Prospective Primary School Teachers' Knowledge of the Ratio Concept

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Abstract: Prospective primary school teachers (PPST) learn about some mathematics concepts in several courses besides the mathematics ones. This happens with the ratio concept which is a cross subject and instrumental concept. Research has shown that this concept is quite hard to master event though it is often used in school as well as in everyday life. This research aims at investigating PPST's knowledge on the ratio concept, namely with regard to their ability to interpret and compare ratios in two different contexts. Data were collected from 50 PPST attending a Portuguese university by means of a questionnaire. Participants were asked to answer to two questions that involve the ratio concept: one of them deals with a pizza division and requires a comparison of homogeneous quantities; the other one deals with the speed concept and involves a comparison of heterogeneous quantities. Both questions require information from a graph to be picked up. Data analysis showed that, in the pizza question, participants in the study tend to use numerical representations under the format of a fraction, which led them to do correct comparison between two ratios. In the case of the speed question, PPST showed more difficulties which seem to have been caused by the physical meaning of speed. Thus, the results suggest that most of these PPST hold a limited and rigid knowledge of the ratio concept that may be due to learning process based on numerical representations and carried out within mathematics courses. An implication of this is that teacher educators need to find ways of developing PPST's cross subject knowledge of the ratio concept so that they can be better prepared to teach this concept to young children embedded into cross disciplinary everyday life contexts.

Keywords: Prospective primary school teachers, ratio comparisons, ratio representations

Introduction

After the implementation of the Bologna process, primary school teacher education in Portugal is a two-step process, including a 180 ECTS three yearlong undergraduate programme (*Licenciatura* in Basic Education) followed by 120 ECTS two yearlong master programmes. These master's curricula depend on whether prospective teacher qualify to teach up to the 4th grade or up to the 6th grade (with a specialization either on science and mathematics or on Portuguese language, history and geography). The undergraduate programme provides teacher candidates training on the diverse subjects they will teach in the future (Portuguese, mathematics, natural and social sciences, arts and physical education) as well as on education. The master programmes provide further training on education and on the subjects prospective teachers prepare to teach, but they concentrate especially on subject specific methods courses and on teaching practice.

Thus, prospective primary school teachers (PPST) learn mathematics (at least 25 ECTS, as dictated by the Portuguese Law) in their undergraduate programme. This encompasses all the mathematics knowledge they would formally acquire to teach the mathematics component to grade 1 to 4 students. As matter of fact, only those PPST that choose a science and mathematics master specialization (that would enable them to teach science and Mathematics to grades 5 and 6) will learn some more mathematics (10ECTS).

Taken together, the undergraduate programme and the master's programme cover all the knowledge components that Shulman (1986) has highlighted as being necessary for a teacher to teach effectively. These components are as follows: content knowledge; pedagogical content knowledge; curricular knowledge and pedagogical knowledge. This paper focuses on content knowledge, more specifically on mathematics knowledge.

According to Shulman (1986), content knowledge includes not only the concepts and principles of the discipline but also its substantive and syntactic structures. The substantive structure refers to the "variety of ways in which the basic concepts and principles of the discipline are organized and incorporate facts." (Shulman 1986, 9). The syntactic structure "is the set of rules in which truth and falsehood, validity or invalidity, are established. […] is the set of rules for determining what is legitimate to say in a disciplinary domain and what breaks the rules." (Shulman 1986, 9). A consequence of this is that, according to Shulman (1986), it is not enough for a teacher to know the concepts and the facts to be taught, that is to know *that* something is so; the teacher must understand *why* it is so in order to be able to support assertions and to separate what is really important from what is peripheral in the discipline.

Teachers' content knowledge interfere not only with the learning tasks that they design for their students but also with the way teachers use those tasks in the classroom and also with why and when they decide to use them (Ball et al 2008; Osana et al 2006; Shulman 1986).

Ball and others (1986) differentiate between two types of mathematics content knowledge: the common one and the specialized one. The former has to do with mathematics knowledge that anyone with a formal background in mathematics holds; the latter has to do with the understanding of the procedures and language relative to a given mathematics concept. This type of content knowledge distinguishes the mathematics teacher from another person with a mathematics background and it enables teachers to use and teach appropriate representations of mathematics concepts.

