The N–matrix completion problem under digraphs assumptions *†

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Abstract

An $n \times n$ matrix is called an N–matrix if all principal minors are negative. In this paper, we are interested in the partial N–matrix completion problem, when the partial N–matrix is non-combinatorially symmetric. In general, this type of partial matrices does not have an N–matrix completion. We prove that a non-combinatorially symmetric partial N–matrix has an N–matrix completion if the graph of its specified entries is an acyclic graph or a cycle. We also prove that there exists the desired completion for partial N–matrices such that in its associated graphs the cycles play an important role.

1 Introduction

A real *partial matrix* is an $n \times n$ array in which some entries are specified, while the remaining entries are free to be chosen. An $n \times n$ partial matrix is said to be *combinatorially symmetric* if the (i, j) entry is specified if and only if the (j, i) entry is and *non-combinatorially symmetric* in other case. A *completion* of a partial matrix is the conventional matrix resulting from a particular choice of values for the unspecified entries. A *matrix completion problem* asks which partial matrices have completions with a given property.

An $n \times n$ real matrix is called an *N–matrix* if all its principal minors are negative. N–matrices arise in the theory of global univalence of functions [4], in multivariate analysis [8] and in linear complementary problems [6, 9]. In [10] the class of N–matrices was also studied in connection with Lemke's algorithm for solving linear and convex quadratic programming problems.

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The submatrix of a matrix A, of size $n \times n$, lying in rows α and columns β, $\alpha, \beta \subseteq \{1, ..., n\}$, is denoted by $A[\alpha|\beta]$, and the principal submatrix $A[\alpha|\alpha]$ is abbreviated to $A[\alpha]$.

The following simple facts are very useful in the study of N–matrices.

Proposition 1.1 *Let $A = (a_{ij})$ be an $n \times n$ N–matrix. Then*

1. *If P is a permutation matrix then, PAP^T is an N–matrix.*

2. *If D is a positive diagonal matrix then, DA and AD are N–matrices.*

3. *If D is a nonsingular diagonal matrix then, DAD^{-1} is an N–matrix.*

4. *$a_{ij} \neq 0$ and $sign(a_{ij}) = sign(a_{ji})$, for all $i, j \in \{1, ..., n\}$.*

5. *If $a_{ii+1} > 0$, $i = 1, 2, \ldots, n-1$, then $A \in \mathcal{S}_n$, where*

$$\mathcal{S}_n = \left\{ A = (a_{ij}) \mid a_{ij} \neq 0 \text{ and } sign(a_{ij}) = (-1)^{i+j+1}, \text{ for all } i, j \in \{1, ..., n\} \right\}.$$

6. *Any principal submatrix of A is an N–matrix.*

It is not difficult to prove, from above properties (3) and (5) that any N–matrix is diagonally similar to an N–matrix in $\mathcal{S}_n$. Moreover, if $A \in \mathcal{S}_n$ is an $n \times n$ N–matrix and $D = \text{diag}(1, -1, 1, -1, \ldots, (-1)^n, (-1)^{n+1})$, then DAD^{-1} is an N–matrix with all entries negative. Therefore, any N–matrix is diagonally similar to a negative N–matrix.

From property (2), we can also suppose that all diagonal entries are equal to -1.

The last property of Proposition 1.1 allows us to give the following definition.

Definition 1.1 *A partial matrix is said to be a partial N–matrix if every completely specified principal submatrix is an N–matrix.*

In this paper we study the *N–matrix completion problem*, that is, when a partial N–matrix has an N–matrix completion.

In the previous proposition we have seen that an N–matrix has no zero entries and is sign-symmetric (the entries in symmetric positions have the same sign). Keeping this in mind, it would not make sense to study the existence of N–matrix completions of partial N–matrices with some zero entry or of non-sign-symmetric partial N–matrices.

On the other hand, taking into account property (5) of Proposition 1.1 we define the set $\mathcal{PS}_n$ of the $n \times n$ partial matrices $A = (a_{ij})$ such that $a_{ij} \neq 0$ and $sign(a_{ij}) = (-1)^{i+j+1}$, for all $i, j \in \{1, ..., n\}$ such that the (i, j) entry is specified.

