About Pascal's tetrahedron with hypercomplex entries

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Abstract. It is evident, that the properties of monogenic polynomials in $(n+1)-$real variables significantly depend on the generators $e_1, e_2, \ldots, e_n$ of the underlying 2^n-dimensional Clifford algebra $C\ell_{0,n}$ over $\mathbb{R}$ and their interactions under multiplication. The case of $n=3$ is studied through the consideration of Pascal's tetrahedron with hypercomplex entries as special case of the general Pascal simplex for arbitrary n, which represents a useful geometric arrangement of all possible products. The different layers $\mathscr{L}_k$ of Pascal's tetrahedron (or pyramid) are built by ordered symmetric products contained in the trinomial expansion of $(e_1+e_2+e_3)^k$, $k=0,1,\ldots$.

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1. INTRODUCTION

The following formulae (2) and (3) are examples of the role that complex (imaginary) entries can play when used in powers of binomials. Since we have

$$(1+i)^{4l+1} = (1+i)^{4l}(1+i) = (2i)^{2l}(1+i) = (-4)^l(1+i) \quad l = 1,2,\ldots, \tag{1}$$

the binomial expansion of the left side of (1) implies immediately the validity of two binomial identities

$$\binom{4l+1}{0} - \binom{4l+1}{2} + \binom{4l+1}{4} - \cdots + \binom{4l+1}{4l} = (-4)^l \tag{2}$$

and

$$\binom{4l+1}{1} - \binom{4l+1}{3} + \binom{4l+1}{5} - \cdots - \binom{4l+1}{4l+1} = (-4)^l \tag{3}$$

after separation of the real and imaginary part.

Working in hypercomplex analysis with *n different non-commutative imaginary units,* the following general question seems natural: What is changing in the ordinary n-dimensional arrangement of multinomial coefficients (*Pascal's simplex*) if the real entries are substituted by n imaginary units $e_1, e_2, \ldots, e_n$?

The fact that the structure of the layer in Pascal's simplex rules the composition of a special set of Appell polynomials in terms of hypercomplex variables was already mentioned in the paper [1]. Due to the relevance of the answer for applications in Clifford analysis, we consider the set $\{e_1, e_2, \ldots, e_n\}$ as being an orthonormal basis of the Euclidean vector space $\mathbb{R}^n$ with a non-commutative product according to the multiplication rules

$$e_k e_l + e_l e_k = -2\delta_{kl}, \; k,l = 1,\ldots,n, \tag{4}$$

where δ_{kl} is the Kronecker symbol. Then the set $\{e_A : A \subseteq \{1,\ldots,n\}\}$ with $e_A = e_{h_1}e_{h_2}\ldots e_{h_r}$, $1 \le h_1 < \cdots < h_r \le n$, $e_\emptyset = e_0 = 1$, forms a basis of the 2^n-dimensional Clifford algebra $C\ell_{0,n}$ over $\mathbb{R}$. We embed $\mathbb{R}^{n+1}$ in $C\ell_{0,n}$ by identifying $(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$ with $x = x_0 + \underline{x} \in \mathscr{A} := \text{span}_{\mathbb{R}}\{1, e_1, \ldots, e_n\} \subset C\ell_{0,n}$. Here $x_0 = \text{Sc}(x)$ and $\underline{x} = \text{Vec}(x) = e_1x_1 + \cdots + e_nx_n$ are the so-called scalar resp. vector part of the paravector $x \in \mathscr{A}$. The conjugate of x is given by $\bar{x} = x_0 - \underline{x}$ and its norm by $|x| = (x\bar{x})^{\frac{1}{2}} = (x_0^2 + x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}$. Obviously, we can identify the case $n=1$ with the complex algebra case by $i := e_1$.