Objective of the Research

The ratio concept is a multifaceted mathematics concept that relates to other mathematics concepts and that is used in other subjects, namely in science. The ratio concept dependency on other concepts and the variety of conceptions and representations that may be associated with it may require teachers to hold a good level of specialized content knowledge if they are expected to appropriately teach this concept.

Thus, the aim of this piece of research was find out how PPST deal with problem situations involving ratio representations and comparisons.

It draws on and adds to previous research dealing with PPST's understanding of ratio as it compares how PPST perform in different problem situations which is an issue that has not yet been tackled.

Theoretical Background

Representations of Mathematics Concepts

The representations of mathematical concepts have concentrated researchers' attention for a long time. Lesh, Post and Behr (1987) state that representations are related to the internal assimilation of mathematics ideas, and therefore they have to do with the mental reproduction of a concept, and to the representation of images, symbols and signals associated with it. Thus, representations may be pictures and diagrams, spoken and written language, manipulative models, and real world situations (Lesh, Post, & Behr 1987), an image or a concrete object (Elia 2004). that is, a representation "is a configuration that may represent something in a certain manner." (Goldin 2002, 208).

Matos and Serrazina (1996) showed that the modes of representation of mathematics concepts relate to Bruner's levels of representation in such a way that: the manipulative mode relates to the enactive level which requires the use of direct experience; the figures relate to the iconic level that has to do with visual tools; the language symbols relate to the symbolic level which is a central one in mathematics.

Several authors (e.g., Dreyfus 2002; Goldin, & Kaput 1996; Janvier, Girardon, & Morand

1993) differentiate internal from external representations. On one hand, the external representations are ways of personalization of ideas and concepts that use written or spoken language and that aim at making the communication about the concept easier (Dreyfus 2002). Maps, tables, graphs, diagrams, models, and formal symbol systems are examples of external representations. According to Janvier, Girardon and Morand (1993), the external representations act as the sense stimulus and they are often conceptualized as ideas and concepts agglomerates. Friedlander and Tabach (2001) distinguish four ways of doing external representations that they believe are at the core of mathematics: (i) verbal representations; (ii) numeric representations; (iii) graphic representations; (iv) algebraic representations. On the other hand, internal representations are the cognitive constructions that are formed in an individual's mind (Goldin, & Kaput 1996). They are often named as mental images and have to do with internal schemes or frameworks through which a person interacts with the external world.

Duval (2003) has classified the representations as mental, internal or computational, and semiotic. Mental representations are the result of the internalization of external representations and therefore they have to do with the set of images and conceptions that an individual holds about an object, a situation or something that is related with the object or the situation. The internal or computational representations have to do with codification of information relative to the automatic performance of a task in such a way as to produce a response that matches the situation. The semiotic representations are external and conscious and they have to do with the use of symbols that belong to a representational system. Geometric figures, a statement in a mother tongue, an algebraic formula or a graph are examples of semiotic representations associated with different semiotic systems. The mental access to mathematics objects requires semiotics representations that are external and conscious from the individual's point of view.

According to Duval (2003), semiotics representations involve three types of cognitive activities: (i) the formation of an identifiable representation; (ii) the treatment that is a transformation that takes place within a given registry; (iii) the conversion that is the transformation of the representation of a mathematics object in another representation of the same object.

The use of several representations of a given concept, object or situation facilitates the transition from a concrete and limited understanding to a more flexible and abstract one (Dreyfus 2002). Lesh, Post and Behr (1987) stated that the understanding of a mathematics idea depends partly on: "(1) the capacity to recognize it when it is absorbed in a variety of different representational systems; (2) the capacity to manipulate the idea in a flexible way through representational systems; and (3) the translation of that idea from one system to the other." (p.36).