In [7] the authors prove that, in general, a partial N–matrix has no N–matrix completion and that being permutation or diagonally similar to a matrix in $\mathcal{PS}_n$ is a necessary condition in order to obtain an N–matrix completion of a partial N–matrix, being also a sufficient condition for matrices of size 3×3.

Observe that when restricting our study of the posed completion problem to partial matrices in $\mathcal{PS}_n$, we are implicitly analyzing the problem for any partial N–matrix that is permutation or diagonally similar to a partial matrix in $\mathcal{PS}_n$.

The specified entries of an $n \times n$ partial matrix A can be described by a graph $G_A = (V, E)$, where the set of vertices V is $\{1, ..., n\}$ and $\{i, j\}$, $i \neq j$, is an edge or arc if and only if the (i, j) entry is specified. A directed graph is associated with a non-combinatorially symmetric partial matrix and, when the partial matrix is combinatorially symmetric, an undirected graph can be used.

A *path* is a sequence of edges $\{i_1, i_2\}$, $\{i_2, i_3\}$, ..., $\{i_{k-1}, i_k\}$ in which all vertices are distinct. A *cycle* is a closed path, that is, a path in which the first and the last vertices coincide.

The N–matrix completion problem in the combinatorially symmetric case was studied in [7]. In this paper we analyze the mentioned problem for non-combinatorially symmetric partial matrices and therefore we work with directed graphs. We make the assumption throughout that all diagonal entries are prescribed and then we omit the loops. Moreover, taking into account Theorem 3.2 of [7] we can suppose, without loss of generality, that the associated graph is strongly connected.

In section 2 we prove that every non-combinatorially symmetric partial N–matrix, whose associated graph is acyclic has an N–matrix completion. In the following section we analyze the posed completion problem when the associated graph to the partial matrix is non acyclic, more concretely, when the graph is a cycle, a semi cycle or a block-cycle.

2 Acyclic Graphs

In this section we prove the existence of an N–matrix completion of non-combinatorially symmetric partial N–matrices whose associated graph is acyclic.

Theorem 2.1 *Every non-combinatorially symmetric partial N–matrix, belonging to $\mathcal{PS}_n$, the graph of whose specified entries is acyclic has an N–matrix completion.*

Proof: Let A be a non-combinatorially symmetric partial N–matrix, belonging to $\mathcal{PS}_n$, the graph of whose specified entries G_A is acyclic. As we have commented in the introduction, we can assume that all specified entries of A are negative and all diagonal elements are equal to -1.

Since the binary relation $\mathcal{R} = E(G_A)$ in set V is such that $G_{\mathcal{R}}$ is acyclic, *the topological order algorithm* gives a total order relation $\mathcal{S}$ that preserves $\mathcal{R}$. From the Hasse diagram of $\mathcal{S}$ (see [3]), we obtain a permutation matrix P such that $\bar{A} = PAP^T$ has the lower triangular part totally unspecified and the upper one has unspecified and known entries.

We may assume, without loss of generality, that $\bar{A}$ has the upper triangular part completely specified. In fact, we can complete each non-prescribed (i, j) entry, $i < j$, with a negative real number. Hence, $\bar{A}$ has the following form

$$\bar{A} = \begin{bmatrix} -1 & -a_{12} & -a_{13} & \cdots & -a_{1n-1} & -a_{1n} \\ ? & -1 & -a_{23} & \cdots & -a_{2n-1} & -a_{2n} \\ ? & ? & -1 & \cdots & -a_{3n-1} & -a_{3n} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ ? & ? & ? & \cdots & -1 & -a_{n-1n} \\ ? & ? & ? & \cdots & ? & -1 \end{bmatrix},$$

where $a_{ij} > 0$, for any $i \in \{1, 2, \ldots, n-1\}$ and any $j \in \{2, 3, \ldots, n\}$ such that $j > i$.

