Inexact Restoration approaches to solve Mathematical Program with Complementarity Constraints

Teófilo Melo[1], M. Teresa T. Monteiro[2] and João Matias[3]

[1] *CIICESI, School of Technology and Management of Felgueiras, Porto Polytechnic*

[2] *Algoritmi Centre, Department of Production and Systems , University of Minho*

[3] *Centre of Mathematics, CM-UTAD*

emails: `tmm@estgf.ipp.pt`, `teresa.monteiro@algoritmi.uminho.pt`,
`j_matias@utad.pt`

Abstract

Mathematical Program with Complementarity Constraints (MPCC) finds application in many fields. As the complementarity constraints fail the standard Linear Independence Constraint Qualification (LICQ) or the Mangasarian-Fromovitz constraint qualification (MFCQ), at any feasible point, the nonlinear programming theory may not be directly applied to MPCC. However, the MPCC can be reformulated as NLP problem and solved by nonlinear programming techniques. One of them, the Inexact Restoration (IR) approach, performs two independent phases in each iteration - the feasibility and the optimality phases.

This work presents two versions of an IR algorithm to solve MPCC. In the feasibility phase two strategies were implemented, depending on the constraints features. One gives more importance to the complementarity constraints, while the other considers the priority of equality and inequality constraints neglecting the complementarity ones. The optimality phase uses the same approach for both algorithm versions.

The algorithms were implemented in MATLAB and the test problems are from MACMPEC collection.

Key words: Mathematical Problem with Complementarity Constraints, Inexact Restoration, NonLinear Programming

1   Introduction

Mathematical Program with Complementarity Constraints is an exciting new application of nonlinear programming techniques. The main application areas are Engineering, Economics and Ecology. In Engineering, problems dealing with contact, obstacle and friction, process modeling and traffic congestion are treated. MPCC problems arise in game theory models like Nash and Stackelberg equilibrium, in finances and taxes problems and in markets competition issues, all related with Economics. There is a growing collection of test problems [13] and Ralph [1] presents some MPCC applications like toll design in traffic networks or communication networks. An important reason why complementarity optimization problems are so pervasive in Engineering and Economics is because the concept of complementarity is tantamount with the notion of system equilibrium. They are very difficult to solve as the usual constraint qualifications, necessary to guarantee the algorithms convergence, fail in all feasible points. This complexity is caused by the disjunctive constraints which lead to some challenging issues that typically are the main concern in the design of efficient solution algorithms. It has been recently shown that MPCC can be efficiently and reliably solved as a nonlinear program (NLP). However this reformulation still continues to violate at any feasible point the same constraint qualifications (MFCQ), *ie*, has no feasible point that strictly satisfies the inequalities. Recent studies of Scheel and Scholthes [2] have proved that the strong stationarity of an MPCC equals the first order optimality conditions of the NLP equivalent. This fact motivates the scientific community to use NLP approaches to deal with MPCC [3].

The point of view of IR approach is that feasibility is an important feature of the problem that must be controlled independently of optimality. Therefore, the methods based on IR consider feasibility and optimality at different phases of a single iteration. A well known drawback of feasible methods is their inability to follow very curved domains, which causes that very short steps might be computed far from the solution. IR methodology tries to avoid this inconvenient using procedures that automatically decrease the tolerance of infeasibility as the solution is approximated [7]. In this way, large steps on an enlarged feasible region are computed at the beginning of the process.

These methods are related to classical restoration algorithms like gradient projection methods, sequential gradient restoration algorithms (SGRA) and generalized reduced gradient (GRG) method but also have singular differences. They generate a sequence of generally infeasible iterates with intermediate iterations that consist of inexactly restored points. The main difference between the different classical methods is the way in which restoration is performed. The convergence theory allows the use of arbitrary algorithms for performing the restoration.

An essential feature of the IR methodology is that one is free to choose different algorithms both for the feasibility and for the optimality phase, so that problem characteristics can be exploited. MPCC has a different type of constraints, the complementarity ones, that

must have a special treatment in the feasibility phase of the IR philosophy.

This paper is organized as follows. Next section presents some topics of IR state of art. Section 3 defines the MPCC problem. Two versions of the IR algorithm are presented in Section 4. Some preliminar conclusions are reported.

