# ON MAPPINGS PRESERVING THE SHARP AND STAR ORDERS

M. BURGOS, A. C. MÁRQUEZ-GARCÍA, AND P. PATRÍCIO

Abstract. The present paper is devoted to the study of linear maps preserving certain relations, such as the sharp partial order and the star partial order in semisimple Banach algebras and C*-algebras.

## 1. Introduction and background

Let $A$ be a Banach algebra. Recall that an element $a \in A$ is *regular* if there is $b \in A$ such that $aba = a$. For a regular element $a \in A$, the set

$$a\{1\} = \{x \in A \colon axa = a\}$$

consists of all $\{1\}$-*inverses* or *inner inverses* of $a$. Notice that if $x$ is a $\{1\}$-inverse of $a$, then $ax$ and $xa$ are idempotents. A $\{1,2\}$-*inverse* or *generalized inverse* of $a$, is a $\{1\}$-inverse of $a$ that is a solution of the equation $xax = x$, that is, it is an element $b \in A$ such that $aba = a$ and $bab = b$.

Note that the condition $x \in a\{1\}$ ensures the existence of a generalized inverse of $a$: in such case, $b = xax$ fulfills the previous identities.

For an element $a$ in $A$, let us consider the left and right multiplication operators $L_a : x \mapsto ax$ and $R_a : x \mapsto xa$, respectively. If $a$ is regular, then so are $L_a$ and $R_a$, and thus their ranges $aA = L_a(A)$ and $Aa = R_a(A)$ are both closed. The unique generalized inverse of $a$ that commutes with $a$ is called the *group inverse* of $a$, whenever it exists. In this case $a$ is said to be *group invertible* and its group inverse is denoted by $a^\sharp$. The set of all group invertible elements of $A$ is denoted by $A^\sharp$.

Even though regularity can be defined in general Banach algebras, this notion has been mostly studied in C*-algebras. Harte and Mbekhta proved in [21] that an element $a$ in a unital C*-algebra $A$ is regular if and only if $aA$ is closed. Given $a$ and $b$ in a C*-algebra $A$, we shall say that $b$ is a *Moore-Penrose inverse* of $a$ if $b$ is a generalized inverse of $a$ and $ab$ and $ba$ are selfadjoint. It is known that every regular element $a$ in $A$ has a unique Moore-Penrose inverse that will be denoted by $a^\dagger$ ([21]). We write $A^\dagger$ for the set of regular elements in the C*-algebra $A$.

Let $M_n(\mathbb{C})$ be the algebra of all $n \times n$ complex matrices. On $M_n(\mathbb{C})$ there are many partial orders, which have been well studied (see [18], [22], [23], [30], [31], [32]). The *star partial order* on $M_n(\mathbb{C})$ was introduced by Drazin in [18], as follows:

$$A \leq_* B \qquad \text{if and only if} \qquad A^*A = A^*B \text{ and } AA^* = BA^*,$$

where as usual $A^*$ denotes the conjugate transpose of $A$. It was proved that $A \leq_* B$ if and only if $A^\dagger A = A^\dagger B$ and $AA^\dagger = BA^\dagger$. Baksalary and Mitra introduced in [5] the *left-star* and *right-star* partial order on $M_n(\mathbb{C})$ , as

$$A* \leq B \qquad \text{if and only if} \qquad A^*A = A^*B \text{ and } \mathrm{Im}A \subseteq \mathrm{Im}B,$$

*Key words and phrases.* Sharp order, star order, orthogonality, zero product, linear preserver, Banach algebra.
AMS classification: 47B48 (primary), 47C15, 47B49, 15A09 (secondary) .

The first author is partially supported by the Spanish Ministry of Science and Innovation, D.G.I. project no. MTM2011-23843, and Junta de Andalucía grant FQM375. The second author is also supported by a Plan Propio de Investigación grant from University of Almería, and Junta de Andalucía grant FQM 3737. The third author is partially supported by FEDER Funds through "Programa Operacional Factores de Competitividade – COMPETE" and by Portuguese Funds through FCT - "Fundação para a Ciência e a Tecnologia", within the project PEst-OE/MAT/UI0013/2014.

and

$$A \leq *B \qquad \text{if and only if} \qquad AA^* = BA^* \text{ and } \operatorname{Im}A^* \subseteq \operatorname{Im}B^*,$$

respectively. Moreover, $A \leq_* B$ if and only if $A* \leq B$ and $A \leq *B$.

Hartwig [22] introduced the *rank sustractivity order* on $M_n(\mathbb{C})$:

$$A \leq_- B \qquad \text{if and only if} \qquad \operatorname{rank}(B - A) = \operatorname{rank}(B) - \operatorname{rank}(A).$$

He proved that

$$A \leq_- B \qquad \text{if and only if} \qquad A^- A = A^- B \text{ and } AA^- = BA^-,$$

where $A^-$ denotes a $\{1\}$-inverse of $A$. This partial order is usually named the *minus partial order*. Later, Mitra used in [30] the group inverse of a matrix to define the *sharp order* on group invertible matrices:

$$A \leq_\sharp B \qquad \text{if and only if} \qquad A^\sharp A = A^\sharp B \text{ and } AA^\sharp = BA^\sharp.$$

In this work, the author compared the star and the sharp order and provided many equivalent formulations to these and other partial orders.

Let $H$ be an infinite-dimensional complex Hilbert space, and $B(H)$ the C$^*$-algebra af all bounded linear operators on $H$. Having into account that an operator $B(H)$ is regular if and only if it has closed range, Šemrl [36] extended the minus partial order from $M_n(\mathbb{C})$ to $B(H)$, finding and appropriate equivalent definition of the minus partial order on $M_n(\mathbb{C})$ which does not involve $\{1\}$-inverses. Following Šemrl's approach, Dolinar and Marovt extended in [17] the star partial order from $M_n(\mathbb{C})$ to $B(H)$. From [17, Theorem 5], for $T, S \in B(H)$, $T \leq_* S$ if and only if, there exist two selfadjoint idempotent operators $P, Q \in B(H)$, such that $\operatorname{Im}(P) = \overline{\operatorname{Im}(T)}$, $\operatorname{Ker}(Q) = \operatorname{Ker}(T)$, $PT = PS$, and $TQ = SQ$.

In [19] Guterman studied additive maps preserving the star, left-star and right-star orders between real and complex matrix algebras. An additive map $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ preserves the star partial order, if $A \leq_* B$ implies that $\phi(A) \leq_* \phi(B)$. Additive maps preserving the left-star and right-star partial order are defined in a similar way. Guterman shows, in particular, that every additive map $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ preserving the star partial order has one of the following forms: for every $A \in M_n(\mathbb{C})$, $\phi(A) = \alpha UAV$, $\phi(A) = \alpha U\overline{A}V$, $\phi(A) = \alpha UA^tV$ or $\phi(A) = \alpha U\overline{A}^tV$ where $\alpha \in \mathbb{C}$, $U$ and $V$ are unitary matrices, $\bar{A}$ denotes the conjugate matrix of $A$ and $A^t$ its transpose.

In [36], Šemrl studied (non necessarily additive) bijective maps preserving the minus partial order. For an infinite-dimensional complex Hilbert space $H$, a mapping $\phi : B(H) \to B(H)$ preserves the minus order if $A \leq_- B$ implies that $\phi(A) \leq_- \phi(B)$. The map $\phi : B(H) \to B(H)$ preserves the minus order in both directions whenever $A \leq_- B$ if and only if $\phi(A) \leq_- \phi(B)$. He proved that a bijective map $\phi : B(H) \to B(H)$ preserving the minus order in both directions has the form $\phi(A) = TAS$ or $\phi(A) = TA^*S$, for some invertible operators $T$ and $S$ (both linear in the first case and both conjugate linear in the second one).

Let $H$ be a separable infinite dimensional complex Hilbert space, and let $K(H)$ be the closed ideal of all compact operators on $H$. Dolinar, Guterman and Marovt, studied in [15] the bijective, additive and continuous mappings on $K(H)$ which preserve the star partial order in both directions. Recently, the authors of [16] bring some results from [19] concerning left and right star partial orders to the infinite-dimensional case, following some techniques from [36]. They show that every bijective additive map $\phi : B(H) \to B(H)$ preserving the left-star partial order in both directions has the form $\phi(A) = UAS$ for all $A \in B(H)$, where $U$ is a unitary operator and $S$ is bijective (note that both $U$ and $S$ can be linear or conjugate linear). The expected conclusions are obtained also for the right-partial order.

The paper is organized as follows. In Section 2 we study linear maps $T : A \to B$ preserving the sharp relation when $A$ is either a unital semisimple Banach algebra having essential socle or a unital real rank zero $C^*$-algebra. We introduce a new relation $(R1)$ which extends the sharp relation to the full algebra:

$$(R1) \qquad a \leq_s b \quad \text{if and only if there exists an idempotent } p \in A \text{ such that } a = pb = bp.$$

we show in Theorem 2.7 that a bijective linear map preserving the sharp order from a unital semisimple Banach algebra with essential socle into a Banach algebra is a Jordan isomorphism multiplied by a invertible central element. A similar result is obtained for the relation $(R1)$ (Theorem 2.16). When $A$ is a unital real rank zero $C^*$-algebra, we prove in Theorem 2.9 that every bounded linear map preserving the sharp order is an appropriate multiple of a Jordan homomorphism. Also an analogous result is proved for the relation $(R1)$ (Theorem 2.17).