We are going to prove that $\bar{A}$ admits an N–matrix completion. For any $x \in \mathbb{R}$, consider the completion

$$\bar{A}_x = \begin{bmatrix} -1 & -a_{12} & -a_{13} & \cdots & -a_{1n-1} & -a_{1n} \\ -x & -1 & -a_{23} & \cdots & -a_{2n-1} & -a_{2n} \\ -x & -x & -1 & \cdots & -a_{3n-1} & -a_{3n} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ -x & -x & -x & \cdots & -1 & -a_{n-1n} \\ -x & -x & -x & \cdots & -x & -1 \end{bmatrix},$$

of $\bar{A}$. It is easy to prove that $\det \bar{A}_x$ is a polynomial of degree $n-1$ in x whose leading coefficient is $-a_{1n} < 0$. Therefore, there exists $M_N \in \mathbb{R}$ such that $\det \bar{A}_x < 0$, for all $x > M_N$.

Let $\alpha \subset N = \{1, 2, \ldots, n\}$ and $|\alpha| = k$. If $k > 1$, the principal submatrix $\bar{A}_x[\alpha]$ has the following form

$$\bar{A}_x[\alpha] = \begin{bmatrix} -1 & -b_{12} & -b_{13} & \cdots & -b_{1k-1} & -b_{1k} \\ -x & -1 & -b_{23} & \cdots & -b_{2k-1} & -b_{2k} \\ -x & -x & -1 & \cdots & -b_{3k-1} & -b_{3k} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ -x & -x & -x & \cdots & -1 & -b_{k-1k} \\ -x & -x & -x & \cdots & -x & -1 \end{bmatrix},$$

and its determinant is a polynomial of degree $k-1$ in x whose leading coefficient is $-b_{1k} < 0$. Consequently, there exists $M_\alpha \in \mathbb{R}$ such that $\det \bar{A}_x[\alpha] < 0$, for all $x > M_\alpha$.

Let $M = \max\{M_\alpha : \alpha \subset N, |\alpha| > 1\}$. It is clear that $\det \bar{A}_x[\alpha] < 0$, for any $x > M$ and any $\alpha \subset N$. Therefore, for any $x > M$, $\bar{A}_x$ is an N–matrix completion of $\bar{A}$ and, hence, A admits an N–matrix completion. □

Since the graph of the specified entries of a partial upper triangular matrix (that is, a partial matrix such that all its entries in positions (i, j), $i \leq j$ are known and the remaining are unspecified) is acyclic, we can establish the following result.

Corollary 2.1 *Let A be a partial upper triangular N–matrix such that $A \in \mathcal{PS}_n$. Then, there exists an N–matrix completion of A.*

3 Non Acyclic Graphs

In this section, we study the posed completion problem when the associated graph to the partial N–matrix is non acyclic. More specifically, we will focus on different types of associated graphs where the cycles play an important role.

Firstly, taking into account Proposition 4.1 of [7], it is easy to prove the following basic results.

Proposition 3.1 *Every non-combinatorially symmetric partial N–matrix belonging to $\mathcal{PS}_n$, the graph of whose specified entries is a cycle, has an N–matrix completion.*

Proposition 3.2 *Every non-combinatorially symmetric partial N–matrix belonging to $\mathcal{PS}_n$, the graph of whose specified entries is a semi cycle (that is the underlying graph is a cycle), has an N–matrix completion.*

When the associated graph to the partial N–matrix is a double cycle with a common vertex, from the Proposition 3.1 and Theorem 3.1 of [7] we can make the following statement.

Proposition 3.3 *Every non-combinatorially symmetric partial N–matrix belonging to $\mathcal{PS}_n$, the graph of whose specified entries is a double cycle with a common vertex, has an N–matrix completion.*

The last result can be generalized for partial N–matrices whose associated graph is a double cycle with one or more common arcs.

