generic integration point are represented in Figure 12.

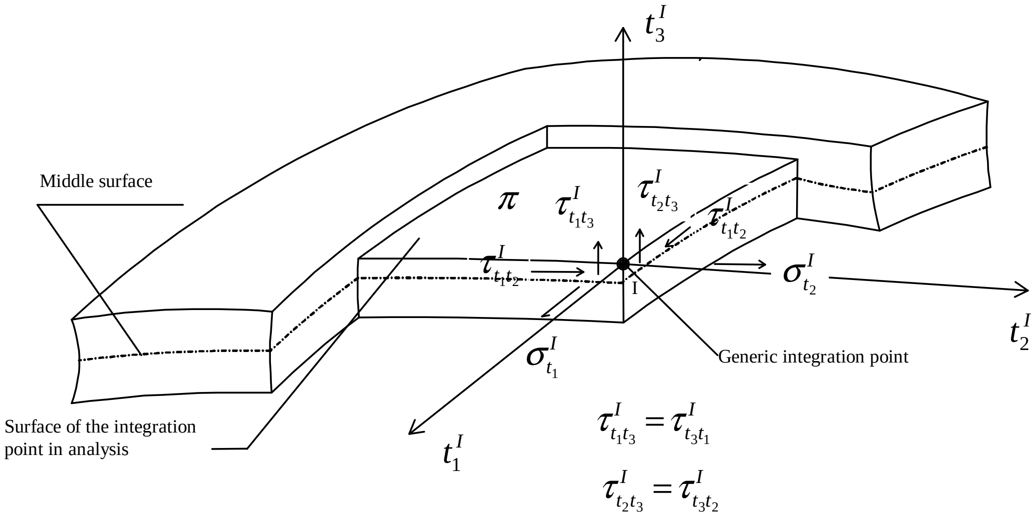


Figure 12 - Stresses in the tangential coordinate system.

In this figure $t_{1,I}$, $t_{2,I}$, $t_{3,I}$ are the axes of the tangential coordinate system of the integration point (IP) *I*. The procedure to define this referential was exposed in section 4.3. The $t_{1,I}$ and $t_{2,I}$ axes define a plane tangent to surface π (of constant s_3) at the IP *I*, while $t_{3,I}$ is orthogonal to this plane at this IP. Hence,

$\sigma^t_{1,I}$ is the stress normal to a plane that is orthogonal to $t_{1,I}$ and passes through IP *I*;
$\sigma^t_{2,I}$ is the stress normal to a plane that is orthogonal to $t_{2,I}$ and passes through IP *I*;
$\tau^t_{12,I}$ is the in-plane shear stress, in the $t_{1,I}$ $t_{2,I}$ plane;
$\tau^t_{23,I}$ is the shear stress in the $t_{3,I}$ direction, in a plane that is orthogonal to $t_{2,I}$ and passes through IP *I*;
$\tau^t_{13,I}$ is the shear stress in the $t_{3,I}$ direction, in a plane that is orthogonal to $t_{1,I}$ and passes through IP *I*;

The stress components in the global coordinate system can be obtained from the the stress components in the tangential coordinate system, performing the following matrix operation,

$$\underline{\sigma}^g = \left[\underline{T}^{gt}\right]^T \underline{\sigma}^t \, \underline{T}^{gt} \tag{63}$$

where,

$$\underline{\bar{\sigma}}^t = \begin{bmatrix} \sigma_1^t & \tau_{12}^t & \tau_{13}^t \\ \tau_{12}^t & \sigma_2^t & \tau_{23}^t \\ \tau_{13}^t & \tau_{23}^t & 0 \end{bmatrix} \tag{64}$$

is the stress tensor in the tangential coordinate system,

$$\underline{\bar{\sigma}}^g = \begin{bmatrix} \sigma_1^g & \tau_{12}^g & \tau_{13}^g \\ \tau_{12}^g & \sigma_2^g & \tau_{23}^g \\ \tau_{13}^g & \tau_{23}^g & \sigma_3^g \end{bmatrix} \tag{65}$$

is the stress tensor in the global coordinate system, and $\underline{T}^{gt}$ is a matrix that converts entities from global to tangential coordinate system, which was defined in expression (32). The components of the stress vector in global coordinate system, $\underline{\sigma}^g$, can be obtained from (65), resulting,

$$\underline{\sigma}^g = \left\{ \sigma_1^g,\, \sigma_2^g,\, \sigma_3^g,\, \tau_{12}^g,\, \tau_{23}^g,\, \tau_{13}^g \right\}^T \tag{66}$$

9 CONSTITUTIVE EQUATIONS FOR LINEAR-ELASTIC MATERIALS

The stress vector, $\underline{\sigma}^t$, of equation (61) can be obtained from the strain vector, $\underline{\varepsilon}^t$, (equation (28)) by the following constitutive equation,

$$\underline{\sigma}^t = \underline{D}\,\underline{\varepsilon}^t \tag{67a}$$

or,

$$\begin{bmatrix} \underline{\sigma}^t_{mb} \\ \underline{\sigma}^t_s \end{bmatrix} = \begin{bmatrix} \underline{D}_{mb} & \underline{0} \\ \underline{0} & \underline{D}_s \end{bmatrix} \begin{bmatrix} \underline{\varepsilon}^t_{mb} \\ \underline{\varepsilon}^t_s \end{bmatrix} \tag{67b}$$

where, for materials with linear elastic behaviour,

$$\underline{D}_{mb} = \frac{E}{1-\upsilon^2} \begin{bmatrix} 1 & \upsilon & 0 \\ \upsilon & 1 & 0 \\ 0 & 0 & \frac{1-\upsilon}{2} \end{bmatrix} \tag{68a}$$

is the in-plane membrane/bending constitutive matrix and,

$$\underline{D}_s = F\ G \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{68b}$$

is the out-of-plane shear constitutive matrix. In (68) E, υ and G are the Young's modulus, the Poisson coefficient and the shear modulus of the concrete. In (69b) $F=5/6$ is a correction shear factor (Barros 1995).