Section 3 is concerned with linear maps between C*-algebras preserving the star order. We connected this problem with that of orthogonality preservers ([11, 12, 13]), and introduce a new relation $(R2)$, which is equivalent to the star order for a large class of C*-algebras. We say that

$$(R2) \qquad a \leq b \quad \text{if and only if} \quad a = pb = bq \text{ for some projections } p, q \in A.$$

We prove in 3.7 that every bijective linear map preserving the sharp order, or equivalently the relation $(R2)$, from a unital $C^*$-algebra with large socle into a C*-algebra is a Jordan *-homomorphism multiplied by an invertible element. If $A$ is a real rank zero $C^*$-algebra, $B$ is a C*-algebra, and $T : A \to B$ is a bounded linear map preserving the relation $(R2)$, then $T$ is a linear map preserving orthogonality (Theorem 3.10). The continuity assumption on $T$ can be dropped when $A$ is in fact linearly spanned by its projections (Theorem 3.9).

## 2. Sharp partial order

Let $A$ be a Banach algebra. Recall that an element $a \in A$ is group invertible if there exists a unique $b \in A$ such that $aba = a$, $bab = b$ and $ab = ba$. In this case, $b$ is called the group inverse of $a$ and denoted by $b = a^\sharp$. Recall also that $A^\sharp$ denotes the set of all group invertible elements in $A$. Let $a \in A^\sharp$ and $b \in A$. We write $a \leq_\sharp b$ when $a^\sharp a = a^\sharp b$ and $aa^\sharp = ba^\sharp$. This relation is a partial order when restricted to $A^\sharp$.

The following lemma collect some useful algebraic properties of the sharp relation.

**Lemma 2.1.** *Let $R$ be a unital (associative) ring. The following assertions hold:*

*(1) $p \in R^\sharp$ is an idempotent if and only if $p \leq_\sharp 1$.*
*(2) The maximal elements respect to the partial order $\leq_\sharp$ in $\mathcal{R}^\sharp$ are precisely the invertible elements.*
*(3) Let $a \in R^\sharp$, $b \in R$ and $u$ a group invertible element commuting with $a$ and $b$. If $a \leq_\sharp b$ then $ua \leq_\sharp ub$.*

*Proof.* The first assertion is clear.

Let $a \in R^\sharp$. It is straightforward to prove that $a \leq_\sharp (a - \mathbf{1} + aa^\sharp)$. If $a$ is a maximal element with respect to $\leq_\sharp$, then $a = (a - \mathbf{1} + aa^\sharp)$, which means that $\mathbf{1} = aa^\sharp = a^\sharp a$. Reciprocally, if $a \in R$ is invertible and $a \leq_\sharp b$, we have $\mathbf{1} = a^{-1}a = a^{-1}b$, which clearly implies that $a = b$.

Finally, pick $a, b \in R$ and $u \in R^\sharp$ with $a \leq_\sharp b$, $ua = au$ and $ub = bu$. Since $u$ also commutes with $a^\sharp$ and $b^\sharp$, and $(ua)^\sharp = u^\sharp a^\sharp$, it follows that

$$(ua)^\sharp ua = u^\sharp ua^\sharp a = u^\sharp ua^\sharp b = (ua)^\sharp ub.$$

Similarly we show that $(ua)(ua)^\sharp = (ub)(ua)^\sharp$. □

A linear map $T : A \to B$ between Banach algebras is a *Jordan homomorphism* if $T(a^2) = T(a)^2$, for all $a \in A$, equivalently $T(a \circ b) = T(a) \circ T(b)$, for every $a, b \in A$, where $\circ$ denotes the usual Jordan product $a \circ b = \frac{1}{2}(ab + ba)$ . If $A$ and $B$ are unital, $T$ is called *unital* if $T(\mathbf{1}) = \mathbf{1}$, where $\mathbf{1}$

is used for the identity element of both $A$ and $B$. It is well known that if $T : A \to B$ is a Jordan homomorphism, then $T$ is a *Jordan triple homomorphism*, that is

$$T(\{a\,b\,c\}) = \{T(a)\,T(b)\,T(c)\}, \quad \text{for all } a, b, c \in A,$$

where $\{a\,b\,c\} = \frac{1}{2}(abc + cba)$ is the Jordan triple product in $A$. A linear (additive) map $T : A \to B$ between Banach algebras *preserves the sharp relation* if $a \leq_\sharp b$ implies $T(a) \leq_\sharp T(b)$.

We begin this section by noticing that every Jordan homomorphism preserves the sharp relation.

**Lemma 2.2.** *Let $A$ and $B$ be Banach algebras and $T : A \to B$ a Jordan homomorphism. Then $T$ preserves the relation $\leq_\sharp$.*

*Proof.* Let $a \in A^\sharp$ and $b \in A$ such that $a \leq_\sharp b$. Let us prove that $T(a) \leq_\sharp T(b)$. Recall that, by [29, Theorem 2.1], if $T$ is a Jordan homomorphism, then $T$ strongly preserves group invertibility, that is, $T(x^\sharp) = T(x)^\sharp$ for every group invertible element $x \in A$. As part of the proof of this result, Mbekhta showed that a Jordan homomorphism preserves the commutativity of $\{1\}$-inverses, that is, if $xyx = x$ and $xy = yx$, then $T(x)T(y) = T(y)T(x)$. Having this facts in mind, since $T(aa^\sharp + a^\sharp a) = T(a)T(a^\sharp) + T(a^\sharp)T(a)$, we obtain $T(aa^\sharp) = T(a)T(a)^\sharp$. Moreover, as $a^\sharp = a^\sharp b a^\sharp$ and $a^\sharp b = ba^\sharp$, the same arguments show that $T(a^\sharp b) = T(a)^\sharp T(b)$. Consequently

$$T(a)^\sharp T(a) = T(a^\sharp a) = T(a^\sharp b) = T(a)^\sharp T(b).$$

The identity $T(a)T(a)^\sharp = T(b)T(a)^\sharp$ can be obtained in the same way and, thus, $T(a) \leq_\sharp T(b)$. □

From Lemmas 2.1 and 2.2 it is clear that every Jordan homomorphism multiplied by an invertible element commuting with its range, also preserves the sharp relation. We address the question whether the reciprocal result holds. First we will study linear preservers of the sharp relation in the environment of semisimple Banach algebras with non zero socle. The *socle* of a semisimple Banach algebra $A$, $\mathrm{Soc}(A)$, is the sum of all minimal left ideals of $A$, or minimal right ideals of $A$, if they exists; otherwise it is zero. Recall that every minimal left ideal of $A$ is of the form $Ae$ for some minimal idempotent $e$, that is, a non-zero idempotent with $eAe = \mathbb{C}e$.

An element $u \in A$ is said to be of *rank-one* if $u \neq 0$, and $u$ belongs to some minimal left ideal of $A$, or equivalently, $uAu = \mathbb{C}u \neq 0$. Every element of the socle is a finite sum of rank-one elements, that is to say that the socle coincides with the set of all finite rank elements. Moreover it is also well known that every element of the socle is regular ([4, 7]).

Given $u \in A$ a rank-one element, there exists $\tau(u) \in \mathbb{C}$ such that $u^2 = \tau(u)u$. Moreover, $\tau(u) = 0$ or $\tau(u)$ is the only non-zero point of the spectrum of $u$.

Thus, if $\tau(u) \neq 0$ then $\tau(u)^{-1}u$ is a minimal idempotent, and $u = \tau(u)(\tau(u)^{-1}u)$. Now, for $\tau(u) = 0$, let $x \in A$, and $\lambda \in \mathbb{C}$ be such that $uxu = u$ and $x - \lambda\mathbf{1}$ is invertible. Therefore, $e_1 = ux$ and $e_2 = u(x - \lambda)$ are minimal idempotents satisfying $u = \lambda^{-1}(e_1 - e_2)$. From this follows that every element of the socle of a semisimple Banach algebra is a linear combination of minimal idempotents (compare with Lemma 1.1 of [14]). Recall also that every rank-one element is *single*, that is, if $u$ is a rank-one element, for every $a, b \in A$, $aub = 0$ implies that $au = 0$ or $bu = 0$.

**Remark 2.3.** *Notice that for every $a \in A^\sharp$ and $b \in A$, $ab = ba = 0$ is equivalent to $a \leq_\sharp (a + b)$. Hence, given $A, B$ Banach algebras and $T : A \to B$ a linear map preserving the sharp relation, for every $a \in A^\sharp$ and $b \in A$, $ab = ba = 0$ implies that $T(a)T(b) = T(b)T(a) = 0$.*

The initial step for the description of zero product preserving linear maps in [14] consists in describing the behaviour of the mapping on Jordan products of minimal idempotents $p, q \in A$. In this sense, our aim is to achieve the identities from [14, Lemma 2.5, Lemma 2.6], through rank-one group invertible elements, that is, rank-one elements with non-zero trace.

