Further results on the inverse along an element in semigroups and rings

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Abstract

In this paper, we introduce a new notion in a semigroup S as an extension of Mary's inverse. Let $a, d \in S$. An element a is called left (resp. right) invertible along d if there exists $b \in S$ such that $bad = d$ (resp. $dab = b$) and $b \leq_{\mathcal{L}} d$ (resp. $b \leq_{\mathcal{R}} d$). An existence criterion of this type inverse is derived. Moreover, several characterizations of left (right) regularity, left (right) π-regularity and left (right) $*$-regularity are given in a semigroup. Further, another existence criterion of this type inverse is given by means of a left (right) invertibility of certain elements in a ring. Finally we study the (left, right) inverse along a product in a ring, and, as an application, Mary's inverse along a matrix is expressed.

Keywords:
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1. Introduction

Throughout this paper, S is a semigroup. An element $a \in S$ is (von Neumann) regular if there exists x in S such that $axa = a$. Such x is called an inner inverse of a. By $a\{1\} = \{x \in S : axa = a\}$ we denote the set of all inner inverses of a. An arbitrary element in $a\{1\}$ is denoted by $a^{(1)}$.

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The element a is left (right) regular (see e.g. [2]) if there exists x such that $a = xa^2$ ($a = a^2x$), and strongly regular if it is both left regular and right regular. It is left (right) π-regular (see e.g. [2]) if there exists x such that $a^n = xa^{n+1}$ ($a^n = a^{n+1}x$) for a positive integer n. If a is both left and right π-regular, then a is strongly π-regular.

Let $*$ be an involution (anti-isomorphism of degree 2) on S, that is, the involution satisfies $(x^*)^* = x$ and $(xy)^* = y^*x^*$ for all $x, y \in S$. Let $a \in S$. We call a left (right) $*$-regular if there is x such that $a = aa^*ax$ ($a = xaa^*a$). A $*$-semigroup S is called left (right) $*$-regular if all elements in S are left (right) $*$-regular. If x satisfies $axa = a$ and $(ax)^* = ax$, then x is a $\{1,3\}$-inverse of a. If y satisfies $aya = a$ and $(ya)^* = ya$, then y is a $\{1,4\}$-inverse of a.

The standard notions of group, Drazin and Moore-Penrose inverse can be referred to the literature [4, 9]. Following [4], an element a is Drazin invertible if and only if it is strongly π-regular. In particular, a is group invertible if and only if it is strongly regular. It is well known that $a \in S$ is Moore-Penrose invertible if and only if $a \in aa^*S \cap Sa^*a$ if and only if it is both $\{1,3\}$ and $\{1,4\}$-invertible. All these inverses, if they exist, are unique. We denote by $a^{\#}$, a^D and $a^{\dagger}$ the group, Drazin and Moore-Penrose inverses of a, respectively.

Mary [6] recently defined a new generalized inverse in a semigroup S called the inverse along an element. Motivated by [6], we introduce in section 2 below a new notion. An existence criterion of this type inverse is derived. Moreover, several characterizations of left (right) regularity, left (right) π-regularity and left (right) $*$-regularity are given in a semigroup. Also, we prove that $a \in S$ is Moore-Penrose invertible if and only if it is left $*$-regular if and only if it is right $*$-regular. In section 3, another existence criterion of this type inverse is given by means of a left (right) invertibility of certain elements in a ring, and as an application, the formula of the inverse along a matrix is expressed.

2. One-sided inverse along an element in semigroups

Green's preorders in a semigroup [5] are defined as followed (S^1 denotes the monoid generated by S)

$a \leq_{\mathcal{L}} b \Leftrightarrow S^1a \subseteq S^1b \Leftrightarrow$ there exists $x \in S^1$ such that $a = xb$.

$a \leq_{\mathcal{R}} b \Leftrightarrow aS^1 \subseteq bS^1 \Leftrightarrow$ there exists $x \in S^1$ such that $a = bx$.

$a \leq_{\mathcal{H}} b \Leftrightarrow a \leq_{\mathcal{L}} b$ and $a \leq_{\mathcal{R}} b$.

We next introduce a notion that is based on Green's preorders in a semigroup.

Definition 2.1. *Let $a, d \in S$. An element a is left invertible along d if there exists $b \in S$ such that $bad = d$ and $b \leq_{\mathcal{L}} d$.*

Any b satisfying the conditions in Definition 2.1 is called a left inverse of a along d.

