Control Charts with Supplementary Runs Rules for Monitoring Bivariate Processes

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Abstract

Control charts based on the T^2 and $|\mathbf{S}|$ statistics are common used for monitoring multivariate processes. However, the T^2 and $|\mathbf{S}|$ statistics are not well known by the users. As an alternative to the use of these charts, in this article we propose a new chart for monitoring the mean vector and the covariance matrix of bivariate processes, named as the MRMAX chart. The monitoring statistic associated to the MRMAX chart is based on the sample means and sample ranges. As the practitioners are, in general, more familiar with means and ranges, they will not have difficult to use the proposed chart. Following the rule proposed by Shewhart, the MRMAX chart signals when a sample point falls in the action region. The MRMAX chart has the classical disadvantage of all the control charts that adopt the rule proposed by Shewhart: it is not efficient to detect small shifts in the process parameter being monitored. This way, we investigate the efficiency of the MRMAX chart in detecting shifts in the mean vector and/or in the covariance matrix under the use of supplementary and synthetic runs rules. In terms of efficiency, we can say that, the MRMAX charts with supplementary or synthetic runs rules have better performance than the MRMAX without runs rules. However, it is important to highlight that the synthetic runs rule is simpler for the users than the supplementary runs rules. Markov chain models are used to obtain the properties of the proposed control charts.

Keywords: control charts; mean vector; covariance matrix; bivariate processes; supplementary runs rules.

1 Introduction

The control charts proposed by Shewhart were designed for monitoring the mean and the variance of univariate processes. Nowadays, an increasing number of processes is requiring the monitoring of two or more than two quality characteristics. In this context, Hotelling (1947) proposed the T^2 chart to detect changes in the mean vector of multivariate processes, and Alt (1985) proposed the generalized variance $|\mathbf{S}|$ chart to detect changes in the covariance matrix $\mathbf{\Sigma}$.

After the works of Hotelling and Alt, a growing number of researchers has been dealing with the statistical control of multivariate processes. For example, Costa and Machado (2008) and Champ and Aparisi (2008) considered the use of the double sampling procedure with the chart proposed by Hotelling. Costa and Machado (2008a, 2009), Machado and Costa (2008) and Machado *et al.* (2008) considered the largest value among the sample variances of the quality characteristics to control the covariance matrix of multivariate processes. Costa and Machado (2011) also proposed a chart based on sample ranges, the RMAX chart, for monitoring the covariance matrix of multivariate processes.

The joint control of the mean vector and the covariance matrix of multivariate processes has also been studied by several authors. Chou *et al.* (2002) have considered the control of the mean vector and the covariance matrix by using log-likelihood ratio statistics. Takemoto and Arizono (2005) considered the Kullback–Leibler information to control multivariate processes.

Khoo (2005) studied the joint properties of the T^2 and $|\mathbf{S}|$ charts. The speed with which these charts signal changes in the mean vector and/or in the covariance matrix was obtained by simulation. Chen *et al.* (2005) proposed a single EWMA chart to control both, the mean vector and the covariance matrix. Their

chart is more efficient than the joint T^2 and |**S**| in signaling small changes in the process. Zhang and Chang (2008) proposed two EWMA charts based on individual observations that are not only fast in signaling but also very efficient in informing how the assignable cause affected the process; if only changing the mean vector or only changing the covariance matrix or changing both.

Machado *et al*. (2009) proposed two statistical tools for monitoring the mean vector and the covariance matrix of bivariate processes: the MVMAX chart, which only requires the computation of statistics familiar to the users, that is, sample means and sample variances and two charts for joint use based on the non-central chi-square statistic (NCS statistic). The joint NCS charts are recommended to identify the out-of-control variable instead of the nature of the disturbance, that is, the one that only affects the mean vector or only affects the covariance matrix or both.

In this article, we propose the MRMAX chart to control the mean vector and the covariance matrix of bivariate processes. The sample points correspond to the maximum among the values of the standardized sample means (in module) and ranges (weighted) of two quality characteristics. User´s familiarity with sample means and sample ranges is a point in favor of the MRMAX chart. Following the rule proposed by Shewhart, the MRMAX chart signals when a sample point falls in the action region.

