On the structure of generalized Appell sequences of paravector valued homogeneous monogenic polynomials

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Abstract. The fact that generalized Appell sequences of monogenic polynomials in the setting of hypercomplex function theory also satisfy a corresponding binomial type theorem allows to obtain their explicit structure. Recently it has been obtained a complete characterization in the case of paravector valued homogeneous polynomials of three real variables. The aim of this contribution is the study of paravector valued homogeneous polynomials of four real variables, where new types of generalized Appell sequences could be detected.

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INTRODUCTION AND BASIC NOTATION

Let $\{e_1, e_2, \cdots, e_n\}$ be an orthonormal base of the Euclidean vector space $\mathbb{R}^n$ with a product according to the multiplication rules $e_k e_l + e_l e_k = -2\delta_{kl},\ k,l = 1,\cdots,n$, where δ_{kl} is the Kronecker symbol. This non-commutative product generates the 2^n-dimensional Clifford algebra $C\ell_{0,n}$ over $\mathbb{R}$ and the set $\{e_A : A \subseteq \{1,\cdots,n\}\}$ with $e_A = e_{h_1} e_{h_2} \cdots e_{h_r}$, $1 \le h_1 < \cdots < h_r \le n$, $e_\emptyset = e_0 = 1$, forms a basis of $C\ell_{0,n}$. The real vector space $\mathbb{R}^{n+1}$ will be embedded in $C\ell_{0,n}$ by identifying the element $(x_0, x_1, \cdots, x_n) \in \mathbb{R}^{n+1}$ with the element $x = x_0 + \underline{x} \in \mathscr{A}_n := \mathrm{span}_{\mathbb{R}}\{1, e_1, \ldots, e_n\} \subset C\ell_{0,n}$. Here, x_0 and $\underline{x} = e_1 x_1 + \cdots + e_n x_n$ are, the so-called, scalar and vector parts of the paravector $x \in \mathscr{A}_n$. The conjugate of x is $\bar{x} = x_0 - \underline{x}$ and the norm $|x|$ of x is defined by $|x|^2 = x\bar{x} = \bar{x}x = x_0^2 + x_1^2 + \cdots + x_n^2$.

In what follows we consider $C\ell_{0,n}$-valued functions defined in some open subset $\Omega \subset \mathbb{R}^{n+1}$, i.e., functions of the form $f(z) = \sum_A f_A(z) e_A$, where $f_A(z)$ are real valued. We suppose that f is hypercomplex differentiable in Ω in the sense of [1], [2], i.e. has a uniquely defined areolar derivative f' in each point of Ω. Then f is real differentiable (even real analytic) and f' can be expressed in terms of the partial derivatives with respect to x_k as $f' = \frac{1}{2}(\partial_0 - \partial_{\underline{x}})f$, where $\partial_0 := \frac{\partial}{\partial x_0}$, $\quad \partial_{\underline{x}} := e_1 \frac{\partial}{\partial x_1} + \cdots + e_n \frac{\partial}{\partial x_n}$. If now $\overline{\partial} := \frac{1}{2}(\partial_0 + \partial_{\underline{x}})$ is the usual generalized Cauchy-Riemann differential operator, then, obviously $f' = \partial f$. Since in [1] it has been shown that a hypercomplex differentiable function belongs to the kernel of $\overline{\partial}$, i.e. satisfies the property $\overline{\partial} f = 0$ (f is a *monogenic function* in the sense of Clifford Analysis), then it follows that in fact $f' = \partial_0 f$ like in the complex case.

HYPERCOMPLEX APPELL POLYNOMIALS

We recall the classical definition of sequences of Appell polynomials [3] adapted to the hypercomplex case: a sequence of monogenic polynomials $(\mathscr{F}_k)_{k\ge 0}$ of exact degree k is called a *generalized Appell sequence* if $\mathscr{F}_0(x) \equiv 1$ and $\mathscr{F}_k' = k\mathscr{F}_{k-1},\ k = 1,2,\ldots.$ (see e.g. [4]). Another equivalent definition of Appell polynomials is provided by the following result ([5]):

Theorem 1 *A monogenic polynomial sequence $(\mathscr{F}_k)_{k\ge 0}$ is an Appell set if and only if it satisfies the binomial-type expansion*

$$\mathscr{F}_k(x) = \mathscr{F}_k(x_0 + \underline{x}) = \sum_{s=0}^{k} \binom{k}{s} \mathscr{F}_{k-s}(x_0)\mathscr{F}_s(\underline{x}),\ x \in \mathscr{A}_n.$$