**Lemma 2.4.** *Let $A$ and $B$ be Banach algebras. Assume that $A$ is unital. Let $T : A \to B$ be a linear map preserving the sharp relation. For every idempotent element $p \in A$, the following holds.*

*(1)* $T(p)^2 = T(p)T(\mathbf{1}) = T(\mathbf{1})T(p)$.

*(2)* $T(p) = T(\mathbf{1})T(\mathbf{1})^\sharp T(p) = T(p)T(\mathbf{1})^\sharp T(\mathbf{1})$

*Proof.* Since $\mathbf{1} \leq_\sharp \mathbf{1}$, $T(\mathbf{1}) \leq_\sharp T(\mathbf{1})$, which in particular implies that $T(\mathbf{1})$ has group inverse.

The first identity follows from Remark 2.3. Indeed, as $p(\mathbf{1}-p) = (\mathbf{1}-p)p = 0$, we have $T(p)\big(T(\mathbf{1}) - T(p)\big) = \big(T(\mathbf{1}) - T(p)\big)T(p) = 0$, which proves (1).

Now, by using that $T(p)^\sharp T(p) = (T(p)^\sharp T(p))^\sharp = (T(p)^\sharp T(\mathbf{1}))^\sharp = T(\mathbf{1})^\sharp T(p)$ we get

$$T(p) = T(p)^2 T(\mathbf{1}) = T(p)T(\mathbf{1})^\sharp T(\mathbf{1}) = T(\mathbf{1})T(\mathbf{1})^\sharp T(p).$$

□

**Proposition 2.5.** *Let $A$ be a unital semisimple Banach algebra with non-zero socle, $B$ a Banach algebra and $T : A \to B$ a linear map preserving the sharp relation. Then*

$$T(p \circ q)T(\mathbf{1}) = T(p) \circ T(q),$$

*for every minimal idempotents $p, q \in A$.*

*Proof.* In order to simplify the notation, we write $h = T(\mathbf{1})$.
Take minimal idempotents $p, q \in A$. Then $pq$ is a rank-one element. We must consider different cases:
*Case 1*: $(pq)^2 \neq 0$, that is, $\tau(pq) = \tau(qp) \neq 0$.

If we assume that $p = pq = qp$, then $p(\mathbf{1}-q) = (\mathbf{1}-q)p = 0$ and, as we have noticed in Remark 2.3, $T(p)\,(h - T(q)) = (h - T(q))\,T(p) = 0$. This leads to $T(p)h = T(p)T(q)$ and $hT(p) = T(q)T(p)$, which in particular gives $T(p \circ q)h = T(p) \circ T(q)$.

Suppose now that either $p \neq pq$ or $p \neq qp$. If $\tau(p(\mathbf{1}-q)) \neq 0$ (for $\tau(q(\mathbf{1}-p)) \neq 0$ the proof is similar), then $pq$, $p(\mathbf{1}-q)$ and $(\mathbf{1}-q)p$ are rank-one group invertible elements. Since $pq\,(\mathbf{1}-q)(\mathbf{1}-p) = (\mathbf{1}-q)(\mathbf{1}-p)\,pq = 0$, $p(\mathbf{1}-q)\,q(\mathbf{1}-p) = q(\mathbf{1}-p)\,p(\mathbf{1}-q) = 0$ and $(\mathbf{1}-q)p\,(\mathbf{1}-p)q = (\mathbf{1}-p)q\,(\mathbf{1}-q)p = 0$, we obtain, respectively:

$$T(pq)h = T(pq)T(p) + T(pq)T(q) - T(pq)T(qp), \tag{2.1}$$

$$T(p)T(q) = T(p)T(qp) + T(pq)T(q) - T(pq)T(qp), \tag{2.2}$$

$$T(q)T(p) = T(q)T(qp) + T(pq)T(p) - T(pq)T(qp). \tag{2.3}$$

From (2.1) and (2.2) it follows that

$$T(pq)h + T(p)T(qp) = T(p)T(q) + T(pq)T(p). \tag{2.4}$$

Analogously (2.1) and (2.3) gives

$$T(pq)h + T(q)T(qp) = T(q)T(p) + T(pq)T(q). \tag{2.5}$$

Note that in (2.1), (2.2) and (2.3), the roles of $p$ and $q$ can be exchanged. Thus, we can process in this way in (2.5) to obtain

$$T(qp)h + T(p)T(pq) = T(p)T(q) + T(qp)T(p). \tag{2.6}$$

From (2.4) and (2.6) we get

$$T(pq+qp)h + T(p)T(pq+qp) = 2T(p)T(q) + T(pq+qp)T(p). \tag{2.7}$$

Using the other side identities of the zero product and proceeding similarly, it follows that

$$hT(pq+qp) + T(pq+qp)T(p) = 2T(q)T(p) + T(p)T(pq+qp). \tag{2.8}$$

Finally, from (2.7) and (2.8) we get $T(p \circ q)h = T(p) \circ T(q)$.

Now, suppose that $\tau(p(\mathbf{1}-q)) = \tau(q(\mathbf{1}-p)) = 0$ ( being $\tau(pq) \neq 0$, and $p \neq pq$ or $p \neq qp$). From $pq\,(\mathbf{1}-q)(\mathbf{1}-p) = (\mathbf{1}-q)(\mathbf{1}-p)\,pq = 0$ and $qp\,(\mathbf{1}-p)(\mathbf{1}-q) = (\mathbf{1}-p)(\mathbf{1}-q)\,qp = 0$ we get

$$T(pq)h = T(pq)T(p) + T(pq)T(q) - T(pq)T(qp),$$
$$T(qp)h = T(qp)T(q) + T(qp)T(p) - T(qp)T(pq).$$

As $\tau(p(\mathbf{1}-q)) = \tau(q(\mathbf{1}-p)) = 0$, we have $p(\mathbf{1}-q)p = q(\mathbf{1}-p)q = 0$, that is, $pqp = p$ and $qpq = p$. When $p = pq$ (the case $p = qp$ is similar), it follows that $qp = q$, and $p \circ q = \frac{1}{2}(p+q)$ is an idempotent. Having into account Lemma 2.4,

$$\begin{aligned} T(p+q)h &= 2T(\frac{1}{2}(p+q))h = 2T(\frac{1}{2}(p+q))^2 \\ &= \frac{1}{2}(T(p)^2 + T(q)^2 + T(p)T(q) + T(q)T(p)). \end{aligned}$$

This yields

$$2T(p)h + 2T(q)h = T(p)^2 + T(q)^2 + T(p)T(q) + T(q)T(p)$$

and, consequently, $T(p \circ q)h = \frac{1}{2}T(p+q)h = T(p) \circ T(q)$. Finally, suppose that $pqp = p$, $qpq = q$, $pq \neq q$ and $qp \neq p$. Then

$$\begin{aligned} (p+pq)^2 &= 2(p+pq), \\ (p+qp)^2 &= 2(p+qp), \\ (q+pq)^2 &= 2(q+pq), \\ (q+qp)^2 &= 2(q+qp). \end{aligned}$$

Therefore

$$\begin{aligned} (p+pq)^\sharp &= \frac{1}{4}(p+pq), \\ (p+qp)^\sharp &= \frac{1}{4}(p+qp), \\ (q+pq)^\sharp &= \frac{1}{4}(q+pq), \\ (q+qp)^\sharp &= \frac{1}{4}(q+qp). \end{aligned}$$

Arguing as above, the following identities are easily obtained:

$$T(p+pq)h = T(p)T(pq) + T(pq)T(p), \tag{2.9}$$

$$T(p+qp)h = T(p)T(qp) + T(qp)T(p), \tag{2.10}$$

$$T(q+pq)h = T(q)T(pq) + T(pq)T(q), \tag{2.11}$$

$$T(q+qp)h = T(q)T(qp) + T(qp)T(q). \tag{2.12}$$

Notice that, for an idempotent $p$ and $x \in A$, such that $pxp = 0$, then

$$p\,(x - px - xp) = (x - px - xp)\,p = 0,$$

and thus

$$T(p)\,(T(x) - T(px) - T(xp)) = (T(x) - T(px) - T(xp))\,T(p) = 0.$$

Applying this fact to $x = 1 - q$ we have:

$$\begin{aligned} T(p)h - T(p)T(q) &= T(p)\,(T(p(\mathbf{1}-q)) + T((\mathbf{1}-q)p)) \\ &= 2T(p)^2 - T(p)T(pq) - T(p)T(qp). \end{aligned}$$

That is,

$$T(p)h = T(p)T(pq) + T(p)T(qp) - T(p)T(q). \tag{2.13}$$

Similarly,

$$hT(p) = T(pq)T(p) + T(qp)T(p) - T(q)T(p). \tag{2.14}$$

From (2.13) and (2.14), we obtain

$$2T(p)h = T(p)T(pq) + T(pq)T(p) + T(p)T(qp) + T(qp)T(p) - (T(p)T(q) + T(q)T(p)). \tag{2.15}$$

The identities (2.15), (2.9) and (2.10), produce $T(p\circ q)h = T(p)\circ T(q)$.

