NONUNIFORM BEHAVIOR AND STABILITY OF HOPFIELD NEURAL NETWORKS WITH DELAY

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Abstract. Based on a new abstract result on the behavior of nonautonomous delayed equations, we obtain a stability result for the solutions of a general discrete nonautonomous Hopfield neural network model with delay. As an application we improve some existing results on the stability of Hopfield models.

1. Introduction

Due to their many applications in various engineering and scientific areas such as signal processing, image processing and pattern classification (see [6, 7]), neural network models are nowadays a subject of active research. One of the most important goals in the study of neural network models is to establish conditions that assure the global stability of equilibrium states [15, 16, 17], of periodic solutions [10, 22] or, more generally, of a particular solution [11].

In the present work we consider a discrete-time nonautonomous neural network with time delay. The relevance of our setting is easily clarified. Although theoretically speaking neural networks should be described by continuous-time models, it is essential to formulate discrete-time versions that can be implemented computationally [17, 18]. It is also important to consider delay in modelling neural networks in order to reproduce the effect of finite transmission speed of signals among neurons (there is a mathematical counterpart of this since time delay may cause instability and oscillation [14]). The nonautonomy is associated to the change of parameters such as neuron charging time, interconnection weights and external inputs in the course of time. This can be translated not only by time-varying parameters, but also by time-varying delays [5, 12, 20, 23]. There are still few stability results in the context of nonautonomous nonperiodic neural network models [21].

The proof presented here to establish our global stability results is different from the usual one. In fact, the classical method of proof used in [7, 9, 15, 17, 21] consists in proving that there is an equilibrium point or a periodic solution and then construct a suitable Lyapunov function that assures the global stability of the particular solution. On the other hand, the technique used here is different from the usual ones. Namely, we see our system as a sufficiently small perturbation of a nonuniform contraction and use Banach's fixed point theorem in some suitable complete metric space to obtain the global stability of our system. This approach allows us to dismiss the requirement of existence of a stationary or, more generally, periodic solution and additionally to consider a more general form for the nonlinear

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part of the model. When we restrict to the particular case of a periodic Hopfield model, our conditions for existence of a globally stable periodic solution generalize results in [22]. For other results of stability of higher order difference equations see [13, 19, 2, 3, 4].

This work is organized in the following way. In section 2 we use the discretization technique in [17] to obtain a discrete version of a generalized neural network model that includes the well known Hopfield neural network models considered in [17, 22] and the bidirectional associative memory neural network models studied in [10, 15]. Next, we state our main stability result and we see that it is a consequence of the abstract result considered in section 3. As a corollary, we get global exponential stability for the models in [21] under distinct hypothesis from the ones assumed in that paper. After, for the periodic model, considering a Poincaré map, we obtain the existence of a periodic solution as a consequence of the global exponential stability. This result improves one of the main results in [22]. To see this we present an illustrative example where our results can be applied but it is not possible to apply the results of Xu and Wu [22], due to an extra hypothesis required in their work. Finally, in section 3, we consider general discrete-time delayed models that include our neural network models as particular cases and obtain the abstract global stability result that we use to prove the stability results in section 2.

2. Hopfield Models

As a generalization of the continuous-time Hopfield neural network models presented in [17, 22] we have

$$x_i'(t) = -a_i(t)x_i(t) + \sum_{j=1}^{N} k_{ij}(t, x_j(t-\alpha_{ij}(t))), \qquad t \geqslant 0,\, i = 1,\dots,N, \tag{1}$$

where $a_i : [0,+\infty[\to [0,+\infty[$, $k_{ij} : [0,+\infty[\times\mathbb{R} \to \mathbb{R}$, and $\alpha_{ij} : [0,+\infty[\to [0,+\infty[$ are continuous functions with α_{ij} bounded and k_{ij} Lipschitz on the second variable. Here $a_i(t)$ is the neuron charging time.

Following the ideas in [17], to obtain a discrete-time analogue of the continuous-time model (1), we consider the following approximation

$$x_i'(t) = -a_i([t/h]h)x_i\,(t) + \sum_{j=1}^{N} k_{ij}\left([t/h]h, x_j\left([t/h]h - \left[\frac{\alpha_{ij}\,([t/h]h)}{h}\right]h\right)\right), \tag{2}$$

$i = 1,\dots,N$, $t \in [mh,(m+1)h[$ for $m \in \mathbb{N}_0$, where h is a fixed positive real number (discretization step size) and $[r]$ denotes the integer part of the real number r. Clearly, for $t \in [mh,(m+1)h[$ we have $[t/h] = m$ and the model (2) has the form

$$x_i'(t) = -a_i(mh)x_i(t) + \sum_{j=1}^{N} k_{ij}\left(mh, x_j\left(\left(m - \left[\frac{\alpha_{ij}(mh)}{h}\right]\right)h\right)\right),$$

which is equivalent to

$$e^{a_i(mh)t}x_i'(t) + a_i(mh)e^{a_i(mh)t}x_i(t) = e^{a_i(mh)t}\sum_{j=1}^{N} k_{ij}\left(mh, x_j\left((m-\tau_{ij}(m))\,h\right)\right), \tag{3}$$

where

$$\tau_{ij}(m) = \left[\frac{\alpha_{ij}(mh)}{h}\right].$$

Integrating (3) over $[mh, t[$, with $t < (m+1)h$, we obtain

$$\int_{mh}^{t} \left[e^{a_i(mh)s} x_i(s)\right]' ds = \left(\frac{e^{a_i(mh)t} - e^{a_i(mh)mh}}{a_i(mh)}\right) \sum_{j=1}^{N} k_{ij}\left(mh, x_j\left((m - \tau_{ij}(m))\, h\right)\right),$$

which is equivalent to

$$x_i(t) = e^{a_i(mh)(mh-t)} x_i(mh) + \left(\frac{1 - e^{a_i(mh)(mh-t)}}{a_i(mh)}\right) \sum_{j=1}^{N} k_{ij}\left(mh, x_j\left((m - \tau_{ij}(m))\, h\right)\right).$$

Letting $t \to (m+1)h$, we obtain

$$x_i((m+1)h) = e^{-a_i(mh)h} x_i(mh) + \left(\frac{1 - e^{-a_i(mh)h}}{a_i(mh)}\right) \sum_{j=1}^{N} k_{ij}\left(mh, x_j\left((m - \tau_{ij}(m))\, h\right)\right). \tag{4}$$

