Crossover to the KPZ equation

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Abstract. We consider the weakly asymmetric simple exclusion process and we show that the density field is governed by an Ornstein-Uhlenbeck process for strength asymmetry $n^{2-\gamma}$ if $\gamma \in (1/2, 1)$, while for $\gamma = 1/2$ it is an energy solution of the KPZ equation.
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1 Introduction

We consider the one-dimensional weakly asymmetric simple exclusion process, i.e. our microscopic dynamic is given by a stochastic lattice gas with hard core exclusion. This process arises as a simple model for the growing of random interfaces. The presence of weak asymmetry breaks down the detailed balance, which implies the system to exhibit a non trivial behavior even in the stationary situation. Using renormalization group techniques, the dynamical scaling exponent has been established as $z = 3/2$ and one of the challenging problems is to establish the limit distribution for the density or current of particles, see Spohn (1991). The weakly asymmetric simple exclusion process was studied in Masi et al (1986) and in Dittrich and Gartner (1991), for $\gamma = 1$; and in Bertini and Giacomin (1997) for $\gamma = 1/2$. The equilibrium density fluctuations (for $\gamma = 1$) are given by an Ornstein-Uhlenbeck process. For $\gamma = 1/2$ (which corresponds to strength asymmetry n^z), Bertini and Giacomin (1997) used the Cole-Hopf transformation to derive the non-equilibrium fluctuations of the current of particles. By removing the drift to the system, there is no effect of the strength of the asymmetry on the limit distribution of the density field. By strengthening the asymmetry the limit distribution "feels" the effect of this strengthening, by developing a non linear term in the limit distribution. In this case the limit density field is a solution of the Kardar-Parisi-Zhang (KPZ) equation. The KPZ equation was proposed in Kardar et al (1986) to model the growth of random interfaces. Denoting by h_t the height of the interface, this equation reads as

$$\partial_t h = D\Delta h + a(\nabla h)^2 + \sigma \mathcal{W}_t,$$

where D, a, σ are related to the thermodynamical properties of the interface and $\mathcal{W}_t$ is a Gaussian space-time white noise with covariance given by $E[\mathcal{W}_t(u)\mathcal{W}_s(v)] = \delta(t-s)\delta(u-v)$. According to z, a non-trivial behavior occurs under re-scaling $h_n(t,x) = n^{-1/2}h(tn^{3/2}, x/n)$. This means, roughly speaking, that in our case, for $\gamma = 1/2$ a non trivial behavior is expected even in the stationary situation and the model belongs to the universality class of the KPZ equation. We provide the characterization of the transition from the Edwards-Wilkinson class to the KPZ class, for the weakly asymmetric simple exclusion process. We prove that the transition depends on the strength of the asymmetry without having any other intermediate state and by establishing precisely the strength in order to have the crossover.

2 The crossover

We consider η_t as the weakly asymmetric simple exclusion process evolving on $\mathbb{Z}$. The state space is $\Omega = \{0,1\}^{\mathbb{Z}}$, and after a mean one exponential time a particle jumps to an empty neighboring site according to a transition rate that has a weak asymmetry to the right. The process is speeded up on the diffusive time scale n^2 so that $\eta_t^n = \eta_{tn^2}$ and the transition rate from x to $x+1$ is $1/2 + 1/n^\gamma$ and from $x+1$ to x is $1/2$. We notice that if we decrease the value of γ, this corresponds to speeding up the asymmetric part of the dynamics on longer time scales as $n^{2-\gamma}$. A stationary state for this process is $\{\nu_\rho : \rho \in [0,1]\}$ the Bernoulli product measure on Ω of parameter ρ. Here we recall briefly the *hydrodynamic limit* for η_t^n, with $\gamma = 1$. For that purpose we introduce the empirical measure as the positive measure in $\mathbb{R}$ defined by

$$\pi_t^n(dx) = \frac{1}{n}\sum_{x\in\mathbb{Z}} \eta_t^n(x)\delta_{x/n}(dx),$$

where for $u \in \mathbb{R}$, δ_u is the Dirac measure at u. Take ρ_0 a strictly positive and piecewise continuous function for which there exists $\rho \in (0,1)$ such that $\int |\rho_0(x) - \rho| dx < +\infty$. Start the process from $\{\mu_n; n \in \mathbb{N}\}$, a product measure in Ω, whose marginal at x is Bernoulli of parameter $\rho_0(x/n)$. In Guo et al (1988) it was shown that $\pi_t^n(dx)$ converges in probability to the deterministic measure $\rho(t,x)dx$, where $\{\rho(t,x); t \geq 0, x \in \mathbb{R}\}$ is the unique weak solution of the viscous Burgers equation

$$\partial_t \rho(t,x) = \frac{1}{2}\Delta\rho(t,x) - \nabla\chi(\rho(t,x))$$

where $\chi(\rho) = \rho(1-\rho)$ is the static compressibility of the system. We establish the fluctuations of the empirical measure from the stationary state ν_ρ. We fix a density ρ and we take η_t^n moving in a reference frame with constant velocity $(1-2\rho)n^{2-\gamma}$. We define the *density fluctuation field* on $H \in \mathcal{S}(\mathbb{R})$ as:

$$\mathcal{Y}_t^{n,\gamma}(H) = \frac{1}{\sqrt{n}}\sum_{x\in\mathbb{Z}} T_t^\gamma H_x\big(\eta_t^n(x) - \rho\big), \tag{1}$$