Example 1 *Standard Appell polynomials: ([4, 5, 6])*

$$\mathscr{P}_k(x_0,\underline{x}) = \sum_{s=0}^{k} \binom{k}{s} c_s(n)\, x_0^{k-s}\, \underline{x}^s, \tag{1}$$

where

$$c_s(n) = \frac{s!!(n-2)!!}{(n+s-1)!!}, \quad \text{if } s \text{ is odd}, \qquad c_s(n) = c_{s-1}(n), \quad \text{if } s \text{ is even}. \tag{2}$$

Example 2 *Pseudo-complex polynomials: ([7, 8])*

$$\mathscr{P}_k(x_0,\underline{x}) = \sum_{s=0}^{k} \binom{k}{s} x_0^{k-s}((i_1x_1+\cdots+i_nx_n)(i_1e_1+\cdots+i_ne_n))^s = (x_0+(i_1x_1+\cdots+i_nx_n)(i_1e_1+\cdots+i_ne_n))^k, \tag{3}$$

where $i_1,\ldots,i_n$ are real parameters such that $i_1^2+\cdots+i_n^2=1$.

These two examples play an important role in the present context, as the following recent result in [8] shows.

Theorem 2 *In $\mathbb{R}^3$, there are exactly two different types of non-trivial Appell polynomials of the form*

$$\mathscr{P}_k(x_0,x_1,x_2) = \sum_{s=0}^{k} \binom{k}{s} d_s\, x_0^{k-s}\,(X_1(x_1,x_2)e_1+X_2(x_1,x_2)e_2)^s, \tag{4}$$

where $X_j = X_j(x_1,x_2)$, $j=1,2$, are two real valued linear functions ($\partial_1X_1\cdot\partial_2X_2\neq 0$) and $d_0=1$:

1. *The 3D standard Appell polynomials (1), corresponding to the case $\partial_1X_2=\partial_2X_1=0$.*
2. *The 3D pseudo-complex polynomials (3), when $\partial_1X_2=\partial_2X_1\neq 0$.*

Observe that, if $\partial_1X_1\cdot\partial_2X_2=0$, then (4) corresponds to trivial Appell polynomials, i.e.

$$\mathscr{P}_k(x_0,x_2) = \sum_{s=0}^{k} \binom{k}{s} x_0^{k-s}(x_2e_2)^s = (x_0+x_2e_2)^k \quad \text{or} \quad \mathscr{P}_k(x_0,x_1) = \sum_{s=0}^{k} \binom{k}{s} x_0^{k-s}(x_1e_1)^s = (x_0+x_1e_1)^k,$$

where we recognize two copies of the ordinary complex power function, namely $z^k=(x+iy)^k$ after setting $x_0:=x$ and $e_j:=i, x_j:=y,\ \ j=1,2$. These cases are trivial in the sense that $e_1\partial_1+e_2\partial_2+e_3\partial_3$ acts only as differential operator with respect to x_1 or x_2 if the components of X are not depending on x_2 or x_1, respectively. More details about Appell sequences in hypercomplex context and the contributions of other authors to this subject can be found in [8, 9].

4D HOMOGENEOUS POLYNOMIALS WITH BINOMIAL EXPANSION

We extend now the result of Theorem 2, by considering $n=3$ and general paravector valued homogeneous polynomials of the form

$$\mathscr{P}_k(x_0,x_1,x_2,x_3) = \sum_{s=0}^{k} \binom{k}{s} d_s\, x_0^{k-s}\, \underline{X}^s, \tag{5}$$

where $\underline{X} := X_1(x_1,x_2,x_3)e_1+X_2(x_1,x_2,x_3)e_2+X_3(x_1,x_2,x_3)e_3$ and $X_j = X_j(x_1,x_2,x_3)$, $j=1,2,3$, are real valued linear functions.

Our purpose here is to choose the linear functions X_j ($j=1,2,3$) and the real coefficients d_s ($s=1,2,\ldots$) in such a way that the corresponding polynomials (5) are monogenic. Since left monogenic functions of the form (5) are also right monogenic, it is enough to solve the real system of first order partial differential equations corresponding to $\overline{\partial}\mathscr{P}_k=0$. The use of Theorem 1 allows to conclude that this procedure leads to a characterization of the set of paravector valued homogeneous monogenic polynomials with the Appell-property.