Thus, identifying $x_i(mh)$ with $x_i(m)$, $a_i(mh)$ with $a_i(m)$ and $k_{ij}(mh, \cdot)$ with $k_{ij}(m, \cdot)$ and defining

$$\theta_i(m) = \frac{1 - e^{-a_i(m)h}}{a_i(m)}, \tag{5}$$

equation (4) becomes

$$x_i(m+1) = e^{-a_i(m)h} x_i(m) + \theta_i(m) \sum_{j=1}^{N} k_{ij}\big(m, x_j(m - \tau_{ij}(m))\big). \tag{6}$$

The model (6) can be rewritten in the following way

$$x_i(m+1) = c_i(m) x_i(m) + \sum_{j=1}^{N} h_{ij}\big(m, x_j(m - \tau_{ij}(m))\big), \tag{7}$$

$i = 1, \ldots, N$, $m \in \mathbb{N}_0$, where $c_i : \mathbb{N}_0 \to]0, 1[$, $\tau_{ij} : \mathbb{N}_0 \to \mathbb{N}_0$ are bounded functions with $\tau := \max\{\tau_{ij}(m) : m \in \mathbb{N}, i, j = 1, \ldots, N\}$, and $h_{ij} : \mathbb{N}_0 \times \mathbb{R} \to \mathbb{R}$ are Lipschitz functions on the second variable, i.e., there exist $H_{ij} : \mathbb{N}_0 \to \mathbb{R}^+$ such that

$$|h_{ij}(m, u) - h_{ij}(m, v)| \leqslant H_{ij}(m)|u - v|, \quad \forall u, v \in \mathbb{R},\, m \in \mathbb{N}_0.$$

In this paper we consider the Hopfield neural network model (7) that generalizes some existent models in the literature [10, 15, 17, 22].

Before stating our main result, we need to introduce some notation. Let

$$\Delta = \left\{(m, n) \in \mathbb{Z}^2 \colon m \geqslant n \geqslant 0\right\}$$

and, given a set $I \subseteq \mathbb{R}$ and a number $r \in \mathbb{Z}^-$, define $I_{\mathbb{Z}} = I \cap \mathbb{Z}$. Consider the space X of the functions

$$\alpha : [r, 0]_{\mathbb{Z}} \to \mathbb{R}$$

equipped with the norm

$$\|\alpha\| = \max_{j=r,\ldots,0} |\alpha(j)|.$$

Given $N \in \mathbb{N}$, we are going to consider the cartesian products X^N and $\mathbb{R}^N$ equipped with the supremum norm, i.e., for $\overline{\alpha} = (\alpha_1, \dots, \alpha_N) \in X^N$ and $\overline{y} = (y_1, \dots, y_N) \in \mathbb{R}^N$, we have

$$\|\overline{\alpha}\| = \max_{i=1,\dots,N} \|\alpha_i\| = \max_{i=1,\dots,N} \left(\max_{j=r,\dots,0} |\alpha_i(j)| \right)$$

and

$$|\overline{y}| = \max_{i=1,\dots,N} |y_i|.$$

Given $n \in \mathbb{N}_0$ and a function $\overline{x} \colon [n+r, +\infty[_{\mathbb{Z}} \to \mathbb{R}^N$ we denote the ith component by x_i, i.e., $\overline{x} = (x_1, \dots x_N)$. For each $m \in \mathbb{N}_0$ such that $m \geqslant n$, we define $\overline{x}_m \in X^N$ by

$$\overline{x}_m(j) = \overline{x}(m+j), \; j = r, r+1, \dots, 0.$$

For each $n \in \mathbb{N}_0$ and each $\overline{\alpha} \in X^N$, we denote by $\overline{x}(\cdot, n, \overline{\alpha})$ the unique solution

$$\overline{x} : [n+r, +\infty[_{\mathbb{Z}} \to \mathbb{R}^N$$

of (7) with initial conditions $\overline{x}_n = \overline{\alpha}$.

We now state our main global stability result for the neural network model given by (7). This theorem furnishes a bound for the distance between solutions of (7) based on bound for the products of consecutive neuron charging times, assuming that the Lipschitz constants of the nonlinear part of the model are sufficiently small. We will use it to obtain several results on the stability of several neural networks.

Theorem 1. *Consider model* (7) *and assume that there exist a double sequence* $(a'_{m,n})_{(m,n)\in\Delta}$ *such that*

$$a^{(i)}_{m,n} := \prod_{s=n}^{m-1} c_i(s) \leqslant a'_{m,n}, \tag{8}$$

for all $i = 1, \dots, N$ *and all* $(m,n) \in \Delta$, *and*

$$\lambda := \max_{i=1,\dots,N} \left[\sup_{(m,n)\in\Delta} \left\{ \frac{1}{a'_{m,n}} \sum_{k=n}^{m-1} a^{(i)}_{m,k+1} a'_{k,n} \sum_{j=1}^{N} H_{ij}(k) \right\} \right] < 1.$$

Then, for every $\overline{\alpha}, \overline{\alpha}^* : [r,0]_{\mathbb{Z}} \to \mathbb{R}^N$ *and every* $(m,n) \in \Delta$, *we have*

$$\|\overline{x}_m(\cdot, n, \overline{\alpha}) - \overline{x}_m(\cdot, n, \overline{\alpha}^*)\| \leqslant \frac{1}{1-\lambda}\, a'_{m,n} \|\overline{\alpha} - \overline{\alpha}^*\|.$$

Proof. Consider $n \in \mathbb{N}_0$ and $\overline{\alpha}, \overline{\alpha}^* : [r,0]_{\mathbb{Z}} \to \mathbb{R}^N$. The change

$$\overline{y}(m) = \overline{x}(m, n, \overline{\alpha}) - \overline{x}(m, n, \overline{\alpha}^*)$$

transforms (7) into the system

$$y_i(m+1) = c_i(m) y_i(m) + \sum_{j=1}^{N} \widetilde{h}_{ij}(m, y_j(m - \tau_{ij}(m))), \tag{9}$$

$i = 1, \dots, N$, $m \geqslant n$, where

$$\widetilde{h}_{ij}(m,u) = h_{ij}(m, u + x_j(m - \tau_{ij}(m), n, \overline{\alpha}^*)) - h_{ij}(m, x_j(m - \tau_{ij}(m), n, \overline{\alpha}^*)).$$