Lemma 3.1 *Let A be a non-combinatorially symmetric partial N–matrix, belonging to $\mathcal{PS}_4$, whose graph is a double cycle with a common arc. Then, there exists an N–matrix completion of A.*

Proof: By permutation and diagonal similarity we only need to consider the following two cases.
(a) A has the form

$$A = \begin{bmatrix} -1 & -1 & -x_{13} & -x_{14} \\ -x_{21} & -1 & -1 & -x_{24} \\ -x_{31} & -x_{32} & -1 & -1 \\ -a_{41} & -x_{42} & -a_{43} & -1 \end{bmatrix}.$$

Consider $c > 1$ and let $x_{21} = c$, $x_{42} = a_{41}$. We denote by A_1 the obtained partial matrix. By applying Proposition 2.2 of [7] to the principal submatrix $C = A_1[\{2,3,4\}]$ we obtain an N–matrix completion

$$C_c = \begin{bmatrix} -1 & -1 & -c_{24} \\ -c_{32} & -1 & -1 \\ -a_{41} & -a_{43} & -1 \end{bmatrix}.$$

It is easy to prove that the matrix obtained from A_1, by replacing $A_1[\{2,3,4\}]$ by C_c and by taking $x_{13} = 1$, $x_{14} = c_{24}$ and $x_{31} = c_{32}$, is an N–matrix completion of A_1 and therefore of A.
(b) A has the form

$$A = \begin{bmatrix} -1 & -1 & -x_{13} & -x_{14} \\ -x_{21} & -1 & -1 & -x_{24} \\ -a_{31} & -x_{32} & -1 & -1 \\ -x_{41} & -a_{42} & -x_{43} & -1 \end{bmatrix}.$$

Let A_x be the following completion of A

$$A_x = \begin{bmatrix} -1 & -1 & -x & -x \\ -x & -1 & -1 & -x \\ -a_{31} & -x & -1 & -1 \\ -x & -a_{42} & -x & -1 \end{bmatrix}.$$

It is easy to prove that $\det A_x[\alpha]$, $\forall \alpha \subseteq \{1,2,3,4\}$, is a polynomial in x of degree $k \leq 4$, whose leading coefficient is negative. Hence, there exists $M > 0$ such that $\det A_x[\alpha] < 0$, for any $x > M$ and $\forall \alpha \subseteq \{1,2,3,4\}$. □

This lemma allows us to establish the following result.

Theorem 3.1 *Let A be a non-combinatorially symmetric partial N–matrix, belonging to $\mathcal{PS}_n$, whose associated graph is a double cycle with a common arc. Then, A admits an N–matrix completion.*

Proof: The proof is by induction on n. If $n = 3$, A admits an N–matrix completion since $A \in \mathcal{PS}_3$. For $n = 4$, see the previous lemma. Let $n > 4$ and A be an $n \times n$ N–matrix, belonging to $\mathcal{PS}_n$, whose associated graph G_A is a double cycle with a common arc. By permutation and diagonal similarity, we may assume that the cycles are Γ_1 : $\{1,2\}, \{2,3\}, \ldots, \{k,k+1\}, \{k+1,1\}$ and Γ_2 : $\{k,k+1\}, \{k+1,k+2\}, \ldots, \{n-1,n\}, \{n,k\}$, with $k+1 \geq n-k+1$, and the matrix A has the form

$$A = \begin{bmatrix} -1 & -1 & \ldots & -x_{1k} & -x_{1k+1} & -x_{1k+2} & \ldots & -x_{1n} \\ -x_{21} & -1 & \ldots & -x_{2k} & -x_{2k+1} & -x_{2k+2} & \ldots & -x_{2n} \\ -x_{31} & -x_{32} & \ldots & -x_{3k} & -x_{3k+1} & -x_{3k+2} & \ldots & -x_{3n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ -x_{k1} & -x_{k2} & \ldots & -1 & -1 & -x_{kk+2} & \ldots & -x_{kn} \\ -a_{k+11} & -x_{k+12} & \ldots & -x_{k+1k} & -1 & -1 & \ldots & -x_{k+1n} \\ -x_{k+21} & -x_{k+22} & \ldots & -x_{k+2k} & -x_{k+2k+1} & -1 & \ldots & -x_{k+2n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ -x_{n1} & -x_{n2} & \ldots & -a_{nk} & -x_{nk+1} & -x_{nk+2} & \ldots & -1 \end{bmatrix}.$$