Following [8], we use the normalization condition $d_0=1$ in order to keep the ordinary real Appell sequence as the restriction on the real line. Therefore, the first degree polynomial in the sequence (5) is $\mathscr{P}_1(x_0,x_1,x_2,x_3)=x_0+d_1\underline{X}$. and it is clear that such polynomial is monogenic if the real linear functions are of the form

$$X_1 = a_1x_1+\lambda_1x_2+\lambda_2x_3, \qquad X_2 = \lambda_1x_1+a_2x_2+\lambda_3x_3, \qquad X_3 = \lambda_2x_1+\lambda_3x_2+a_3x_3$$

and the coefficient d_1 is given by

$$d_1 = \frac{1}{a_1+a_2+a_3}, \qquad \text{for} \quad a_1+a_2+a_3 \neq 0. \tag{6}$$

We point out that the special cases

$$\lambda_1 = \lambda_2 = a_1 = 0 \qquad \text{or} \qquad \lambda_1 = \lambda_3 = a_2 = 0 \qquad \text{or} \qquad \lambda_2 = \lambda_3 = a_3 = 0 \tag{7}$$

correspond to 3D polynomials and therefore Theorem 2 holds, i.e. (5) are, in fact, 3D standard Appell polynomials, 3D pseudo-complex polynomials or complex powers. Our interest lies, of course, on the other cases. We proceed now by examining under what conditions the second degree polynomial $\mathscr{P}_2(x_0,x_1,x_2,x_3) = x_0 + 2d_1\underline{X} + d_2\underline{X}^2$, with d_1 given by (6) is a monogenic polynomial. This problem is equivalent, for independent x_1, x_2 and x_3, to solve the nonlinear system

$$\begin{cases} a_1 - d_2(a_1^2+\lambda_1^2+\lambda_2^2)(a_1+a_2+a_3) = 0 \\ a_2 - d_2(\lambda_1^2+a_2^2+\lambda_3^2)(a_1+a_2+a_3) = 0 \\ a_3 - d_2(\lambda_2^2+\lambda_3^2+a_3^2)(a_1+a_2+a_3) = 0 \\ \lambda_1 - d_2(\lambda_1(a_1+a_2)+\lambda_2\lambda_3)(a_1+a_2+a_3) = 0 \\ \lambda_2 - d_2(\lambda_2(a_1+a_3)+\lambda_1\lambda_3)(a_1+a_2+a_3) = 0 \\ \lambda_3 - d_2(\lambda_3(a_2+a_3)+\lambda_1\lambda_2)(a_1+a_2+a_3) = 0 \end{cases} \tag{8}$$

Cumbersome, but straightforward calculations lead to the following cases:

1. $\lambda_1 = \lambda_2 = \lambda_3 = 0$

We can assume that $a_1a_2a_3 \neq 0$, since the trivial case where one of the a_i's is zero was already considered in (7). The solution of (8) is:

$$a_1 = a_2 = a_3 = \alpha \ \ (\alpha \neq 0) \qquad \text{and} \qquad d_2 = \frac{1}{3\alpha^2}.$$

Since in this case, $d_1 = \frac{1}{3\alpha}$, we obtain

$$\underline{X} = \alpha(x_1e_1+x_2e_2+x_3e_3) \qquad \text{and} \qquad \mathscr{P}_1(x) = x_0 + \tfrac{1}{3}\underline{x}.$$

Applying Theorem 2 of [6], it is clear that

$$d_k = \frac{c_k(3)}{\alpha^k},$$

where $c_k(3)$ are the coefficients (2), for the dimension $n = 3$. In other words, this case corresponds to the standard Appell polynomials (cf (1)) in $\mathbb{R}^4$.

2. $\lambda_1 \neq 0 \quad \text{or} \quad \lambda_2 \neq 0 \quad \text{or} \quad \lambda_3 \neq 0$