If uniform and differential temperature variation (ΔT_u and ΔT_d, respectively) induce strain components represented by the $\underline{\varepsilon}^t_{\Delta T_u}$ and $\underline{\varepsilon}^t_{\Delta T_d}$ vectors, respectively, the strain vector due to the element nodal displacement, $\underline{\varepsilon}^t_{mb}$ in (67), should be replaced by $\underline{\varepsilon}^t_{mb} - \underline{\varepsilon}^t_{\Delta T_u} - \underline{\varepsilon}^t_{\Delta T_d}$, resulting,

$$\begin{bmatrix} \underline{\sigma}_{mb}^{t} \\ \underline{\sigma}_{s}^{t} \end{bmatrix} = \begin{bmatrix} \underline{D}_{mb} & \underline{0} \\ \underline{0} & \underline{D}_{s} \end{bmatrix} \begin{bmatrix} \underline{\varepsilon}_{mb}^{t} - \underline{\varepsilon}_{\Delta T_u}^{t} - \underline{\varepsilon}_{\Delta T_d}^{t} \\ \underline{\varepsilon}_{s}^{t} \end{bmatrix} \tag{69}$$

10 STIFFNESS MATRIX FOR LINEAR-ELASTIC MATERIALS

The stiffness matrix of a shell finite element is obtained applying the principle of virtual work,

$$\delta W_{\text{int}} = \int_V \left[\delta \underline{\varepsilon}^t \right]^T \underline{\sigma}^t \, dV \tag{70}$$

Introducing (60) and (67) into (70) results,

$$\delta W_{\text{int}} = \delta \underline{U}^T \int_V \left[\underline{B}^t \right]^T \underline{D} \, \underline{B}^t \, dV \, \underline{U} \tag{71}$$

from which it can be extracted the element stiffness matrix,

$$\underline{K} = \int_V \left[\underline{B}^t \right]^T \underline{D} \, \underline{B}^t \, dV \tag{72}$$

The lines and columns of $\underline{K}$ corresponding to the sixth degree of freedom of coplanar nodes are composed by null terms. To avoid the occurrence of null pivot when solving the system of equilibrium equations of the structure, a non-null term is introduced in the $k_{6i,6i}$ of the i^{th} coplanar node. If the system of equations is pre-treated in order to separate the free (subscript *f*) and the prescribed (subscript *p*) degrees of freedom,

$$\begin{bmatrix} \underline{K}_{E,ff} & \underline{K}_{E,fp} \\ \underline{K}_{E,pf} & \underline{K}_{E,pp} \end{bmatrix} \begin{bmatrix} \underline{U}_{E,f} \\ \underline{U}_{E,p} \end{bmatrix} = \begin{bmatrix} \underline{Q}_{E,f} \\ \underline{Q}_{E,p} + \underline{R}_E \end{bmatrix} \tag{73}$$

The sixth degree of freedom of coplanar nodes should be declared as a prescribed degree of freedom in order to avoid null pivot, since, in this way, only the free degrees of freedom are part of the system equations to solve,

$$\begin{aligned} \underline{K}_{E,ff} \, \underline{U}_{E,f} &= \underline{Q}_{E,f} - \underline{K}_{E,fp} \, \underline{U}_{E,p} \\ &= \underline{\bar{Q}}_{E,f} \end{aligned} \tag{74}$$

11 LOAD VECTOR

11.1 INTRODUCTION

A shell structure can be submitted to the following load cases:

- Point loads;
- Gravity load;
- Generalized forces per unit length;
- Generalized forces per unit area;
- Uniform temperature variation;
- Differential temperature variation;
- Prescribed displacements.

For plates and plane shells the procedures to obtain the nodal forces equivalent to these load cases were described elsewhere (Barros 2000). In this chapter only the specificities for Ahmad shell structures are dealt with.

11.2 POINT LOADS

In the coplanar nodes, three force components in the global coordinate system and two moment components in the nodal coordinate system can actuate. In the kink nodes, three force components and three moment components can be applied, all of them, in the global coordinate system.

11.3 GRAVITY LOAD

Acceleration components in the global coordinate system can be applied to simulate the gravity load. Rotational accelerations are not so important in static analysis of Civil Engineering structures, as far as, they are not considered for the gravity load.