Now, $\overline{y} = 0$ is an equilibrium point of (9) and, by Theorem 5, we obtain, for all function $\overline{\beta} : [r,0]_{\mathbb{Z}} \to \mathbb{R}^N$,

$$\|\overline{y}_m(\cdot, n, \overline{\beta})\| \leqslant \frac{1}{1-\lambda}\, a'_{m,n} \|\overline{\beta}\|$$

for all $m \geqslant n$. Letting $\overline{\beta} = \overline{\alpha} - \overline{\alpha}^*$, we conclude that

$$\|\overline{x}_m(\cdot, n, \overline{\alpha}) - \overline{x}_m(\cdot, n, \overline{\alpha}^*)\| = \|\overline{y}_m(\cdot, n, \overline{\alpha} - \overline{\alpha}^*)\| \leqslant \frac{1}{1-\lambda}\, a'_{m,n} \|\overline{\alpha} - \overline{\alpha}^*\|$$

for all $m \geqslant n$. □

We stress that Theorem 1 includes situations that are nonuniform and even nonexponential. In fact, our setting is sufficiently general to allow situations where the neuron charging times $c_i(m)$ leads to a sequence $a'_{m,n}$ with a more general dependence on m and n than the usual uniform exponential behavior $a'_{m,n} = D\,\mathrm{e}^{-\mu(m-n)}$.

Example 1. *Choosing*

$$c_i(m) = \mathrm{e}^{-\nu_i + m\varepsilon[1-(-1)^m]/2 - (m+1)\varepsilon\left[1-(-1)^{m+1}\right]/2}$$

where $\nu_i > 0$, for $i = 1, \dots, N$, $\varepsilon > 0$ and $h_{ij} : \mathbb{N}_0 \times \mathbb{R} \to \mathbb{R}$ Lipschitz functions on the second variable , i.e.,

$$|h_{ij}(m, u) - h_{ij}(m, v)| \leqslant H_{ij}(m)|u - v|, \qquad \forall u, v \in \mathbb{R},\ m \in \mathbb{N}_0,$$

with

$$H_{ij}(m) = \frac{1 - \mathrm{e}^{-\varepsilon}}{2N}\, \mathrm{e}^{-\nu_i - \varepsilon(m+1)\left[1-(-1)^{m+1}\right]/2 - \varepsilon m},$$

then

$$\|\bar{x}_m(\cdot, n, \bar{\alpha}) - \bar{x}_m(\cdot, n, \bar{\alpha}^*)\| \leqslant 2\,\mathrm{e}^{-\mu(m-n)+\varepsilon n}\, \|\bar{\alpha} - \bar{\alpha}^*\|.$$

In fact,

$$\begin{aligned} a^{(i)}_{m,n} &= \prod_{s=n}^{m-1} c_i(s) = \mathrm{e}^{-\nu_i(m-n) + \varepsilon n[1-(-1)^n]/2 - \varepsilon m[1-(-1)^m]/2} \\ &\leqslant \mathrm{e}^{-\nu_i(m-n)+\varepsilon n} \leqslant \mathrm{e}^{-\mu(m-n)+\varepsilon n} := a'_{m,n}, \end{aligned}$$

where $\mu = \min_i\{\nu_i\}$. Taking into account that for $m > n \geqslant 0$ we have

$$\begin{aligned} \frac{a^{(i)}_{m,k+1} a'_{k,n}}{a'_{m,n}} &= \mathrm{e}^{(\mu-\nu_i)(m-k) - \varepsilon m[1-(-1)^m]/2 + \nu_i + \varepsilon(k+1)\left[1-(-1)^{k+1}\right]/2} \\ &\leqslant \mathrm{e}^{\nu_i + \varepsilon(k+1)\left[1-(-1)^{k+1}\right]/2} \end{aligned}$$

and

$$\begin{aligned} \sum_{j=1}^{N} H_{i,j}(k) &= \sum_{j=1}^{N} \frac{1-\mathrm{e}^{-\varepsilon}}{2N}\, \mathrm{e}^{-\nu_i - \varepsilon(k+1)\left[1-(-1)^{k+1}\right]/2 - \varepsilon k} \\ &= \frac{1-\mathrm{e}^{-\varepsilon}}{2}\, \mathrm{e}^{-\nu_i - \varepsilon(k+1)\left[1-(-1)^{k+1}\right]/2 - \varepsilon k}, \end{aligned}$$

it follows that

$$\begin{aligned} \lambda &= \max_{i=1,\dots,N} \left[\sup_{(m,n)\in\Delta} \left\{ \frac{1}{a'_{m,n}} \sum_{k=n}^{m-1} a^{(i)}_{m,k+1} a'_{k,n} \sum_{j=1}^{N} H_{ij}(k) \right\} \right] \\ &\leqslant \max_{i=1,\dots,N} \left[\sup_{(m,n)\in\Delta} \left\{ \frac{1-\mathrm{e}^{-\varepsilon}}{2} \sum_{k=n}^{m-1} \mathrm{e}^{-\varepsilon k} \right\} \right] = \frac{1-\mathrm{e}^{-\varepsilon}}{2} \sum_{k=0}^{+\infty} \mathrm{e}^{-\varepsilon k} = \frac{1}{2}. \end{aligned}$$

A nonexponential and nonuniform example can be obtained choosing

$$c_i(m) = \left(\frac{m+2}{m+1}\right)^{-\nu_i} \frac{(m+1)^{\varepsilon[1-(-1)^m]/2}}{(m+2)^{\varepsilon[1-(-1)^{m+1}]/2}}$$

and

$$H_{ij}(m) = \frac{1-\mathrm{e}^{-\varepsilon}}{2N} (k+2)^{-\nu_i-\varepsilon\left[1-(-1)^{k+1}\right]/2} \mathrm{e}^{-\varepsilon k}.$$

With this choice it follows that

$$\|\bar{x}_m(\cdot,n,\bar{\alpha}) - \bar{x}_m(\cdot,n,\bar{\alpha}^*)\| \leqslant 2\left(\frac{m+1}{n+1}\right)^{-\mu} (n+1)^{\varepsilon} \|\bar{\alpha}-\bar{\alpha}^*\|.$$

Despite the generality of our main result, when we apply it to the Hopfield neural network models existing in the literature, we are able to improve some of the known results.