These cases mean that we assume that either one of the parameters is nonzero and the other two vanish or all the parameters are nonzero. The situation where two of the parameters are nonzero and the other one is zero is not possible (see equations 4-6 of (8)). System (8) has now two different solutions:

i. For the first one we have that

$$d_2 = \frac{1}{(a_1+a_2+a_3)^2} = d_1^2$$

and $\lambda_1,\ \lambda_2,\ \lambda_3$ are roots of

$$\lambda_1^2 = a_1a_2, \qquad \lambda_2^2 = a_1a_3, \qquad \lambda_3^2 = a_2a_3, \tag{9}$$

chosen so that $\lambda_1\lambda_2\lambda_3 = a_1a_2a_3$. We point out that, due to (9), if a_1, a_2 and a_3 are nonzero, then they must have the same sign. Following the technique of ([8]), we introduce now the real parameters

$$i_1^2 := \tfrac{a_1}{a_1+a_2+a_3}, \qquad i_2^2 := \tfrac{a_2}{a_1+a_2+a_3}, \qquad i_3^2 := \tfrac{a_3}{a_1+a_2+a_3}, \tag{10}$$

which allow to write

$$\mathscr{P}_1(x) = x_0 + (i_1x_1+i_2x_2+i_3x_3)\,(i_1e_1+i_2e_2+i_3e_3)\,,$$

for a chosen triplet of those roots i_1, i_2 and i_3 as defined in (10). Notice that from this relation we obtain $i_1^2 + i_2^2 + i_3^2 = 1$ and it follows immediately that $(i_1 e_1 + i_2 e_2 + i_3 e_3)^2 = -1$. This shows the isomorphism of the structure of $\mathscr{P}_k(x)$ with z^k and it implies also that $\mathscr{P}_k(x) = (\mathscr{P}_1(x))^k$. In other words, we obtain the 4D-version of the pseudo-complex polynomials (3), if $\lambda_1 \lambda_2 \lambda_3 \neq 0$ and the 3D-version, in the other cases.

(ii) System (8) also admits another solution, namely

$$d_2 = \frac{2}{(a_1 + a_2 + a_3)^2} = 2d_1^2$$

and $\lambda_1, \ \lambda_2, \ \lambda_3$ are roots of

$$\lambda_1^2 = \frac{1}{4} A_1 A_2, \qquad \lambda_2^2 = \frac{1}{4} A_1 A_3, \qquad \lambda_3^2 = \frac{1}{4} A_2 A_3, \tag{11}$$

where $A_1 = -a_1 + a_2 + a_3, A_2 = a_1 - a_2 + a_3$ and $A_3 = a_1 + a_2 - a_3$ are real parameters which must have the same sign, if none of them vanishes. In this case, the roots λ_i must be chosen in order to verify the relation $-8\lambda_1 \lambda_2 \lambda_3 = A_1 A_2 A_3$. We proceed by adapting the procedure used in case **2.i**. More precisely, we define the real parameters

$$j_1^2 := \tfrac{A_1}{A_1 + A_2 + A_3}, \qquad j_2^2 := \tfrac{A_2}{A_1 + A_2 + A_3}, \qquad j_3^2 := \tfrac{A_3}{A_1 + A_2 + A_3}, \tag{12}$$

which verify the relation $j_1^2 + j_2^2 + j_3^2 = 1$. Denoting by $\underline{\tilde{X}}$ the pseudo-complex first degree polynomial

$$\underline{\tilde{X}} := (j_1 x_1 + j_2 x_2 + j_3 x_3)(j_1 e_1 + j_2 e_2 + j_3 e_3),$$

we can write, after some manipulation,

$$\underline{X} = \frac{1}{2d_1}(\underline{x} - \underline{\tilde{X}}) \qquad \text{and} \qquad \mathscr{P}_1(x_0, x_1, x_2, x_3) = x_0 + \tfrac{1}{2}(\underline{x} - \underline{\tilde{X}}).$$

Repeating the same process for $\mathscr{P}_3$, $\mathscr{P}_4, \ldots$, we conclude that

$$d_s = \binom{s}{\lfloor \frac{s}{2} \rfloor} d_1^s, \quad s = 0, 1, \ldots,$$

where $\lfloor \cdot \rfloor$ is the floor function. Therefore a new type of Appell polynomial is obtained:

$$\mathscr{P}_k(x) = \sum_{s=0}^{k} \binom{k}{s} \binom{s}{\lfloor \frac{s}{2} \rfloor} x_0^{k-s} (d_1 \underline{X})^s = \sum_{s=0}^{k} \binom{k}{s} \binom{s}{\lfloor \frac{s}{2} \rfloor} x_0^{k-s} \tfrac{1}{2^s} (\underline{x} - \underline{\tilde{X}})^s = \sum_{s=0}^{k} \binom{k}{s} x_0^{k-s} c_s(2) (\underline{x} - \underline{\tilde{X}})^s.$$

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