Definition 2.2. *Let $a, d \in S$. An element a is right invertible along d if there exists b such that $dab = d$ and $b \leq_{\mathcal{R}} d$.*

In [6], Mary defined a new generalized inverse in a semigroup as follows: An element b is an inverse of a along d if $bad = d = dab$ and $b \leq_{\mathcal{H}} d$. This type inverse is unique, if it exists and denoted by $a^{\|d}$. Mary showed in particular that $a^{\#}$, a^D and $a^{\dagger}$ are the inverses of a along a, a^n and a^* respectively ([6, Theorem 11]). In [3], Drazin introduced (b, c)-inverse in a semigroup. It follows that (d, d)-inverse of a is an inverse of a along d (Mary's inverse). Hence, group inverse, Drazin inverse, Moore-Penrose inverse and Mary's inverse of a are instances of left or right inverse of a along d.

Next, we present an existence criterion of a left inverse along an element.

Theorem 2.3. *Let $a, d \in S$. Then a is left invertible along d if and only if $d \leq_{\mathcal{L}} dad$.*

PROOF. "$\Rightarrow$" Suppose that a is left invertible along d. Then there exists b such that $bad = d$ and $b \leq_{\mathcal{L}} d$. From $b \leq_{\mathcal{L}} d$, it follows that $b = xd$ for some $x \in S^1$. Hence, $d = bad = xdad$, which implies $d \leq_{\mathcal{L}} dad$.

"$\Leftarrow$" $d \leq_{\mathcal{L}} dad$ implies $d = ydad$ for some $y \in S$. Take $b = yd$. Then $b \leq_{\mathcal{L}} d$ and $bad = d$. $\square$

Dually, we can obtain an equivalence for the existence of a right inverse along an element.

Theorem 2.4. *Let $a, d \in S$. Then a is right invertible along d if and only if $d \leq_{\mathcal{R}} dad$.*

Applying Theorems 2.3 and 2.4 and [7, Theorem 2.2], we get the following corollaries.

Corollary 2.5. *Let $a, d \in S$. Then a is invertible along d if and only if it is left and right invertible along d.*

Corollary 2.6. *Let d_l, d_r and d be such that $S^1 d_l = S^1 d$ and $d_r S^1 = dS^1$. Then a is invertible along d if and only if it is left invertible along d_l and right invertible along d_r.*

We consider now the relations between left invertibility along d and left invertibility, left regularity, left π-regularity and left $*$-regularity.

Theorem 2.7. *Let $a \in S$.*

(i) *If S is a monoid, then a is left invertible along 1 if and only it is left invertible.*

(ii) *a is left invertible along a if and only if it is left regular.*

(iii) *There exists $n \in \mathbb{N}$ such that a is left invertible along a^n if and only if it is left π-regular.*

(iv) *If S is a $*$-semigroup, then a is left invertible along a^* if and only if it is left $*$-regular.*

Proof. (i) Suppose that a is left invertible. Then there exists $b \in S$ such that $1 = ba$. Also, as $b = b * 1$, then $b \leq_{\mathcal{L}} 1$ and a is left invertible along 1.

Conversely, if a is left invertible along 1, then there exists $b \in S$ such that $ba = 1$ and a is left invertible.

(ii) Assume that a is left regular. Then exists b in S, $a = ba^2$ hence $a = b^2 a^3$ and $a \leq_{\mathcal{L}} a^3$. By Theorem 2.3, a is left invertible along a.

Conversely, if a is left invertible along a, then there is b in S such that $baa = a$ and a is left regular.

(iii) Let a be left π-regular. Then there exist b in S and an integer n such that $a^n = ba^{n+1}$, and by induction $a^n = b^2 a^{n+2} = \cdots = b^{n+1} a^{2n+1}$. Hence $a \leq_{\mathcal{L}} a^{2n+1}$ and a is left invertible along a^n by Theorem 2.3.

The converse part is straightforward.

(iv) Assume that a is left $*$-regular. Then there exists $x \in S$ such that $a = aa^*ax$ and hence $a^* = x^*a^*aa^*$, which implies that a is left invertible along a^* by Theorem 2.3.