In order to take most profit from the different representations, Dreyfus (2002) argued for the complementarity of the processes of abstraction and representation and added that they should develop in four stages as follows: (i) use of a single representation; (ii) use of more than one representation; (iii) establishment of connexions between representations; and (iv) integration of representations and establishment of flexible relationships among them. This way of conceiving the formation of mathematics concepts from multiple representations aims facilitating the attainment of an ever increasing abstract conception of the mathematics concepts. The process that underpins the links among representations is the translation which means that a change from the formulation of a mathematics relationship to another one is taking place.

The Ratio Concept

The ratio concept, taken as a comparison between quantities, is a multifaceted concept that is related with several other mathematics concepts, including rational number, proportionality and similarity. Besides, there are several conceptions of ratio and consequently there is no consensual way of defining it.

The ratio concept has connections with the rational number concept and Lamon (2007) includes ratio in one of his constructs. On one hand, the part-whole interpretation of rational numbers has to do with measure, and operator. Going through this type of interpretation processes facilitates the development of an understanding of measurement units and equivalent fractions. On the other hand, the quotient interpretation of rational numbers has to do with ratio and rates which are necessary to compare and to sum and subtract fractions. Besides, the operator interpretation has to do with multiplication and division of fractions.

Lamon (2007) assumes that a ratio is a comparison between two quantities and he distinguishes between internal or homogeneous ratios and external or heterogeneous ratios. The former involve comparison between quantities of the same magnitude (variable); the latter involve comparisons between quantities of different magnitudes (variables). Besides, According to Suggate, Davis and Goulding (2006) there are three types of ratio comparisons: part-part (e.g., Joseph eats two parts of the cake and Maria eats three parts of it); part-whole (e.g., Joseph has eaten two of the three parts of a cake); and whole-whole (e.g.; 1m in the map corresponds to 1 000 000m in reality). However, Viana and Miranda (2016) found out that when students are asked to compare ratios, they only use the equivalence of fractions (by reducing to the same denominator) or the transformation of a ratio into a decimal number or a percentage.

Besides, there is some empirical evidence that textbooks are sparse to support teaching of ratio in the primary school (Stafford, Oldham, & O'Dowd 2015). These results my at least in part be responsible for the fact that primary school teachers show a limited domain of mathematics' content knowledge an do not perceive the difference between fraction and ratio and proportional reasoning (Livy, & Vale 2011). However, it is worth noting that this lack of ratio content knowledge may also due to the fact that initial teacher education does not include the formal study of the ratio concept (Stafford, Oldham, & O'Dowd 2015) in some countries, including Portugal.

Berenson and others (2013) investigated PPST's and prospective mathematics and science teachers' conceptions and representations of the ratio concept in order to infer how prepared they were to teach it to their future students. The authors found out that whatever the country and grade level teachers were qualifying to teach, the participants in the study defined ratio as a comparison/relationship or as a fraction/percentage/proportion/division. As far as the representations of ratio through mathematical symbols are concerned, they used column (:), either isolated or integrated in expressions like: X:Y or 3:2, and fractions. In what concerns representations about how fractions are used, they made drawings, diagrams or other pictorial representations showing comparisons or numerical representations (usually in everyday settings) or geometric properties (e.g., similarity) or statistical representations (e.g., bar graphs); very few participants in the study showed drawings (e.g., map scales in architecture or design settings) illustrating applications of ratio. Finally, other prospective teachers showed numerical, algebraic or verbal representations that were similar to those given in mathematics symbols.

Price (2014) also collected data through the same questionnaire with American elementary and secondary school teachers and she concluded that even though the results were similar to those reported by Berenson and others (2013), the elementary school teachers showed more diversity in their answers that the secondary school teachers did probably because the latter's training focused on a narrower content knowledge area and therefore they had more limited choices for their answers.

Data collected from prospective kindergarten teachers and PPST teachers (Fernandes, & Leite 2015) through the same questionnaire mentioned in Berenson and others (2013) showed that subjects conceptualize ratio as a comparison or a relationship between magnitudes or as a mathematical operation but they do not make it explicit the type of comparison involved in a ratio. Besides, they hardly relate it to rational numbers. Also, even though they used mathematical symbols to represent ratios, the majority used operations with letters or with constants or even the operation signals only. However, when they were asked to describe how they would explain ratios, they still used mathematical symbols but diagrams and graphs were the most used types of representations.