In [22] the authors considered the following discretization of a nonautonomous continuous-time Hopfield neural network model, which is a particular case of (7),

$$x_i(m+1) = x_i(m)\,\mathrm{e}^{-a_i(m)h} + \theta_i(m)\left[\sum_{j=1}^{N} b_{ij}(m) f_j(x_j(m-\tau(m))) + I_i(m)\right], \quad (10)$$

$i = 1,\dots,N$, where $a_i, b_{ij}, I_i\colon \mathbb{N}_0 \to \mathbb{R}$ and $\tau\colon \mathbb{N}_0 \to \mathbb{N}_0$ are bounded functions with $a_i(m) > 0$, $0 \leqslant \tau(m) \leqslant \tau$, $f_j : \mathbb{R} \to \mathbb{R}$ are Lipschitz functions with Lipschitz constant $F_j > 0$, $\theta_i(m)$ is given by (5) and $h > 0$ (h is the discretization step size). We are going to use the following notation

$$a_i^- = \inf_m a_i(m) \quad \text{and} \quad b_{ij}^+ = \sup_m |b_{ij}(m)| \quad \text{and} \quad \theta_i^+ = \sup_m \theta_i(m).$$

We have the following result that establishes the global exponential stability of all solutions of (10).

Corollary 2. *If*

$$a_i^- > \sum_{j=1}^{N} b_{ij}^+ F_j. \quad (11)$$

for every $i = 1,\dots,N$, then model (10) is globally exponentially stable, i.e., there are constants $\mu > 0$ and $C > 1$ such that

$$\|\overline{x}_m(\cdot,n,\overline{\alpha}) - \overline{x}_m(\cdot,n,\overline{\alpha}^*)\| \leqslant C\,\mathrm{e}^{-\mu(m-n)} \|\overline{\alpha}-\overline{\alpha}^*\|$$

for every $\overline{\alpha},\overline{\alpha}^* : [-\tau,0]_{\mathbb{Z}} \to \mathbb{R}^N$ and every $(m,n) \in \Delta$.

Proof. We will show that we are in the conditions of Theorem 1. Defining $\nu_i = a_i^- h$, by (11) there is positive number $\mu < \min_i \nu_i$ such that

$$\frac{\mathrm{e}^{\nu_i-\mu}-1}{\mathrm{e}^{\nu_i}-1} a_i^- > \sum_{j=1}^{N} b_{ij}^+ F_j, \quad \forall i = 1,\dots,N. \quad (12)$$

Putting $a'_{m,n} = \mathrm{e}^{-\mu(m-n)}$, condition (8) is trivially satisfied because

$$a_{m,n}^{(i)} \leqslant \mathrm{e}^{-\nu_i(m-n)} \leqslant \mathrm{e}^{-\mu(m-n)}.$$

Since $\theta_i^+ = \dfrac{1-\mathrm{e}^{-\nu_i}}{a_i^-}$, we have by (12) that

$$\begin{aligned}
\lambda &= \max_{i=1,\dots,N}\left[\sup_{(m,n)\in\Delta}\left\{\frac{1}{a'_{m,n}}\sum_{k=n}^{m-1} a_{m,k+1}^{(i)} a'_{k,n}\,\theta_i(k)\sum_{j=1}^{N}|b_{ij}(k)|F_j\right\}\right]\\
&\leqslant \max_{i=1,\dots,N}\left[\sup_{(m,n)\in\Delta}\left\{\mathrm{e}^{\mu(m-n)}\sum_{k=n}^{m-1}\mathrm{e}^{-\nu_i(m-k-1)-\mu(k-n)}\right\}\theta_i^+\sum_{j=1}^{N} b_{ij}^+F_j\right]\\
&< \max_{i=1,\dots,N}\left[\sup_{(m,n)\in\Delta}\left\{\sum_{k=n}^{m-1}\mathrm{e}^{(\nu_i-\mu)(k-m)}\right\}\mathrm{e}^{\nu_i}\;\frac{1-\mathrm{e}^{-\nu_i}}{a_i^-}\,\frac{\mathrm{e}^{\nu_i-\mu}-1}{\mathrm{e}^{\nu_i}-1}a_i^-\right]\\
&= \max_{i=1,\dots,N}\left[\sup_{(m,n)\in\Delta}\left\{\frac{1-\mathrm{e}^{(\nu_i-\mu)(n-m)}}{\mathrm{e}^{\nu_i-\mu}-1}\right\}\frac{(\mathrm{e}^{\nu_i}-1)(\mathrm{e}^{\nu_i-\mu}-1)}{(\mathrm{e}^{\nu_i}-1)a_i^-}a_i^-\right]\\
&= \max_{i=1,\dots,N}\left[\sup_{(m,n)\in\Delta}\left\{1-\mathrm{e}^{(\nu_i-\mu)(n-m)}\right\}\right]\\
&= 1
\end{aligned}$$

and this proves the corollary. □

In the next corollary we slightly improve condition (11) in the last corollary. To do that we need to define the concept of an M-matrix. We say that a square real matrix is an M-matrix if the off-diagonal entries are nonpositive and all the eigenvalues have positive real part.

Now consider the $N\times N$-matrix $\mathcal{M}$ defined by

$$\mathcal{M} = diag(a_1^-,\dots,a_N^-) - \left[b_{ij}^+F_j\right]$$

Corollary 3. *If $\mathcal{M}$ is an M-matrix, then the model* (10) *is global exponential stable, i.e., there are $\mu>0$ and $C>1$ such that*

$$\|\overline{x}_m(\cdot,n,\overline{\alpha})-\overline{x}_m(\cdot,n,\overline{\alpha}^*)\| \leqslant C\,\mathrm{e}^{-\mu(m-n)}\,\|\overline{\alpha}-\overline{\alpha}^*\|.$$

for every $\overline{\alpha},\overline{\alpha}^*:[-\tau,0]_{\mathbb{Z}}\to\mathbb{R}^N$ and every $(m,n)\in\Delta$.