*Case 2*: $\tau(pq)=\tau(qp)=0$. In this case $pqp=qpq=0$ and, since every rank-one element is single it must be $pq=0$ or $qp=0$.

Suppose that $pq=0$ and $qp\neq 0$. As

$$p\big(q(\mathbf{1}-p)\big)=\big(q(\mathbf{1}-p)\big)p \quad \text{and} \quad q\big((\mathbf{1}-q)p\big)=\big((\mathbf{1}-q)p\big)q=0,$$

we obtain, respectively

$$\begin{aligned} T(p)T(q)&=T(p)T(qp), \qquad & T(q)T(p)&=T(qp)T(p),\\ T(q)T(p)&=T(q)T(qp), \qquad & T(p)T(q)&=T(qp)T(q). \end{aligned}$$

As $pq=0$, $p+q-qp$ is an idempotent element. Hence by Lemma 2.4, $T(p+q-qp)^2=T(p+q-qp)h$, that is

$$\begin{gathered} T(p)^2+T(q)^2+T(p)T(q)+T(q)T(p)+T(qp)^2-T(p)T(qp)-T(q)T(qp)\\ -T(qp)T(p)-T(qp)T(q)=T(p)h+T(q)h-T(qp)h. \end{gathered}$$

Having in mind the previous identities we deduce that

$$T(qp)h=T(p)T(q)+T(q)T(p)-T(qp)^2.$$

It only remains to prove that $T(qp)^2=0$. To this end, we will prove that, for every rank-one element $u\in A$ with $\tau(u)=0$, we have $T(u)^2=0$. As we know, given $x\in A$ and $\lambda\in\mathbb{C}$ such that $uxu=u$ and $x-\lambda\mathbf{1}$ is invertible, $e_1=ux$ and $e_2=u(x-\lambda)$ are minimal idempotents such that $\lambda u=e_1-e_2$, $e_1e_2=e_2$ and $e_2e_1=e_1$. Therefore

$$\begin{aligned} \lambda^2T(u)^2 &= T(e_1-e_2)^2=T(e_1)^2+T(e_2)^2-T(e_1)T(e_2)-T(e_2)T(e_1)\\ &= T(e_1+e_2)h-(T(e_1)T(e_2)+T(e_2)T(e_1)). \end{aligned}$$

As $e_1e_2=e_2$ and $e_2e_1=e_1$, then $(e_1+e_2)^\sharp=\frac{1}{4}(e_1+e_2)$ and

$$T(e_1+e_2)h=T(e_1e_2+e_2e_1)h=T(e_1)T(e_2)+T(e_2)T(e_1).$$

Hence, $T(u)^2=0$ as wanted. □

Since every element of the socle is a linear combination of minimal idempotents, once we have obtained Lemma 2.4 and Proposition 2.5, it can be checked that the rest of calculations shown in Lemma 2.5, Lemma 2.6 and Theorem 2.7 in [14] still work for our setting.

**Proposition 2.6.** *Let $A$ and $B$ be Banach algebras. Assume that $A$ is unital, with non-zero socle. Let $T:A\to B$ be a linear map preserving the sharp relation. Let $h=T(\mathbf{1})$. For $a\in A$ and $x,y\in Soc(A)$, the following identities hold.*

(i) $T(x)h=hT(x)$.
(ii) $T(a\circ x)h=T(a)\circ T(x)$.
(iii) $T(x)hT(a)=T(x)T(a)h$, *and* $T(a)hT(x)=hT(a)T(x)$.
(iv) $\{T(x),\,T(a),\,T(y)\}=T(\{x,\,a,\,y\})h^2$.
(v) $\{T(x),\,T(a)^2,\,T(y)\}=T(\{x,\,a^2,\,y\})h^3$.

Recall that a non-zero ideal $I$ of $A$ is called *essential* if it has non-zero intersection with every non-zero ideal of $A$. For a semisimple Banach algebra $A$ this is equivalent to the condition $aI=0$, for $a\in A$, implies $a=0$. It is well known, that if $a\in A$ verifies $xax=0$ for all $x\in\mathrm{Soc}(A)$, then $xa=ax=0$ for every $x\in\mathrm{Soc}(A)$.Thus if $\mathrm{Soc}(A)$ is essential, from $xax=0$ for all $x\in\mathrm{Soc}(A)$, it follows that $a=0$.

**Theorem 2.7.** *Let $A$ and $B$ be unital Banach algebras, $A$ having essential socle. Let $T:A\to B$ be a bijective linear map. Then, the following conditions are equivalent:*

*(1) $T$ preserves the sharp relation,*

*(2) $T$ is a Jordan isomorphism multiplied by a central invertible element.*

*Proof.* Assume that $T$ preserves the sharp relation. Taking into account the preceding proposition, we argue as in [14, Theorem 2.7].

Let us first make an easy observation: we know from Lemma 2.4 that $T(p) = hh^\sharp T(p) = T(p)h^\sharp h$, for every minimal idempotent $p \in A$, where $h = T(\mathbf{1})$. By linearizing, the same holds for every elements in the socle, that is

$$T(x) = T(x)h^\sharp h \tag{2.16}$$

for all $x \in \mathrm{Soc}(A)$. Let $a \in A$, and $x \in \mathrm{Soc}(A)$. By the surjectivity of $T$, there exists $b \in A$ such that $T(b) = T(a)h - hT(a)$. From Proposition 2.6 (iii), it follows that $T(xbx)h^2 = 0$, or equivalently (multiplying by $h^\sharp$), $T(xbx)h = 0$. From Equation (2.16) we deduce that $T(xbx) = 0$. Since $T$ is injective, it follows that $xbx = 0$, for all $x \in \mathrm{Soc}(A)$, and thus, $b = 0$. This proves that

$$T(a)h = hT(a), \qquad \text{for every } a \in A.$$

Similarly, since $T(x)h^\sharp hT(a) = T(x)T(a)$, and $T(x)hT(a^2)T(x) = hT(x)T(a^2)T(x) = h^3T(xa^2x) = T(x)T(a)^2T(x)$, for every $x \in \mathrm{Soc}(A)$, we can prove that

$$T(a) = T(a)h^\sharp h \quad \text{and} \quad T(a^2)h = T(a)^2,$$

for every $a \in A$ (compare if necessary with the proof of [14, Theorem 2.7]). By the surjectivity of $T$, it is clear that $h$ is invertible, and that $h^{-1}T$ is a Jordan isomorphism.

The reciprocal statement follows from Lemmas 2.1 and 2.2. □

Recall that a $C^*$-algebra $A$ is of *real rank zero* if the set of all real linear combinations of orthogonal projections is dense in the set of all hermitian elements of $A$ (see [9]). Notice that every von Neumann algebra, and, in particular, the algebra of all bounded linear operators on a complex Hilbert space $H$ is of real rank zero.

The following observation has become a standard tool in the study of Jordan homomorphisms (see [3] or [24]).

**Lemma 2.8.** *Let $A$ be a real rank zero $C^*$-algebra, $B$ a Banach algebra and $T : A \to B$ a bounded linear mapping sending projections to idempotents. Then $T$ is a Jordan homomorphism. Moreover, if $B$ is a $C^*$-algebra and $T$ sends projections to projections, it is in fact a *-Jordan homomorphism.*

**Theorem 2.9.** *Let $A$ and $B$ be unital Banach algebras. Assume that $A$ has real rank zero $C^*$-algebra. Let $T : A \to B$ be a bounded linear map. The following conditions are equivalent:*

*(1) $T$ preserves the relation $\leq_\sharp$,*
*(2) $T = T(\mathbf{1})S$ where $S$ is a Jordan homomorphism, $T(\mathbf{1})$ is group invertible and it commutes with $S(A)$.*

*Proof.* Let $h = T(\mathbf{1})$. Suppose that $T$ preserves the sharp relation. Since $1 \leq_\sharp 1$, we have $h \leq_\sharp h$ and, thus, $h$ is group invertible. From Lemma 2.4 we know that

$$T(p)^2 = T(p)h = hT(p)$$

and

$$T(p) = hh^\sharp T(p) = T(p)h^\sharp h,$$

for every idempotent $p \in A$. As every selfadjoint element in $A$ can be approximated by real linear combinations of (orthogonal) idempotents, and $T$ is bounded, we get $hT(x) = T(x)h$, and $T(x) = hh^\sharp T(x)$ for every selfadjoint element $x \in A$. Moreover, since for every $x \in A$ there exists $x_1, x_2 \in A$ selfadjoint elements such that $x = x_1 + ix_2$, it is clear that $hT(x) = T(x)h$ and $T(x) = hh^\sharp T(x)$ for every $x \in A$.

Now, from $hT(p) = T(p)^2$, multiplying by $\left(h^\sharp\right)^2$ and taking into account the commutativity of $h$ (which implies the commutativity of $h^\sharp$), we deduce that $h^\sharp T(p) = \left(h^\sharp T(p)\right)^2$. Let $: A \to B$ be the map defined as $S(x) := h^\sharp T(x)$, for all $x \in A$. The previous identity gives $S(p) = S(p)^2$, for every idempotent $p \in A$. Lemma 2.8 guarantees that $S$ is a Jordan homomorphism. Finally, note that $T = hh^\sharp T = hS$.

