

A Parabolic Quasi-Variational Inequality Arising in a Superconductivity Model

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Abstract. We consider the existence of solutions for a parabolic quasilinear problem with a gradient constraint which threshold depends on the solution itself. The problem may be considered as a quasi-variational inequality and the existence of solution is shown by considering a suitable family of approximating quasilinear equations of p -Laplacian type. *A priori* estimates on the time derivative of the approximating solutions and on the nonlinear diffusion coefficients are used in the passage to the limit, as well as a suitable sequence of convex sets with variable gradient constraint. The asymptotic behaviour as $t \rightarrow \infty$ is also considered, and the solutions of the quasi-variational inequality are shown to converge, at least for subsequences, to a solution of a stationary quasi-variational inequality. These results can be applied to the critical-state model of type-II superconductors in longitudinal geometry.

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1. – Introduction

In a critical-state model of type-II superconductors with a longitudinal geometry, the main unknown is the magnetic field $H = (0, 0, u(x, t))$, where $x = (x_1, x_2) \in \Omega \subseteq \mathbb{R}^2$. By Maxwell's equations,

$$\begin{cases} \mu H_t + \nabla \times E = 0, \\ \nabla \times H = J, \end{cases}$$

where E denotes the electric field, the unknown density of the induced current J is given by

$$J = \left(\frac{\partial u}{\partial x_2}, -\frac{\partial u}{\partial x_1}, 0 \right) = \nabla^\perp u .$$

In some superconductivity models (see [3]), it is assumed that the electric field E inside the superconductor depends on the current density J through

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a power law (which generalizes the classical Ohm’s law $E = \rho J$) with the scalar resistivity given by $\rho = \rho(|J|) = \rho_0|J|^{p-2} = \rho_0|\nabla u|^{p-2}$, where $\rho_0 > 0$ is a constant and $p \geq 2$. Hence, the Maxwell’s equation for the longitudinal component of the magnetic field implies that $u = u(x, t)$ satisfies the two-dimensional quasilinear parabolic equation

$$\mu u_t - \rho_0 \nabla \cdot (|\nabla u|^{p-2} \nabla u) = 0.$$

In Bean’s critical-state model (see [11]), the current density cannot exceed the critical value $j_c > 0$ and it has been suggested that this threshold may also depend on $|H| = |u|$ (see Kim *et al.* [6]).

The constitutive relation for E is then modified to

$$E = \begin{cases} \rho_0 |\nabla u|^{p-2} \nabla^\perp u & \text{if } |\nabla u| < j_c(|u|), \\ (\rho_0 j_c^{p-2} + \lambda) \nabla^\perp u & \text{if } |\nabla u| = j_c(|u|), \end{cases}$$

being $\lambda \geq 0$ an unknown Lagrange multiplier.

Imposing initial and boundary conditions (in a bounded domain Ω in \mathbb{R}^2), it may be easily shown that this problem is equivalent to the following quasi-variational inequality (for details about this derivation see [11], where only the degenerate case $\rho_0 = 0$ was considered) to a new variable $h = u - h_e$,

$$\left\{ \begin{array}{l} h(t) \in \mathbb{K}_{h(t)} \text{ for a.e. } t \in [0, T], \quad h(0) = h_0 \in \mathbb{K}_{h_0}, \\ \int_{\Omega} h_t(t) (v - h(t)) + \frac{\rho_0}{\mu} \int_{\Omega} |\nabla h(t)|^{p-2} \nabla h(t) \cdot \nabla (v - h(t)) \\ \geq \int_{\Omega} f(t) (v - h(t)), \quad \forall v \in \mathbb{K}_{h(t)} \text{ for a.e. } t \in [0, T], \end{array} \right.$$

being

$$\mathbb{K}_{h(t)} = \left\{ v \in W_0^{1,p}(\Omega) : |\nabla v| \leq j_c(|h(t) + h_e(t)|) \text{ a.e.} \right\}$$

where $h_e = h_e(t)$ is related to the density of external currents, as well as f .

This model is the main motivation for the study of this new type of quasi-variational inequalities. This kind of problems seem not well studied in the literature and are not considered, for instance, in the classical references on quasi-variational inequalities [1] or [2]. Recent works in this area are, for example, the modelling works [10], [11] or a very special case of a one-dimensional quasi-variational inequality considered in [7].

In Section 2 we present the main results of this paper, a result on the existence of solutions for the quasi-variational inequality and a result about the asymptotic behaviour in time of these solutions.

In Section 3 we consider a family of approximating solutions and we establish *a priori* estimates. Section 4 includes the passage to the limit in the family of solutions of the approximated problems, concluding with the proof of the existence of at least a solution for the quasi-variational inequality in the N -dimensional case.

Section 5 studies the asymptotic behaviour in time of the solutions, which, in particular, also yields the existence of a solution to the corresponding elliptic quasi-variational inequality, extending a result of [8].

2. – Main results

Let Ω be a bounded, open subset of \mathbb{R}^N , with smooth boundary $\partial\Omega$, and let $T \in \mathbb{R}^+$, $I =]0, T[$, $Q_T = \Omega \times I$, $\Sigma = \partial\Omega \times I$, $\Omega_0 = \overline{\Omega} \times \{0\}$. Let $\nabla = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_N})$ denote the spatial gradient and suppose that F , f and h are given functions such that

- (1) $F \in C^0(\mathbb{R})$, $\exists m > 0: F \geq m$,
- (2) $f \in L^\infty(Q_T)$; $f_t \in M(Q_T)$,
- (3) $h \in W_0^{1,\infty}(\Omega)$, $|\nabla h| \leq F(h)$ a.e. in Ω , $\Delta_p h \in M(\Omega)$,

where $\Delta_p h = \nabla \cdot (|\nabla h|^{p-2} \nabla h)$ denotes the p -Laplacian, $1 < p < \infty$ and $M(\Omega)$, $M(Q_T)$ denote the spaces of bounded measures in Ω and Q_T , respectively, $M(\Omega) = [C^0(\overline{\Omega})]'$ and $M(Q_T) = [C^0(\overline{Q_T})]'$. We also denote by $W_0^{1,\infty}(\Omega)$ the space of Lipschitz functions that vanish on $\partial\Omega$.