Proof. If $\mathcal{M}$ is an M-matrix, then (see Fiedler [8, Theorem 5.1]) there is $\overline{d} = (d_1,\dots,d_N)>0$ such that $\mathcal{M}\,\overline{d}>0$, i.e.,

$$d_i a_i^- > \sum_{j=1}^{N} d_j b_{ij}^+ F_j. \tag{13}$$

The change $y_i(m)=d_i^{-1}x_i(m)$, $m\in\mathbb{N}_0$ and $i=1,\dots,N$, transforms (10) into

$$y_i(m+1) = y_i(m)\,\mathrm{e}^{-a_i(m)h} + \theta_i(m)\left[\sum_{j=1}^{N}\tilde{b}_{ij}(m)\tilde{f}_j(y_j(m-\tau(m)))+\tilde{I}_i(m)\right],$$

where

$$\tilde{b}_{ij}(m)=d_i^{-1}b_{ij}(m),\quad \tilde{f}_j(u)=f_j(d_ju),\quad\text{and}\quad \tilde{I}_i(m)=d_i^{-1}I_i(m),$$

for $m \in \mathbb{N}_0$ and $u \in \mathbb{R}$. As f_j are Lipschitz functions with constant F_j, then $\tilde{f}_j$ are also Lipshitz functions with constant $\tilde{F}_j = d_j F_j$. From (13) we have

$$a_i^- > \sum_{j=1}^{N} d_i^{-1} b_{ij}^+ d_j F_j$$

which is equivalent to

$$a_i^- > \sum_{j=1}^{N} \tilde{b}_{ij}^+ \tilde{F}_j$$

and the result follows from the Corollary 2. □

Now we improve [22, Theorem 4.2], which proves the existence and global stability of the periodic solution of the Hopfield neural network model (10) with periodic coefficients. Let $\omega \in \mathbb{N}$ and consider the model (10) where $a_i, b_{ij}, I_i \colon \mathbb{N}_0 \to \mathbb{R}$ and $\tau : \mathbb{N}_0 \to \mathbb{N}_0$ are ω-periodic functions.

Theorem 4. *If $\mathcal{M}$ is an M-matrix, then the model* (10) *has a unique ω-periodic solution which is globally exponentially stable.*

Proof. Let $n \in \mathbb{N}_0$. From Corollary 3, there are $\mu > 0$ and $C > 1$ such that

$$\|\overline{x}_m(\cdot, n, \overline{\alpha}) - \overline{x}_m(\cdot, n, \overline{\alpha}^*)\| \leqslant C\,\mathrm{e}^{-\mu(m-n)}\,\|\overline{\alpha} - \overline{\alpha}^*\|, \tag{14}$$

for all $m \geqslant n$ and all $\overline{\alpha}, \overline{\alpha}^* \in X^N$. Now, choosing an integer $k \in \mathbb{N}$ such that

$$C\,\mathrm{e}^{-\mu k \omega} < 1 \tag{15}$$

and defining the map $P : X^N \to X^N$ by $P(\overline{\alpha}) = \overline{x}_{n+\omega}(\cdot, n, \overline{\alpha})$. For $\overline{\alpha}, \overline{\alpha}^* \in X^N$, we have

$$\begin{aligned}
&\|P^k(\overline{\alpha}) - P^k(\overline{\alpha}^*)\| \\
&= \|P(P^{k-1}(\overline{\alpha})) - P(P^{k-1}(\overline{\alpha}^*))\| \\
&= \|\overline{x}_{n+\omega}(\cdot, n, P^{k-1}(\overline{\alpha})) - \overline{x}_{n+\omega}(\cdot, n, P^{k-1}(\overline{\alpha}^*))\| \\
&= \|\overline{x}_{n+\omega}(\cdot, n, \overline{x}_{n+\omega}(\cdot, n, P^{k-2}(\overline{\alpha}))) - \overline{x}_{n+\omega}(\cdot, n, \overline{x}_{n+\omega}(\cdot, n, P^{k-2}(\overline{\alpha}^*)))\|,
\end{aligned}$$

and, as the model (10) is ω-periodic, from (14),

$$\begin{aligned}
\|P^k(\overline{\alpha}) - P^k(\overline{\alpha}^*)\| &= \|\overline{x}_{n+2\omega}(\cdot, n, P^{k-2}(\overline{\alpha})) - \overline{x}_{n+2\omega}(\cdot, n, P^{k-2}(\overline{\alpha}^*))\| \\
&= \|\overline{x}_{n+k\omega}(\cdot, n, \overline{\alpha}) - \overline{x}_{n+k\omega}(\cdot, n, \overline{\alpha}^*)\| \\
&\leqslant C\,\mathrm{e}^{-\mu k \omega}\,\|\overline{\alpha} - \overline{\alpha}^*\|.
\end{aligned}$$

From (15), the map P^k is a contraction on X^N. As X^N is a Banach space, we conclude that there is a unique point $\overline{\varphi} \in X^N$ such that $P^k(\overline{\varphi}) = \overline{\varphi}$. Noting that

$$P^k(P(\overline{\varphi})) = P(P^k(\overline{\varphi})) = P(\overline{\varphi}),$$

we have $P(\overline{\varphi}) = \overline{\varphi}$ which means $\overline{x}_{n+\omega}(\cdot, n, \overline{\varphi}) = \overline{\varphi}$.

Finally, as $\overline{x}(m, n, \overline{\varphi})$ is a solution of (10) with a_i, b_{ij}, I_i, τ ω-periodic functions, we know that $\overline{x}(m + \omega, n, \overline{\varphi})$ is also a solution of (10) and

$$\overline{x}(m, n, \overline{\varphi}) = \overline{x}(m, n, \overline{x}_{n+\omega}(\cdot, n, \overline{\varphi})) = \overline{x}(m + \omega, n, \overline{\varphi})$$

for every $m \geqslant n$. Therefore $\overline{x}(m, n, \overline{\varphi})$ is a ω-periodic solution of (10) and, from (14), all other solutions converge to it with exponential rates. □

In [22, Theorem 4.2] the authors use the extra hypothesis:

$$\min_{1\leqslant i\leqslant N} \left\{ \widehat{a}_i - F_i \sum_{j=1}^{N} \left| \widehat{b}_{ji} \right| \right\} > 0 \tag{16}$$

where

$$\widehat{a}_i = \frac{1}{\omega} \sum_{m=0}^{\omega-1} a_i(m) \qquad \text{and} \qquad \widehat{b}_{ij} = \frac{1}{\omega} \sum_{m=0}^{\omega-1} b_{ij}(m).$$

Note that hypothesis $(H3)$ in [22] is equivalent to $\mathcal{M}$ being an M-matrix (see Fiedler [8, Theorem 5.1]).

Next we present an example similar to the example of Xu and Wu [22] that does not verify (16).