Conversely, if a is left invertible along a^*, it follows from Theorem 2.3 that $a^* \leq_{\mathcal{L}} a^*aa^*$. Hence, $a = aa^*ay$ for some $y \in S$ and a is left $*$-regular. $\square$

Applying Theorems 2.3 and 2.7, we give some characterizations of left invertibility and left generalized invertibilities in the following corollary.

Corollary 2.8. *Let $a \in S$. Then*

(i) *If S is a monoid, a is left invertible if and only if $1 \leq_{\mathcal{L}} a$.*

(ii) *a is left regular if and only if $a \leq_{\mathcal{L}} a^3$.*

(iii) *a is left π-regular if and only if $a^m \leq_{\mathcal{L}} a^{2m+1}$, for a positive integer m.*

(iv) *If S is a $*$-semigroup, then a is left $*$-regular if and only if $a^* \leq_{\mathcal{L}} a^*aa^*$.*

Dually, we have the following result.

Theorem 2.9. *Let $a \in S$. Then*

(i) *If S is a monoid, a is right invertible along 1 if and only it is right invertible.*

(ii) *a is right invertible along a if and only if it is right regular.*

(iii) *a is right invertible along a^m if and only if it is right π-regular.*

(iv) *If S is a $*$-semigroup, then a is right invertible along a^* if and only if it is right $*$-regular.*

By Theorems 2.4 and 2.9, we have

Corollary 2.10. *Let $a \in S$. Then*

(i) *If S is a monoid, a is right invertible if and only if $1 \leq_{\mathcal{R}} a$.*

(ii) *a is right regular if and only if $a \leq_{\mathcal{R}} a^3$.*

(iii) *a is right π-regular if and only if $a^m \leq_{\mathcal{R}} a^{2m+1}$, for a positive integer m.*

(iv) *If S is a $*$-semigroup, then a is right $*$-regular if and only if $a^* \leq_{\mathcal{R}} a^*aa^*$.*

Remark 2.11. Let S be a non Dedekind finite ring with $ab = 1 \neq ba$. Then a is right invertible along a (a^n) by Theorem 2.4, but one can show that it is not left invertible along a (a^n). However, in a $*$-semigroup, we prove that every right $*$-regular element is left $*$-regular (see Theorem 2.16 below).

We present characterizations of $\{1,3\}$-inverse, $\{1,4\}$-inverse, left $*$-regularity and right $*$-regularity of an element in a $*$-semigroup with an identity element.

The conditions (i) and (ii) in Proposition 2.12 below were essentially proved in [11, Lemma 2.2] in a ring with involution case.

Proposition 2.12. *Let S be a $*$-semigroup and let $a \in S^1$. Then*

(i) *a has a $\{1,3\}$-inverse if and only if $S^1a = S^1a^*a$.*

(ii) *a has a $\{1,4\}$-inverse if and only if $aS^1 = aa^*S^1$.*

(iii) *a is left $*$-regular if and only if $aS^1 = aa^*aS^1$.*

(iv) *a is right $*$-regular if and only if $S^1a = S^1aa^*a$.*

Remark 2.13. Proposition 2.12 does not hold in the case that there is no identity element. Indeed, let S be a null semigroup ($xy = 0$, $\forall x, y \in S$) distinct from $\{0\}$. Then 0 is the only von Neumann regular element but $(\forall a \in S)$ $Sa = 0 = Saa^*a = Sa^*a$ for instance.

Remark 2.14. If a is left $*$-regular, then a has a $\{1,4\}$-inverse by Proposition 2.12. But the converse does not necessarily hold. Let $S = M_2(\mathbb{C})$ and the involution is the transpose. Take $A = \left(\begin{smallmatrix} 1 & 0 \\ i & 0 \end{smallmatrix}\right)$ and $A^* = \left(\begin{smallmatrix} 1 & i \\ 0 & 0 \end{smallmatrix}\right)$. Then $AA^*S = AS$, which implies that A is $\{1,4\}$-invertible. However $AA^*AS = 0$. So, A is not left $*$-regular.

Now, we construct a $*$-semigroup to illustrate various relations in Proposition 2.12.

Example 2.15. Let $A = \{1, 2, 3\}$. Then every map from A to A can be written as $\begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix}$, where $i, j, k \in A$. If S is a semigroup generated by $x = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 3 \end{pmatrix}$ and $y = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix}$, then $S = \{x, x^2, y, xy, yx\}$. Set $x^* = x$, $(x^2)^* = x^2$, $y^* = y$, $(xy)^* = yx$ and $(yx)^* = xy$, then $*$ is an involution on S. Moreover, we get

(i) x is regular but neither $\{1,3\}$ nor $\{1,4\}$-invertible.