Given $u \in L^\infty(\Omega)$, we can define a convex set

$$(4) \quad \mathbb{K}_u = \left\{ v \in W_0^{1,p}(\Omega) : |\nabla v| \leq F(u) \text{ a.e. in } \Omega \right\}$$

and the quasi-variational problem as follows:

To find u in an appropriate class of functions, such that:

$$(5) \quad \begin{cases} u(t) \in \mathbb{K}_{u(t)} \cap L^\infty(\Omega) \text{ for a.e. } t \in I, & u(0) = h, \\ \int_\Omega u_t(t) (v - u(t)) + \int_\Omega |\nabla u(t)|^{p-2} \nabla u(t) \cdot \nabla (v - u(t)) \\ \geq \int_\Omega f(t) (v - u(t)), \quad \forall v \in \mathbb{K}_{u(t)}, \text{ for a.e. } t \in I. \end{cases}$$

Our first aim is to prove existence of a solution to the problem (5). For that, we consider, for $\varepsilon > 0$, f_ε , h_ε and F_ε smooth functions approximating, respectively, f , h and F in the norms of $L^q(Q_T)$, $W_0^{1,q}(\Omega)$, for any $q < \infty$ and in $C^0(\mathbb{R})$, with $h^\varepsilon|_{\partial\Omega} = 0$, $\|\Delta_p h^\varepsilon + \varepsilon \Delta h^\varepsilon\|_{L^1(\Omega)} \leq C$ (C independent of ε) and a C^2 function $k_\varepsilon: \mathbb{R} \rightarrow \mathbb{R}$, non decreasing and such that

$$(6) \quad k_\varepsilon = 1 \text{ if } s \leq 0, \quad k_\varepsilon = e^{\frac{s}{\varepsilon}} \text{ if } s \geq \varepsilon.$$

We consider a sequence of the following approximated problems

$$(7) \quad \begin{cases} u_t^\varepsilon - \nabla \cdot \left[k_\varepsilon(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon)) (|\nabla u^\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla u^\varepsilon + \varepsilon \nabla u^\varepsilon \right] = f_\varepsilon \text{ in } Q_T, \\ u^\varepsilon(0) = h^\varepsilon, \\ u^\varepsilon|_\Sigma = 0. \end{cases}$$

We noticed that a similar approximation was introduced by Gerhardt (see [5]) in the treatment of the elastic-plastic torsion problem with multiply connected cross-section, which is an elliptic variational inequality and it was also used for studying a parabolic variational inequality with gradient constraint in [12].

THEOREM 1. *With the assumptions (1), (2) and (3), for any $1 < p < \infty$, the problem (5) has at least a solution u belonging to $L^\infty(0, T; W_0^{1,\infty}(\Omega)) \cap C^0(\overline{Q}_T)$, such that $u_t \in L^\infty(0, T; M(\Omega))$. In addition, u is the weak limit in $L^q(0, T; W_0^{1,q}(\Omega))$ (for any $q < \infty$) of a subsequence $\{u^{\varepsilon_n}\}_n$ of solutions of the family of approximated problems (7), $u^{\varepsilon_n} \rightharpoonup u$ in $C^0(\overline{Q}_T)$ and also $u_t^{\varepsilon_n} \rightharpoonup u_t$ in $L^\infty(0, T; M(\Omega))$ weak-**.

REMARK 1. Since u is continuous and bounded, for each $t \in [0, T]$, $\mathbb{K}_{u(t)} \subset W_0^{1,\infty}(\Omega) \subset C^0(\overline{\Omega})$ and the first integral of the left hand side of (5) should be understood in the duality sense between $M(\Omega)$ and $C^0(\overline{\Omega})$, taking into account that $u_t(t)$ is a.e. a bounded measure.

Consider now the corresponding stationary quasi-variational inequality:

To find

$$(8) \quad \begin{cases} u^\infty \in \mathbb{K}_{u^\infty} \cap L^\infty(\Omega) , \\ \int_\Omega |\nabla u^\infty|^{p-2} \nabla u^\infty \cdot \nabla (v - u^\infty) \geq \int_\Omega f^\infty (v - u^\infty) , \quad \forall v \in \mathbb{K}_{u^\infty} , \end{cases}$$

where we assume $f^\infty \in L^1(\Omega)$, F satisfies (1) and

$$\mathbb{K}_{u^\infty} = \left\{ v \in W_0^{1,p}(\Omega) : |\nabla v| \leq F(u^\infty) \text{ a.e. in } \Omega \right\} .$$

Existence of a solution of the problem (8) may be proved as in [3].

We may also consider the problem (5) with $T = +\infty$. Then we have the following

THEOREM 2. *Suppose that the assumption (2) is satisfied for $T = +\infty$, i.e., there exists $C > 0$ such that*

$$(9) \quad \sup_{0 < t < \infty} \|f(t)\|_{L^\infty(\Omega)} + \int_0^{+\infty} \int_\Omega |f_t| \leq C ,$$

and, in addition,

$$(10) \quad \frac{1}{t} \int_0^t \int_\Omega |f_t(\tau)| \tau d\tau \longrightarrow 0 \quad \text{when } t \rightarrow \infty ,$$

$$(11) \quad f(t) \rightharpoonup f_\infty \text{ in } L^1(\Omega)\text{-weak, when } t \rightarrow +\infty .$$

Let u denote a solution of the problem (5). Then there exists a sequence $\{t_n\}_n$, $t_n \xrightarrow{n} +\infty$, and a function $u^\infty \in W_0^{1,\infty}(\Omega)$ such that

$$\begin{cases} u(t_n) \xrightarrow{n} u^\infty \text{ in } W_0^{1,\infty}(\Omega) \text{ weak-* and uniformly in } \overline{\Omega}, \\ u^\infty \text{ is a solution of the problem (8).} \end{cases}$$

3. – The approximating problems

Let us consider the family of approximated problems defined by (7).

PROPOSITION 1. *With the assumptions (1), (2) and (3), the problem (7) has a unique solution $u^\varepsilon \in C_{\alpha, \alpha/2}^{2,1}(Q_T) \cap C^0(\overline{Q}_T)$, $0 < \alpha < 1$.*

PROOF. This is a direct consequence of well known results for quasilinear parabolic equations (see [9]). □

We shall establish some a priori estimates for the solutions of the approximated problems.