11.4 GENERALIZED FORCES PER UNIT LENGTH

Any edge of a finite element can be submitted to three forces and two moments distributed per unit length. These generalized forces are defined in the nodal-edge coordinate system, as it is represented in Figure 13. The l_1 axis is tangent to the edge at the node, and its direction is defined from the numeration order given to the nodes of the edge. The l_2 axis is in the plane tangent to the middle surface of shell element at the node, and it is pointing to the interior of the element. For a generic edge coordinate, s_p, of a loaded edge, the unit vectors of l_1 ($\hat{l}_1$) and l_2 ($\hat{l}_2$) are defined from the following expression

$$\left[\hat{l}_1 \quad \hat{l}_2\right]_{s_p} = \begin{bmatrix} \dfrac{\partial x_1/\partial s_1}{\|\underline{l}_1\|} & \dfrac{\partial x_1/\partial s_2}{\|\underline{l}_2\|} \\ \dfrac{\partial x_2/\partial s_1}{\|\underline{l}_1\|} & \dfrac{\partial x_2/\partial s_2}{\|\underline{l}_2\|} \\ \dfrac{\partial x_3/\partial s_1}{\|\underline{l}_1\|} & \dfrac{\partial x_3/\partial s_2}{\|\underline{l}_2\|} \end{bmatrix}_{s_p} \tag{75}$$

where $\|\underline{l}_1\|$ and $\|\underline{l}_2\|$ are the norms of the vectors $\underline{l}_1$ and $\underline{l}_2$, respectively. The $\hat{l}_3$ unit vector results from the cross product of vector $\hat{l}_1$ by vector $\hat{l}_2$ at the s_p coordinate,

$$\hat{l}_{3,s_p} = \hat{l}_{1,s_p} \times \hat{l}_{2,s_p} \tag{76}$$

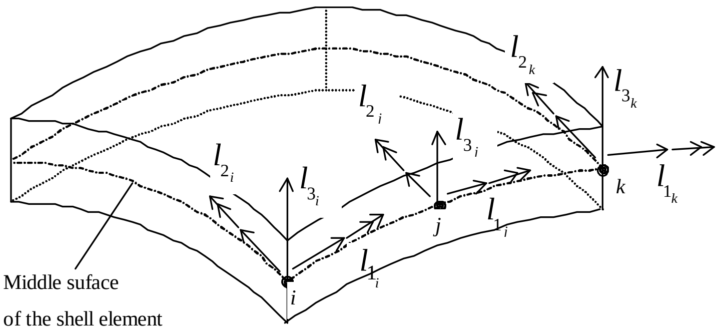


Figure 13 - Coordinate systems for the generalized forces per unit length of a shell finite element.

Therefore, the procedure to determine the nodal-edge coordinate system is equal to the one of the tangential coordinate system, apart that the last one is evaluated in the element IP, while the former in determined in the edge IP.

The distributed generalized forces are transferred from the nodal-edge coordinate system, $\underline{q}_L^e$, to the global coordinate system, $\underline{q}_L^g$, using the following relationship,

$$\left(\underline{q}_L^g\right)_{s_p} = \left[\underline{T}^{ge}\right]_{s_p}^T \left(\underline{q}_L^e\right)_{s_p} \tag{77}$$

where $\underline{T}_{s_p}^{ge}$ is the transformation matriz from global to nodal-edge coordinate systems, at s_p edge coordinate. The format of $\underline{T}_{s_p}^{ge}$ is dependent of the nodal type. In case of kink node $\underline{T}_{s_p}^{ge}$ has the following format,

$$\left[\underline{T}^{ge}\right]_{(s_p)} = \begin{bmatrix} \hat{l}_1 & \underline{0} \\ \hat{l}_2 & \underline{0} \\ \hat{l}_3 & \underline{0} \\ \underline{0} & \hat{l}_1 \\ \underline{0} & \hat{l}_2 \end{bmatrix}_{(s_p)} \tag{78}$$

If coplanar node, after have been transferred to global coordinate system, the moment components of the generalized force vector should be transferred from the global coordinate system to the nodal coordinate system,

$$\begin{bmatrix} m_{1L}^n \\ m_{2L}^n \end{bmatrix}_{(s_p)} = \begin{bmatrix} \hat{n}_1^T \\ \hat{n}_2^T \end{bmatrix}_{(s_p)} \begin{bmatrix} m_{1L}^g \\ m_{2L}^g \\ m_{3L}^g \end{bmatrix}_{(s_p)} \tag{79}$$

where $\begin{bmatrix} m_{1L}^n & m_{2L}^n \end{bmatrix}_{(s_p)}^T$ and $\begin{bmatrix} m_{1L}^g & m_{2L}^g & m_{3L}^g \end{bmatrix}_{(s_p)}^T$ are the vectors of the distributed moments in the nodal and global coordinate systems at s_p edge coordinate, and $\hat{n}_1$, $\hat{n}_2$ are the unit vectors of n_1 and n_2 axes of the nodal coordinate system. The moment components in the nodal- edge coordinate system can be transformed to moment components in the global coordinate system from,

$$\begin{bmatrix} m_{1L}^{g} \\ m_{2L}^{g} \\ m_{3L}^{g} \end{bmatrix}_{(s_p)} = \begin{bmatrix} \hat{l}_1 & \hat{l}_2 \end{bmatrix}_{(s_p)} \begin{bmatrix} m_{1L}^{e} \\ m_{2L}^{e} \end{bmatrix}_{(s_p)} \tag{80}$$

Introducing (80) into (79) yields,

$$\begin{bmatrix} m_{1L}^{n} \\ m_{2L}^{n} \end{bmatrix}_{(s_p)} = \begin{bmatrix} \hat{n}_1^{T} \\ \hat{n}_2^{T} \end{bmatrix}_{(s_p)} \begin{bmatrix} \hat{l}_1 & \hat{l}_2 \end{bmatrix}_{(s_p)} \begin{bmatrix} m_{1L}^{e} \\ m_{2L}^{e} \end{bmatrix}_{(s_p)} \tag{81}$$

From which the moment components in the nodal-edge coordinate system, m_{1L}^{e}, m_{2L}^{e}, is transferred to nodal coordinate system, m_{1L}^{n}, m_{2L}^{n}.