In order to obtain the desired completion take $x_{k+12} = a_{k+11}$ and $x_{21} = c$ such that $c > 1$. We call A_1 to the resulting partial matrix. The principal submatrix $C = A_1[\{2,3,\ldots,n\}]$ is an $(n-1)\times(n-1)$ partial N–matrix whose graph is a double cycle, with a common arc. By induction hypothesis there exists an N–matrix completion $C_c = (-c_{ij})_{i,j=2}^n$ of C. We consider the completion A_c of A obtained by replacing the principal submatrix $A[\{2,3,\ldots,n\}]$ by C_c and by choosing

$$\begin{array}{ll} x_{1j} = c_{2j}, & j = 3,4,\ldots,n \\ x_{i1} = c_{i2}, & i = 3,4,\ldots,n, \quad i \neq k+1. \end{array}$$

From the proof of Proposition 3.1 of [7], it follows that A_c is an N–matrix. □

We are now interested in the N–matrix completion problem for $n \times n$ partial matrices, whose associated directed graph is a double cycle with h common arcs, $h > 1$.

Firstly, consider the following particular case.

Lemma 3.2 *Let A be a non-combinatorially symmetric partial N–matrix, belonging to $\mathcal{PS}_{h+2}$, whose graph is a double cycle with h common arcs, $h \geq 2$. Then, there exists an N–matrix completion A_c of A.*

Proof: The proof is by induction on h. If $h = 2$, we can assume, without loss of generality, that matrix A has the form

$$A = \begin{bmatrix} -1 & -1 & -x_{13} & -x_{14} \\ -x_{21} & -1 & -1 & -x_{24} \\ -x_{31} & -x_{32} & -1 & -1 \\ -a_{41} & -a_{42} & -x_{43} & -1 \end{bmatrix}.$$

Given a positive real number x, consider the following completion

$$A_x = \begin{bmatrix} -1 & -1 & -x & -x \\ -x & -1 & -1 & -x \\ -x & -x & -1 & -1 \\ -a_{41} & -a_{42} & -x & -1 \end{bmatrix}$$

of A. It is easy to prove that, for each $\alpha \subseteq \{1,2,3,4\}$, $\det A_x[\alpha]$ is a polynomial in x of degree less than or equal to 4, whose leading coefficient is negative. Hence, there exists $M > 0$ such that $\det A_x[\alpha] < 0$, for any $x > M$ and $\forall \alpha \subseteq \{1,2,3,4\}$.

Suppose now that the result is true for $h - 1$ and let us prove it for h.

By permutation and diagonal similarity, we can assume that matrix A has the form

$$A = \begin{bmatrix} -1 & -1 & -x_{13} & \dots & -x_{1h} & -x_{1h+1} & -x_{1h+2} \\ -x_{21} & -1 & -1 & \dots & -x_{2h} & -x_{2h+1} & -x_{2h+2} \\ -x_{31} & -x_{32} & -1 & \dots & -x_{3h} & -x_{3h+1} & -x_{3h+2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ -x_{h1} & -x_{h2} & -x_{h3} & \dots & -1 & -1 & -x_{hh+2} \\ -x_{h+11} & -x_{h+12} & -x_{h+13} & \dots & -x_{h+1h} & -1 & -1 \\ -a_{h+21} & -a_{h+22} & -x_{h+23} & \dots & -x_{h+2h} & -x_{h+2h+1} & -1 \end{bmatrix}.$$

Take $x_{h+2h+1} = c$, with $c > 1$, $x_{h+11} = a_{h+21}$ and $x_{h+12} = a_{h+22}$. We denote by A_1 the resulting partial matrix. By applying the induction hypothesis to the principal submatrix $C = A_1[\{1,2,\dots,h+1\}]$ we obtain an N–matrix completion $C_c = (-c_{ij})$ of C. Now, the desired completion A_c of A is obtained by replacing the principal submatrix $A[\{1,2,\dots,h+1\}]$ by C_c and by taking

$$\begin{aligned} x_{h+2j} &= c_{h+1j}, \qquad j = 3,4,\dots,h \\ x_{ih+2} &= c_{ih+1} \qquad i = 1,2,\dots,h. \end{aligned}$$

Once again from the proof of Proposition 3.1 of [7], it follows that A_c is indeed an N–matrix. □

We can extend this result in the following way.