LEMMA 1. *Suppose that (1), (2) and (3) are satisfied and that u^ε is the solution of the problem (7). Then there exists a constant $M > 0$ independent of $\varepsilon \in]0, 1[$ such that*

$$(12) \quad |u^\varepsilon(x, t)| \leq M \quad \text{for all } (x, t) \in \overline{Q}_T .$$

PROOF. This estimate is a consequence of the assumptions and global boundedness for weak solutions of the Dirichlet problem for quasilinear degenerate parabolic equations (see [4] and its references). We need only to remark that in (7) $k_\varepsilon \geq 1$ and this is the crucial fact for deriving the weak maximum principle estimate. □

REMARK 2. We remark that the constant M that appears in (12) depends on T only through $\|f(t)\|_{L^\infty(\Omega)}$, and therefore it will be independent of T if $f \in L^\infty(0, \infty; L^\infty(\Omega)) = L^\infty(Q_\infty)$, i.e., under the assumption (9).

LEMMA 2. *Suppose that (1), (2) and (3) are satisfied and u^ε is the solution of the problem (7). Then there exists a constant $C > 0$ independent of $\varepsilon \in]0, 1[$ such that*

$$\|u_t^\varepsilon\|_{L^\infty(0,T;L^1(\Omega))} \leq C .$$

PROOF. We differentiate the equation of the approximated problem (7) in order to t and set $v = u_t^\varepsilon$. Then

$$(13) \quad \begin{aligned} & v_t - \nabla \cdot \left[k'_\varepsilon \left(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon) \right) \left(2 \nabla u^\varepsilon \cdot \nabla v - 2 F_\varepsilon F'_\varepsilon v \right) \left(|\nabla u^\varepsilon|^2 + \varepsilon \right)^{\frac{p-2}{2}} \nabla u^\varepsilon \right. \\ & + k_\varepsilon \left(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon) \right) \frac{p-2}{2} \left(|\nabla u^\varepsilon|^2 + \varepsilon \right)^{\frac{p-4}{2}} 2 \nabla u^\varepsilon \cdot \nabla v \nabla u^\varepsilon \\ & \left. + k_\varepsilon \left(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon) \right) \left(|\nabla u^\varepsilon|^2 + \varepsilon \right)^{\frac{p-2}{2}} \nabla v + \varepsilon \nabla v \right] = f_t^\varepsilon . \end{aligned}$$

Consider the sign function, denoted by sgn^0 and defined as follows:

$$\text{sgn}^0(\tau) = \begin{cases} -1 & \text{if } \tau < 0, \\ 0 & \text{if } \tau = 0, \\ 1 & \text{if } \tau > 0. \end{cases} \quad (\tau \in \mathbb{R}) ,$$

Let $s_\delta: \mathbb{R} \rightarrow \mathbb{R}$, $\delta > 0$, be a sequence of C^2 functions, approximating pointwise sgn^0 when $\delta \rightarrow 0$, such that

$$s_\delta|_{\{0\} \cup \mathbb{R} \setminus]-\delta, \delta[} = \text{sgn}^0|_{\{0\} \cup \mathbb{R} \setminus]-\delta, \delta[}, \quad s'_\delta \geq 0.$$

Notice that, if $\zeta_\delta(\tau) = \int_0^\tau s_\delta(\sigma) d\sigma$, then

$$\forall \tau \in \mathbb{R} \quad \lim_{\delta \rightarrow 0} \tau s'_\delta(\tau) = 0, \quad \lim_{\delta \rightarrow 0} \zeta_\delta(\tau) = |\tau|.$$

Multiply the equation (13) by $s_\delta(v)$ and integrate over Q_t . Then, since $v|_{\partial\Omega \times]0, t[} = 0$ and $s_\delta(v)|_{\partial\Omega \times]0, t[} = 0$, we have

$$\begin{aligned} & \int_{\Omega} \zeta_\delta(v(t)) - \int_{\Omega} \zeta_\delta(v(0)) + \int_{Q_t} 2k'_\varepsilon(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon)) (\nabla u^\varepsilon \cdot \nabla v)^2 (|\nabla u^\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} s'_\delta(v) \\ & + \int_{Q_t} -2k'_\varepsilon(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon)) F_\varepsilon F'_\varepsilon(|\nabla u^\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla u^\varepsilon \cdot \nabla v \, v s'_\delta(v) \\ (14) \quad & + \int_{Q_t} k_\varepsilon(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon)) (p-2) (|\nabla u^\varepsilon|^2 + \varepsilon)^{\frac{p-4}{2}} (\nabla u^\varepsilon \cdot \nabla v)^2 s'_\delta(v) \\ & + \int_{Q_t} k_\varepsilon(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon)) (|\nabla u^\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} |\nabla v|^2 s'_\delta(v) \\ & + \int_{Q_t} k_\varepsilon(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon)) \varepsilon |\nabla v|^2 s'_\delta(v) = \int_{Q_t} f_t^\varepsilon s_\delta(v). \end{aligned}$$

Notice that, since k_ε is monotone nondecreasing,

$$\begin{aligned} & \int_{Q_t} 2k'_\varepsilon(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon)) (\nabla u^\varepsilon \cdot \nabla v)^2 (|\nabla u^\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} s'_\delta(v) \geq 0 \\ & \int_{Q_t} k_\varepsilon(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon)) \varepsilon |\nabla v|^2 s'_\delta(v) \geq 0, \end{aligned}$$

and

$$\int_{Q_t} -2k'_\varepsilon(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon)) F_\varepsilon F'_\varepsilon(|\nabla u^\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla u^\varepsilon \cdot \nabla v \, v s'_\delta(v) \xrightarrow{\delta \rightarrow 0} 0.$$

On the other hand,

$$\begin{aligned} & \int_{Q_t} k_\varepsilon \left(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon) \right) (p-2) \left(|\nabla u^\varepsilon|^2 + \varepsilon \right)^{\frac{p-4}{2}} (\nabla u^\varepsilon \cdot \nabla v)^2 s'_\delta(v) \\ & \quad + \int_{Q_t} k_\varepsilon \left(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon) \right) \left(|\nabla u^\varepsilon|^2 + \varepsilon \right)^{\frac{p-2}{2}} |\nabla v|^2 s'_\delta(v) \\ & = \int_{Q_t} k_\varepsilon \left(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon) \right) \left(|\nabla u^\varepsilon|^2 + \varepsilon \right)^{\frac{p-4}{2}} \\ & \quad \times s'_\delta(v) \left[(p-2) (\nabla u^\varepsilon \cdot \nabla v)^2 + \left(|\nabla u^\varepsilon|^2 + \varepsilon \right) |\nabla v|^2 \right]. \end{aligned}$$