(ii) y and x^2 are projectors and hence Moore-Penrose invertible.

(iii) xy is $\{1,4\}$-invertible but neither $\{1,3\}$-invertible nor left $*$-regular.

(iv) yx is $\{1,3\}$-invertible but neither $\{1,4\}$-invertible nor right $*$-regular.

Theorem 2.16. *Let S be a $*$-semigroup and let $a \in S$. Then the following conditions are equivalent*:

(i) *a is Moore-Penrose invertible.*

(ii) *a is left $*$-regular.*

(iii) *a is right $*$-regular.*

Proof. (i)⇒(ii) Let $a^\dagger$ be the Moore-Penrose inverse of a. Then $a = a(a^\dagger a)^* = aa^*(a^\dagger aa^\dagger)^* = aa^*aa^\dagger(a^\dagger)^*$ and hence a is left $*$-regular.

(ii)⇔(iii) Assume that a is left $*$-regular. There exists $x \in S$ such that $a = aa^*ax$ and hence $a^* = x^*a^*aa^*$. Since $(ax)^*a = (ax)^*aa^*(ax)$, it follows that $(ax)^*a = [(ax)^*a]^* = a^*ax$. Hence, we have $a = aa^*ax = a(ax)^*a = ax^*a^*a = (ax^*x^*a^*)aa^*a$. So, a is right $*$-regular.

The converse part follows by a similar way.

(iii)⇒(i) Let a be right $*$-regular and hence left $*$-regular. We have $a \in aa^*S \cap Sa^*a$. Thus, a is Moore-Penrose invertible. $\square$

Recall that a semigroup S is called $*$-regular if all elements in S are Moore-Penrose invertible. Hence, we get

Corollary 2.17. *Let S be a $*$-semigroup. Then S is $*$-regular if and only every element in S is left (right) $*$-regular.*

The following lemma was given by Penrose in complex matrices (see [9, p. 407]), it indeed holds in a $*$-semigroup.

Lemma 2.18. *Let S be a $*$-semigroup and let $a \in S$. If $axa = a = aya$, $(ax)^* = ax$ and $(ya)^* = ya$ for some $x, y \in S$. Then a is Moore-Penrose invertible and $a^\dagger = yax$.*

We now present the formula of the Moore-Penrose inverse of a left (right) $*$-regular element.

Theorem 2.19. *Let S be a $*$-semigroup and let $a \in S$. If $a = aa^*ax$ for some $x \in S$, then a is Moore-Penrose invertible and $a^\dagger = a^*ax^2a^*$.*

Proof. If $a = aa^*ax$, then $(ax)^*$ is a $\{1,4\}$-inverse of a according to [11, Lemma 2.2]. By the proof (ii)⇔(iii) in Theorem 2.16, it is known that $a = (ax^*x^*a^*a)a^*a$, and $(ax^*x^*a^*a)^*$ is a $\{1,3\}$-inverse of a. By virtue of Lemma 2.18, it follows that a is Moore-Penrose invertible and $a^\dagger = (ax)^*a(ax^*x^*a^*a)^* = a^*ax^2a^*$. $\square$

Dually, we have the following result.

Theorem 2.20. *Let S be a $*$-semigroup and let $a \in S$. If $a = yaa^*a$ for some $y \in S$, then a is Moore-Penrose invertible and $a^\dagger = a^*y^2aa^*$.*

We then recover and improve some known characterizations of generalized invertibility in a semigroup.

Corollary 2.21. [6, Theorem 11] *Let $a \in S$. Then*

(i) *a is invertible if and only if it is invertible along 1. In this case, $a^{-1} = a^{\|1}$.*

(ii) *a is group invertible if and only if it is invertible along a. In this case, $a^{\#} = a^{\|a}$.*

(iii) *a is Drazin invertible if and only if there exists an integer n, a is invertible along a^n. In this case, $a^D = a^{\|a^n}$.*

(iv) *a is Moore-Penrose invertible if and only if it is left (right) invertible along a^*. In this case, $a^\dagger = a^{\|a^*}$.*

3. One-sided inverse along a product in rings

In this section, we present equivalent conditions for the existence of one-sided inverse along a product in a ring R. In what follows, R is always an associative ring with unity 1.