2. PASCAL'S TETRAHEDRON WITH HYPERCOMPLEX ENTRIES

As mentioned in the beginning, the consideration of an arbitrary value of n leads to an arrangement of multinomial coefficients following the multinomial expansion theorem. But essential and non trivial effects of the non-commutative multiplication can already be seen in the case of $n = 3$. The corresponding 3-simplex is the Pascal's tetrahedron (see [2]) with hypercomplex entries. The different layers $\mathscr{L}_k$ of it are built by the elements of the trinomial expansion of $(e_1 + e_2 + e_3)^k,\ k = 0, 1, \ldots$. As example, let us consider the case $k = 3$, i.e. the construction of the third layer $\mathscr{L}_3$. By taking into account non-commutativity the expansion can explicitly be written in the following order:

$$\begin{aligned}(e_1+e_2+e_3)^3 = e_1^3 & \\ & + (e_1e_1e_2 + e_1e_2e_1 + e_2e_1e_1) + (e_1e_1e_3 + e_1e_3e_1 + e_3e_1e_1) \\ & + (e_1e_2e_2 + e_2e_1e_2 + e_2e_2e_1) \\ & \qquad + (e_1e_2e_3 + e_1e_3e_2 + e_2e_1e_3 + e_2e_3e_1 + e_3e_1e_2 + e_3e_2e_1) + (e_1e_3e_3 + e_3e_1e_3 + e_3e_3e_1) \\ & + e_2^3 + (e_2e_2e_3 + e_2e_3e_2 + e_3e_2e_2) + (e_2e_3e_3 + e_3e_2e_3 + e_3e_3e_2) + e_3^3. \end{aligned} \tag{5}$$

This expansion corresponds to the case $k = 3$ in Pascal's tetrahedron for real entries, given in the general form (cf. [2])

$$(a+b+c)^k = \sum_{m=0}^{k}\sum_{s=0}^{m} \binom{k}{m}\binom{m}{s} a^{k-m} b^{m-s} c^s, \quad a, b, c \in \mathbb{R}. \tag{6}$$

The corresponding layer $\mathscr{L}_3$ written in ordered rows as arrangement of the different monomials corresponding to the increasing row-index m[1] has the form[2]:

$$\begin{array}{lccccccc}
m=0 & & & & \binom{3}{0}a^3 & & & \\
m=1 & & & \binom{3}{1}\binom{1}{0}a^2b & & \binom{3}{1}\binom{1}{1}a^2c & & \\
m=2 & & \binom{3}{2}\binom{2}{0}ab^2 & & \binom{3}{2}\binom{2}{1}abc & & \binom{3}{2}\binom{2}{2}ac^2 & \\
m=3 & \binom{3}{3}\binom{3}{0}b^3 & & \binom{3}{3}\binom{3}{1}b^2c & & \binom{3}{3}\binom{3}{2}bc^2 & & \binom{3}{3}\binom{3}{3}c^3
\end{array} .$$

The differences between $(e_1 + e_2 + e_3)^3$ and $(a + b + c)^3$ are obvious and due to the non-commutativity of the hypercomplex imaginary units we cannot obtain (5) by substituting $a = e_1$, $b = e_2$, $c = e_3$ in (6). Nevertheless, it exists a way to describe the trinomial expansion of $(e_1 + e_2 + e_3)^k$ formally in the same way as that of $(a+b+c)^3$. To do so one has to use the following (cf. [3])

Definition 1 (Symmetric Product) *Let $V_{+,\cdot}$ be a commutative or non-commutative ring, $a_k \in V$, $k = 1, \ldots, n$, then the "$\times$"-product is defined by*

$$a_1 \times a_2 \times \cdots \times a_n = \frac{1}{n!} \sum_{\pi(s_1,\ldots,s_n)} a_{s_1} a_{s_2} \ldots a_{s_n} \tag{7}$$

*where the sum runs over **all** permutations of all $(s_1, \ldots, s_n)$.*

together with the

Convention: If the factor a_j occurs μ_j-times in (7), we briefly write

$$\underbrace{a_1 \times \cdots \times a_1}_{\mu_1} \times \cdots \times \underbrace{a_n \times \cdots \times a_n}_{\mu_n} = a_1{}^{\mu_1} \times a_2{}^{\mu_2} \times \cdots \times a_n{}^{\mu_n} \tag{8}$$

and set parentheses if the powers are understood in the ordinary way.

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[1] The index s is increasing in NW-SE diagonal direction, whereas $(m-s)$ increases in NE-SW diagonal direction. Since $\binom{k}{m}\binom{m}{s} = \binom{k}{m}\binom{m}{m-s} = \binom{k}{k-m}\binom{m}{s}$ all elements with the same corresponding index in this three directions are the same (symmetry property).

[2] To be exact, Pascal's tetrahedron includes only the trinomial coefficients $\binom{k}{m}\binom{m}{s}$. This is the case if $a = b = c = 1$.