The converse can be checked straightforwardly combining Lemmas 2.2 and 2.1. □

One may think whether the above result is true for general C*-algebras.

**Example 2.10.** *Let $A = C([0,1])$, $B = M_2(\mathbb{C})$ and $T : A \to B$ the map given by*

$$T(f) = \begin{pmatrix} f(0) & f(1) \\ 0 & 0 \end{pmatrix}.$$

*Notice that $A^\sharp = A^{-1} \cup \{0\}$. Hence $0 \leq_\sharp g$ for every $g \in A$ and for $f \in A^{-1}$, $f \leq_\sharp g$ if and only if $f = g$. Trivially, $T$ fulfills $T(0) \leq_\sharp T(f)$ for every $f \in A$. To see that $T(f) \leq_\sharp T(f)$ for every $f \in A^\sharp$ it is enough to show that $T$ sends invertible functions to group invertible matrices. This last assert follows from the fact that every matrix*

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$$

*with $a \neq 0$ has group inverse*

$$\begin{pmatrix} a^{-1} & a^{-2}b \\ 0 & 0 \end{pmatrix}$$

*for every $b \in \mathbb{C}$. Finally, $T = T(\mathbf{1})^\sharp T$ and it can be easily seen that $T$ is not a Jordan homomorphism. However, it can be checked that $TT(\mathbf{1})^\sharp$ is a Jordan homomorphism.*

**Remark 2.11.** *Let $A$ and $B$ be unital semisimple Banach algebras and $T : A \to B$ be a surjective linear map. Notice that, in view of Lemma 2.1 (3), if $T$ preserves the sharp relation in both directions, that is,*

$$a \leq_\sharp b \quad \text{if and only if} \quad T(a) \leq_\sharp T(b),$$

*then $T$ preserves invertibility in both directions. Obviously, $T$ is injective, cause $T(a) = 0$ implies that $a \leq_\sharp 0$ and hence $a = 0$. Therefore, the mapping $S = T(\mathbf{1})^{-1}T$ is a unital bijective linear map preserving invertibility in both directions. In this case, it is well known that, if $A$ has essential socle ([8]) or $A$ has real rank zero ([3],[24]), then $S$ is a Jordan isomorphism.*

Let $R$ be a (unital associative) ring. The sharp relation $a \leq_\sharp b$ makes sense only when $a$ is group invertible. Note that, from $a^\sharp a = a^\sharp b = ba^\sharp$, we get $a = aa^\sharp b = ba^\sharp a$, that is, $a = bp = pb$ where $p = aa^\sharp = a^\sharp a$ is an idempotent element. So it make sense to extend the sharp relation to the whole ring, in the following way.

**Definition 2.12.** *Let $A$ be a (unital associative) ring and let $a, b \in A$. We say that*

$$(R1) \qquad a \leq_s b \quad \text{if and only if there exists an idempotent } p \in A \text{ such that } a = pb = bp.$$

This last definition provides a natural extension of $\leq_\sharp$ in the following sense: if $a \leq_s b$ and $a$ is group invertible, then $a \leq_\sharp b$. Indeed, if $a = bp = pb$ for some idempotent $p \in A$, then $a = ap = pa$. As $a$ is a group invertible element, we get $a^\sharp = pa^\sharp = a^\sharp p$. Thus, $a^\sharp a = a^\sharp pb = a^\sharp b$ and, similarly, $aa^\sharp = ba^\sharp$. Observe also that $a \in A$ is group invertible element if and only if $a \leq_s u$ for some invertible element $u$. Indeed, if $a$ is group invertible, then $a - \mathbf{1} + aa^\sharp$ is invertible and $a \leq_s (a - \mathbf{1} + aa^\sharp)$. Reciprocally, if if $a \leq_s u$ for some invertible element $u$, then $a = pu = up$, for certain idempotent element $p \in A$. Therefore, $a^\sharp = u^{-1}p$. Notice also that for every $a, b \in A$, and $u \in A$ commuting with $a$ and $b$, if $a \leq_s b$ then $ua \leq_s ub$. Indeed, let $p$ be an idempotent in $A$ such that $a = bp = pb =$. It follows that $ua = u(bp) = (ub)p$ and $ua = au = (pb)u = p(bu) = p(ub)$.

In the next lemma, we prove that every Jordan homomorphism preserves the relation $(R1)$; whence so does every Jordan homomorphism multiplied by an element commuting with its range.

**Lemma 2.13.** *Let $A$ and $B$ be Banach algebras and $T : A \to B$ a Jordan homomorphism. If $a \leq_s b$, then $T(a) \leq_s T(b)$.*

*Proof.* For every idempotent $p \in A$, $T(p) = T(p)^2$ holds. Let $a, b \in A$ and suppose that $a \leq_s b$, that is, there exists an idempotent element $p \in A$ such that $a = bp = pb$. As $a = ap = pa$, we get

$$2T(a) = T(pa + ap) = T(p)T(a) + T(a)T(p).$$

Multiplying this last equation by $T(p)$ on the left and on the right, respectively, and combining their results, it can be obtained that $T(a)T(p) = T(p)T(a)$, which yields $T(a) = T(p)T(a)$. From $a = bp = pb$ we can also write

$$2T(a) = T(bp + pb) = T(b)T(p) + T(p)T(b).$$

We multiply this expression by $T(p)$ on the right, to produce

$$2T(a) = 2T(a)T(p) = T(b)T(p) + T(p)T(b)T(p).$$

Since $T$ preserves triple products, it follows

$$2T(a) = T(b)T(p) + T(pbp) = T(b)T(p) + T(a),$$

which finally gives $T(a) = T(b)T(p)$. The identity $T(a) = T(p)T(b)$ can be obtained similarly. This proves that $T(a) \leq_s T(b)$ as desired. $\square$

It is a natural question to ask if multiples of Jordan homomorphisms arise from linear maps preserving the relation $(R1)$. We focus on the two settings that we are already dealing with, that is, unital semisimple Banach algebras with large socle and real rank zero C*-algebras.

**Lemma 2.14.** *Let $A$ and $B$ be Banach algebras and $T : A \to B$ a linear map preserving the relation $(R1)$. Then, for every $a \in A^\sharp$, $b \in B$, the condition $ab = ba = 0$ implies $T(a)T(b) = T(b)T(a) = 0$.*

*Proof.* Take $a \in A^\sharp$, $b \in B$. Then it is clear that

$$ab = ba = 0 \quad \text{if and only if} \quad a^\sharp b = ba^\sharp = 0 \quad \text{if and only if} \quad a \leq_s a + b.$$

Therefore, if $ab = ba = 0$, then $T(a) \leq_s T(a) + T(b)$, that is,

$$T(a) = pT(a) = T(a)p = p(T(a) + T(b)) = (T(a) + T(b))p$$

for some idempotent $p \in B$. In particular, $pT(b) = T(b)p = 0$, which gives $T(a)T(b) = T(a)pT(b) = 0$ and $T(b)T(a) = T(b)pT(a) = 0$, as desired. $\square$

**Remark 2.15.** *Let $A$ and $B$ be Banach algebras and $T : A \to B$ a linear map preserving the relation $(R1)$. We assume that $A$ is unital. Let $p$ be an idempotent element in $A$. The previous lemma implies, in particular, that*

$$T(p)^2 = T(p)T(\mathbf{1}) = T(\mathbf{1})T(p).$$

*Moreover, since $T(p) \leq_s T(\mathbf{1})$, there exists an idempotent $q \in B$ such that $T(p) = T(\mathbf{1})q = qT(\mathbf{1})$. If we assume moreover that $T(\mathbf{1})$ is group invertible, then it is clear that*

$$T(p) = T(\mathbf{1})T(\mathbf{1})^\sharp T(p) = T(p)T(\mathbf{1})^\sharp T(\mathbf{1}).$$

From Lemma 2.14 it follows that the conclusions in Propositions 2.5 and 2.6 still hold. Having in mind these facts and the previous remark, the proof of the next theorem runs in the same way as that of Theorem 2.7.

**Theorem 2.16.** *Let $A$ and $B$ be unital Banach algebras, $A$ having essential socle. Let $T : A \to B$ be a bijective linear map. Assume that $T(\mathbf{1})$ has group inverse. Then, the following conditions are equivalent:*

*(1) $T$ preserves the relation $(R1)$,*
*(2) $T$ is a Jordan isomorphism multiplied by a central invertible element.*

To conclude this section, we consider a continuous linear mapping defined on a unital real rank zero C*-algebra that preserves the relation $(R1)$.