Example 2. *In the model* (10) *with* $N = 2$*, let*

$$\begin{aligned}
&a_1(m) = 25 + \cos\frac{m\pi}{\omega}, \quad b_{11}(m) = 16 + \cos\frac{m\pi}{\omega}, \quad b_{12}(m) = 4 + \cos\frac{m\pi}{\omega}\\
&a_2(m) = 29 + \sin\frac{m\pi}{\omega}, \quad b_{21}(m) = 16 + \sin\frac{m\pi}{\omega}, \quad b_{22}(m) = 8 + \sin\frac{m\pi}{\omega}\\
&I_1(m) = \cos\frac{m\pi}{\omega}, \quad I_2(m) = \sin\frac{m\pi}{\omega}, \quad \tau(m) = (1 + (-1)^m)/2\\
&f_1(u) = \arctan u, \quad f_2(u) = \tanh u, \quad \omega = 10.
\end{aligned}$$

Clearly $F_1 = F_2 = 1$ *and since*

$$\mathcal{M} = \begin{bmatrix} a_1^- & 0 \\ 0 & a_2^- \end{bmatrix} - \begin{bmatrix} b_{11}^+ F_1 & b_{12}^+ F_2 \\ b_{21}^+ F_1 & b_{22}^+ F_2 \end{bmatrix} = \begin{bmatrix} 7 & -5 \\ -17 & 19 \end{bmatrix}$$

is an M*-matrix (the eigenvalues are* 2 *and* 24*), by Theorem 4, this example has a unique* ω*-periodic solution which is globally exponentially stable.*

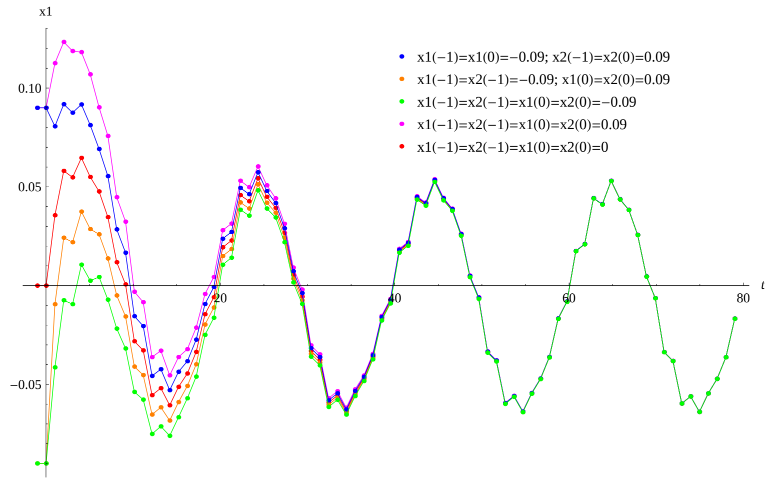
However, since

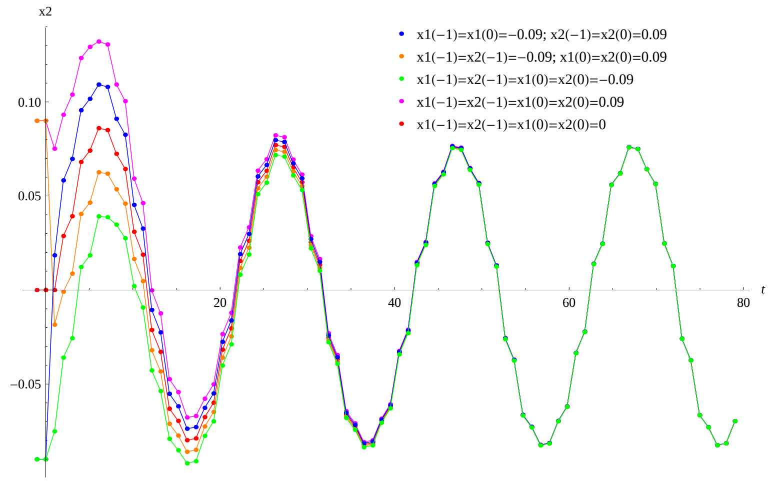
$$\widehat{a}_1 - F_1 \left(|\widehat{b}_{11}| + |\widehat{b}_{21}| \right) = -7,$$

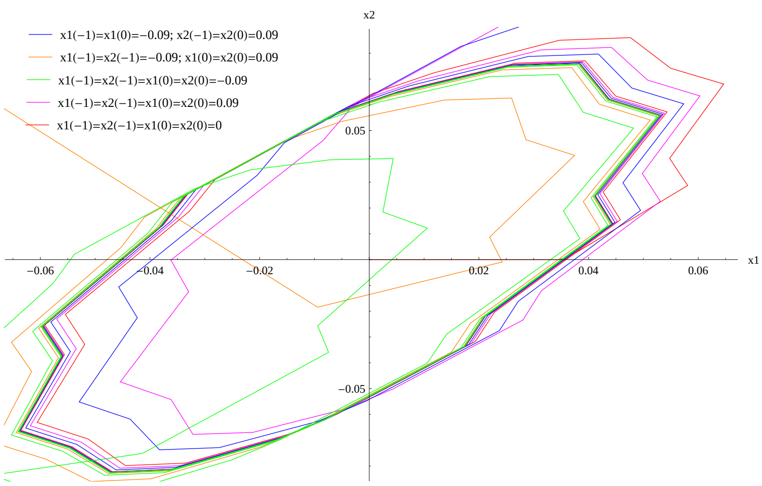
(16) *is not satisfied and thus the result of Xu and Wu* [22] *cannot be applied to this example.*