A follow up study, with an improved version of the questionnaire, mainly for language issues, was carried out with Irish prospective mathematics teachers (Oldham, & Shuilleabhain 2014). Even though the authors got more extensive responses, the results were also similar to the ones obtained by Berenson and others (2013). Afterwards, another research study (Oldham, Stafford, & O'Dowd 2015) focusing on primary school Irish prospective teachers highlighted their lack of readily available mathematics content knowledge to answer to the ratio questions, probably because ratio has not given enough attention in the teacher education the curricula.

Ilany, Keret and Bem-Chaim (2004) concluded that the use of investigative activities that include tasks focused on settings that are familiar to prospective Mathematics teachers and that require the use and the relationship of concepts that are relevant for a good understanding of the ratio concept may promote prospective teachers content knowledge as well as pedagogic content knowledge. Oldham and Shuilleabhain (2013) have pointed out that ratio has been largely accepted as intuitively understood by students and teachers alike. However, the results of the research reviewed show that this is not the case. Rather, research involving PPST reveals conceptual problems that emerge independently of curricular and pedagogical traditions, while also pointing to approaches reflected in responses from one country that may be helpful in another (Oldham, Stafford, & O'Dowd 2015). Thus, further understanding of prospective teachers' conceptual difficulties with the ratio concept is necessary if action is to be taken by teacher educators to improve prospective teachers' content knowledge on the ratio concept.

Methodology

To attain the objective of this study, 81 PPST attending the 6th (that is the last) semester of a three yearlong undergraduate studies programme (*Licenciatura* in Basic education) in a University of the north of Portugal answered to a questionnaire on ratio. The subjects' age range was 20 to 34 years old with an average of 22,0 years old. All of them had studied a few mathematics courses at university. However, some of them had taken Mathematics in secondary school while other had not. It is worth mentioning that they had not studied the ratio concept at university and had no teaching experience. This means that none of them had experience of explaining the ratio concept to someone else.

In the literature it is possible to find some questionnaires on ratio (e.g.: Berenson et al 2013; Oldham, & Shuilleabhain 2014) but they were not appropriate for the purpose of this study as they were not focused on subjects' performance in different ratio situations. Thus, two problem situations dealing with representations and comparisons of ratio were designed: the pizza question, involving part-whole comparisons and homogeneous magnitudes; the speed question, dealing with part-part comparisons and heterogeneous magnitudes. The Portuguese word for ratio (that is *razão*) was not used in order to avoid language troubles as reported in Berenson and others (2013). The problem situations were reviewed by science education and mathematics education specialists in order to avoid technical language inaccuracies or some other type of mistake and also to check their consistency with the objective of the research. The versions of the problems that were used in this study are given in figures 1 and 2, after translation from Portuguese to English.

The *Licenciatura* in Basics Education programme director gave permission to the authors to carry out this research. Afterwards, PPST were invited by one of the authors, in a face to face basis, to participate in the study during a class time. They were informed about the objective of the study and the anonymous character of the questionnaire as well as that they would be allowed to withdraw at any time even after initiating the process of answering to the questionnaire. All of them volunteered to participate. Data were collected under exam conditions by one of the authors who was teaching them the final mathematics course of their undergraduate studies. This have may lead them to engage more deeply into the problem solving process with the aim of responding to the questionnaire.

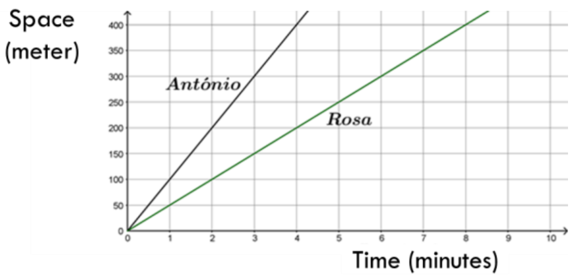
João and Maria bought a certain amount of pizza each one, as shown in the picture. The two pizzas are equal and so are the parts in which each pizza is divided.