11.5 GENERALIZED FORCES PER UNIT AREA

The procedure for obtaining the nodal forces equivalent to generalized forces distributed in the finite element area is described elsewhere (Barros 2000) for the case of plane shells. For Ahmad shell elements special care should be taken with the coplanar nodes. In this type of nodes, the moment components should be transferred to the nodal coordinate system,

$$\begin{bmatrix} m_{1A}^{n} \\ m_{2A}^{n} \end{bmatrix}_{(s_1^p, s_2^p)} = \begin{bmatrix} \hat{n}_1^{T} \\ \hat{n}_2^{T} \end{bmatrix}_{(s_1^p, s_2^p)} \begin{bmatrix} \hat{r}_1 & \hat{r}_2 \end{bmatrix}_{(s_1^p, s_2^p)} \begin{bmatrix} m_{1A}^{r} \\ m_{2A}^{r} \end{bmatrix}_{(s_1^p, s_2^p)} \tag{82}$$

where $\begin{bmatrix} m_{1A}^{r} & m_{2A}^{r} \end{bmatrix}_{(s_1^p, s_2^p)}^{T}$ are the moment components in the user-defined coordinate system, r_i, at the element coordinates s_1^p, s_2^p, $\hat{r}_1$ and $\hat{r}_2$ are the unit vectors of the r_1 and r_2 user-defined coordinate system, and m_{1A}^{n}, m_{2A}^{n} are the moment components at the nodal coordinate system.

In kink nodes, the generalized forces should be transferred from user-defined coordinate system to global coordinate system.

11.6 UNIFORM TEMPERATURE VARIATION

Figure 14 represents a shell element submitted to uniform temperature variation of ΔT_u .

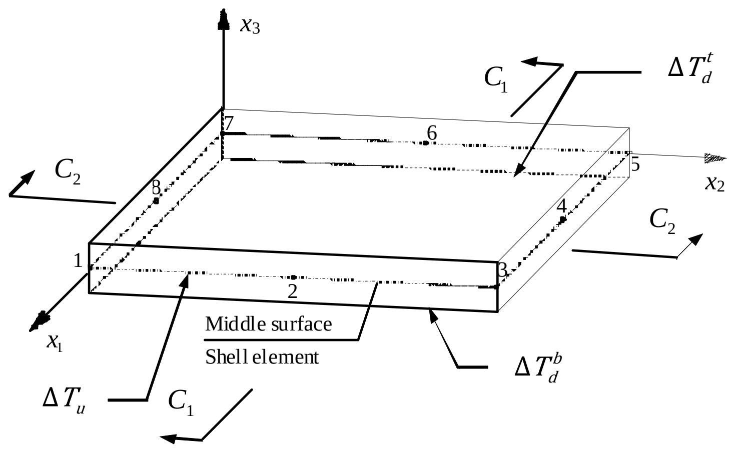


Figure 14 - Deformation due to uniform temperature variation.

Due to ΔT_u this element undergoes a strain of $\varepsilon_{\Delta T_u}^t$ in t_1 and t_2 axis of the tangential coordinate system (the one where the stresses due to temperature variation are calculated), resulting the following strain vector,

$$\underline{\varepsilon}_{\Delta T_u}^t = \alpha \Delta T_u [1 \quad 1 \quad 0]^T \tag{83}$$

where α is the coefficient of thermal expansion. In this approach it is assumed that temperature uniform variation does not introduces in-plane shear deformations and out-of-plane shear deformations. The stresses due to ΔT_u are obtained from the in-plane constitutive relationship,

$$\underline{\sigma}_{\Delta T_u}^t = \underline{D}_{mb} \ \underline{\varepsilon}_{\Delta T_u}^t \tag{84}$$

The internal forces due to uniform temperature variation are calculated from the following expression,

$$\underline{Q}_{\Delta T_u} = \int_V B_{mb}^T \, \underline{\sigma}_{\Delta T_u}^t \, dV = \sum_{i=1}^{IPs1} \sum_{j=1}^{IPs2} \sum_{k=1}^{IPs3} \left(\underline{B}_{mb}^T \, \underline{\sigma}_{\Delta T_u}^t \, \underline{J} \right)_{(s_i, s_j, s_k)} W_i \, W_j \, W_k \tag{85}$$

11.7 DIFERENTIAL TEMPERATURE VARIATION

Consider a shell submited to diferential temperature variation, of ΔT_d^t and ΔT_d^b at top and bottom surfaces of the shell ($\left|\Delta T_d^t\right| = -\left|\Delta T_d^b\right|$). Figure 15 represents a cross section of length dt_1 and orthogonal to t_2 axis of the tangential coordinate system.