The following graphs illustrate our example with the initial conditions

$$\begin{aligned}
&x_1(-1) = x_2(-1) = x_1(0) = x_2(0) = 0,\\
&x_1(-1) = x_2(-1) = x_1(0) = x_2(0) = 0.09,\\
&x_1(-1) = x_1(0) = -0.09 \quad \textit{and} \quad x_2(-1) = x_2(0) = 0.09,\\
&x_1(-1) = x_1(0) = 0.09 \quad \textit{and} \quad x_2(-1) = x_2(0) = -0.09,\\
&x_1(-1) = x_1(0) = -0.09 \quad \textit{and} \quad x_2(-1) = x_2(0) = -0.09.
\end{aligned}$$


x1
t
x1(−1)=x1(0)=−0.09; x2(−1)=x2(0)=0.09
x1(−1)=x2(−1)=−0.09; x1(0)=x2(0)=0.09
x1(−1)=x2(−1)=x1(0)=x2(0)=−0.09
x1(−1)=x2(−1)=x1(0)=x2(0)=0.09
x1(−1)=x2(−1)=x1(0)=x2(0)=0


x2
t
x1(−1)=x1(0)=−0.09; x2(−1)=x2(0)=0.09
x1(−1)=x2(−1)=−0.09; x1(0)=x2(0)=0.09
x1(−1)=x2(−1)=x1(0)=x2(0)=−0.09
x1(−1)=x2(−1)=x1(0)=x2(0)=0.09
x1(−1)=x2(−1)=x1(0)=x2(0)=0


x2
x1
x1(−1)=x1(0)=−0.09; x2(−1)=x2(0)=0.09
x1(−1)=x2(−1)=−0.09; x1(0)=x2(0)=0.09
x1(−1)=x2(−1)=x1(0)=x2(0)=−0.09
x1(−1)=x2(−1)=x1(0)=x2(0)=0.09
x1(−1)=x2(−1)=x1(0)=x2(0)=0

3. Stability of nonuniform contractions

Let Y be a Banach space and denote by X the space of functions

$$\alpha : [r, 0]_{\mathbb{Z}} \to Y$$

equipped with the norm $\|\alpha\| = \max_{j \in [r,0]_{\mathbb{Z}}} |\alpha(j)|$, where $|\cdot|$ is the norm in Y. Given $N \in \mathbb{N}$, we are going to consider the cartesian products X^N and Y^N equipped with the supremum norm, i.e., for $\overline{\alpha} = (\alpha_1, \dots, \alpha_N) \in X^N$ and $\overline{y} = (y_1, \dots, y_N) \in Y^N$, we have

$$\|\overline{\alpha}\| = \max_{i=1,\dots,N} \|\alpha_i\| = \max_{i=1,\dots,N} \left(\max_{j=r,\dots,0} |\alpha_i(j)| \right)$$

and

$$|\overline{y}| = \max_{i=1,\dots,N} |y_i|.$$

Given a function $\overline{x}\colon [r, +\infty[_{\mathbb{Z}} \to Y^N$ we define, for each $m \in \mathbb{N}_0$, $\overline{x}_m \in X^N$ by

$$\overline{x}_m(j) = \overline{x}(m + j), \; j = r, r + 1, \dots, 0.$$

Let $\overline{f}_m : X^N \to Y^N$ be Lipschitz perturbations such that

$$\overline{f}_m(0) = 0 \tag{17}$$

for every $m \in \mathbb{N}_0$.

We are going to consider the following nonlinear delay difference equation

$$\overline{x}(m + 1) = \overline{L}_m \overline{x}_m + \overline{f}_m(\overline{x}_m), \; m \in \mathbb{N}_0, \tag{18}$$

where, for each $m \in \mathbb{N}_0$, $\overline{L}_m\colon X^N \to Y^N$ is given by

$$\overline{L}_m \overline{\alpha} = \left(L_m^{(1)} \alpha_1, L_m^{(2)} \alpha_2, \dots, L_m^{(N)} \alpha_N \right), \tag{19}$$

with $L_m^{(i)}\colon X \to Y$ a bounded linear operator for $i = 1, \dots, N$. For each $n \in \mathbb{N}_0$ and $\overline{\alpha} \in X^N$, we obtain a unique function

$$\overline{x}\colon [n + r, +\infty[_{\mathbb{Z}} \to Y^N,$$

denoted by $\overline{x}(\cdot, n, \overline{\alpha})$, such that $\overline{x}_n = \overline{\alpha}$ and (18) holds. Consequently, for each $(m, n) \in \Delta$, we can define the operator $\overline{\mathcal{F}}_{m,n} : X^N \to X^N$ given by

$$\overline{\mathcal{F}}_{m,n}(\overline{\alpha}) = \overline{x}_n(\cdot, n, \overline{\alpha}), \; \overline{\alpha} \in X^N.$$

Associated with equation (18), we will consider the linear difference equation

$$v_i(m + 1) = L_m^{(i)}(v_{i,m}) \tag{20}$$

$i = 1, \dots, N$, where $v_{i,m} \in X$ is defined, as usual, by $v_{i,m}(j) = v_i(m + j)$, $j = r, r + 1, \dots, 0$ and $L_m^{(i)}$ is given by (19). For each $n \in \mathbb{N}_0$ and $\alpha_i \in X$, we obtain a unique function $v_i : [n + r, +\infty[_{\mathbb{Z}} \to Y$, denoted by $v_i(\cdot, n, \alpha_i)$, such that $v_{i,n} = \alpha_i$ verifying (20).

For each $(m, n) \in \Delta$ and $i = 1, \dots, N$, we define the operator $\mathcal{A}_{m,n}^{(i)} : X \to X$ by

$$\mathcal{A}_{m,n}^{(i)} \alpha_i = v_{i,m}(\cdot, n, \alpha_i), \; \alpha_i \in X.$$

We can easily verify that

a) $\mathcal{A}_{m,n}^{(i)}$ is linear for each $(m, n) \in \Delta$;
b) $\mathcal{A}_{m,m}^{(i)} = \mathrm{Id}$;
c) $\mathcal{A}_{l,m}^{(i)} \mathcal{A}_{m,n}^{(i)} = \mathcal{A}_{l,n}^{(i)}$ for $(l, m), (m, n) \in \Delta$.

It is easy to prove by induction in m (see [1]) that

$$\overline{\mathcal{F}}_{m,n}(\overline{\alpha}) = \left(\mathcal{F}^{(1)}_{m,n}(\overline{\alpha}), \ldots, \mathcal{F}^{(N)}_{m,n}(\overline{\alpha})\right),$$

where, for $i = 1, \ldots, N$,

$$\mathcal{F}^{(i)}_{m,n}(\overline{\alpha}) = \mathcal{A}^{(i)}_{m,n}\alpha_i + \sum_{k=n}^{m-1} \mathcal{A}^{(i)}_{m,k+1}\Gamma f^{(i)}_k(\overline{x}_k),$$

$\overline{\alpha} = (\alpha_1, \ldots, \alpha_N)$, $\overline{f}_k(\overline{x}_k) = \left(f^{(1)}_k(\overline{x}_k), \ldots, f^{(N)}_k(\overline{x}_k)\right)$ and $\Gamma\colon Y \to X$ is defined by

$$\begin{aligned} \Gamma\, u\colon [r,0]_{\mathbb{Z}} &\to Y \\ j &\mapsto \Gamma u(j) = \begin{cases} u & \text{if } j = 0, \\ 0 & \text{if } j < 0, \end{cases} \end{aligned}$$

for all $u \in Y$.