Give other representations (as much as you can) of the amounts of pizza that João and Maria have bought. Explain those representations.

Who bought the largest amount of pizza, João or Maria? Explain your answer.

Figure 1: The Pizza Problem

António and Rosa went out for a walk. Their movement is shown in the following graph:



How long did it take for António to walk 250 meters?

Figure 2: The speed problem

Data were content analysed based on a set of categories emerging from answers obtained in each question. Afterwards, absolute frequencies and percentages were computed for each category. To promote data reliability, content analysis was done by two of the authors, separately. Discrepant cases were discussed with a third author and final classification was done.

Findings

Results Relative to the Pizza Problem

PPST were asked to give other representations (different from the one given in figure 1) of the amounts of pizza that João and Maria have bought and to explain those representations. Table 1 shows that they gave three general types of representations (diagrammatic, numeric, and numeric line representations), with some specific subtypes of representations in the diagrammatic and the numeric types. The types of representations obtained are three of the four types of external representations that Friedlander and Tabach (2001) consider as essential in mathematics.

Table 1: Types of other representations given by the PPST

(N=81)

Types of representation of Ratio		**f**	**%**
Diagrammatic representation	Continuous model	61	75
	Discrete model	6	7
Numerical representation	Fraction	76	94
	Decimal	24	30
	Percentage	8	10
Numerical line		3	4
Do not answer		1	1

The numeric representation was the one given by larger number of PPST. Almost all students (94%) used the fraction representation subtype and some of them also used the decimal (30%) and the percentage (10%) ways of representing a ratio. Most of the students that used decimal and percentage subtypes of representation started by doing a fraction representation and afterwards they transformed the fraction into the other subtypes of representation, through operative processes. According to Lesh, Post and Behr (1987), the ability to translate an idea from one system of representation to the other provides evidence of understanding of the idea that is at stake. In fact, the fraction could be obtained directly from reading the representation provided in the problem but the others needed to be computed and this requires understanding. Figure 3 shows an example of an answer that uses these three subtypes of numeric representation and even adds a diagrammatic representation which is different from the one given in the pizza problem (see figure 1). However, attention was given to the number of parts only and not to the length of the rectangles (which is different in the two cases).

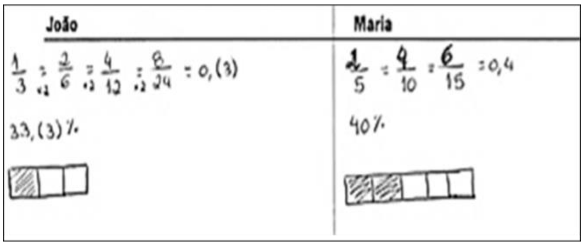
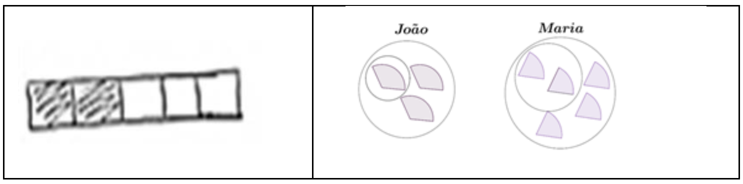


Figure 3: representations of ratio done by A14

As shown by figures given in table 1, large percentages of students gave diagrammatic representations, being the continuous (75%) subtype of representation much more frequent than the discrete one (7%). The larger use of the continuous representations may be due to the fact that it is usually used in classes to work with fractions and therefore it was probably very familiar to PPST. Figure 4 shows an example of a continuous (a) and a discrete representation (b).



(a) – A 14 (b)- A ??

Figure 4: Continuous (a) and discrete (b) representations of ratio

The numeric line representations were the least frequent (4%) and they ???. Figure 5 shows an example of this type of representation, done by A ??.