Theorem 3.2 *Let A be a non-combinatorially symmetric partial N–matrix, belonging to $\mathcal{PS}_n$, whose graph is a double cycle with h common arcs, $h \geq 2, n \geq h+2$. Then, there exists an N–matrix completion A_c of A.*

Proof: The proof follows by induction on n. The case in which $n = h+2$ is solved in the previous lemma. Suppose that the result is true for $n-1$ and let A be a non-combinatorially symmetric partial N–matrix, belonging to $\mathcal{PS}_n$, whose graph is a double cycle with h common arcs. Without loss of generality, we can assume that the cycles are Γ_1

$$\{1,2\},\{2,3\},\ldots,\{k,k+1\},\ldots,\{k+h-1,k+h\},\{k+h,1\}$$

and Γ_2

$$\{k,k+1\},\ldots,\{k+h-1,k+h\},\{k+h,k+h+1\},\ldots,\{n-1,n\},\{n,k\},$$

and the matrix has the form

$$A = \begin{bmatrix} -1 & -1 & \ldots & -x_{1k} & -x_{1k+1} & \ldots & -x_{1k+h} & \ldots & -x_{1n} \\ -x_{21} & -1 & \ldots & -x_{2k} & -x_{2k+1} & \ldots & -x_{2k+h} & \ldots & -x_{2n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ -x_{k1} & -x_{k2} & \ldots & -1 & -1 & \ldots & -x_{kk+h} & \ldots & -x_{kn} \\ -x_{k+11} & -x_{k+12} & \ldots & -x_{k+1k} & -1 & \ldots & -x_{k+1k+h} & \ldots & -x_{k+1n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ -a_{k+h1} & -x_{k+h2} & \ldots & -x_{k+hk} & -x_{k+hk+1} & \ldots & -1 & \ldots & -x_{k+hn} \\ -x_{k+h+11} & -x_{k+h+12} & \ldots & -x_{k+h+1k} & -x_{k+h+1k+1} & \ldots & -x_{k+h+1k+h} & \ldots & -x_{k+h+1n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ -x_{n1} & -x_{n2} & \ldots & -a_{nk} & -x_{nk+1} & \ldots & -x_{nk+h} & \ldots & -1 \end{bmatrix}.$$

Consider $c > 1$ and take $x_{21} = c$ and $x_{k+h2} = a_{k+h1}$. Denote by A_1 the obtained partial matrix. The principal submatrix $C = A_1[\{2,\ldots,n\}]$ is an $(n-1)\times(n-1)$ partial N–matrix whose associated graph is a double cycle with h common arcs. By the induction hypothesis, there exists an N–matrix completion $C_c = (-c_{ij})$ of C. Consider the completion A_c of A obtained by replacing the principal submatrix $A[\{2,\ldots,n\}]$ by C_c and by taking

$$\begin{array}{ll} x_{i1} = c_{i2}, & i = 3,4,\ldots,n, \quad i \neq k+h \\ x_{1j} = c_{2j} & j = 3,4,\ldots,n. \end{array}$$

The proof of Proposition 3.1 of [7] allows us to conclude that A_c is an N–matrix. □

In light of the proceeding results for double cycles, we present a generalization of this graph structure.

Definition 3.1 *A block cycle is a graph formed by a collection of cycles $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$, such that Γ_i and Γ_j have a common vertex or $h_{ij} \geq 1$ common arcs if and only if $j = i-1$ or $j = i+1$.*

Theorem 3.3 *Every non-combinatorially symmetric partial N–matrix, belonging to $\mathcal{PS}_n$, whose graph of the specified entries is a block cycle, has an N–matrix completion.*

Proof: The proof is by induction on the number k of cycles in the block cycle. For $k = 2$, see the previous results for double cycles with one common vertex or h common arcs.