For $p \geq 2$, obviously, the second member of this equation is nonnegative. For $1 < p < 2$, using Cauchy-Schwartz inequality, we have

$$\begin{aligned} & \int_{Q_t} k_\varepsilon \left(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon) \right) \left(|\nabla u^\varepsilon|^2 + \varepsilon \right)^{\frac{p-4}{2}} s'_\delta(v) \left[(p-2) (\nabla u^\varepsilon \cdot \nabla v)^2 + \left(|\nabla u^\varepsilon|^2 + \varepsilon \right) |\nabla v|^2 \right] \\ & \geq \int_{Q_t} k_\varepsilon \left(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon) \right) \left(|\nabla u^\varepsilon|^2 + \varepsilon \right)^{\frac{p-4}{2}} s'_\delta(v) |\nabla u^\varepsilon|^2 |\nabla v|^2 (1 - (2-p)) \geq 0. \end{aligned}$$

So, since $\int_{Q_t} f_t^\varepsilon s_\delta(v) \leq \int_{Q_t} |f_t^\varepsilon|$, neglecting the nonnegative terms of (14) and letting $\delta \rightarrow 0$ in (14), we obtain

$$(15) \quad \int_{\Omega} |v(t)| - \int_{\Omega} |v(0)| \leq \int_{Q_t} |f_t^\varepsilon|,$$

and so

$$\|u_t^\varepsilon(t)\|_{L^1(\Omega)} \leq \|\Delta_p h^\varepsilon + \varepsilon \Delta h^\varepsilon\|_{L^1(\Omega)} + \|f_t^\varepsilon\|_{L^1(Q_T)} \leq C,$$

where $C > 0$ may be chosen independent of ε . □

LEMMA 3. *Suppose that assumptions (1), (2) and (3) are satisfied and u^ε is the solution to the problem (7). Then there is a constant $C > 0$ independent of $\varepsilon \in]0, 1[$ such that*

$$(16) \quad \left\| k_\varepsilon \left(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon) \right) \right\|_{L^1(Q_T)} \leq C.$$

PROOF. We consider the first equation of the problem (7), multiply it by u^ε and integrate over Q_T . Then

$$\int_{Q_T} u_t^\varepsilon u^\varepsilon + \int_{Q_T} k_\varepsilon \left(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon) \right) \left[\left(|\nabla u^\varepsilon|^2 + \varepsilon \right)^{p-2} |\nabla u^\varepsilon|^2 + \varepsilon |\nabla u^\varepsilon|^2 \right] = \int_{Q_T} f^\varepsilon u^\varepsilon.$$

Since f^ε and u^ε are uniformly bounded, independently of ε , in $L^\infty(Q_T)$,

$$\left| \int_{Q_T} f^\varepsilon u^\varepsilon \right| \leq C, \quad C \text{ independent of } \varepsilon$$

and, since u_t^ε is uniformly bounded in $L^\infty(0, T; L^1(\Omega))$, by Lemma 2,

$$\left| \int_{Q_T} u_t^\varepsilon u^\varepsilon \right| \leq C, \quad C \text{ independent of } \varepsilon$$

and so, it follows that

$$\int_{Q_T} k_\varepsilon \left(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon) \right) |\nabla u^\varepsilon|^p \leq C', \quad C' \text{ constant independent of } \varepsilon.$$

But, on the other hand,

$$\begin{aligned} \int_{Q_T} k_\varepsilon \left(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon) \right) &= \int_{\{|\nabla u^\varepsilon|^2 \leq F_\varepsilon^2(u^\varepsilon) + \varepsilon\}} k_\varepsilon \left(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon) \right) \\ &\quad + \int_{\{|\nabla u^\varepsilon|^2 > F_\varepsilon^2(u^\varepsilon) + \varepsilon\}} k_\varepsilon \left(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon) \right) \\ &\leq |Q_T| e + \int_{Q_T} k_\varepsilon \left(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon) \right) \frac{|\nabla u^\varepsilon|^p}{m^p} \end{aligned}$$

because in $\{|\nabla u^\varepsilon|^2 \leq F_\varepsilon^2(u^\varepsilon) + \varepsilon\}$ we have $k_\varepsilon(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon)) \leq e$ and in $\{|\nabla u^\varepsilon|^2 \geq F_\varepsilon^2(u^\varepsilon) + \varepsilon\}$ we have $\frac{|\nabla u^\varepsilon|}{m} \geq 1$.

So, we conclude

$$\begin{aligned} \int_{Q_T} k_\varepsilon \left(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon) \right) &\leq |Q_T| e + \frac{1}{m^p} \int_{Q_T} k_\varepsilon \left(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon) \right) |\nabla u^\varepsilon|^p \\ &\leq |Q_T| e + \frac{C'}{m^p}, \quad \text{independently of } \varepsilon. \quad \square \end{aligned}$$

LEMMA 4. *Suppose that the assumptions (1), (2) and (3) are satisfied and u^ε is the solution of the problem (7). Then, for any $q \in [1, +\infty[$ there exists a constant $C_q > 0$ independent of $\varepsilon \in]0, 1[$ such that*

$$\|\nabla u^\varepsilon\|_{L^q(Q_T)} \leq C_q.$$

PROOF. We know, from (16), that there exists a constant C , independent of ε , such that, for any $\varepsilon \in]0, 1[$,

$$\int_{Q_T} k_\varepsilon \left(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon) \right) \leq C.$$

So, we have

$$C \geq \int_{\{|\nabla u^\varepsilon|^2 > F_\varepsilon^2(u^\varepsilon) + \varepsilon\}} k_\varepsilon \left(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon) \right) = \int_{\{|\nabla u^\varepsilon|^2 > F_\varepsilon^2(u^\varepsilon) + \varepsilon\}} e^{\frac{|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon)}{\varepsilon}} .$$