First, we begin with a well-known lemma.

Lemma 3.1. *Let $a, b, c \in R$.*

(i) *If $(1+ab)c = 1$, then $(1+ba)(1-bca) = 1$.*

(ii) *If $c(1+ab) = 1$, then $(1-bca)(1+ba) = 1$.*

It follows from Lemma 3.1 that $1+ab$ is (left, right) invertible if and only if $1+ba$ is (left, right) invertible and $(1+ba)^{-1} = 1 - b(1+ab)^{-1}a$. This result is known as Jacobson's Lemma.

Let $a \in R$. By a_l^{-1} and a_r^{-1} we denote a left inverse and a right inverse of a, respectively. Next, we present an existence criterion of a left inverse along a product by means of one-sided invertibility of certain elements.

Theorem 3.2. *Let $p, a, q, m \in R$ with m regular. If $m \leq_{\mathcal{L}} pm$ and $m \leq_{\mathcal{R}} mq$, then the following conditions are equivalent:*

(i) *a is left invertible along pmq.*

(ii) *$u = mqap + 1 - mm^{(1)}$ is left invertible.*

(iii) *$v = qapm + 1 - m^{(1)}m$ is left invertible.*

In this case, $pu_l^{-1}mq$ is a left inverse of a along pmq.

Proof. It follows from Lemma 3.1 that (ii)$\Leftrightarrow$(iii).

(i)$\Rightarrow$(ii) Suppose that a is left invertible along pmq. From Theorem 2.3, we get $pmq \leq_{\mathcal{L}} pmqapmq$. Hence, there exists $x \in R$ such that

$$pmq = xpmqapmq. \qquad (*)$$

By $m \leq_{\mathcal{R}} mq$, there exists $q' \in R$ such that $m = mqq'$. Similarly, $m \leq_{\mathcal{L}} pm$ guarantees that $m = p'pm$ for some $p' \in R$. Multiplying the equation $(*)$ by q' on the right yields $pm = xpmqapm$. Set $y = mm^{(1)}p'xpmm^{(1)} + 1 - mm^{(1)}$, we obtain $y(mqapmm^{(1)} + 1 - mm^{(1)}) = 1$. Indeed, we have

$$\begin{aligned}
& y(mqapmm^{(1)} + 1 - mm^{(1)}) \\
= \ & (mm^{(1)}p'xpmm^{(1)} + 1 - mm^{(1)})(mqapmm^{(1)} + 1 - mm^{(1)}) \\
= \ & mm^{(1)}p'xpmqapmm^{(1)} + 1 - mm^{(1)} \\
= \ & mm^{(1)}p'pmm^{(1)} + 1 - mm^{(1)} \\
= \ & mm^{(1)} + 1 - mm^{(1)} \\
= \ & 1.
\end{aligned}$$

Consequently, $mqapmm^{(1)} + 1 - mm^{(1)}$ is left invertible. Again, Lemma 3.1 ensures that $mqap + 1 - mm^{(1)}$ is left invertible.

(ii)$\Rightarrow$(i) Suppose now that u is left invertible. Then there is u' such that $u'u = 1$. Since $um = mqapm$, it follows that $m = u'mqapm$. Also, by $m \leq_{\mathcal{L}} pm$, there exists $p' \in R$ such that $p'pm = m$ and hence $pmq = pu'mqapmq = pu'p'pmqapmq$. Take $b = pu'p'pmq$, then $b \leq_{\mathcal{L}} pmq$, that is, a is left invertible along pmq.

Hence, $b = pu_l^{-1}mq$ is a left inverse of a along pmq. □

As a special corollary of Theorem 3.2, we get

Corollary 3.3. *Let $a, m \in R$ with m regular. Then the following conditions are equivalent*:

(i) *a is left invertible along m.*

(ii) *$u = ma + 1 - mm^{(1)}$ is left invertible.*

(iii) *$v = am + 1 - m^{(1)}m$ is left invertible.*

In this case, $u_l^{-1}m$ is a left inverse of a along m.