Suppose that the result is true for block cycles with less than k cycles.

Let A be a non-combinatorially symmetric partial N–matrix, belonging to $\mathcal{PS}_n$, whose graph of the specified entries is a block cycle with k cycles $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$.

If there is a cycle Γ_i such that Γ_i has exactly one vertex in common with Γ_{i+1}, then we can complete the block cycles $\Gamma_1, \ldots, \Gamma_i$ and $\Gamma_{i+1}, \ldots, \Gamma_k$ by applying the induction hypothesis, and the graph of the resulting partial N–matrix is 1-chordal (see [2]). Hence, A admits N–matrix completions (see [7]).

Suppose, then, that each cycle $\Gamma_{i-1}, \Gamma_{i+1}$ has more than a common vertex with Γ_i, $i = 2, ..., k-1$. We can suppose, without loss of generality, that Γ_1 and Γ_2 have $p \geq 1$ common arcs, that Γ_1 is the cycle

$$\{1,2\}, \{2,3\}, \ldots, \{q_1, q_1+1\}, \ldots, \{q_1+p-1, q_1+p\}, \{q_1+p, 1\}$$

and Γ_2 is the cycle

$$\{q_1, q_1+1\}, \ldots, \{q_1+p-1, q_1+p\}, \{q_1+p, q_1+p+1\}, \ldots, \{q_1+q_2, q_1\}.$$

Since the associated graph of the partial N–matrix $A[\{1, 2, \ldots, q_1+p\}]$ is a cycle, there exists an N–matrix completion $B_c = (-b_{ij})$ of it. Let A_1 be the partial N–matrix obtained from A by replacing $A[\{1, 2, \ldots, q_1+p\}]$ by B_c, by specifying the $(q_1+q_2, q_1+p,)$ entry with $-a_{q_1+q_2,q_1}/b_{q_1+p,q_1}$ and unspecifying the position (q_1+q_2, q_1). The matrix $C = A_1[\{q_1+p, q_1+p+1, \ldots, n\}]$ is a partial N–matrix whose associated graph is a block cycle with $k-1$ cycles. Then, by applying the induction hypothesis we obtain an N–matrix completion C_c of C.

Let A_2 be the partial N–matrix obtained from A_1 by replacing $C = A_1[\{q_1+p, q_1+p+1, \ldots, n\}]$ by C_c. Since the associated graph of A_2 is a 1-chordal graph we can obtain an N–matrix completion of A_2, and therefore of A, whose entry in position (q_1+q_2, q_1) is $-a_{q_1+q_2,q_1}$. □

Finally, the following result allows us to give another type of partial N–matrices which have N–matrix completions.

Proposition 3.4 *Let A be a partial N–matrix, belonging to $\mathcal{PS}_n$, partitioned as follows*

$$A = \begin{bmatrix} A_{11} & A_{12} & X_{13} & \ldots & X_{1p-1} & X_{1p} \\ X_{21} & A_{22} & A_{23} & \ldots & X_{2p-1} & X_{2p} \\ X_{31} & X_{32} & A_{33} & \ldots & X_{3p-1} & X_{3p} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ X_{p-11} & X_{p-12} & X_{p-13} & \ldots & A_{p-1p-1} & A_{p-1p} \\ X_{p1} & X_{p2} & X_{p3} & \ldots & X_{pp-1} & A_{pp} \end{bmatrix},$$

where each diagonal block A_{ii} is an $n_i \times n_i$ completable partial N–matrix, each block A_{ii+1} has exactly one specified entry and each X_{ij} is totally unspecified. Then, there exists an N–matrix completion A_c of A.