Recalling that

$$\forall s \in \mathbb{R}^+ \quad \forall j \in \mathbb{N} \quad e^s \geq \frac{s^j}{j!}$$

we obtain, for each $j \in \mathbb{N}$,

$$(17) \quad \int_{\{|\nabla u^\varepsilon|^2 > F_\varepsilon^2(u^\varepsilon) + \varepsilon\}} \left[|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon) \right]^j \leq j! \varepsilon^j \int_{\{|\nabla u^\varepsilon|^2 > F_\varepsilon^2(u^\varepsilon) + \varepsilon\}} e^{\frac{|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon)}{\varepsilon}} \\ \leq j! \varepsilon^j C .$$

Given $q \in \mathbb{N}$, we have

$$(18) \quad \int_{Q_T} |\nabla u^\varepsilon|^{2q} = \int_{\{|\nabla u^\varepsilon|^2 \leq F_\varepsilon^2(u^\varepsilon) + \varepsilon\}} |\nabla u^\varepsilon|^{2q} + \int_{\{|\nabla u^\varepsilon|^2 > F_\varepsilon^2(u^\varepsilon) + \varepsilon\}} |\nabla u^\varepsilon|^{2q} ,$$

Since there exists a constant $M > 0$, not depending on ε , such that $\|u^\varepsilon\|_{L^\infty(Q_T)} \leq M$, we can estimate the first integral in the second member of (18) as follows:

$$\int_{\{|\nabla u^\varepsilon|^2 \leq F_\varepsilon^2(u^\varepsilon) + \varepsilon\}} |\nabla u^\varepsilon|^{2q} \leq \int_{Q_T} \left[F_\varepsilon^2(u^\varepsilon) + \varepsilon \right]^q \leq \max_{s \in [-M, M]} \left\{ F^2(s) + 1 \right\}^q |Q_T| .$$

On the other hand the second integral of the second member of (18) satisfies

$$\int_{\{|\nabla u^\varepsilon|^2 > F_\varepsilon^2(u^\varepsilon) + \varepsilon\}} |\nabla u^\varepsilon|^{2q} \\ = \int_{\{|\nabla u^\varepsilon|^2 > F_\varepsilon^2(u^\varepsilon) + \varepsilon\}} \left[|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon) + F_\varepsilon^2(u^\varepsilon) \right]^q \\ \leq \int_{\{|\nabla u^\varepsilon|^2 > F_\varepsilon^2(u^\varepsilon) + \varepsilon\}} \sum_{j=0}^q \binom{q}{j} \max_{x \in [-M, M]} \left[F_\varepsilon^2(x) \right]^{q-j} \left[|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon) \right]^j \\ \leq \sum_{j=0}^p D_j \int_{\{|\nabla u^\varepsilon|^2 > F_\varepsilon^2(u^\varepsilon) + \varepsilon\}} \left[|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon) \right]^j ,$$

where D_j is a constant independent of ε .

So, using (17), we easily conclude that

$$\|\nabla u^\varepsilon\|_{L^q(Q_T)} \leq C_q , \quad C_q \text{ depending on } q, \text{ but not on } \varepsilon . \quad \square$$

4. – The limiting process and further remarks

In this section we will let $\varepsilon \rightarrow 0$ and prove that if we define

$$(19) \quad \mathbb{K}_{u^\varepsilon(t)} = \left\{ v \in W_0^{1,p}(\Omega) : |\nabla v| \leq F(u^\varepsilon(t)) \text{ a.e. in } \Omega \right\}$$

we have, at least for a subsequence $\varepsilon \rightarrow 0$, for any given $v(t) \in \mathbb{K}_{u(t)}$ for every $t \in I$, the convergence of a sequence of functions v^ε , with $v^\varepsilon(t) \in \mathbb{K}_{u^\varepsilon(t)}$, to the function v , when $\varepsilon \rightarrow 0$.

Recalling Lemma 4, let u be the weak limit in $L^q(0, T; W_0^{1,q}(\Omega))$, $\forall q < \infty$, of a subsequence of $\{u^\varepsilon\}_{\varepsilon \in]0,1[}$ as $\varepsilon \rightarrow 0$, i.e.,

$$(20) \quad u^\varepsilon \rightharpoonup_\varepsilon u \text{ in } L^q(0, T; W_0^{1,q}(\Omega)) .$$

Then

LEMMA 5. *Suppose that the assumptions (1), (2) and (3) are satisfied and that u^ε is the solution of the problem (7). Let $\mathbb{K}_{u^\varepsilon(t)}$ be the convex set defined in (19) and u the function defined in (20), weak limit in $L^q(0, T; W_0^{1,q}(\Omega))$ of the subsequence $\{u^\varepsilon\}_\varepsilon$.*

Then for any $v \in L^\infty(0, T; W_0^{1,p}(\Omega))$, $1 \leq p \leq \infty$ such that $v(t) \in \mathbb{K}_{u(t)}$ for a.e. $t \in [0, T]$, there exists a $v^\varepsilon \in L^\infty(0, T; W_0^{1,p}(\Omega))$ with $v^\varepsilon(t) \in \mathbb{K}_{u^\varepsilon(t)}$ for a.e. $t \in [0, T]$ such that

$$v^\varepsilon \rightarrow_\varepsilon v \quad \text{in } L^\infty(0, T; W_0^{1,p}(\Omega)), \quad 1 \leq p \leq \infty.$$

PROOF. Since there are constants C_1 and C_q such that

$$\|u^\varepsilon\|_{L^\infty(0,T;L^1(\Omega))} \leq C_1 \quad \text{and} \quad \|u^\varepsilon\|_{L^q(0,T;W_0^{1,q}(\Omega))} \leq C_q$$

(for $q > N$), by a well known compactness theorem ([13], page 84), $\{u^\varepsilon\}_\varepsilon$ is relatively compact in $C([0, T]; C(\overline{\Omega}))$ and so, at least for another subsequence of the subsequence in (20), still denoted by $\varepsilon \rightarrow 0$, we have

$$(21) \quad u^\varepsilon \rightarrow_\varepsilon u \quad \text{uniformly in } C^0(\overline{Q}_T)$$

and therefore also

$$F(u^\varepsilon) \rightarrow_\varepsilon F(u) \quad \text{uniformly in } C^0(\overline{Q}_T) .$$

Let $\alpha_\varepsilon(t) = \|F(u^\varepsilon(t)) - F(u(t))\|_{L^\infty(\Omega)}$ and notice that

$$\alpha_\varepsilon(t) \rightarrow_\varepsilon 0 \quad \text{uniformly in } t \in [0, T] .$$