2 IR state of art

Feasible methods are very important in many real life situations, feasible nonoptimal points have a meaning and useful whereas nonfeasible points have no meaning. In [6] is presented an algorithm in which in each iteration demands, first reducing the norm of the constraints and after the reduction of the Lagrangian function. The new point is calculated by means of a merit function that combines feasibility and optimality. Modern Inexact Restoration methods for Nonlinear Programming begin with the algorithm of Martínez and Pillota [4]. In the feasibility phase, given the current iterate x^k, an intermediate more feasible point y^k is computed using an arbitrary procedure, which is chosen according to the problem characteristics. The trial point z is computed on "tangent set" that passes through y^k in such way that an optimality measure improves in z with respect to y^k . If the point z is acceptable for a criterion that combines feasibility and optimality, one defines the new iterate $x^{k+1} = z$. Otherwise, the trial point is chosen in a smaller trust region around y^k. The convergence theory of IR methods [8] is related with the convergence theory of SQP algorithms, the main analogies between IR and SQP method presented in [5] are: both are trust-region methods; every iteration is composed by two phases, the first related to feasibility and the second to optimality; the optimal phase seeks a "more optimal" point in a "tangent approximation" to the constraints.
However, there exist very important differences, in both methods in the restoration and optímality phase IR deals with the true function and constraints, while SQP deals with a model of both; the trust region in SQP is centered in the current point, instead in IR is centered in the restored point. Because of these differences, which allow one to relate IR to the classical feasible methods.
The IR approach has been the subject of interest to researchers in very different areas [12], [11], is a strategy that can be easily combined with other techniques [10]. Recently [9] presented a new general scheme for IR methods for nonlinear programming, this differs from previous methods, in which the tangent phase needs both line search based on the objective function and a confirmation based on a penalty function or a filter decision scheme.

3   MPCC definition

We consider Mathematical Program with Complementarity Constraints (MPCC):

$$
\begin{aligned}
min \quad & f(x) \\
s.t. \quad & c_i(x) = 0, \ i \in E, \\
& c_i(x) \geq 0, \ i \in I, \\
& 0 \leq x_1 \perp x_2 \geq 0,
\end{aligned}
\tag{1}
$$

where f and c are the nonlinear objective function and the constraint functions, respectively, assumed to be twice continuously differentiable. E and I are two disjoined finite index sets with cardinality p and m, respectively. A decomposition $x = (x_0, x_1, x_2)$ of the variables is used where $x_0 \in \mathbb{R}^n$ (control variables) and $(x_1, x_2) \in \mathbb{R}^{2q}$ (state variables). The expressions $0 \leq x_1 \perp x_2 \geq 0 : \mathbb{R}^{2q} \to \mathbb{R}^q$ are the q complementarity constraints, whose set is denoted by Q. The notation $x_1 \perp x_2$ means that $x_{1i}x_{2i} = 0$, for $i \in Q$, ie, the complementarity condition owns the disjunctive nature - $x_{1i} = 0$ or $x_{2i} = 0$, for $i \in Q$. This formulation doesn't exclude complementarity constraints like $0 \leq G(x) \perp H(x) \geq 0$. With this kind of complementarity constraints, the problem can be reformulated, by introducing the slack variables x_1 and x_2. Grouping all the equality constraints in $c_i(x) = 0$, the complementarity constraints have the form $0 \leq x_1 \perp x_2 \geq 0$ and the problem presents the formulation (1). In this formulation all the properties like constraint qualifications or second order conditions are preserved. This formulation makes easy the properties theoretical study.

One attractive way of solving (1) is to replace the complementarity constraints by a set of nonlinear inequalities, such as $x_{1i}\, x_{2i} \leq 0, i \in Q$, and then solve the equivalent nonlinear program (NLP):

$$
\begin{aligned}
min \quad & f(x) \\
s.t. \quad & c_i(x) = 0, \ i \in E, \\
& c_i(x) \geq 0, \ i \in I, \\
& x_{1i}x_{2i} \leq 0, \ i \in Q, \\
& x_1 \geq 0, \ x_2 \geq 0.
\end{aligned}
\tag{2}
$$

4   Inexact Restoration approach

In the IR approach, two independent phases are performed in each iteration - the feasibility and optimality phases. The first one, given a point x^k, finds an intermediate point y^k with less constraints violation. The goal is to compute a search direction towards a region more feasible. The optimality phase, starts from y^k and it aims to optimize the objective function $f(x)$ into the satisfied constraints space finding a "more optimal " point x^{k+1}.

The IR philosophy is described in the high level fluxograma.