Dually, we have

Theorem 3.4. *Let $p, a, q, m \in R$ with m regular. If $m \leq_{\mathcal{L}} pm$ and $m \leq_{\mathcal{R}} mq$, then the following conditions are equivalent*:

(i) *a is right invertible along pmq.*

(ii) *$u = mqap + 1 - mm^{(1)}$ is right invertible.*

(iii) *$v = qapm + 1 - m^{(1)}m$ is right invertible.*

In this case, $pmv_r^{-1}q$ is a right inverse of a along pmq.

Corollary 3.5. *Let $a, m \in R$ with m regular. Then the following conditions are equivalent*:

(i) *a is right invertible along m.*

(ii) *$u = ma + 1 - mm^{(1)}$ is right invertible.*

(iii) *$v = am + 1 - m^{(1)}m$ is right invertible.*

In this case, mv_r^{-1} is a right inverse of a along m.

An involution $*$ in a ring R is an anti-isomorphism of degree 2 which satisfies $(a^*)^* = a$, $(ab)^* = b^*a^*$ and $(a+b)^* = a^* + b^*$, for all $a, b \in R$.

Let S be a ring with involution in Theorem 2.16. We have

Corollary 3.6. *Let R be a ring with involution and let $a \in R$. Then*

(i) *a is left $*$-regular if and only if it is right $*$-regular.*

(ii) *R is $*$-regular if and only if every element in R is left (right) $*$-regular.*

Recall that a ring R is called strongly π-regular if each element $a \in R$ is left (right) π-regular (see e.g. [1]). In particular, R is called strongly regular if each element $a \in R$ is left (right) regular. We next give new characterizations of strongly (π-) regular rings, $*$-regular rings, by one-sided invertibility along an element.

Corollary 3.7. *Let $a \in R$. Then*

(i) *R is a strongly regular ring if and only if every element a is left (right) invertible along a.*

(ii) *R is a strongly π-regular ring if and only if every element a is left (right) invertible along a^n for some positive n.*

(iii) *R is a $*$-regular ring if and only if every element a is left (right) invertible along a^*.*

We have seen that a is both left and right invertible along pmq if and only if it is invertible along pmq. Moreover, the inverse of a along pmq is unique (Corollary 2.5). Hence we have

Corollary 3.8. ([10, Theorem 2.2] *Let $p, a, q, m \in R$ with m regular. If $m \leq_{\mathcal{L}} pm$ and $m \leq_{\mathcal{R}} mq$, then the following conditions are equivalent*:

(i) $a^{\|pmq}$ *exists.*

(ii) $u = mqap + 1 - mm^{(1)}$ *is invertible.*

(iii) $v = qapm + 1 - m^{(1)}m$ *is invertible.*

In this case,

$$a^{\|pmq} = pu^{-1}mq = pmv^{-1}q.$$

By taking $p = q = 1$ we get

Corollary 3.9. ([7, Theorem 3.2] and [8, Theorem 1.3]) *Let $m \in R$ be regular. Then the following are equivalent*:

(i) a *is invertible along* m.

(ii) $u = ma + 1 - mm^{(1)}$ *is invertible.*

(iii) $v = am + 1 - m^{(1)}m$ *is invertible.*

In this case,

$$a^{\|m} = u^{-1}m = mv^{-1}.$$

We finally give some applications of the inverse along a product by its existence criterion. More results on the inverse along a matrix can be referred to references [8, 10]. By the symbol $R_{2\times 2}$ we denote the ring of 2×2 matrices over a ring R.

Let $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, $D = \begin{bmatrix} d_1 & d_3 \\ d_2 & d_4 \end{bmatrix} \in R_{2\times 2}$ with D regular and assume that d_4 in matrix D is invertible. Then we have the following decomposition

$$D = \begin{bmatrix} d_1 & d_3 \\ d_2 & d_4 \end{bmatrix} = \begin{bmatrix} 1 & d_3 d_4^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s & 0 \\ 0 & d_4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ d_4^{-1} d_2 & 1 \end{bmatrix} =: PMQ,$$

where $s = d_1 - d_3 d_4^{-1} d_2$ is the Schur complement of d_4 in the matrix D. It is well known that D is regular if and only if M is regular. Similarly, if d_1 is invertible, $d_4 - d_2 d_1^{-1} d_3$ is called the Schur complement of d_1 in the matrix D.