Define $\psi_\varepsilon(t) = 1 + \frac{\alpha_\varepsilon(t)}{m}$ and, given $v \in L^\infty(0, T; W_0^{1,p}(\Omega))$ such that $v(t) \in \mathbb{K}_{u(t)}$ for a.e. $t \in [0, T]$, define $v^\varepsilon(t) = \frac{1}{\psi_\varepsilon(t)} v(t) \in L^\infty(0, T; W_0^{1,p}(\Omega))$. Then,

$$|\nabla v^\varepsilon(t)| = \frac{1}{\psi_\varepsilon(t)} |\nabla v(t)| \leq \frac{1}{\psi_\varepsilon(t)} F(u(t)) \leq F(u^\varepsilon(t))$$

because

$$\begin{aligned} \psi_\varepsilon(t) &= 1 + \frac{\alpha_\varepsilon(t)}{m} \geq 1 + \frac{\alpha_\varepsilon(t)}{F(u^\varepsilon(t))} \\ &\geq \frac{F(u^\varepsilon(t)) + F(u(t)) - F(u^\varepsilon(t))}{F(u^\varepsilon(t))} = \frac{F(u(t))}{F(u^\varepsilon(t))}. \end{aligned}$$

So, we have $v^\varepsilon(t) \in \mathbb{K}_{u^\varepsilon(t)}$ for a.e. $t \in [0, T]$ and $v^\varepsilon \xrightarrow{\varepsilon} v$ in $L^\infty(0, T; W_0^{1,p}(\Omega))$ since

$$\|v^\varepsilon(t) - v(t)\|_{W_0^{1,p}(\Omega)} = \left| \frac{1}{\psi_\varepsilon(t)} - 1 \right| \|v\|_{L^\infty(0,T;W_0^{1,p}(\Omega))} \xrightarrow{\varepsilon} 0$$

uniformly in $t \in [0, T]$. □

PROOF OF THEOREM 1. Recalling (20) and (21), for some subsequence $\varepsilon \rightarrow 0$ we have

$$\begin{aligned} u^\varepsilon &\xrightarrow{\varepsilon} u \quad \text{uniformly in } C^0(\overline{Q}_T), \\ u^\varepsilon &\rightharpoonup u \quad \text{weakly in } L^q(0, T; W^{1,q}(\Omega)), \quad \forall q < \infty, \end{aligned}$$

and, besides that, this subsequence may be chosen in such a way that

$$u_t^\varepsilon \xrightarrow{\varepsilon} u_t \quad \text{in } L^\infty(0, T; M(\Omega)) \text{ weak-}^* .$$

Now, given $w \in \mathbb{K}_{u^\varepsilon(t)}$ for a.e. $t \in [0, T]$, we have

$$\begin{aligned} &\int_\Omega (|\nabla w|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla w \cdot \nabla(w - u^\varepsilon(t)) \\ &= \int_\Omega k_\varepsilon (|\nabla w|^2 - F^2(u^\varepsilon(t))) (|\nabla w|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla w \cdot \nabla(w - u^\varepsilon(t)) \\ &\geq \int_\Omega k_\varepsilon (|\nabla u^\varepsilon(t)|^2 - F^2(u^\varepsilon(t))) (|\nabla u^\varepsilon(t)|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla u^\varepsilon(t) \cdot \nabla(w - u^\varepsilon(t)) \end{aligned}$$

since, by monotonicity,

$$\begin{aligned} & \int_{\Omega} k_{\varepsilon} \left(|\nabla w|^2 - F^2(u^{\varepsilon}(t)) \right) \left(|\nabla w|^2 + \varepsilon \right)^{\frac{p-2}{2}} \nabla w \cdot \nabla(w - u^{\varepsilon}(t)) \\ & - \int_{\Omega} k_{\varepsilon} \left(|\nabla u^{\varepsilon}(t)|^2 - F^2(u^{\varepsilon}(t)) \right) \left(|\nabla u^{\varepsilon}(t)|^2 + \varepsilon \right)^{\frac{p-2}{2}} \nabla u^{\varepsilon}(t) \cdot \nabla(w - u^{\varepsilon}(t)) \geq 0. \end{aligned}$$

Given $v \in L^{\infty}(0, T; W_0^{1,p}(\Omega))$ such that $v(t) \in \mathbb{K}_{u(t)}$, for a.e. $t \in [0, T]$, we consider $v^{\varepsilon} \in L^{\infty}(0, T; W_0^{1,p}(\Omega))$ as in Lemma 5. Since $v^{\varepsilon}(t) \in \mathbb{K}_{u^{\varepsilon}(t)}$, multiplying the equation (7) by $v^{\varepsilon}(t) - u^{\varepsilon}(t)$ and integrating over $\Omega \times]s, t[$, $0 < s < t < T$, we have

$$\int_s^t \int_{\Omega} u_t^{\varepsilon} (v^{\varepsilon} - u^{\varepsilon}) + \int_s^t \int_{\Omega} \left(|\nabla v^{\varepsilon}|^2 + \varepsilon \right)^{\frac{p-2}{2}} \nabla v^{\varepsilon} \cdot \nabla(v^{\varepsilon} - u^{\varepsilon}) \geq \int_s^t \int_{\Omega} f^{\varepsilon}(v^{\varepsilon} - u^{\varepsilon})$$

and letting $\varepsilon \rightarrow 0$, since $v^{\varepsilon} \rightarrow v$ in $L^{\infty}(0, T; W_0^{1,p}(\Omega))$ strongly, we obtain

$$\int_s^t \int_{\Omega} u_t (v - u) + \int_s^t \int_{\Omega} |\nabla v|^{p-2} \nabla v \cdot \nabla(v - u) \geq \int_s^t \int_{\Omega} f(v - u).$$

Since s and t are arbitrary, we may conclude

$$\begin{aligned} (22) \quad & \int_{\Omega} u_t(t) (v - u(t)) + \int_{\Omega} |\nabla v|^{p-2} \nabla v \cdot \nabla(v - u(t)) \\ & \geq \int_{\Omega} f(t) (v - u(t)) \quad , \quad \forall v \in \mathbb{K}_{u(t)} \quad \text{for a.e. } t \in I. \end{aligned}$$

It still remains to prove that $|\nabla u(t)| \leq F(u(t))$ a.e. in Ω , for a.e. $t \in I$. If we prove this, a variant of Minty's lemma will show that (22) implies (5).