The MRMAX chart has the classical disadvantage of all charts that adopt the rule proposed by Shewhart: it is not efficient to detect small changes. This handicap of Shewhart control charts was recognized at the very beginning by the Western Eletric Company who suggested in 1956 the adoption of supplementary runs rules in order to make the charts more sensitive to small changes. Since then, various authors have studied the properties of the control charts with supplementary runs rules. For the univariate case, the most referenced are Champ and Woodall (1987), Champ (1992), Zhang and Wu (2005), Celano *et al.* (2006), Koutras *et al.* (2007), Kim *et al.* (2009), Antzoulakos and Rakitzis (2008) and Riaz *et al.* (2011). Regarding to the multivariate case, Aparisi *et al.* (2004) and Koutras *et al.* (2006) are the most referenced. The numerical evaluation of the performance of such rules can be easily achieved by the aid of Markov Chains, as indicated in Champ and Woodall (1987) and Champ (1992).

This way, we study the MRMAX chart with supplementary runs rules. We also compare the supplemented MRMAX chart with the synthetic MRMAX chart.

The paper is organized as follows. In Section 2 we present the properties of the MRMAX chart. We also present the properties of the synthetic MRMAX chart and the MRMAX chart with runs rules. The proposed charts are compared in Section 3. Conclusions are in Section 4.

2 The MRMAX Chart

In this section we propose a new chart based on sample means and sample ranges for monitoring the mean vector $\boldsymbol{\mu}$ and/or the covariance matrix $\boldsymbol{\Sigma}$ of two quality characteristics that follow a bivariate normal distribution. The sample point plotted on the proposed chart corresponds to the largest value among $(|Z_1|,|Z_2|,W_1,W_2)$, where $Z_i = \sqrt{n}(\overline{X}_i - \mu_i)/\sigma_i$ and $W_i = kR_i/\sigma_i$, $i = 1,2$. As the monitoring statistic is the maximum value among standardized sample means and weighted standardized sample ranges, the chart is called the MRMAX chart.

The parameter *k* is required to attend the imposed condition that, during the in-control period, the statistics $(|Z_1|,|Z_2|,W_1,W_2)$ have the same probability to exceed *CL*, the control limit of the MRMAX chart.

The process is considered to start with the mean vector and the covariance matrix on target ($\boldsymbol{\mu} = \boldsymbol{\mu}_0$ and $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_0$), where $\boldsymbol{\mu}_0' = (\mu_1;\mu_2)$ and $\boldsymbol{\Sigma}_0 = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$. The occurrence of the assignable cause changes the mean vector from $\boldsymbol{\mu}_0'$ to $\boldsymbol{\mu}_1' = (\mu_1 + \delta_1\sigma_1;\mu_2 + \delta_2\sigma_2)$ and/or the covariance matrix from $\boldsymbol{\Sigma}_0$ to

$\mathbf{\Sigma}_1 = \begin{bmatrix} a_1 \cdot a_1 \cdot \sigma_{11} & a_1 \cdot a_2 \cdot \sigma_{12} \\ a_1 \cdot a_2 \cdot \sigma_{21} & a_2 \cdot a_2 \cdot \sigma_{22} \end{bmatrix}$. The correlation coefficient $\rho = \dfrac{\sigma_{12}}{\sigma_1 \sigma_2}$ is not affected by the assignable cause. After the occurrence of the assignable cause it is assumed that at least one $|\delta_i|$ becomes larger than zero and/or at least one a_i becomes larger than one, $i = 1,2$.

If the MRMAX statistic falls beyond the control limit, *CL*, the control chart signals an out-of-control condition. Once the MRMAX chart signals, the user can immediately examine the sample means and the sample ranges of the two quality characteristics to discover which variable was affected by the assignable cause, that is, the one with the sample mean and/or the sample range larger than the control limit.