According to Corollary 3.8, it is known that $A^{\|D}$ exists if and only if $U = MQAP + I - MM^{(1)}$ is invertible. One can get $I - MM^{(1)} = \begin{bmatrix} 1 - ss^{(1)} & 0 \\ 0 & 0 \end{bmatrix}$ by a direct calculation.

We also get that $MQAP = \begin{bmatrix} sa & \alpha \\ d_2a + d_4b & \beta \end{bmatrix}$, where

$$\begin{aligned} \alpha &= s(ad_3d_4^{-1} + c), \\ \beta &= (d_2a + d_4b)d_3d_4^{-1} + d_2c + d_4d. \end{aligned}$$

Hence, it follows that $U = \begin{bmatrix} u & \alpha \\ d_2a + d_4b & \beta \end{bmatrix}$, where $u = sa + 1 - ss^{(1)}$.

If $a^{\|s}$ exists, applying Corollary 3.9, it follows that u is invertible.

Using the Schur complement, we have

$$U = \begin{bmatrix} u & \alpha \\ d_2a + d_4b & \beta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ (d_2a + d_4b)u^{-1} & 1 \end{bmatrix} \begin{bmatrix} u & 0 \\ 0 & \xi \end{bmatrix} \begin{bmatrix} 1 & u^{-1}\alpha \\ 0 & 1 \end{bmatrix},$$

where $\xi = \beta - (d_2a + d_4b)a^{\|s}(ad_3d_4^{-1} + c)$. Moreover, U is invertible if and only if ξ is invertible.

In this case,

$$U^{-1} = \begin{bmatrix} 1 & -u^{-1}\alpha \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u^{-1} & 0 \\ 0 & \xi^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -(d_2a + d_4b)u^{-1} & 1 \end{bmatrix}.$$

Thus, $A^{\|D}$ exists if and only if $\xi = \beta - (d_2a + d_4b)a^{\|s}(ad_3d_4^{-1} + c)$ is invertible. Moreover, we get

$$A^{\|D} = PU^{-1}MQ = \begin{bmatrix} x_1s + x_3d_2 & x_3d_4 \\ x_2s + \xi^{-1}d_2 & \xi^{-1}d_4 \end{bmatrix}, \text{ where}$$

$$\begin{aligned} x_1 &= u^{-1} + (u^{-1}\alpha - d_3d_4^{-1})\xi^{-1}(d_2a + d_4b)u^{-1}, \\ x_2 &= -\xi^{-1}(d_2a + d_4b)u^{-1}, \\ x_3 &= d_3d_4\xi^{-1} - u^{-1}\alpha\xi^{-1}. \end{aligned}$$

Remark 3.10. Even if $a^{\|s}$ does not exist, $A^{\|D}$ may exist. For instance, take $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $D = \begin{bmatrix} d_1 & d_3 \\ d_2 & d_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in R_{2\times 2}$. Since $s = d_1 - d_3d_4^{-1}d_2 = 1$, it follows that $sa + 1 - ss^{(1)} = 0$. Hence, $a^{\|s}$ does not exist by Corollary 3.9. However, A is invertible along D since they are both invertible.

We close this section with some further remarks:

(i) In Theorem 3.2, since $v_l^{-1}(1 + (qap - m^{(1)})m) = 1$, it follows that $1 - mv_l^{-1}(qap - m^{(1)})$ is a left inverse of u by Lemma 3.1. Hence, we can give the representation of a left inverse of a along pmq by v_l^{-1}.

(ii) We give another proof for Corollary 3.6(i). Assume that a is left $*$-regular (we have $a = aa^*ax$ for some $x \in R$). Then it is left invertible along a^* according to Theorem 2.7. Moreover, a is regular, and $(ax)^*$ is an inner inverse (indeed a $\{1, 4\}$-inverse) of a. Indeed, it follows that $[(ax)^*a]^* = a^*ax = (ax)^*a$ and $a(ax)^*a = aa^*ax = a$ since $a^*ax = (aa^*ax)^*ax = (ax)^*aa^*ax = (ax)^*a$. By Corollary 3.3, $u = a^*a + 1 - a^*(a^*)^{(1)} = a^*a + 1 - (a^{(1)}a)^*$ is left invertible. Hence, we can pick an inner inverse $(ax)^*$ of a such that $a^{(1)}a$ is symmetric. Then $u = u^*$ is right invertible, and by Corollary 3.5, it follows that a is right invertible along a^*.

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