To prove that $|\nabla u(t)| \leq F(u(t))$ a.e. in Ω , for a.e. $t \in I$ is equivalent to prove that the set $\{(x, t) \in Q_T : |\nabla u(x, t)| > F(u(x, t))\}$ has measure zero.

We recall from Lemma 3 that

$$\exists C > 0 \quad \forall \varepsilon \in]0, 1[\quad \int_{Q_T} k_{\varepsilon} \left(|\nabla u^{\varepsilon}|^2 - F_{\varepsilon}^2(u^{\varepsilon}) \right) \leq C \quad .$$

Given $\varepsilon \in]0, 1[$ and $M \in \mathbb{R}^+$, $M \geq \varepsilon$, we define

$$A_{M,\varepsilon} = \left\{ (x, t) \in Q_T : |\nabla u^{\varepsilon}(x, t)|^2 - F_{\varepsilon}^2(u^{\varepsilon}(x, t)) \geq M \right\}$$

and then, since $k_{\varepsilon}(|\nabla u^{\varepsilon}|^2 - F^2(u^{\varepsilon})) \geq e^{\frac{M}{\varepsilon}}$ in $A_{M,\varepsilon}$, we have

$$\forall \varepsilon \in]0, 1[\quad \forall M \in [\varepsilon, +\infty[\quad |A_{M,\varepsilon}| e^{\frac{M}{\varepsilon}} \leq \int_{A_{M,\varepsilon}} k_{\varepsilon} \left(|\nabla u^{\varepsilon}|^2 - F_{\varepsilon}^2(u^{\varepsilon}) \right) \leq C \quad ,$$

where $|A|$ denotes the Lebesgue measure of the set A in \mathbb{R}^{N+1} .

So we have

$$|A_{M,\varepsilon}| \leq C e^{-\frac{M}{\varepsilon}}$$

and, choosing $M = \sqrt{\varepsilon}$, we get

$$|A_{\sqrt{\varepsilon},\varepsilon}| \leq C e^{-\frac{1}{\sqrt{\varepsilon}}}$$

and

$$\begin{aligned} \int_{Q_T} (|\nabla u|^2 - F_\varepsilon^2(u))^+ &\leq \liminf_{\varepsilon \rightarrow 0} \int_{Q_T} (|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon) - \sqrt{\varepsilon})^+ \\ &= \liminf_{\varepsilon \rightarrow 0} \int_{A_{\sqrt{\varepsilon},\varepsilon}} (|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon) - \sqrt{\varepsilon}) \\ &\leq \lim_{\varepsilon \rightarrow 0} D |A_{\sqrt{\varepsilon},\varepsilon}|^{\frac{1}{2}} = 0, \end{aligned}$$

being D a constant that bounds from above $\left[\int_{Q_T} (|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon) - \sqrt{\varepsilon})^2 \right]^{1/2}$, independently of ε .

So, $(|\nabla u|^2 - F^2(u))^+ = 0$ a.e. in Q_T or, equivalently,

$$|\nabla u| \leq F(u) \quad \text{a.e. in } Q_T.$$

Substitute now in (22), for a.e. $t \in I$, v by $u(t) + \theta(w - u(t))$, where w belongs to $\mathbb{K}_{u(t)}$, $\theta \in]0, 1]$ and divide both members by θ . We obtain

$$\begin{aligned} \int_{\Omega} u_t(t)(w - u(t)) + \int_{\Omega} \left| \nabla (u(t) + \theta(w - u(t))) \right|^{p-2} \nabla (u(t) + \theta(w - u(t))) \cdot \nabla (w - u(t)) \\ \geq \int_{\Omega} f(t)(w - u(t)), \quad \forall w \in \mathbb{K}_{u(t)} \quad \text{for a.e. } t \in I. \end{aligned}$$

Letting $\theta \rightarrow 0$ we conclude that u is a solution to the quasivariational inequality (5). □

5. – Asymptotic behaviour in time

Consider $T = +\infty$. We first show that there exists a global solution of the quasivariational inequality (5) when $T = +\infty$ and then, in some sense, there exists a limit function, when $t \rightarrow +\infty$, solution of a stationary quasivariational inequality.

LEMMA 6. *Suppose that (9) is satisfied. Then the problem (5) has a solution u such that*

$$u \in L^\infty(0, +\infty; W_0^{1,\infty}(\Omega)) \cap C^0(\mathbb{R}_0^+ \times \bar{\Omega}), \quad u_t \in L^\infty(0, +\infty; M(\Omega)),$$

being u the weak limit in $L^q(0, +\infty; W^{1,q}(\Omega))$ (for any $q < \infty$) of a subsequence $\{u^\varepsilon\}_\varepsilon$ of solutions of the family of approximated problems (7), $u_t^\varepsilon \rightharpoonup u_t$ in $L^\infty(0, +\infty; M(\Omega))$ weak-* and, for all $T > 0$, there exists a subsequence of $\varepsilon \rightarrow 0$, called $\varepsilon_T \rightarrow 0$, such that $u^{\varepsilon_T} \xrightarrow{\varepsilon_T} u$ in $C^0(\bar{Q}_T)$. In addition, we have the estimate

$$(23) \quad t \int_\Omega |u_t(t)| \leq \int_0^t \left(\int_\Omega |f_t(\tau)| \right) (1 + \tau) d\tau, \quad \text{for } 0 < t < \infty.$$

PROOF. The assumption (9) implies that the estimates of Lemmas 1 and 2 may be obtained when $T = +\infty$. The L^∞ boundedness of $\{u^\varepsilon\}_\varepsilon$, independently of ε and T and its boundedness in $L^\infty(0, +\infty; W^{1,q}(\Omega))$ can also be obtained easily from the boundedness of $k_\varepsilon(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon))$ in $L^\infty(0, +\infty; L^1(\Omega))$, which can be obtained as in Lemma 3, integrating only on Ω , for a.e. $t \in [0, T]$. So, we can pass to the limit in a subsequence $\varepsilon \rightarrow 0$ and get a function $u \in L^\infty(0, +\infty; W^{1,\infty}(\Omega))$, $u_t \in L^\infty(0, +\infty; M(\Omega))$ which is the weak-limit of u^ε in these spaces.