2.1 The Properties of the MRMAX Chart

To obtain the false alarm risk α and the power of detection *P* of the MRMAX chart, we use the property that the sample means are independent of the sample ranges. Therefore, the MRMAX chart has probability *P* of signaling:

$$P = P_M + P_D - P_M P_D \tag{1}$$

Regarding to the sample means, the probability P_M of $|Z_1|$ and/or $|Z_2|$ exceeding the control limit is given by:

$$P_M = 1 - \int_{-CL+\delta_1\sqrt{n}}^{CL+\delta_1\sqrt{n}} \int_{-CL+\delta_2\sqrt{n}}^{CL+\delta_2\sqrt{n}} f(Z_1, Z_2) d_{Z_1} d_{Z_2} \tag{2}$$

where $f(Z_1, Z_2)$ is a standardized bivariate normal distribution function, reminding that *CL* is the control limit of the MRMAX chart.

Regarding to the sample ranges, the probability P_D of W_1 and/or W_2 exceeding the control limit is:

$$P_D = 1 - \Pr[(W_1 < CL) \cap (W_2 < CL)] \tag{3}$$

where $W_1 = kR_1 / \sigma_1$ and $W_2 = kR_2 / \sigma_2$, being $R_i = \max[x_{i1}, x_{i2}, \ldots, x_{in}] - \min[x_{i1}, x_{i2}, \ldots, x_{in}]$, $i = 1,2$ the sample ranges of two quality characteristics X_1 and X_2.

We have that $x_i \sim N(\mu_i, a_i \sigma_i)$, with $i = 1,2$. Then,

$$P_D = 1 - F(w_1, w_2) \tag{4}$$

being $w_1 = CL/a_1$ and $w_2 = CL/a_2$. *CL* was divided by a_i to hold the condition that $x_i / a_i \sigma_i \sim N(\mu_i / a_i \sigma_i, 1)$, with $i = 1,2$.

Costa and Machado (2011) present the probability P_D for the bivariate and trivariate cases.

Table 1 shows the average run length (*ARL*) for the MRMAX chart when ρ = 0.0, 0.5, 0.7, a_1 and a_2 =1.0; 1.25; 1.5 and δ_1 and δ_2 =0.0; 0.5; 0.75; 1.0. A type I risk of 0.5% is adopted. The correlation coefficient has

minor influence on the properties of the MRMAX chart. The other Tables of this article were built fixing ρ = 0.5.

Table 1: The effect of $\rho^{\bullet}$ on the *ARL* for the MRMAX chart (*n*=5).

Shifts (covariance matrix)			Shifts (mean vector)												
			$\lvert\delta_1\rvert$	0	0	0.5	0.5	0	0.75	0.5	0.75	0.75	0	1.0	1.0
a_1	a_2	$\rho^{\bullet}$	$\lvert\delta_2\rvert$	0	0.5	0	0.5	0.75	0	0.75	0.5	0.75	1.0	0	1.0
1.0	1.0	0.0		200.0	47.3	47.3	27.0	15.6	15.6	12.6	12.6	8.4	6.1	6.1	3.4
		0.5		200.0	47.3	47.3	27.0	15.6	15.6	12.6	12.6	8.4	6.1	6.1	3.4
		0.7		200.0	47.3	47.3	27.0	15.6	15.6	12.6	12.6	8.4	6.1	6.1	3.4
1.25	1.0	0.0		32.2	21.4	15.0	12.3	11.3	7.9	8.2	7.1	5.6	5.4	4.3	2.8
		0.5		32.2	21.4	15.1	12.3	11.3	7.9	8.2	7.1	5.6	5.4	4.3	2.8
		0.7		32.1	21.3	15.0	12.2	11.3	7.9	8.2	7.1	5.6	5.4	4.3	2.8
1.5	1.0	0.0		9.0	7.9	6.3	5.8	6.1	4.5	4.8	4.3	3.7	3.9	3.2	2.4
		0.5		9.0	7.9	6.3	5.8	6.1	4.5	4.8	4.3	3.4	3.9	3.2	2.4
		0.7		8.9	7.9	6.3	5.8	6.1	4.5	4.8	4.3	3.4	3.9	3.2	2.4
1.25	1.25	0.0		17.7	11.0	11.0	8.1	6.7	6.7	5.5	5.5	4.3	4.0	4.0	2.5
		0.5		18.1	11.1	11.1	8.1	6.7	6.7	5.5	5.5	4.3	4.0	4.0	2.5
		0.7		18.5	11.3	11.3	8.2	6.8	6.8	5.6	5.6	4.3	4.0	4.0	2.5
1.25	1.5	0.0		7.4	5.6	6.0	4.8	4.1	4.5	3.7	3.8	3.2	3.0	3.2	2.1
		0.5		7.6	5.6	6.1	4.8	4.2	4.6	3.7	3.9	3.2	3.0	3.2	2.1
		0.7		7.7	5.7	6.2	4.9	4.2	4.6	3.8	3.9	3.2	3.0	3.2	2.1
1.5	1.5	0.0		4.8	4.0	4.0	3.5	3.3	3.3	2.9	2.9	2.6	2.6	2.6	1.9
		0.5		5.0	4.2	4.2	3.6	3.4	3.4	3.0	3.0	2.6	2.6	2.6	1.9
		0.7		5.2	4.3	4.3	3.7	3.4	3.4	3.1	3.1	2.6	2.7	2.7	1.9