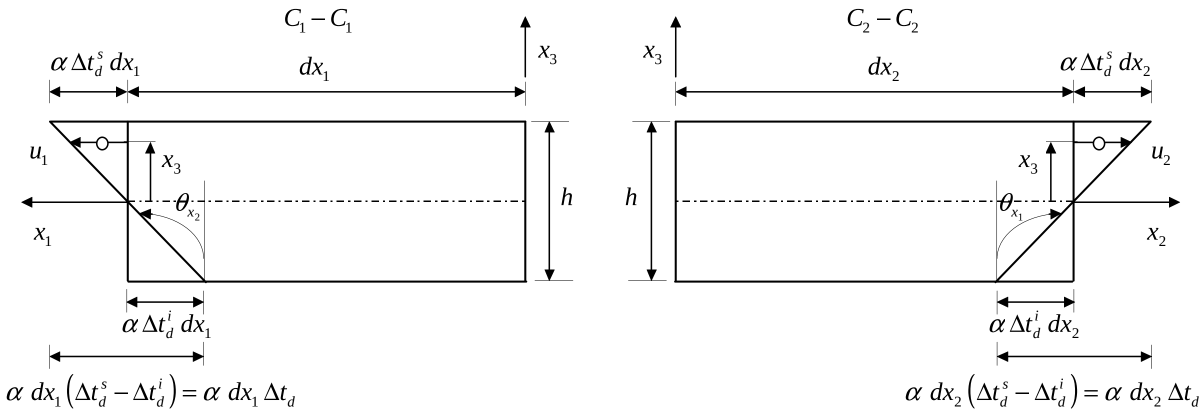


Figure 15 - Deformation due to diferential temperature variation (while this figure is not been updated, *t* should be replaced by *T* and *x* by *t*).

When submitted to ΔT_d^t and ΔT_d^b the reference fiber rotates in turn of t_2 axis,

$$d\theta_2^t = \frac{\alpha \, dt_1 \, \Delta T_d}{h} \tag{86}$$

where $\Delta T_d = \Delta T_d^t + \Delta T_d^b$ and h is the thickness of the shell cross section at the IP where the stresses due to differential temperature variation are evaluated. The displacement and the corresponding strain of any longitudinal fibre at s_3 position are determined from the following expressions (see Figure 15),

$$
\begin{aligned}
du_1^t &= \frac{h}{2}\; s_3\, d\theta_2^t \\
&= \frac{\Delta T_d}{2}\; s_3\, \alpha\, dt_1
\end{aligned}
\tag{87}
$$

$$
\begin{aligned}
\varepsilon_1^t &= \frac{du_1^t}{dt_1} \\
&= s_3\, \alpha \frac{\Delta T_d}{2}
\end{aligned}
\tag{88}
$$

In the t_2 direction (see Figure 15),

$$
\begin{aligned}
du_2^t &= -\frac{h}{2}\; s_3\, d\theta_1^t \\
&= -\frac{\Delta T_d}{2}\; s_3\, \alpha\, dt_1
\end{aligned}
\tag{89}
$$

$$
\begin{aligned}
\varepsilon_2^t &= \frac{du_2^t}{dt_2} \\
&= -\, s_3\, \alpha \frac{\Delta T_d}{2}
\end{aligned}
\tag{90}
$$

Since differential temperature variation do not induce in-plane shear strains and out-of-plane shear strains in isotropic materials, the strain vector due to differential temperature variation has the following format,

$$
\underline{\varepsilon}_{\Delta T_d}^t = s_3\, \alpha \frac{\Delta T_d}{2} \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}
\tag{91}
$$

The internal forces due to differential temperature variation are calculated from the following expression,

$$
\begin{aligned}
\underline{Q}_{\Delta T_d} &= \int_V B_{mb}^T\; \underline{\sigma}_{\Delta T_d}^t\; dV \\
&= \sum_{i=1}^{IPs1} \sum_{j=1}^{IPs2} \sum_{k=1}^{IPs3} \left(\underline{B}_{mb}^T\; \underline{\sigma}_{\Delta T_d}^t\; \underline{J} \right)_{\left(s_i, s_j, s_k\right)} W_i\, W_j\, W_k
\end{aligned}
\tag{92}
$$

11.8 PRESCRIBED DISPLACEMENTS

The prescribed displacement should be directly introduced in the prescribed displacement vector, $\underline{U}_{E,f}$, of the system of equilibrium equations (73).

12 RESULTANT STRESSES

Like the stress components, the resultant forces (per unit length) are calculated in the tangential coordinate system. The resultant forces are determined in the IP of the middle surface of the shell element. The generalized resultant forces in a shell element comprise membrane, bending and out-of-plane shear forces.

The membrane forces result from the integration in the shell thickness of the in-plane stress components (σ_{t_1}, σ_{t_2} and $\tau_{t_1t_2}$),

$$\begin{aligned}\underline{\bar{\sigma}}_m^t = \left\{N_{t_1},\ N_{t_2},\ N_{t_1t_2}\right\}^T &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \left\{\sigma_{t_1},\ \sigma_{t_2},\ \sigma_{t_1t_2}\right\}^T dt_3 \\ &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[\underline{\sigma}_{mb}^t\right]^T dt_3\end{aligned} \qquad (93a)$$

where h is the thickness at the integration point in analysis. The bending moments are obtained integrating, in the shell thickness, the moments produced by the in-plane stress components (σ_{t_1}, σ_{t_2} and $\tau_{t_1t_2}$),

$$\begin{aligned}\underline{\bar{\sigma}}_b^t = \left\{M_{t_2},\ M_{t_1},\ M_{t_1t_2}\right\}^T &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \left\{\sigma_{t_1},\ \sigma_{t_2},\ \sigma_{t_1t_2}\right\}^T t_3\, dt_3 \\ &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[\underline{\sigma}_{mb}^t\right]^T t_3 dt_3\end{aligned} \qquad (93b)$$