High level IR fluxograma
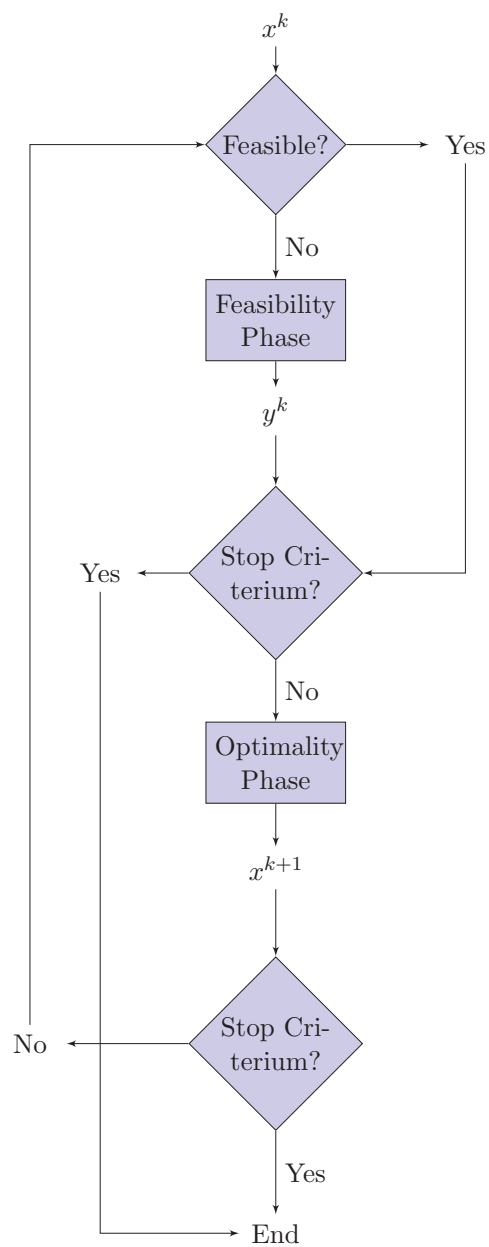
4.1 Application to MPCC

The feasibility phase aims to minimize the sum of the constraints violation in the feasible set. The IR philosophy allows to exploit the specificity of the MPCC constraints. In these context, two versions of the feasibility phase are implemented, denoted by A1 and A2.

A1 version gives priority to the complementarity constraints. The idea is to minimize the violation of the linearized complementarity constraints, satisfying the constraints that are already satisfied, solving the following linear problem:

$$\min_{d_{fea}{}^k \in \mathbb{R}^n} \quad \sum_{i \in Q_v} A_i^k d_{fea}{}^k$$

$$s.t. \quad \begin{aligned} A_i^k d_{fea} + c_i^k &= 0, \ i \in E \\ A_i^k d_{fea} + c_i^k &\geq 0, \ i \in I \\ A_i^k d_{fea} + c_i^k &\leq 0, \ i \in Q_s \end{aligned} \tag{3}$$

where $A^k = \nabla c(x^k)^T$ represents the constraints Jacobian matrix, Q_v and Q_s are the sets of the violated and satisfied complementarity constraints, respectively.

A2 version enforces the violation decrease of the equality and inequality constraints. Complementarity constraints violation is not prioritized. The following linear problem is solved:

$$\min_{d_{fea}{}^k \in \mathbb{R}^n} \quad \sum_{i \in \{E_v \cup I_v\}} A_i^k d_{fea}{}^k$$

$$s.t. \quad \begin{aligned} A_i^k d_{fea} + c_i^k &= 0, \ i \in E_s \\ A_i^k d_{fea} + c_i^k &\geq 0, \ i \in I_s \\ A_i^k d_{fea} + c_i^k &\leq 0, \ i \in Q \end{aligned} \tag{4}$$

where E_v, E_s, I_v, and I_s are the sets of violated and satisfied equality constraints, and violated and satisfied inequality constraints, respectively. The search direction $d_{fea}{}^k$ is the solution for the linear problems (3) and (4) - these problems are solved using the `linprog` routine from the MATLAB Optimization toolbox. The point $y^k = x^k + \alpha d_{fea}{}^k$ is obtained by a line search procedure.

The optimality phase starts from y^k, solves a quadratic optimization problem whose objective function is a quadratic approximation of $f(x)$ and linear approximations of the constraints:

$$\min_{d_{opt}{}^k \in \mathbb{R}^n} \quad \tfrac{1}{2}(d_{opt}{}^k)^T W^k d_{opt}{}^k + (d_{opt}{}^k)^T g^k$$

$$s.t. \quad \begin{aligned} A_i^k d_{fea} + c_i^k &= 0, \ i \in E \\ A_i^k d_{fea} + c_i^k &\geq 0, \ i \in I \\ A_i^k d_{fea} + c_i^k &\leq 0, \ i \in Q \end{aligned} \tag{5}$$

were $g^k = \nabla f(y^k)$ is the gradient of the objective function, $A^k = \nabla c(y^k)^T$ is the Jacobian matrix of the constraints and $W^k = \nabla^2 L(y^k)$ is the Hessian matrix of the Lagrangian function. The solution of the quadratic problem (5) is the search direction $d_{opt}{}^k$. This search direction will determine the new point x^{k+1}. To solve this quadratic subproblem the `quadprog` routine from the MATLAB Optimization toolbox is used.

The algorithms were implemented in MATLAB. Numerical experiences using a set of AMPL test problems from MacMPEC [13] are ongoing.

5 Conclusions

Two versions of an iterative algorithm using the IR philosophy were implemented in MAT-LAB to solve MPCC. The differences between the versions are only in the feasibility phase - the first one gives priority to the complementarity constraints while the second version considers the equality and inequality constraints more important. The preliminar ongoing numerical results show that the first version performs better than the second one. The algorithms are still in an improvement phase and more numerical experiments with larger dimension problems will be performed.

Acknowledgements

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