2.2 The synthetic MRMAX Chart

The signaling rule of the synthetic chart requires a second consecutive point beyond the control limit not far from the first one. The number of samples between them can not exceed *L*, a pre-specified value. The growing interest in using this rule may be explained by the fact that many practitioners prefer waiting until the occurrence of a second point beyond the control limits before looking for an assignable cause (see Wu and Spedding (2000, 2000a), Calzada and Scariano (2001), Wu *et al.* (2001, 2001a), Davis and Woodall (2002), Costa and Rahim (2006), Machado *et al.* (2008), Aparisi and De Luna (2009), Costa *et al.* (2009), Castagliola and Khoo (2009), Khoo *et al.* (2009), Khilare and Shirke (2010), Pawar and Shirke (2010), Zhang *et al.* (2010), Khoo *et al.* (2011, 2011a) and Zhang *et al.* (2011)).

According to Davis and Woodall (2002) the proper parameter to measure the performance of the synthetic chart is the steady-state *ARL*, that is, the *ARL* value obtained when the process remains in-control for a long time before the occurrence of the assignable cause.

The following transition matrix of the Markov chain is used to obtain the steady-state *ARL*.

$$\begin{array}{c|cccccccc}
 & 00..00 & 0..001 & 0..010 & 0..100 & \dots & 010..0 & 100..0 & Signal \\
\hline
00...00 & A & B & 0 & 0 & \dots & 0 & 0 & 0 \\
0..001 & 0 & 0 & A & 0 & \dots & 0 & 0 & B \\
0..010 & 0 & 0 & 0 & A & \dots & 0 & 0 & B \\
\dots & .. & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
001..0 & 0 & 0 & 0 & 0 & \dots & A & 0 & B \\
010..0 & 0 & 0 & 0 & 0 & \dots & 0 & A & B \\
100..0 & A & 0 & 0 & 0 & \dots & 0 & 0 & B \\
Signal & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1
\end{array} \tag{5}$$

The transient states describe the position of the last *L* sample points; "1" means the sample point fell beyond the control limits, and "0" means the sample point fell in the central region. For instance, the transient state (010..0) is reached when the second of the last *L* points falls in the action region and all others points fall in the central region. The events "0" and "1" occur with probabilities A and B=1-A, respectively.

The steady-state *ARL* is given by

$$\mathbf{S} \times \mathbf{ARL} \tag{6}$$

where **S** is the vector with the stationary probabilities of being in each nonabsorbing state and **ARL** is the vector of *ARL*s taking each nonabsorbing state as the initial state.

We have that

$$\mathbf{ARL} = (\mathbf{I} - \mathbf{R})^{-1}\mathbf{1}, \tag{7}$$

where **I** is an (*L*+1) by (*L*+1) identity matrix, **R** is the transition matrix given in (5) with the last row and column removed, and **1** is an (*L*+1) by one vector of ones. The vector $\mathbf{S}' = (1/C; B/C, \dots, B/C)$, with *C*=1+*LB*, was obtained by solving the system of linear equations:

$$\mathbf{S} = \mathbf{R}_{\mathbf{adj}}\mathbf{S}, \text{ constrained to } \mathbf{S1} = 1. \tag{8}$$