Finally, the out-of-plane shear forces are calculated from the integration in the shell thickness of the ou-of-plane shear stresses ($\tau_{t_2t_3}$ and $\tau_{t_1t_3}$),

$$\underline{\bar{\sigma}}_s^t = \left\{ V_{t_2t_3},\, V_{t_1t_3} \right\}^T = \int_{-\frac{h}{2}}^{\frac{h}{2}} \left\{ \tau_{t_2t_3},\, \tau_{t_1t_3} \right\}^T dt_3 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[\underline{\sigma}_s^t \right]^T dt_3 \qquad (93c)$$

Thererofe, the vector of the resultant forces in a shell element is composed by the following components,

$$\underline{\bar{\sigma}}^t = \left\{ \underline{\bar{\sigma}}_m^t,\, \underline{\bar{\sigma}}_b^t,\, \underline{\bar{\sigma}}_s^t \right\}^T = \left\{ N_{t_1},\, N_{t_2},\, N_{t_1t_2},\, M_{t_2},\, M_{t_1},\, M_{t_1t_2},\, V_{t_2t_3},\, V_{t_1t_3} \right\}^T \qquad (94)$$

The resultant stress components are represented in Figure 12, for a generic IP.

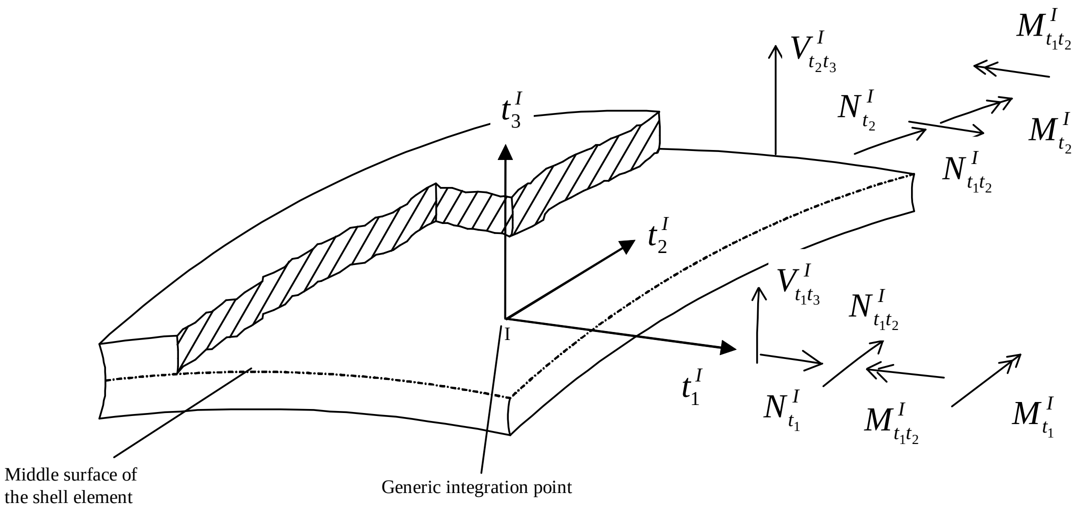


Figure 11 - Resultant forces in the tangential coordinate system.

Hence:

$N_{t_1}^I$ is the membrane force normal to a plane that is orthogonal to t_1^I;
$N_{t_2}^I$ is the membrane force normal to a plane that is orthogonal to t_2^I;
$N_{t_1t_2}^I = N_{t_2t_1}^I$ is the membrane shear force in the t_1t_2 plane;
$M_{t_2}^I = M_{t_1t_3}^I$ is the bending moment in turn of t_2^I axis (bending the t_3t_1 plane);

$M_{t_1}^I = M_{t_2t_3}^I$ is the bending moment in turn of t_1^I axis (bending the t_3t_2 plane);

$M_{t_1t_2}^I$ is the twisting moment in the t_1t_2 plane;

$V_{t_2t_3}^I$ is the resultant out-of-plane shear force in the t_3^I direction, in a plane that is orthogonal to t_2^I;

$V_{t_1t_3}^I$ is the resultant out-of-plane shear force in the t_3^I direction, in a plane that is orthogonal to t_1;

To evaluate the membrane forces, the in-plane stress vector in (93a), $\underline{\sigma}_{mb}^t$, is replaced by $\underline{D}_{mb}\,\underline{\varepsilon}_{mb}^t$, according to the in-plane part of the constitutive relationship indicated in (69), resulting,

$$\begin{aligned}\underline{\bar{\sigma}}_m^t = \left\{N_{t_1},\, N_{t_2},\, N_{t_1t_2}\right\}^T &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[\underline{\sigma}_{mb}^t\right]^T dt_3 \\ &= \int_{-1}^{+1} \underline{D}_{mb}\left(\underline{\varepsilon}_{mb}^t - \underline{\varepsilon}_{\Delta T_u}^t - \underline{\varepsilon}_{\Delta T_d}^t\right)\frac{h}{2}\,ds_3 \\ &= \frac{1}{2}\underline{D}_{mb}\int_{-1}^{+1}\left(\underline{\varepsilon}_{mb}^t - \underline{\varepsilon}_{\Delta T_u}^t - \underline{\varepsilon}_{\Delta T_d}^t\right)h\,ds_3\end{aligned} \tag{95}$$