The uniform convergence of a subsequence of u^ε is obtained only in $C^0(\bar{Q}_T)$, for each T , which is enough to pass to the limit in the approximated problem and to obtain that u is a solution of the quasivariational inequality for almost every $t \in \mathbb{R}^+$

In order to show (23) we multiply the approximated equation (7) by $ts_\delta(u_t^\varepsilon)$ and, arguing as in the proof of Lemma 2, we obtain, after letting $\delta \rightarrow 0$,

$$\begin{aligned} \int_\Omega (t|v(t)| - |v(0)| + |v(t)|) &= \int_0^t \left(\int_\Omega \frac{d|v|}{d\tau} \right) \tau d\tau \\ &\leq \int_0^t \left(\int_\Omega |f_t^\varepsilon| \right) \tau d\tau, \quad \text{for any } 0 < t < \infty. \end{aligned}$$

Using (15) we obtain

$$t \int_\Omega |v(t)| \leq \int_0^t \left(\int_\Omega |f_t^\varepsilon| \right) d\tau + \int_0^t \left(\int_\Omega |f_t^\varepsilon| \right) \tau d\tau,$$

and letting $\varepsilon \rightarrow 0$ we conclude (23). □

PROOF OF THEOREM 2. Let u be a solution of the problem (5), with $T = +\infty$.

Since

$$\exists M > 0 \quad \|u\|_{L^\infty(\mathbb{R}^+ \times \Omega)} \leq M$$

and

$$|\nabla u(x, t)| \leq F(u(x, t)) \leq \max_{|s| \leq M} \{F(s)\} \quad \text{for a.e. } (x, t) \in \mathbb{R}^+ \times \Omega$$

we have

$$\exists C > 0 \quad \|u\|_{L^\infty(\mathbb{R}^+; W^{1,\infty}(\Omega))} \leq C$$

and so, for a subsequence $(t_n)_n$, $t_n \nearrow +\infty$ and some function $u^\infty \in W_0^{1,\infty}(\Omega)$, we have

$$\begin{aligned} u(t_n) &\rightharpoonup_n u^\infty \quad \text{in } W^{1,\infty}(\Omega) \text{ weak-}^* , \\ u(t_n) &\rightarrow_n u^\infty \quad \text{in } C^0(\overline{\Omega}) . \end{aligned}$$

Since for every measurable subset $\omega \subset \Omega$ we have $(1 < q < \infty)$

$$\int_\omega |\nabla u^\infty|^q \leq \liminf_n \int_\omega |\nabla u(t_n)|^q \leq \lim_n \int_\omega F^q(u(t_n)) = \int_\omega F^q(u^\infty),$$

we find that

$$|\nabla u^\infty| \leq F(u^\infty) \quad \text{a.e. in } \Omega,$$

and, therefore, we have

$$(24) \quad u^\infty \in \mathbb{K}_{u^\infty} .$$

It remains to show that u^∞ satisfies the inequality in (8).

By monotonicity, we rewrite (5) for a.e. $t \in \mathbb{R}^+$ in the form

$$(25) \quad \int_\Omega u_t(v-u(t)) + \int_\Omega |\nabla v|^{p-2} \nabla v \cdot \nabla(v-u(t)) \geq \int_\Omega f(t)(v-u(t)), \quad \forall v \in \mathbb{K}_{u(t)}.$$

We consider a given function φ satisfying

$$\varphi \in C^1(\mathbb{R}_0^+), \quad 0 \leq \varphi(t) \leq 1, \quad \varphi(0) = 1, \quad \varphi(t) \xrightarrow{t \rightarrow +\infty} 0, \quad \varphi_t(t) \xrightarrow{t \rightarrow +\infty} 0,$$

and we define, for $0 < \delta < 1$,

$$v^\delta(x, t) = \varphi(t) u(x, t) + (1 - \delta)(1 - \varphi(t)) w^\infty$$

with w^∞ arbitrarily given in \mathbb{K}_{u^∞} .

We have

$$|\nabla v^\delta(t)| \leq \varphi(t) F(u(t)) + (1 - \delta)(1 - \varphi(t)) F(u^\infty),$$

and, since $F(u(t_n)) \rightarrow_n F(u^\infty)$ uniformly, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{F(u^\infty)}{F(u(t_n))} \leq \frac{1}{1 - \delta} \quad \text{for } n \geq n_0$$

and so,

$$|\nabla v^\delta(t_n)| \leq \varphi(t_n) F(u(t_n)) + (1 - \varphi(t_n)) F(u(t_n)) = F(u(t_n)) \quad \text{for } n \geq n_0.$$

Substituting in (25), at time t_n , v by $v^\delta(t_n) \in \mathbb{K}_{u(t_n)}$ we obtain

$$\begin{aligned} \int_{\Omega} u_t(t_n) (v^\delta(t_n) - u(t_n)) + \int_{\Omega} |\nabla v^\delta(t_n)|^{p-2} \nabla v^\delta(t_n) \cdot \nabla (v^\delta(t_n) - u(t_n)) \\ \geq \int_{\Omega} f(t_n) (v^\delta(t_n) - u(t_n)) \end{aligned}$$

and, letting $t_n \rightarrow +\infty$, we obtain

$$\int_{\Omega} |(1-\delta) \nabla w^\infty|^{p-2} (1-\delta) \nabla w^\infty \cdot \nabla ((1-\delta) w^\infty - u^\infty) \geq \int_{\Omega} f^\infty ((1-\delta) w^\infty - u^\infty),$$

since $f(t_n) \xrightarrow{n} f^\infty$ in $L^1(\Omega)$ -weak and $\int_{\Omega} |u_t(t_n)| \xrightarrow{n} 0$ by (23) and the assumptions (9) and (10). Letting now $\delta \rightarrow 0$, we get

$$\int_{\Omega} |\nabla w^\infty|^{p-2} \nabla w^\infty \cdot \nabla (w^\infty - u^\infty) \geq \int_{\Omega} f^\infty (w^\infty - u^\infty), \quad \forall w^\infty \in \mathbb{K}_{u^\infty}$$

and, applying Minty's lemma, we see that u^∞ is a solution to the problem (8), since we have already proved that $u^\infty \in \mathbb{K}_{u^\infty}$. \square

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