The matrix $\mathbf{R}_{\mathbf{adj}}$ is an adjusted version of **R**, where A and B in the first row are held and the remaining As are switched by 1 s. The matrix $\mathbf{R}_{\mathbf{adj}}$ follows:

$$\begin{pmatrix} A & B & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \tag{9}$$

Table 2 shows the influence of the design parameter *L* on the MRMAX performance. As *L* increases the speed with which the synthetic MRMAX chart signals also increases. The gain in speed is more significant when *L* increases from 1 to 6. For instance, when $a_1 = a_2 = 1$ and $\delta_1 = \delta_2 = 0.5$, the *ARL* reduces 18.0% (from 21.4 to 17.5). On the other hand, as *L* increases from 6 to 10, the *ARL* reduces 1.0% (from 17.5 to 17.3), see Table 2. For this reason, Table 3 in Section 3 was built fixing *L* equal to 7.

Table 2: The influence of *L* on the synthetic MRMAX chart's performance (ρ = 0.5, *n*=5).

				L=	1	2	3	4	5	6	7	8	9	10
				CL=	2.350	2.469	2.536	2.582	2.617	2.644	2.667	2.687	2.703	2.719
a_1	a_2	$\lvert\delta_1\rvert$	$\lvert\delta_2\rvert$						*ARL*					
1.0	1.0	0	0		200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0
		0	0.5		45.3	40.9	39.1	37.9	37.2	36.8	36.3	36.1	35.8	35.8
		0.5	0.5		21.4	19.2	18.4	17.9	17.7	17.5	17.4	17.3	17.3	17.3
1.25	1.25	0	0		15.4	13.6	12.9	12.5	12.3	12.1	12.1	12.0	12.0	11.9
		0	0.5		9.67	8.43	7.98	7.74	7.60	7.5	7.5	7.4	7.4	7.4
		0.5	0.5		6.95	6.06	5.75	5.59	5.50	5.5	5.4	5.4	5.4	5.4
1.5	1.0	0	0		9.82	8.28	7.69	7.36	7.17	7.0	7.0	6.9	6.8	6.8
		0	0.5		7.42	6.40	6.03	5.84	5.73	5.7	5.6	5.6	5.6	5.6
		0.5	0		7.02	5.90	5.49	5.27	5.14	5.1	5.0	5.0	4.9	4.9
		0.5	0.5		5.68	4.89	4.60	4.46	4.38	4.3	4.3	4.3	4.3	4.2

2.3 The MRMAX Chart with Supplementary Runs Rules

In this article we also propose the MRMAX chart with runs rules. A runs rule causes a signal if *s* of the last *m* values of the statistic being plotted fall in the interval (w, CL), where $1 \le s \le m$ and $w < CL$, being *w* the warning limit and *CL* the control limit. The notation $T:(s, m, w, CL)$ is used to represent this runs rule. According to this notation, the basic Shewhart rule (C_1) can be expressed by C_1: $(1;1;CL;\infty)$ or $(1;1;-\infty;-CL)$. This article focuses on the performance of two supplemented MRMAX charts. The first one with the supplementary runs rule C_2:(2;3;*w*;*CL*) and the second with the supplementary runs rule C_3:(3;4;*w*;*CL*).

Figure 1 shows the three points positions that lead the MRMAX chart with the supplementary runs rule C_2 to signal.