Introducing the in-plane part of (60) into (95) yields,

$$\begin{aligned}\underline{\bar{\sigma}}_m^t = \left\{N_{t_1},\, N_{t_2},\, N_{t_1t_2}\right\}^T &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[\underline{\sigma}_{mb}^t\right]^T dt_3 \\ &= \frac{h}{2}\underline{D}_{mb}\int_{-1}^{+1}\left(\underline{B}_{mb}^t\,\underline{U} - \underline{\varepsilon}_{\Delta T_u}^t - \underline{\varepsilon}_{\Delta T_d}^t\right) ds_3 \\ &= \frac{h}{2}\underline{D}_{mb}\int_{-1}^{+1} \underline{B}_{mb}^t\,\underline{U}\,ds_3 - 2\underline{\varepsilon}_{\Delta T_u}^t\end{aligned} \tag{96}$$

since (see Equation (91))

$$\int_{-1}^{+1} \underline{\varepsilon}_{\Delta T_d}^t \; ds_3 = 0 \tag{97}$$

The integral in (96) can be determined by numerical integration,

$$\begin{aligned}\underline{\bar{\sigma}}_m^t = \left\{ N_{t_1},\, N_{t_2},\, N_{t_1 t_2} \right\}^T &= \frac{h}{2}\underline{D}_{mb}\left\{ \sum_{k=1}^{nGP3}\left(\underline{B}_{mb}^t\left(s_1, s_1, P_k\right)\underline{U}\, W_k \right)\right\} - h\,\underline{D}_{mb}\,\underline{\varepsilon}_{\Delta T_u}^t \\ &= \frac{h}{2}\underline{D}_{mb}\left\{ \sum_{k=1}^{nGP3}\left(\underline{B}_{mb}^t\left(s_1, s_1, P_k\right)\underline{U}\, W_k \right)\right\} - \underline{N}_{\Delta T_u}^t\end{aligned} \tag{98}$$

where

$$\begin{aligned}\underline{N}_{\Delta T_u}^t &= h\,\underline{D}_{mb}\,\underline{\varepsilon}_{\Delta T_u}^t \\ &= h\,\underline{\sigma}_{\Delta T_u}^t\end{aligned} \tag{99}$$

are the membrane forces due to uniform temperature variation.

To evaluate the bending moments, the in-plane stress vector in (93a), $\underline{\sigma}_{mb}^t$, is replaced by $\underline{D}_{mb}\,\underline{\varepsilon}_{mb}^t$, according to the in-plane part of the constitutive relationship indicated in (69), resulting,

$$\begin{aligned}\underline{\bar{\sigma}}_b^t = \left\{ M_{t_2},\, M_{t_1},\, M_{t_1 t_2} \right\}^T &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[\underline{\sigma}_{mb}^t\right]^T t_3\,dt_3 \\ &= \int_{-1}^{+1} \underline{D}_{mb}\left(\underline{\varepsilon}_{mb}^t - \underline{\varepsilon}_{\Delta T_u}^t - \underline{\varepsilon}_{\Delta T_d}^t \right)\frac{h}{2} s_3 \frac{h}{2} ds_3 \\ &= \frac{h^2}{4}\underline{D}_{mb}\int_{-1}^{+1}\left(\underline{\varepsilon}_{mb}^t - \underline{\varepsilon}_{\Delta T_u}^t - \underline{\varepsilon}_{\Delta T_d}^t \right) s_3\; ds_3 \\ &= \frac{h^2}{4}\underline{D}_{mb}\int_{-1}^{+1}\left(\underline{B}_{mb}^t\,\underline{U} - \underline{\varepsilon}_{\Delta T_u}^t - \underline{\varepsilon}_{\Delta T_d}^t \right) s_3\; ds_3 \\ &= \frac{h^2}{4}\underline{D}_{mb}\left\{ \sum_{k=1}^{nGP3}\left(\underline{B}_{mb}^t\left(s_1, s_1, P_k\right)\underline{U}\, s_3\, W_k \right) - \frac{2}{3}\underline{\bar{\varepsilon}}_{\Delta T_d}^t \right\} \\ &= \frac{h^2}{4}\underline{D}_{mb}\sum_{k=1}^{nGP3}\left(\underline{B}_{mb}^t\left(s_1, s_1, P_k\right)\underline{U}\, s_3\, W_k \right) - \frac{h^2}{6}\underline{D}_{mb}\,\underline{\bar{\varepsilon}}_{\Delta T_d}^t \\ &= \frac{h^2}{4}\underline{D}_{mb}\sum_{k=1}^{nGP3}\left(\underline{B}_{mb}^t\left(s_1, s_1, P_k\right)\underline{U}\, s_3\, W_k \right) - \underline{M}_{\Delta T_d}^t\end{aligned} \tag{100}$$

since

$$\int_{-1}^{+1} \underline{\varepsilon}^{t}_{\Delta T_u} \; s_3 \; ds_3 \; = 0 \tag{101}$$

and

$$\int_{-1}^{+1} \underline{\varepsilon}^{t}_{\Delta T_d} \; s_3 \; ds_3 = \int_{-1}^{+1} \underline{\overline{\varepsilon}}^{t}_{\Delta T_d} \; s_3^2 \; ds_3 \; = \frac{2}{3} \underline{\overline{\varepsilon}}^{t}_{\Delta T_d} \tag{102}$$

In (100) $\underline{M}^{t}_{\Delta T_d}$ is the bending moment vector due to differential temperature variation.