Figure 5: Numeric line representation of ratio

Thus, PPST were able to do part-whole representations of ratio that differ from the one given in the problem. Besides, as the values of the percentages obtained for the most frequent subtypes of representations indicate, several PPST gave more than one type of representation. In fact they have combined numeric representations (mainly of the fraction and/or decimal subtypes) with diagrammatic representations (mainly of the continuous subtype). These types of representations had been found in previous studies focusing on PPST (e.g., Berenson et al 2013; Stafford, Oldham, & O'Dowd 2015) even though Fernandes and Leite (2015) found that the diagrammatic representations were more popular to explain ratio to someone else than to just represent ratios.

Table 2 shows that the number of representations given by each student ranged between 1 (21%) and five (1%). Besides, almost half of them gave two representations (46%) and about one fifth of them gave either one (21%) or three (22%) representations. Only 9% gave four representations and only 1% gave five representations.

Table 2: Number of representations per student

(N=81)

Number of representations	**f**	**%**
One	17	21
Two	37	46
Three	18	22
Four	7	9
Five	1	1
Do not answer	1	1

It should be noted that, as Dreyfus (2002) and Duval (2003) emphasized, the number of representations of a concept that a person is able to do is an indicator or the understanding he/she holds o the concept.

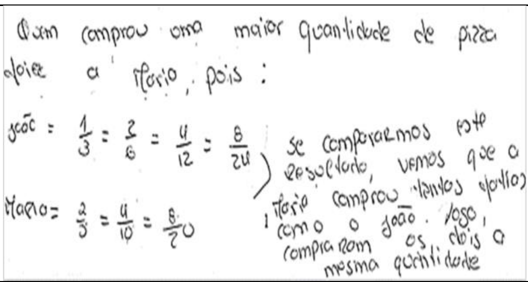
Afterwards, PPST were asked to compare the two ratios and to identify the biggest one. To do so, they would need to select and use a type of representation. Even though all but one PPST gave a correct final answer, table 3 shows that about one third (31%) of them did not explain how they reached that answer or why it is the correct answer. The remaining 68% used numerical representations to do the required ratio comparison.

Table 3: Types of comparisons of ratio done by the students

(N=81)

Types of comparisons		**f**	**%**
Fractions comparison	Reducing to same numerator/denominator	21	26
	Intuition	12	15
Percentage comparison		4	5
Decimal comparison		18	22
No justification		25	31
Do not answer		1	1

Most of the PPST that used fractions computed equivalent fractions by reducing fractions to the same denominator or to de same numerator, that is they used a strategy previously described by Viana and Miranda (2016). However, some of the students that have opted for reducing fraction to the same numerator found some difficulties that are illustrated by the answer given by A77 (figure 6). In fact, by focusing on the numerator, they forgot about the denominators (which were different) and drew a wrong conclusion. Besides, in some cases, they gave answers containing internal contradictions (which the authors seem to have not perceived). It is the case of the answer given in figure 6 that started by stating that Maria had bought the largest amount of pizza and that ended by concluding that both João and Maria bought the same amount of pizza.


Quem comprou uma maior quantidade de pizza foi a Maria, pois:
João: $\frac{1}{3} = \frac{2}{6} = \frac{4}{12} = \frac{8}{24}$
Maria: $\frac{2}{5} = \frac{4}{10} = \frac{8}{20}$
Se compararmos este resultado, vemos que a Maria comprou tantas fatias como o João. Logo, compraram os dois a mesma quantidade

Who bought the largest amount of pizza was Maria, because:

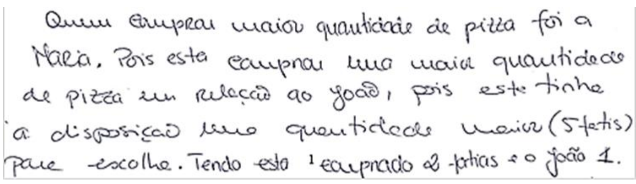
João: $\frac{1}{3} = \frac{2}{6} = \frac{4}{12} = \frac{8}{24}$

Maria: $\frac{2}{5} = \frac{4}{10} = \frac{8}{20}$

If we compare these results, we will note that Maria has bought as many slices as João. Then, they bought the same amount.