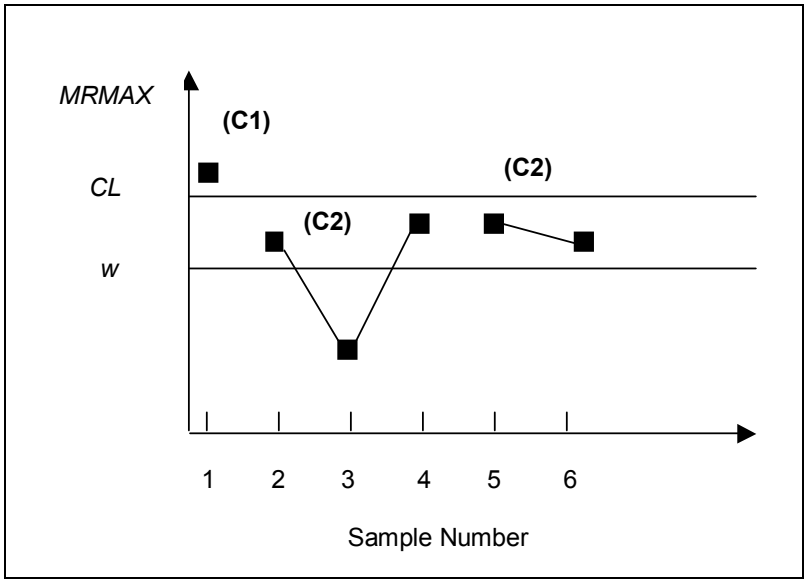

Figure 1: MRMAX chart with the supplementary runs rule C_2.

The steady-state *ARL* of the MRMAX chart with supplementary runs rules C_2 can be obtained with the aid of the Markov Chain represented by the matrix (10). It is a Markov Chain with 3 transient states (A, A1, B) and one absorbing state (C). These states are defined according to the position of the sample points on the control chart:

State A: the last two sample points belong to the interval (0;*w*);

State A1: the previous sample belongs to the interval (*w*;*CL*) and the current one belongs to the interval (0;*w*);

State B: the previous sample belongs to the interval (0;*w*) and the current one belongs to the interval (*w*;*CL*);

State C: the absorbing state *C* is reached when the last transient state is state *A* and the current sample falls beyond *CL* or the last transient state is state A_1 or *B* and the current sample falls beyond *w*.

Let p_1=*P*[next observed sample point will be in the region (0;*w*)], p_2= *P*[next observed sample point will be in the region (*w*;*CL*)] and p_3 =1-p_1-p_2 =*P*[next observed sample point will be in the region (CL,∞)]. Thus, we have that:

$$\begin{array}{c} States \\ A \\ A1 \\ B \\ C \end{array} \begin{array}{c} \begin{array}{cccc} A & A1 & B & C \end{array} \\ \begin{bmatrix} p_1 & 0 & p_2 & (1-p_1-p_2) \\ p_1 & 0 & 0 & (1-p_1) \\ 0 & p_1 & 0 & (1-p_1) \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \tag{10}$$

The vector **S** consists of three non-absorbing states, that is, $\mathbf{S}' = (S_1, S_2, S_3)$, and it can be obtained by solving the system of linear equations in (8). With **S**, we only need to compute the inverse of (**I**-**R**) in order to obtain the steady-state *ARL*, see expressions (6) and (7).

Following the same approach, we obtain the vector **S** for the runs rule C_3.

In the next section we will compare the MRMAX with supplementary runs rules with the synthetic and standard MRMAX charts. The performance of the supplemented MRMAX chart depends on the values of p_1 and p_2. We investigated for which values of p_1 and p_2 the MRMAX chart has a better overall performance and we obtained the *ARL* values under this condition.

3 Comparing Charts

In this section we compare the standard MRMAX (MRMAX-1), the supplemented MRMAX charts based on the runs rules C_2 and C_3 (MRMAX-2 and MRMAX-3, respectively) and the synthetic MRMAX chart (MRMAX-4).

Table 3 presents the *ARL*s for the MRMAX charts, where ρ = 0.5, a_1 and a_2 =1.0; 1.25; 1.5 and δ_1 and δ_2 =0.0; 0.5; 0.75; 1.0. A type I risk of 0.5% is adopted.

The MRMAX charts with the supplementary or the synthetic runs rules have better performance than the standard MRMAX.

If it is well known that the assignable cause only affects the covariance matrix, then the MRMAX-2 and MRMAX-3 charts are the best option and present similar performances. The same behavior is observed for small changes in the mean vector. In the other cases, the MRMAX-4 chart is similar in performance to the MRMAX-2 and MRMAX-3 charts. However, it is important to highlight that the synthetic runs rule is simpler for the users than the supplementary runs rules.

Table 3: *ARL* for the MRMAX-1, MRMAX-2, MRMAX-3 and MRMAX-4 charts (n=5 and ρ = 0.5).