To evaluate the out-of-plane shear forces, the out-of-plane shear stress vector in (93c), $\underline{\sigma}^{t}_{s}$, is replaced by $\underline{D}_s \, \underline{\varepsilon}^{t}_{s}$, according to the in-plane part of the constitutive relationship indicated in (70), resulting,

$$\begin{aligned}
\underline{\overline{\sigma}}^{t}_{s} = \left\{ V_{t_2 t_3} , V_{t_1 t_3} \right\}^T &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[\underline{\sigma}^{t}_{s} \right]^T dt_3 \\
&= \int_{-1}^{+1} \underline{D}_s \, \underline{\varepsilon}^{t}_{s} \frac{h}{2} ds_3 \\
&= \frac{h}{2} \underline{D}_s \int_{-1}^{+1} \underline{B}^{t}_{s} \, \underline{U} \, ds_3 \\
&= \frac{h}{2} \underline{D}_s \sum_{k=1}^{nGP3} \left(\underline{B}^{t}_{s} \left(s_1, s_1, P_k \right) \underline{U} \, W_k \right)
\end{aligned} \tag{103}$$

13 REFERENCES

Ahmad, S., Irons, B.M., Zienkiewicz, O.C., (1970), “Analysis of thick thin shell structures by curved finite elements”, Int. J. Num. Meth. Engng., 2, pp. 419-451.

Azevedo, A.F.M.; Barros, J.A.O.; Sena-Cruz, J.M.; Gouveia, A.V., "Software no ensino e no projecto de estruturas", III Congresso Luso Moçambicano de Engenharia, pp. 81-92, 19-21 Agosto 2003.

Barros, J.A.O., “Lajes e cascas segundo o método dos elementos finitos”, Relatório 00-DEC/E-2, 131 pags., Março 2000.

Barros, J.A.O., “Comportamento do betão reforçado com fibras - análise experimental e simulação numérica”, Tese de Doutoramento, Faculdade de Engenharia da Universidade do Porto, pp. 502, Dezembro 1995.

Figueiras, J.A. (1983), “Ultimate load analysis of anisotropic and reinforced concrete plates and shells”, Ph.D. Thesis, C/Ph/72/83, Uni. College of Swansea.

Fung, Y.C. (1965), “Foundations of solid mechanics”, Prentice-Hall, Englwood Cliffs, New Jersey, U.S.A..

Oñate, E., Cálculo de estructuras por el método de elementos finitos - Análisis estático lineal, CIMNE, UPC, 1995.

Póvoas, R.H.C.F. (1991), “Modelos não-lineares de análise e dimensionamento de estruturas laminares de betão incluindo efeitos diferidos”, Tese de Doutoramento, Faculdade de Engenharia da Universidade do Porto.

Sena-Cruz, J.M.; Barros, J.A.O., Azevedo, A.F.M., "Elasto-plastic smeared crack model for concrete", Relatório técnico 04-DEC/E-05, Departamento de Engenharia Civil, Universidade do Minho, Abril 2004.

Zienkiewicz, O. C.; Taylor, R. L., The Finite Element Method, 4th Edition, McGraw-Hill, 1989.

ANNEX I

Knowing the nodal displacements of the structure, $\underline{U}_E$, from solving the system of equilibrium equations, the strain vector in a generic IP can be obtained from (60) where $\underline{U}$ is the vector of the element nodal displacements in the global coordinate system. To obtain the strain vector

$$\begin{bmatrix} \frac{\partial w_1}{\partial t_1} & \frac{\partial w_2}{\partial t_1} & \frac{\partial w_3}{\partial t_1} \\ \frac{\partial w_1}{\partial t_2} & \frac{\partial w_2}{\partial t_2} & \frac{\partial w_3}{\partial t_2} \\ \frac{\partial w_1}{\partial t_3} & \frac{\partial w_2}{\partial t_3} & \frac{\partial w_3}{\partial t_3} \end{bmatrix} = \begin{bmatrix} b_{11} & \cdots & b_{1,18n} \\ b_{21} & \cdots & b_{2,18n} \\ b_{31} & \cdots & b_{3,18n} \end{bmatrix} \begin{bmatrix} \begin{matrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \\ \theta_{1,1} \\ \theta_{2,1} \\ \theta_{3,1} \\ \vdots \\ u_{1,n} \\ u_{2,n} \\ u_{3,n} \\ \theta_{1,n} \\ \theta_{2,n} \\ \theta_{3,n} \end{matrix} & \underline{0} & \underline{0} \\ \underline{0} & \begin{matrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \\ \theta_{1,1} \\ \theta_{2,1} \\ \theta_{3,1} \\ \vdots \\ u_{1,n} \\ u_{2,n} \\ u_{3,n} \\ \theta_{1,n} \\ \theta_{2,n} \\ \theta_{3,n} \end{matrix} & \underline{0} \\ \underline{0} & \underline{0} & \begin{matrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \\ \theta_{1,1} \\ \theta_{2,1} \\ \theta_{3,1} \\ \vdots \\ u_{1,n} \\ u_{2,n} \\ u_{3,n} \\ \theta_{1,n} \\ \theta_{2,n} \\ \theta_{3,n} \end{matrix} \end{bmatrix} \underline{J}^{-1} \left[\underline{T}^{gt} \right]^T \qquad \text{(A1.1)}$$