Figure 6: Fractions comparison based on reduction to the same numerator (A77 original answer | translated answer)

PPST that made intuitive comparisons described procedures that may not lead to a correct answer. In fact, some of them did an intuitive comparison of the areas of pizza bought by João and Maria based on the number of parts of pizza bought by each of them. They compared the fractions 1/3 and 2/5 and concluded that 2/5 would represent the largest amount as it corresponds to two parts while the 1/3 corresponds to only one part. They just ignored that the areas of each slice were different in the two cases and drew a conclusion without taking it into account. This reasoning is illustrated in figure 7 by the answer given by A 03. Even though 2/5 is larger than 1/3 and the result is correct, this type of reasoning (that ignores the denominators) does not give any systematic guarantee of reaching a correct answer through it.


Quem comprou maior quantidade de pizza foi a Maria. Pois esta comprou uma maior quantidade de pizza em relação ao João, pois este tinha à disposição uma quantidade maior (5 fatias) para escolher. Tendo esta comprado 2 fatias e o João 1.

Who bought the largest amount of pizza was Maria. Because she bought more pizza compared with João, as she had a larger amount of pizza (5 slices) to choose from. And she bought 2 slices and João bought only 1.

Figure 7: Intuitive comparison based on the numerator only (A 03 original answer | translated answer)

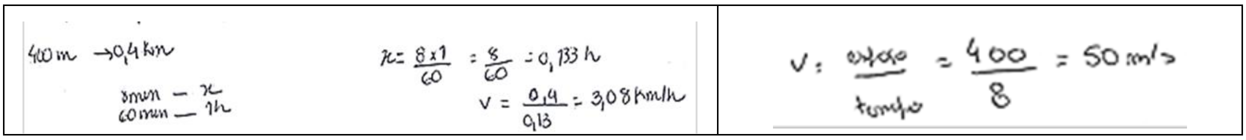
Thus, a considerable amount of PPST used decimals (22%) but only a few used percentages (5%) to compare the parts of pizza that João and Maria have bought. Most of them did correct answers showing ability to do what Lemon (2007) and Suggate, Davis and Goulding (2006) call part-whole comparisons.

Results Relative to the Speed Problem

As the speed problem involves a graph (see figure 2), and heterogeneous ratios in Lamon’s (2007) sense, before concentrating on the part-whole comparison issue, a question was asked to PPST in order to check whether they were able (or not) to read a graph and to compute a speed, using data from the graph. This was necessary procedure given the difficulties with graphs that have been reported in the literature.

As far the ability to read the graph is concerned, 89% were able to correctly indicate the time that António took to walk 250m even though they have written it differently: 2 minutes and a half; 2 minutes and 30 seconds; 2,5 minutes. The remaining 11% gave incorrect answers not because they did not know how to read the graph but because they did not know how to write 2, 5 minutes or because they ignored parts of a minute. Then, in the former case, they wrote things like “2 minutes and 50 seconds. Hence, it can be stated that almost all the students know how to read a graph space *versus* time.

With regard to speed calculation, 75% were able to calculate Rosa’s speed and gave the result in m/min (60%), as it could be read in the graph, or in m/s (11%) or km/h (4%). Thus, they had to do units conversion (e.g., A78), as shown in figure 8a. However, a few of them did not pay attention to the units associated to the variables in the graph and gave a wrong result in terms of units (e.g., A 50), as illustrated in figure 8b.



(a) A 78 (b) A 50

Figure 8: Rosa’s speed computation: (a) graph reading followed by units conversion; (b) graph reading without attention to units

A few students gave answers that suggest that they have conceptual difficulties with speed. In fact, they did not use the correct mathematics formula to calculate speed; they rather inverted it, having considered the ratio time/space. Other six percent of the PPST gave the time that Rosa took to walk 400m or 200m instead of her speed. These incorrect responses suggest an insufficient understanding of the speed concept, which had already been pointed out by ???. Also, 6% did not answer or did not give an understandable answer to the question. Therefore, more than 70% of the PPST know how to compute a speed value from a graph space *versus* time which involves a ratio between heterogeneous magnitudes.