Proof: Let $\tilde{A}$ be the partial N–matrix obtained from A by replacing each diagonal block A_{ii}, of size $n_i \times n_i$, by an N–matrix completion A_{ii_c}. Without loss of generality, we can assume that all the specified entries of $\tilde{A}$ are negative. If the (i,j) entry is specified, we denote by $-a_{ij}$ the element in that position. By left diagonal multiplication, we can also assume that all diagonal entries are equal to -1.

The proof is by induction on the number p of diagonal blocks.

Firstly, consider the case in which $p = 2$. Let (i, j) be the specified entry of A_{12}. Consider the $n \times n$ partial N–matrix $\bar{A}$ such that $\bar{A}[1, \dots, n_1] = A_{11_c}$, $\bar{A}[n_1 + 1, \dots, n] = A_{22_c}$, the elements in positions $(n_1, n_1 + 1)$ and $(n_1 + 1, n_1)$ are, respectively, $-a_{ij}/a_{in_1}a_{n_1+1j}$ and $-c$, with $c > a_{in_1}a_{n_1+1j}/a_{ij}$, and all the remaining entries are unspecified. Observe that the directed graph associated to $\bar{A}$ is 1-chordal, with 3 maximal cliques (see [2]). By applying Proposition 3.2 of [7] to the principal submatrix $C = \bar{A}[n_1, \dots, n]$, we obtain an N–matrix completion C_c of C, whose element in position (n_1, j) is $-a_{ij}/a_{in_1}$. Let $\hat{A}$ be the partial N–matrix obtained from $\bar{A}$ by replacing C by C_c. By applying Proposition 3.2 of [7] to $\hat{A}$, we obtain an N–matrix completion A_c of $\hat{A}$, whose element in position (i, j) is $-a_{ij}$. Hence, A_c is also an N–matrix completion of A.

Now, suppose that the result is true for $p - 1$ diagonal blocks and we are going to prove it for p diagonal blocks.

Firstly, we complete the submatrix $C = A[1, \dots, n_1 + n_2]$ of A to an N–matrix C_c, using the reasoning presented for case $p = 2$. Let $\bar{A}$ be the partial N–matrix obtained from A by replacing C by C_c. $\bar{A}$ is a partial N–matrix that admits the partition

$$\bar{A} = \begin{bmatrix} C_c & \bar{A}_{12} & Y_{13} & \dots & Y_{1p-1} \\ Y_{21} & A_{33} & \bar{A}_{23} & \dots & Y_{2p-1} \\ Y_{31} & Y_{32} & A_{44} & \dots & Y_{3p-1} \\ \vdots & \vdots & \vdots & & \vdots \\ Y_{p-11} & Y_{p-12} & Y_{p-13} & \dots & A_{pp} \end{bmatrix},$$

where each diagonal block is a completable partial N–matrix, each block $\bar{A}_{ii+1}$ has exactly one specified entry and each Y_{ij} is totally unspecified. Note that there are $p - 1$ diagonal blocks. Therefore, by the induction hypothesis, there exists an N–matrix completion A_c of $\bar{A}$ and, consequently, of A. □

We conclude this work with the following open problem, which generalizes the result given in the last proposition.

A partial matrix A, of size $n \times n$, is called *reducible* if there is a permutation matrix P

such that

$$PAP^T = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1p} \\ X_{21} & A_{22} & \dots & A_{2p} \\ \vdots & \vdots & & \vdots \\ X_{p1} & X_{p2} & \dots & A_{pp} \end{bmatrix},$$

where X_{ij} is a totally unspecified rectangular matrix, for $i = 2, \dots, p$, $j = 1, 2, \dots, i-1$ and A_{ij}, of size $n_i \times n_j$, for $i = 1, 2, \dots, p$, $j = i, i+1, \dots, p$, are partial matrices or conventional matrices. A partial matrix is *irreducible* if it is not reducible.

Question 3.1 *Let A be a partial N–matrix, of size $n \times n$, such that every irreducible principal submatrix has an N–matrix completion. Then, does there exist an N–matrix completion of A?*

From Proposition 2.2 of [7] the above question has an afirmative answer for matrices of size 3×3.

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