**Theorem 2.17.** *Let $A$ be a unital real rank zero C*-algebra and $B$ be a Banach algebra. Let $T : A \to B$ be a continuous linear map. Assume that $T(\mathbf{1})$ is group invertible. Then, the following conditions are equivalent:*

*(1) $T$ preserves the relation $(R1)$,*
*(2) $T = T(\mathbf{1})S$ where $S$ is a Jordan homomorphism, and $T(\mathbf{1})$ commutes with $S(A)$.*

*Proof.* From Remark 2.15 we know that

$$T(p)^2 = T(p)h = hT(p),$$

and

$$T(p) = hh^\sharp T(p) = T(p)h^\sharp h,$$

for every idempotent $p$ in $A$. For every real linear combination of mutually orthogonal idempotents $x = \sum_{k=1}^n \lambda_k p_k$, we have

$$hT(x^2) = hT\left(\sum_{k=1}^n \lambda_k^2 p_k\right) = \sum_{k=1}^n \lambda_k^2 hT(p_k) = \sum_{k=1}^n \lambda_k^2 T(p_k)^2 = T(x)^2.$$

Since $A$ has real rank zero and $T$ is continuous, it is clear that

$$T(a)h = hT(a), \qquad T(a) = hh^\sharp T(a),$$

and

$$hT(a^2) = T(a)^2,$$

for every selfadjoint element $a$ in $A$. Since for every element $a \in A$ , there exists selfadjoint elements $x, y \in A$ such that $a = x + iy$ and

$$a^2 = (x+iy)^2 = x^2 - y^2 + i\big((x+y)^2 - x^2 - y^2\big),$$

we have

$$T(a)h = hT(a), \qquad T(a) = hh^\sharp T(a),$$

and

$$hT(a^2) = hT((x+iy)^2) = hT\big(x^2 - y^2 + i\big((x+y)^2 - x^2 - y^2\big)\big) = T(a)^2.$$

From these identities, it is clear that $S(x) = h^\sharp T(x)$ is a Jordan homomorphism, and $T(x) = hS(x)$ for every $x \in A$ (compare with Theorem 2.9). □

## 3. Star partial order and orthogonality

Recall that every C*-algebra $A$ can be endowed with Jordan triple product defined by

$$\{a, b, c\} := \frac{1}{2}(ab^*c + cb^*a).$$

An element $e \in A$ satisfying $\{e, e, e\} = e$ is called *tripotent* or *partial isometry*. Every tripotent $e$ in $A$ gives rise to the so-called Pierce decomposition of $A$:

$$A = A_2(e) \oplus A_1(e) \oplus A_0(e),$$

where $A_2(e) = ee^*Ae^*e$, $A_1(e) = (1 - ee^*)Ae^*e \oplus ee^*A(1 - e^*e)$ and $A_0(e) = (1 - ee^*)A(1 - e^*e)$. The Pierce subspace $A_2(e)$ admits a structure of a unital JB*-algebra with unit $e$, product given by $x \bullet y = \{x, e, y\}$ and involution given by $e^\diamond = \{e, x, e\}$ (see [20]).

For each element $a$ in a von Neumann algebra $W$ there exists a unique partial isometry $r(a)$ in $W$ such that $a = r(a)|a|$, and $r(a)^*r(a)$ is the support projection of $|a|$, where $|a| = (a^*a)^{\frac{1}{2}}$. It is also known that $r(a)a^*r(a) = \{r(a), a, r(a)\} = a$ (see [35, §1.12] for details). The element $r(a)$ will be called the *range partial isometry* of $a$.

Two elements $a, b$ in a $C^*$-algebra $A$ are called *orthogonal* (denoted by $a \perp b$) if $ab^* = b^*a = 0$. For a subset $S$ of $A$, we write

$$S_A^{\perp} := \{y \in A : y \perp x, \forall x \in S\}.$$

A linear mapping $T : A \to B$ between $C^*$-algebras is said to be *orthogonality preserving* if $T(a) \perp T(b)$ whenever $a \perp b$. The map $T$ is *biorthogonality preserving* if $T(a) \perp T(b)$ if and only if $a \perp b$. A linear mapping $T : A \to B$ between $C^*$-algebras *preserves the star order* if $a \leq_* b$ implies that $T(a) \leq_* T(b)$.

Notice that given $a, b$ in a C*-algebra $A$, $a \leq_* b$ if and only if $a \perp (a + b)$, so the problem of preserving star partial order is in fact equivalent to that of preserving orthogonality.

The study of orthogonality preserving bounded linear maps between $C^*$-algebras began with the work of W. Arendt [1] in the setting of unital abelian $C^*$-algebras. Orthogonality preserving linear operators between general $C^*$-algebras were first considered by M. Wolff in [37]. He proved that every orthogonality preserving bounded symmetric linear map between $C^*$-algebras is a multiple of a Jordan $^*$-homomorphism. Recall that a linear map $T : A \to B$ is called *selfadjoint* (or *symmetric*) if $T(a^*) = T(a)^*$, for every $a \in A$. Selfadjoint Jordan homomorphisms are named *Jordan $^*$-homomorphism*.

Under continuity assumptions, orthogonality preserving (bounded) linear maps between general C*-algebras completely described in [11] and [12]:

**Theorem 3.1.** [11, Theorem 17 and Corollary 18] *Let $T : A \to B$ be a bounded linear mapping between two $C^*$-algebras. For $h = T^{**}(\mathbf{1})$ and $r = r(h)$ the following assertions are equivalent.*

*a) $T$ is orthogonality preserving.*

*b) There exists a unique Jordan $^*$-homomorphism $S : A \to B_2^{**}(r)$ satisfying that $S^{**}(\mathbf{1}) = r$ and $T(z) = hr^*S(z) = S(z)r^*h$ for all $z \in A$.*

*c) $T$ preserves zero triple products, that is, $\{T(x), T(y), T(z)\} = 0$ whenever $\{x, y, z\} = 0$.* □

In [13], J. Garcés, A. Peralta and the first author of this note proved that every biorthogonality preserving surjective linear map between compact $C^*$-algebras or von Neumann algebras is automatically continuous. As every symmetric linear mapping between $C^*$-algebras is orthogonality preserving whenever it preserves zero products, it follows that every symmetric biseparanting linear map between von Neumann algebras is automatically continuous. In [10] it is proved that every biorthogonality preserving surjective linear map $T : A \to B$, from a unital $C^*$-algebra having essential socle to a general $C^*$-algebra is automatically bounded and a Jordan $^*$-isomorphism multiplied by an invertible element.

Note that in the setting of complex matrix algebras the star partial order can be stated as follows:

$$A \leq_* B \quad \text{if and only if} \quad A = PB = BQ,$$

for some selfadjoint idempotent matrices $P, Q$. This characterization is still true for the more general context of Rickart C*-algebras, [27]. A C*-algebra is called *Rickart C*-algebra* if the left annihilator (respectively, right annihilator) of any element $a \in A$ is generated by a projection. It is well known that every von Neumann algebra is a Rickart $C^*$-algebra, and that every Rickart $C^*$-algebra has real rank zero ([6],[35]). If $A$ is a Rickart C*-algebra, then for every element $a \in A$, there exists a unique projection $p$ such that

$$\mathrm{ann}_l(a) = \{x \in A \colon xa = 0\} = A(\mathbf{1} - p).$$

We denote it by $p = \mathrm{lp}(a)$. Similarly, we denote by $q = \mathrm{rp}(a)$ the unique projection such that

$$\mathrm{ann}_r(a) = \{x \in A \colon ax = 0\} = (\mathbf{1} - q)A.$$

If $a$ and $b$ are elements in a Rickart C*-algebra $A$, such that $a \leq_* b$, or equivalently $a \perp (a+b)$, then $(a + b)^* \in \mathrm{ann}_l(a)$ and $(a + b)^* \in \mathrm{ann}_r(a)$, which show that $(a + b)^*\mathrm{lp}(a) = 0$ and $\mathrm{rp}(a)(a + b)^* = 0$.

Having in mind that $a = \mathrm{lp}(a)a = a\mathrm{rp}(a)$, we conclude that $a = \mathrm{lp}(a)b = b\mathrm{rp}(a)$. (Compare with [27, Theorem 1].)

Motivated by the previous characterization, we will study the following relation.

**Definition 3.2.** *Let $A$ be a $C^*$-algebra. We define*

$$(R2) \qquad a \leq b \quad \text{if and only if} \quad a = pb = bq \text{ for some projections } p, q \in A.$$

Even for non Rickart C$^*$-algebras, the notion just presented is deeply related to the star partial order. As a matter of fact, if $a = pb = bq$ for some projections $p, q \in A$, then $a^*a = b^*ppb = b^*pb = a^*b$ and $aa^* = bqqb^* = bqb^* = ba^*$. As consequence, $a \leq_* b$. Reciprocally, if $a \in A$ is regular and $a^*a = a^*b$, then it can be checked that $a^\dagger a = a^\dagger b$. Hencefore $a = aa^\dagger a = aa^\dagger b$, where $p = aa^\dagger$ is a projection. Similarly, from $aa^* = ba^*$ we get $a = ba^\dagger a$, where $q = a^\dagger a$ is a projection. We have proved the following:

**Lemma 3.3.** *Let $A$ be a C$^*$-algebra. Then $a \leq b$ implies $a \leq_* b$. If $a$ is regular, $a \leq_* b$ implies $a \leq b$.*

The previous lemma shows that, for a regular element $a$ in a C$^*$-algebra $A$, $a \perp b$ if and only if $a \leq (a + b)$.