Table 4 shows the types of strategies that PPST used to find out who moved faster (António or Rosa). It should be emphasised that do succeed in doing so they were required to calculate ratios (speed values) and to do part-part comparisons.

Table 4: Types of strategies used to obtain the highest speed

(N=81)

Type of strategy	**f**	**%**
Compute speeds to choose the highest	28	35
Compare times needed to follow a certain path	42	52
Compare spaces followed in a given time interval	7	9
Do not answer	4	5

PPST that computed the highest speed started by calculating António’s and Rosa’s speeds, afterwards they compared them and finally they identified the highest speed. This was the case of A36, as shown in figure 9. It should be noted that as no information was provided on the units that should be used for speed, most PPST used m/min, as it was in the graph but a few used the international units system speed unit (m/s), as A36 did. A few PPST reached a wrong result because they took wrong values from the graph or they did not pay attention to units that were used in the graph.

$$r_{Rosa} = \frac{400}{480} \approx \frac{40}{48} = \frac{20}{24} = \frac{10}{12} = \frac{5}{6} \simeq 0,8(3) \text{ m/s}$$

$8 \text{ min} = 480 s$

$$r_{António} = \frac{400}{240} = \frac{40}{24} = \frac{20}{12} = \frac{10}{6} = \frac{5}{3} = 1,(6) \text{ m/s}$$

$4 \text{ min} = 240 \text{ s}$

$$\frac{5}{3} > \frac{5}{6}$$

Figure 9: Computation of the highest speed through speeds comparison (A36)

As far as the comparison of times needs to cover a certain path, is concerned, PPST compared the times used by António and by Rosa to cover a certain path and they concluded that the highest speed belongs to the one that used less time to do it. Most of these students took as reference 400m that is the highest space value given in the graph (figure 10). It would be interesting to understand the reasoning underpinning this choice. Half of the others PPST used other space values and the remaining just did a qualitative comparison without mentioning any specific space value.

O António, pois percorreu 400 m em 4 minutos e a Rosa percorreu 400 m em 8 minutos, ou seja, demorou o dobro do tempo a percorrer a mesma distância do António.	António because he walked 400 m in 4 minutes and Rosa walked 400 m in 8 minutes that is, she doubled the time used by António to walk the same distance.

Figure 10: Computation of the highest speed through times comparison (A13 original answer | translated answer)

PPST that opted by comparing the space covered in a given time by António and Rosa, they took the values from the graph to conclude that the person that walked faster was the one that walked more meter. This time, they concentrated on 1 min or on 2 min. that are the lower times explicitly shown in the graph. However, one did it for several time instants. These types of strategies have been reported in the literature and they have been found even among in-service science /maths teachers.

Hence, more than half of the PPST succeeded on solving the speed problem by comparing times or spaces for the same space or the same time instead of calculating and comparing heterogeneous ratios that is instead of doing part-part comparisons based on the speed mathematics formula.

Conclusions and Implications for Teacher Education

Results indicate that most PPST that participated in the study were able to solve problems that require operating with ratio in two different contexts: a context with homogeneous magnitudes and part whole comparisons; a context with heterogeneous magnitudes and part-part comparisons. However, nearly half of them were able to give two types of representation only, about one third was not able to explain ration comparisons, and only about one third were able to compare space/time ratios.

Thus, the results of this study suggest that a considerable number of PPST that participated in the study reported in this paper may lack content knowledge on the ratio concept. They also suggest that initial primary school teacher education needs to pay more attention to the ratio concept, by formally integrating it in the teacher education curriculum and by approaching it explicitly. Besides, the complex nature of the concept together with the difference between the results obtained in the two problem situations suggest that this concept should be approached in several contexts (and not only in the mathematics one) from a conceptual and a representational points of view. This means that ratio content knowledge and ratio pedagogical content knowledge should be approached in an integrated manner to trigger each other. Finally, this may require mathematics' teacher educators and mathematics educators to work together to foster the development of PPST specialized knowledge of ratio.

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