Consequently, the multinomial theorem for entries of a commutative or non-commutative ring, V is obtained in the same form as the ordinary multinomial theorem for real entries (cf. [3]).

Theorem 1 (General multinomial theorem) *Using the symmetric product (1) together with the convention (8), the powers of a sum of n different elements $a_1,\dots a_n$ of an arbitrary commutative or non-commutative ring V can be expanded in the form*

$$(a_1+a_2+\cdots+a_n)^k = \sum_{|\mu|=k} \binom{k}{\mu} \vec{a}^{\mu}$$

where, as usual, $\binom{k}{\mu} = \frac{k!}{\mu!}$, $|\mu| = \mu_1+\mu_2+\cdots+\mu_n$ *and* $\vec{a}^{\mu} = a_1{}^{\mu_1} \times a_2{}^{\mu_2} \times \cdots \times a_n{}^{\mu_n}$, $k \in \mathbb{N}$.

It follows straightforward that the trinomial expansion (5) can now be rewritten as

$$(e_1+e_2+e_3)^k = \sum_{m=0}^{k}\sum_{s=0}^{m} \binom{k}{m}\binom{m}{s} e_1^{k-m} \times e_2^{m-s} \times e_3^{s}.$$

From (4) it follows, for example, that the central element in the layer $\mathscr{L}_3$ of Pascal's tetrahedron is given by $\binom{3}{2}\binom{2}{1} e_1 \times e_2 \times e_3 = (e_1[e_2e_3+e_3e_2]+e_2[e_1e_3+e_3e_1]+e_3[e_1e_2+e_2e_1]) = 0$ and the final form of $\mathscr{L}_3$ is

$$\begin{array}{ccccccc} & & & -e_1 & & & \\ & & -e_2 & & -e_3 & & \\ & -e_1 & & 0 & & -e_1 & \\ -e_2 & & -e_3 & & -e_2 & & -e_3 \end{array}$$

3. THE COMPLETE CHARACTERIZATION OF PASCAL'S HYPERCOMPLEX TETRAHEDRON

For the complete characterization of Pascal's tetrahedron built from hypercomplex entries for arbitrary values of k we use the fact that $(e_1+e_2+e_3)$ is a paravector and therefore $(e_1+e_2+e_3)^{2l} = (-1)^l(1+1+1)^l$, $l = 0,1,2,\dots$. This means that in the case of even k ($k = 2l$) the $\mathscr{L}_k$ is filled with the multinomial numbers of $(1+1+1)^l$ multiplied by $(-1)^l$, but the rows, (NE-SW)- resp. (NW-SE)-diagonals with odd indices contain only zeros. For the case of odd k ($k = 2l+1$) we use the fact that $(e_1+e_2+e_3)^{2l+1} = (-1)^l(1+1+1)^l(e_1+e_2+e_3)$ which shows (without any calculation of the concrete value of $e_1^{k-m} \times e_2^{m-s} \times e_3^{s}$) that a layer of odd degree contains only real multiples of the hypercomplex generators e_1 or e_2 or e_3, with the exception of $k = 1$. Taking this into account we can prove the following

Theorem 2 *Let* $\mathscr{E}^3_{(k,m,s)} := \binom{k}{m}\binom{m}{s} e_1^{k-m} \times e_2^{m-s} \times e_3^{s}$, $k = 0,1,2,\dots$; $m = 0,1,\dots,k$; $s = 0,1,\dots,m$ *and*

$$(e_1+e_2+e_3)^k = \sum_{m=0}^{k}\sum_{s=0}^{m} \binom{k}{m}\binom{m}{s} e_1^{k-m} \times e_2^{m-s} \times e_3^{s} = \sum_{m=0}^{k}\sum_{s=0}^{m} \mathscr{E}_{(k,m,s)}.$$

Then the entries $\mathscr{E}_{(k,m,s)}$ of Pascal's hypercomplex tetrahedron are given in the following form:

I. *If k is even then*

$$\mathscr{E}_{(k,m,s)} = \begin{cases} (-1)^{\frac{k}{2}} \binom{\frac{k}{2}}{\frac{m}{2}} \binom{\frac{m}{2}}{\frac{s}{2}}, & m \text{ even, } s \text{ even,} \\ 0, & \text{otherwise.} \end{cases}$$