Cause every element in the socle $C^*$-algebra $A$ is regular, we can employ the techniques on orthogonality preserving maps on $C^*$-algebras with large socle in order to determine the structure of linear maps preserving the relation $(R2)$, due to the crucial role played by the regular elements within our proofs.

**Lemma 3.4.** *Let $A$ and $B$ be C$^*$-algebras. Assume that $A$ is unital with non-zero socle. Let $T : A \to B$ be a linear map preserving the relation $(R2)$. Let $h = T(\mathbf{1})$. For every $a \in A$ and $x, y \in Soc(A)$, the following identities hold:*

*(1) $T(x)h^* = hT(x^*)^*$ and $h^*T(x) = T(x^*)^*h$,*
*(2) $T(ax + xa)h^* = T(a)T(x^*)^* + T(x)T(a^*)^*$ and*
*$h^*T(ax + xa) = T(x^*)^*T(a) + T(a^*)^*T(x)$,*
*(3) $T(x)h^*T(a) = T(x)T(a^*)^*h$, and*
*$T(a)h^*T(x) = hT(a^*)^*T(x)$,*
*(4) $\{T(x)\,T(a)\,T(y)\} = T(\{x\,a\,y\})h^*h$,*
*(5) $\{T(x)\,\{T(a)\,h\,T(a)\}\,T(y)\} = \{h\,\{h\,T(\{x\,a^2\,y\})\,h\}\,h\}$.*

*Proof.* Let $a$ be a regular element in $A$ and $a \perp b$. As we have pointed out, $a \leq (a + b)$. By hypothesis, $T(a) \leq T(a) + T(b)$, and hence $T(a) \perp T(b)$. That is, $T$ sends mutually orthogonal elements into mutually orthogonal elements, when one of them is regular. A quickly inspection of the proof of [10, Lemma 2.1], allows us to see that one of the elements appearing in all the orthogonality relations is always regular. Hence, the identities obtained there hold when orthogonality is replaced by the relation $(R2)$. Thus $(1), (2)$ and $(3)$ are clear. We deduce $(4)$ and $(5)$ from them arguing as in [10, Proposition 2.2]. □

**Proposition 3.5.** *Let $A$ and $B$ be C$^*$-algebras. Assume that $A$ is unital with non-zero socle. Let $T : A \to B$ be a linear map preserving orthogonality or the relation $(R2)$. Then, for $x \in Soc(A)$, the condition $T(x) \perp T(\mathbf{1})$ implies $T(x) = 0$.*

*Proof.* For the sake of simplicity, let us denote $h = T(\mathbf{1})$. Pick $x \in \mathrm{Soc}(A)$ satisfying $T(x) \perp T(\mathbf{1})$. By [10, Lemma 2.1], if $T$ preserves orthogonality, or by the previous lemma if $T$ preserves the relation $(R2)$, we get

$$0 = T(x)h^* = hT(x^*)^*,$$
$$0 = h^*T(x) = T(x^*)^*h,$$

and hence $T(x^*) \perp h$. Moreover

$$T(xx^* + x^*x)h^*h = T(x)T(x)^*h + T(x^*)T(x^*)^*h = 0,$$

or equivalently $T(xx^*+x^*x)h^*=0$. This leads to

$$T(x)T(x)^*+T(x^*)T(x^*)^*=0,$$

which clearly implies that $T(x)=T(x^*)=0$. □

**Proposition 3.6.** *Let $A$ and $B$ be C*-algebras, where $A$ is unital and has essential socle. Let $T:A\to B$ be an injective linear map preserving orthogonality or preserving the relation $(R2)$. Then $T(A)\cap\{T(\mathbf{1})\}^{\perp}=\{0\}$.*

*Proof.* Let $a\in A$ be such that $T(a)\perp h$. We claim that, for every $x\in \mathrm{Soc}(A)$, we have:

(1) $T(a\circ x)\perp h$,
(2) $T(a)\perp T(x)$.

Indeed, given $x\in\mathrm{Soc}(A)$, taking into account [10, Lemma 2.1] or Lemma3.4, we obtain

$$T(ax+xa)h^*h=T(a)T(x^*)^*h+T(x)T(a^*)^*h$$

$$=T(a)h^*T(x)+T(x)h^*T(a)=0.$$

Similarly, $hh^*T(ax+xa)=0$, which proves (1).

In order to show that (2) holds, let $p$ be a minimal projection in $A$. From (1)

$$0=T(ap+pa)h^*h=T(a)T(p)^*h+T(p)T(a^*)^*h=T(a)T(p)^*h.$$

In the same way, we prove $hT(p)^*T(a)=0$. Now, as $T(p)T(p)^*=T(p)h^*=hT(p)^*$ and $T(p)^*T(p)=T(p)^*h$, it follows that

$$T(a)T(p)^*T(p)=T(p)T(p)^*T(a)=0,$$

and by cancellation, $T(a)\perp T(p)$. Since $\mathrm{Soc}(A)$ is linearly spanned by its minimal projections, we get (2).

Finally, take $a\in A$ such that $T(a)\perp h$. Again by [10, Lemma 2.1] (respectively, Lemma3.4)

$$hh^*T(\{x,a,y\})=T(\{x,a,y\})h^*h=T(x)T(a)^*T(y)+T(y)T(a)^*T(x)=0,\quad \text{for every } x,y\in\mathrm{Soc}(A).$$

That is, $T(\{x,a,y\})\perp h$ for every $x,y\in\mathrm{Soc}(A)$. By the previous proposition, $T(\{x,a,y\})=0$, and since $T$ is injective, $\{x,a,y\}=0$ for every $x,y\in\mathrm{Soc}(A)$. The essentiality of the socle of $A$ gives $a=0$ and finishes the proof. □

The next result improves the main conclusion of [10, Theorem 3.2]. Notice that one we have shown in Proposition 3.6 that the orthogonal of $\{T(\mathbf{1})\}$ does not contains elements of the image of $T$, the rest of the proof of Theorem 3.2 in [10] runs in the same way.

**Corollary 3.7.** *Let $A$ and $B$ be C*-algebras. Suppose that $A$ is unital and has essential socle. Let $T:A\to B$ be a bijective linear map preserving orthogonality or the relation $(R2)$. Then $B$ is unital and $T$ is a Jordan *-homomorphism multiplied by an invertible element.*

In order to describe linear preservers of the relation $(R2)$, we consider under what circumstances we can obtain a bounded linear map preserving orthogonality, so that [11, Theorem 17 and Corollary 18] can be used to conclude its description.

**Lemma 3.8.** *Let $A$ and $B$ be C*-algebras, where $A$ is unital, and $T:A\to B$ a bounded linear map. Suppose that*

*(1) $T(x)h^*=hT(x^*)^*$,*
*(2) $T(xy+yx)h^*=T(x)T(y^*)^*+T(y)T(x^*)^*$,*

*for every $x,y\in A$, where, $h=T(\mathbf{1})$. Then $T$ is bounded and preserves orthogonality.*

*Proof.* Indeed, it is clear that $T$ is bounded: From the second identity, if follows that

$$T(x^2)h^* = T(x)T(x^*)^* \qquad (x \in A).$$

Therefore, the linear mapping $S : A \to B$, given by $S(x) = T(x)h^*$, is positive, and hence continuous. So $T$ is also bounded.

Let us write $k = h^*h$. For every $x, y, z$ in $A$,

$$\begin{aligned} 2T\big((x \circ y^*) \circ z\big)k &= \big(T(x \circ y^*)T(z^*)^* + T(z)T(x^* \circ y)^*\big)h \\ &= T(x \circ y^*)h^*T(z) + T(z)T(x^* \circ y)^*h \\ &= \frac{1}{2}\big(T(x)T(y)^*T(z) + T(y^*)T(x^*)^*T(z) \\ &+ T(z)T(x^*)^*T(y^*) + T(z)T(y)^*T(x)\big) \\ &= \{T(x), T(y), T(z)\} + \{T(y^*), T(x^*), T(z)\}. \end{aligned}$$

Similarly

$$2T\left((z \circ y^*) \circ x\right)k = \{T(x), T(y), T(z)\} + \{T(y^*), T(z^*), T(x)\},$$
$$2T\left((x \circ z) \circ y^*\right)k = \{T(x), T(z^*), T(y^*)\} + \{T(z), T(x^*), T(y^*)\}.$$

From these equalities we get

$$\begin{aligned} T(\{x, y, z\})k &= T\big((x \circ y^*) \circ z + (z \circ y^*) \circ x - (x \circ z) \circ y^*\big) \\ &= \{T(x), T(y), T(z)\}, \end{aligned}$$

for $x, y, z \in A$.

It is clear now that $T$ preserves zero triple products or, equivalently, orthogonality (Theorem 3.1). $\square$

In many C*-algebras, every element can be expressed as a finite linear combination of projections: the Bunce-Deddens algebras, the irrotational rotation algebras; simple, unital AF C*-algebras with finitely many extremal states; UHF C*-algebras; unital, simple C*-algebras of real rank zero with no tracial states; properly infinite C*- and von Neumann algebras;... (see [26], [28], [34], and the references therein).