where $T_t^\gamma H(\cdot) = H(\cdot - (1-2\rho)tn^{1-\gamma})$. Here and from now on we use the denotation $H_x = H(x/N)$. For $\gamma = 1$, it is not hard to show that $\{\mathcal{Y}_t^{n,\gamma}; n \in \mathbb{N}\}$ converges to $\mathcal{Y}_t^\gamma$ solution of

$$d\mathcal{Y}_t^\gamma = \frac{1}{2}\Delta\mathcal{Y}_t^\gamma dt + \sqrt{\chi(\rho)}\nabla d\mathcal{W}_t, \tag{2}$$

where $\mathcal{W}_t$ is a space-time white noise, that is, a Gaussian process of mean zero and covariance $\delta(x-x')\delta(t-t')$. So, for $\gamma = 1$ the system belongs to the Edwards-Wilkinson class.

In order to see the effect of the asymmetry in the limit density field we increment the strength of the asymmetry by decreasing the value of γ. As discussed in Bertini and Giacomin (1997), the effect of the asymmetry is presented in the limit field when $\gamma = 1/2$ and in that case $\mathcal{Y}_t^\gamma$ has a very different qualitatively behavior from the one obtained for $\gamma = 1$, namely the solution of (2). Here we characterize the limit field $\mathcal{Y}_t^\gamma$ for the intermediate state, i.e. for $\gamma \in (1/2, 1)$, by showing that it solves (2) and in this case the system still belongs to the Edwards-Wilkinson class. The idea of the proof is to use Dynkin's formula together with simple computations, to write down

$$\mathcal{Y}_t^{n,\gamma}(H) = M_t^{n,\gamma}(H) + \mathcal{Y}_0^{n,\gamma}(H) + \mathcal{I}_t^{n,\gamma}(H) + \mathcal{A}_t^{n,\gamma}(H)$$

where $M_t^{n,\gamma}(H)$ is a martingale with respect to the natural filtration,

$$\mathcal{I}_t^{n,\gamma}(H) = \int_0^t \frac{1}{2\sqrt{n}}\sum_{x\in\mathbb{Z}} \Delta^n T_s^\gamma H_x(\eta_s^n(x) - \rho)ds,$$

$$\mathcal{A}_t^{n,\gamma}(H)=\int_0^t \frac{n^{1-\gamma}}{\sqrt{n}}\sum_{x\in\mathbb{Z}}\nabla^n T_s^\gamma H_x\Big\{\eta_s^n(x)(1-\eta_s^n(x+1))-\chi(\rho)-(1-2\rho)(\eta_s^n(x)-\rho)\Big\}ds,$$

and Δ^n, ∇^n are the discrete laplacian and the discrete derivative, respectively. Then we need to analyze the asymptotic behavior of the martingale and the integral terms. The hard programme is to analyze $\mathcal{A}_t^{n,\gamma}(H)$. We can show that this term vanishes as $N\to+\infty$ as a consequence of the stronger Boltzmann-Gibbs principle given in Corollary 7.4 of Gonçalves (2008). There the result was obtained for the symmetric simple exclusion but is also true for the weakly asymmetric version. In fact the result can be stated as: if $\psi:\Omega\to\mathbb{R}$ is a local function, $\gamma\in(1/2,1)$ and if $H\in\mathcal{S}(\mathbb{R})$ then

$$\lim_{n\to\infty}\mathbb{E}_{\nu_\rho}\Big[\Big(\int_0^t \frac{n^{1-\gamma}}{\sqrt{n}}\sum_{x\in\mathbb{Z}}H_x\Big\{\tau_x\psi(\eta_s^n)-E_{\nu_\rho}[\psi(\eta)]-\partial_\rho E_{\nu_\rho}[\psi(\eta)](\eta_s^n(x)-\rho)\Big\}ds\Big)^2\Big]=0. \quad (3)$$

Last result together with some computations on the quadratic variation of the martingale, gives us that $\mathcal{Y}_t^\gamma$ is solution of (2).

So, if we want to see the effect of the asymmetry in the limit field, we go towards decreasing the value of γ, which, as mentioned above, corresponds to speeding up the asymmetric part of the dynamics. This is in agreement with the result of Bertini and Giacomin (1997) which says that for $\gamma=1/2$ that is indeed the case. Recently in Jara and Gonçalves (2010), it was shown that for $\gamma=1/2$, $\{\mathcal{Y}_t^{n,\gamma};n\in\mathbb{N}\}$ is tight and any limit point is a weak solution of the KPZ equation:

$$d\mathcal{Y}_t^\gamma=\frac{1}{2}\Delta\mathcal{Y}_t^\gamma dt+\nabla(\mathcal{Y}_t^\gamma)^2dt+\sqrt{\chi(\rho)}\nabla d\mathcal{W}_t. \quad (4)$$

Since we are in the presence of a stronger asymmetry, the result in (3) is no longer true. In order to establish last result, a second order Boltzmann-Gibbs principle was derived in Jara and Gonçalves (2010). The ingredients invoked in order to derive this stronger replacement is a multi-scale argument introduced in Gonçalves (2008) combined with some fundamental features of the model: as a sharp spectral gap bound for the dynamics restricted to finite boxes, plus a second order expansion on the equivalence of ensembles.

The results beyond the hydrodynamic time scale, the crossover at $\gamma=1/2$ and the KPZ class, are in fact true for a general class of weakly asymmetric exclusion processes see Gonçalves and Jara (2010).

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