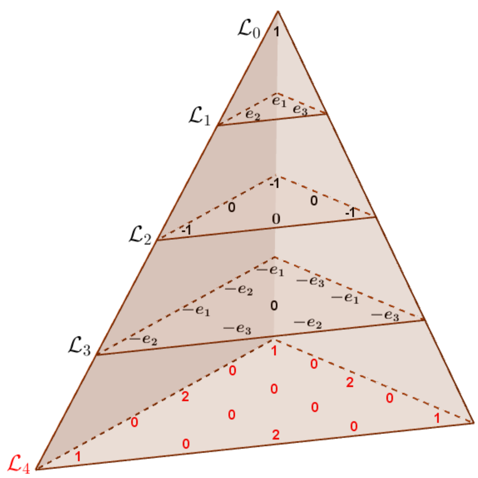
II. *If k is odd then*

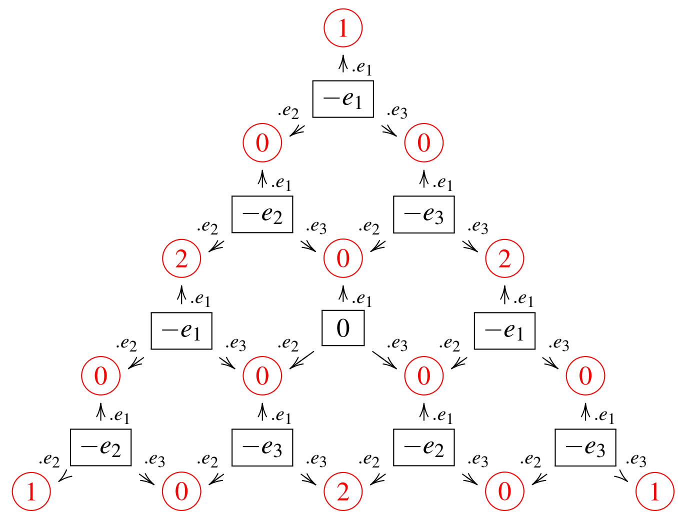
$$\mathscr{E}_{(k,m,s)} = \begin{cases} 0, & m \text{ even, } s \text{ odd,} \\ (-1)^{\frac{k-1}{2}} \binom{\frac{k-1}{2}}{\frac{m}{2}} \binom{\frac{m}{2}}{\frac{s}{2}} e_1, & m \text{ even, } s \text{ even,} \\ (-1)^{\frac{k-1}{2}} \binom{\frac{k-1}{2}}{\frac{m-1}{2}} \binom{\frac{m-1}{2}}{\frac{s-1}{2}} e_3, & m \text{ odd, } s \text{ odd,} \\ (-1)^{\frac{k-1}{2}} \binom{\frac{k-1}{2}}{\frac{m-1}{2}} \binom{\frac{m-1}{2}}{\frac{s}{2}} e_2, & m \text{ odd, } s \text{ even.} \end{cases}$$

Moreover, following the recurrence properties of Pascal's tetrahedron with real entries (cf. [2]) it can be shown that the numbers on every $k-$th layer are the sum of the three adjacent numbers in the $(k-1)-$th layer (the layer above the $k-$th layer in the tetrahedron), each one multiplied by e_1 or e_2 or e_3, respectively, i. e. we have

$$\mathscr{E}_{(k,m,s)} = \mathscr{E}_{(k-1,m-1,s-1)} e_3 + \mathscr{E}_{(k-1,m-1,s)} e_2 + \mathscr{E}_{(k-1,m,s)} e_1.$$

The following pictures try to illustrate Pascal's tetrahedron as well as the mentioned recurrence relation for the case $k = 4$.





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REFERENCES

1. M. I. Falcão, and H. Malonek, "Clifford Analysis between Continuous and Discrete," in *Numerical Analysis and Applied Mathematics*, ed. by T. E. Simos et al., AIP Conference Proceedings 1048, American Institute of Physics, New York, 2008, pp. 682–685.
2. Boris A. Bondarenko, *Generalized Pascal triangles and pyramids, their fractals, graphs, and applications.* Transl. from the Russian by Richard C. Bollinger. (English) Santa Clara, CA: The Fibonacci Association, vii, 253 p. (1993).
3. H. R. Malonek, "Selected topics in hypercomplex function theory", in *Clifford algebras and potential theory* (ed. S.-L. Eriksson), Report 7, University of Joensuu, 2004, pp. 111–150.