In the next Theorem, we show that if $A$ is one of these kind of C*-algebras, and $T : A \to B$ preserves the relation $(R2)$, then $T$ satisfies the conditions in Lemma 3.8 and hence $T$ preserves orthogonality. In particular, from [13, Theorem 14], $T$ is automatically bounded.

**Theorem 3.9.** *Let $A$ be a unital C*-algebra linearly spanned by its projections, $B$ a C*-algebra and $T : A \to B$ be a linear map preserving the relation $(R2)$. Then $T$ preserves orthogonality.*

*Proof.* For any projections $p, q \in A$, it is easy to show that

$$qp \leq qp + (1-q)(1-p) \quad \text{and} \quad q(1-p) \leq q(1-p) + (1-q)p.$$

By hypothesis,

$$T(qp) \leq T(qp) + T((1-q)(1-p)) \quad \text{and} \quad T(q(1-p)) \leq T(q(1-p)) + T((1-q)p).$$

In particular

$$T(qp) \perp T((1-q)(1-p)) \quad \text{and} \quad T(q(1-p)) \perp T((1-q)p).$$

With these identities in mind, we can argue as in [13, Theorem 14] to obtain

(1) $T(x)h^* = hT(x^*)^*$,
(2) $T(xy + yx)h^* = T(x)T(y^*)^* + T(y)T(x^*)^*$,

for every $x, y \in A$. The conclusion follows by applying Lemma 3.8. $\square$

It is not difficult to realize that if $A$ is not linearly spanned by its projections but it has enough projections, in the sense that $A$ has real rank zero, and the map $T$ is assumed to be continuous, then the previous line of arguments provides the following result.

**Theorem 3.10.** *Let $A$ be a unital real rank zero C*-algebra, $B$ a C*-algebra and $T : A \to B$ a bounded linear map preserving the relation* $(R2)$. *Then $T$ preserves orthogonality.*

**Remark 3.11.** *Let $A$ be a von Neumann algebra, $B$ a C*-algebra and $T : A \to B$ a bijective linear map preserving the relation* $(R2)$. *As every von Neumann algebra is a Rickart C*-algebra, the relations* $(R2)$ *and* $\leq_*$ *are equivalent in $A$. Hence, $T$ preserves orthogonality. From* [33, Corollary 4.6] *$T$ is automatically bounded. Hence Theorem 3.1 implies that $T$ is an appropriate multiple of a Jordan $^*$-homomorphism.*

## References


[1] W. Arendt, *Spectral properties of Lamperti operators*, Indiana Univ. Math. J. **32** (1983) 199–215.

[2] B. Aupetit, *A Primer on Spectral Theory* Springer-Verlag (1991).

[3] B. Aupetit, *Spectrum-preserving linear mappings between Banach algebras or Jordan-Banach algebras*, J. London Math. Soc. **62** (2000) 917–924.

[4] B. Aupetit, H. du Mouton, H. *Trace and determinant in Banach algebras*, Studia Math. **121** (1996), 115–136.

[5] J. K. Baksalary, S. M. Mitra, *Left-star and right-star partial orderings*, Linear Algebra Appl. **149** (1991), 73–89.

[6] B. Blackadar, *Projections in C*-algebras*, C*-algebras:1943-1993. A Fifty Year Celebration (1994), 131–150.

[7] M. Brešar, P. Seňrl, *Finite rank elements in semisimple Banach algebras*, Studia Math. **128** (1998), 287–298.

[8] M. Brešar, A. Fošner, and P. Šemrl, *A note on invertibility preservers on Banach algebras.* Proc. Amer. Math. Soc. **131** (2003), 3833?3837.

[9] L. G. Brown, G. K. Pedersen, *C*-algebras of real rank zero*, J. Funct. Anal. **99** (1991) 131-149.

[10] M. Burgos, *Orthogonality preserving linear maps on C*-algebras with non-zero socles*, J. Math. Anal. Appl. **401** (2013) 479–487.

[11] M. Burgos, F.J. Fernández-Polo, J.J. Garces, J. Martinez Moreno, A.M. Peralta, *Orthogonality preservers in C*-algebras, JB*-algebras and JB*-triples*, J. Math. Anal. Appl. **348** (2008) 220–233.

[12] M. Burgos, F.J. Fernández-Polo, J.J. Garcs, A.M. Peralta, *Orthogonality preservers revisited*, Asian-Eur. J. Math. **2** (2009) 387–405.

[13] M. Burgos, J. Garcés, A. Peralta, *Automatic continuity of biorthogonality preservers between compact C*-algebras and von Neumann algebras*, J. Math. Anal. Appl. **376** (2011) 221–230.

[14] M. Burgos, J. Sánchez-Ortega, *On maps preserving zero products*, Lin. Mult. Alg. **61** (3) (2013) 323–335.

[15] G. Dolinar, A. Guterman, J. Marovt, *Automorphisms of $K(H)$ with respect to the star partial order*, Operators and Matrices, **7** (1) (2013) 225–239.

[16] G. Dolinar, A. Guterman, J. Marovt, *Monotone transformations on B(H) with respect to the left-star and the right-star partial order*, Math. Ineq. Appl. **17** (2) (2014), 573-589.

[17] G. Dolinar, J. Marovt, *Star partial order on $B(H)$*, Linea Algebra Appl. **2011** (2011), 319-326.

[18] M. P. Drazin, *Natural structures on semigroups with involution*, Bull. Amer. Nath. Soc. **84** (1978) 139–141.

[19] A. E. Guterman, *Monotone additive transformations on matrices*, Mat. Zametki **81** (2007) 681–692.

[20] H. Hanche-Olsen, E. Størmer, *Jordan Operator Algebras*, Monogr. Stud. Math. **21** (1984), Pitman (Advanced Publishing Program), Boston, MA.

[21] R. Harte, M. Mbekhta, *On generalized inverses in C*-algebras*, Studia Math. 103 (1992) 71–77.

[22] R. E. Hartwig, *How to partially order regular elements*, Math. Japon. **25** (1980), 1–13.

[23] R. E. Hartwig, G. P. H. Styan *On some characterizations of the "star"partial ordering for matrices and rank substractivity*, Linear Algebra Appl. **82** (1986) 145–161.

[24] I. Kovacs, *Invertibility-preserving maps of C*-algebras with real rank zero*, Abstr. Appl. Anal. Volume 2005, Number **6** (2005) 685–689.

[25] L. Lebtahi, P. Patricio, N. Thome, *The diamond partial order in rings*, Lin. Mult. Alg., **62(3)** (2014), 386–395.

[26] Y.-F. Lin and M. Mathieu, *Jordan isomorphism of purely infinite $C^*$-algebras*, Quart. J. Math., **58**, 249-253 (2007).

[27] J. Marovt, D. S. Rakić, D. S. Djordjević, *Star, Left-star, and right-star partial orders in Rickart *-rings*, Lin. Mult. Alg., **63(2)** (2015) 343–365.

[28] L. W. Marcoux, *On the Linear Span of the Projections in Certain Simple $C^*$-algebras*, Indiana Univ. Math. J. **51**, 753–771 (2002).

[29] M. Mbekhta, *A Hua type theorems and linear preserver strongly problems*, Proc. Roy. Irish Acad. **109A** (2) (2009) 109–121.

[30] S. K. Mitra, *On Group Inverses and the Sharp Order*, Linear Algebra Appl. **92** (1987) 17–37.
[31] S.K. Mitra, *Matrix partial order through generalized inverses: unified theory*, Linear Algebra Appl. **148** (1991) 237–263.
[32] S. K. Mitra, P. Bhimasankaram, S. B. Malik, Matrix partiar orders, shorted operators and applications. Hackensack, NJ World Scientific Publishing Company (2010).
[33] T. Oikhberg, A. M. Peralta, M. Ramrez, *Automatic continuity of M-norms on C*-algebras*, J. Math. Anal. Appl. **381** (2011) 799–811.
[34] C. Pearcy and D. Topping, *Sums of small numbers of idempotents*, Michigan Math. J. **14**, 453–465 (1967).
[35] S. Sakai, *C*- and W*-algebras*, Springer-Verlag, Berlin-New York, 1971.
[36] P. Semrl, *Automorphisms of B(H) with respect to minus partial order*, J. Math. Anal. Appl. **369** (1) (2010) 205–213.
[37] M. Wolff, *Disjointness preserving operators in C.-algebras*, Arch. Math. 62 (1994) 248–253.

Campus de Jerez, Facultad de Ciencias Sociales y de la Comunicación Av. de la Universidad s/n, 11405 Jerez, Cádiz, Spain
*E-mail address*: `maria.burgos@uca.es`

Departamento Matemáticas, Universidad de Almería, 04120 Almería, Spain
*E-mail address*: `acmarquez@ual.es`

Departamento de Matemática e Aplicações & CMAT – Centro de Matemática, Universidade do Minho, Braga, Portugal
*E-mail address*: `pedro@math.uminho.pt`