		Shifts (Mean vector)												
Shifts (covariance matrix)		$\lvert\delta_1\rvert$	0	0	0.5	0.5	0	0.75	0.5	0.75	0.75	0	1.0	1.0
a_1	a_2	$\lvert\delta_2\rvert$	0	0.5	0	0.5	0.75	0	0.75	0.5	0.75	1.0	0	1.0
1.0	1.0		200.0(1)	47.3	47.3	27.0	15.6	15.6	12.6	12.6	8.4	6.1	6.1	3.4
			200.0(2)	36.3	36.3	17.4	10.8	10.8	7.9	7.9	5.1	4.3	4.3	2.3
			200.0(3)	34.8	34.8	16.3	10.8	10.8	7.8	7.8	5.3	4.6	4.6	2.6
			200.0(4)	34.0	34.0	15.3	10.5	10.5	7.5	7.5	5.1	4.6	4.6	2.7
1.25	1.0		32.2	21.4	15.1	12.3	11.3	7.9	8.2	7.1	5.6	5.4	4.3	2.8
			25.2	14.0	11.4	8.2	7.1	6.0	5.2	5.0	3.8	3.6	3.4	2.1
			22.8	13.6	11.0	8.0	7.2	6.0	5.3	5.1	3.9	3.8	3.6	2.3
			22.8	13.2	10.8	7.8	7.0	6.0	5.2	5.0	3.9	3.8	3.6	2.4
1.5	1.0		9.0	7.9	6.3	5.8	6.1	4.5	4.8	4.3	3.4	3.9	3.2	2.4
			7.0	5.6	5.0	4.3	4.1	3.7	3.4	3.3	2.8	2.7	2.7	1.9
			5.5	4.7	4.2	3.7	3.6	3.3	3.1	3.0	2.6	2.6	2.5	1.9
			5.5	4.6	4.2	3.7	3.6	3.3	3.1	3.0	2.7	2.6	2.5	2.0

1.25	1.25	18.1	11.1	11.1	8.1	6.7	6.7	5.6	5.6	4.3	4.0	4.0	2.5
		12.1	7.5	7.5	5.4	4.7	4.7	3.9	3.9	3.1	3.0	3.0	2.0
		10.9	7.1	7.1	5.3	4.7	4.7	3.9	3.9	3.2	3.1	3.1	2.1
		10.8	7.0	7.0	5.3	4.7	4.7	3.9	3.9	3.2	3.1	3.1	2.2
1.25	1.5	7.6	5.6	6.1	4.8	4.2	4.6	3.7	3.9	3.2	3.0	3.2	2.1
		5.4	4.2	4.3	3.5	3.2	3.3	2.8	2.8	2.4	2.5	2.4	1.8
		4.4	3.5	3.6	3.1	2.9	2.9	2.6	2.6	2.3	2.3	2.3	1.8
		4.3	3.5	3.6	3.1	2.9	2.9	2.6	2.7	2.3	2.3	2.3	1.8
1.5	1.5	5.0	4.2	4.2	3.6	3.4	3.4	3.0	3.0	2.6	2.6	2.6	1.9
		3.6	3.1	3.1	2.7	2.6	2.6	2.3	2.3	2.1	2.1	2.1	1.6
		2.8	2.5	2.5	2.3	2.2	2.2	2.0	2.0	1.9	1.8	1.8	1.5
		2.8	2.5	2.5	2.3	2.2	2.2	2.1	2.1	1.9	1.9	1.9	1.5

(1) MRMAX-1; (2) MRMAX-2 (L=7); (3) MRMAX-3 (p_2=0.03); (4) MRMAX-4 (p_2 = 0.07).

4 Conclusions

In this article we proposed a single chart based on the standardized sample means and sample ranges (MRMAX chart) for monitoring the mean vector and the covariance matrix of bivariate processes.

User's familiarity with the computation of these statistics is a point in favor of the MRMAX chart. We compared the supplemented MRMAX chart and the synthetic MRMAX chart with the standard MRMAX chart.

The supplementary and the synthetic runs rules enhance the performance of the MRMAX chart. However, the synthetic runs rule is simpler to administer if compared with the supplementary runs rules.

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