Theorem 5. *Let $\overline{f}_m\colon X^N \to Y^N$ be Lipschitz functions such that* (17) *is satisfied and consider equation* (18)*. Let $\left(a^{(i)}_{m,n}\right)_{(m,n)\in\Delta}$, $i = 1, \ldots, N$, and let $\left(a'_{m,n}\right)_{(m,n)\in\Delta}$ be double sequences such that*

$$\|\mathcal{A}^{(i)}_{m,n}\| \leqslant a^{(i)}_{m,n} \leqslant a'_{m,n}$$

for all $(m,n) \in \Delta$, where $\mathcal{A}^{(i)}_{m,n}$ are the evolutions operators of equation (20) *derived from equation* (18)*. Assume that*

$$\lambda := \max_{i=1,\ldots,N} \left[\sup_{(m,n)\in\Delta} \left\{ \frac{1}{a'_{m,n}} \sum_{k=n}^{m-1} a^{(i)}_{m,k+1} \operatorname{Lip}(f^{(i)}_k) a'_{k,n} \right\} \right] < 1.$$

Then

$$\|\overline{\mathcal{F}}_{m,n}(\overline{\alpha})\| \leqslant \frac{1}{1-\lambda}\, a'_{m,n} \|\overline{\alpha}\|$$

for every $(m,n) \in \Delta$.

Proof. Given $n \in \mathbb{N}_0$ and $\overline{\alpha} \in X^N \setminus \{0\}$, let $\mathcal{C}_{n,\overline{\alpha}}$ be the space of functions

$$\overline{x}\colon [n+r, +\infty[_{\mathbb{Z}} \to Y^N$$

such that

$$\begin{aligned} &\overline{x}_n = \overline{\alpha} \\ &|\overline{x}|_{\mathcal{C}_{n,\overline{\alpha}}} = \sup\left\{ \frac{\|\overline{x}_m\|}{a'_{m,n}\|\overline{\alpha}\|} : m \geqslant n \right\} < +\infty. \end{aligned}$$

It is clear that $\mathcal{C}_{n,\overline{\alpha}}$ is a complete metric space with the metric defined by

$$d(\overline{x}, \overline{y}) = |\overline{x} - \overline{y}|_{\mathcal{C}_{n,\overline{\alpha}}} = \sup\left\{ \frac{\|\overline{x}_m - \overline{y}_m\|}{a'_{m,n}\,\|\overline{\alpha}\|} : m \geqslant n \right\}.$$

For every $\overline{x} \in \mathcal{C}_{n,\overline{\alpha}}$ we define

$$(J\overline{x})_m = \begin{cases} \overline{\alpha} & \text{if } m = n \\ (\xi^{(1)}_m, \ldots, \xi^{(N)}_m) & \text{if } m > n. \end{cases}$$

where, for each $i = 1, \dots, N$,

$$\xi_m^{(i)} = \mathcal{A}_{m,n}^{(i)}\alpha_i + \sum_{k=n}^{m-1} \mathcal{A}_{m,k+1}^{(i)}\Gamma f_k^{(i)}(\overline{x}_k).$$

Since for $m > n$ we have

$$\begin{aligned}
\|\,(J\overline{x})_m\,\| &\leqslant \max_{i=1,\dots,N}\left\{\|\mathcal{A}_{m,n}^{(i)}\alpha_i\| + \sum_{k=n}^{m-1}\|\mathcal{A}_{m,k+1}^{(i)}\|\|\Gamma f_k^{(i)}(\overline{x}_k)\|\right\}\\
&\leqslant \max_{i=1,\dots,N}\left\{a_{m,n}^{(i)}\|\alpha_i\| + \sum_{k=n}^{m-1} a_{m,k+1}^{(i)}\operatorname{Lip}(f_k^{(i)})\|\overline{x}_k\|\right\}\\
&\leqslant \max_{i=1,\dots,N}\left\{a'_{m,n}\|\alpha_i\| + \sum_{k=n}^{m-1} a_{m,k+1}^{(i)}\operatorname{Lip}(f_k^{(i)})a'_{k,n}\|\overline{\alpha}\|\,|\overline{x}|_{\mathcal{C}_{n,\overline{\alpha}}}\right\}\\
&\leqslant \left(1+\lambda\,|\overline{x}|_{\mathcal{C}_{n,\overline{\alpha}}}\right)a'_{m,n}\|\overline{\alpha}\|
\end{aligned}$$

and this implies that $J\overline{x}$ belongs to $\mathcal{C}_{n,\overline{\alpha}}$ and

$$|J\overline{x}|_{\mathcal{C}_{n,\overline{\alpha}}} \leqslant 1 + \lambda\,|\overline{x}|_{\mathcal{C}_{n,\overline{\alpha}}}\,. \tag{21}$$

Hence $J\colon \mathcal{C}_{n,\overline{\alpha}} \to \mathcal{C}_{n,\overline{\alpha}}$.

Now we prove that J is a contraction. For every $\overline{x}, \overline{y} \in \mathcal{C}_{n,\overline{\alpha}}$ and every $m > n$ we have

$$\begin{aligned}
\|\,(J\overline{x} - J\overline{y})_m\,\| &= \|\,(J\overline{x})_m - (J\overline{y})_m\,\|\\
&\leqslant \max_{i=1,\dots,N}\left\{\sum_{k=n}^{m-1}\|\mathcal{A}_{m,k+1}^{(i)}\|\|\Gamma f_k^{(i)}(\overline{x}_k) - \Gamma f_k^{(i)}(\overline{y}_k)\|\right\}\\
&\leqslant \max_{i=1,\dots,N}\left\{\sum_{k=n}^{m-1} a_{m,k+1}^{(i)}\operatorname{Lip}(f_k^{(i)})\|\overline{x}_k - \overline{y}_k\|\right\}\\
&\leqslant \max_{i=1,\dots,N}\left\{\sum_{k=n}^{m-1} a_{m,k+1}^{(i)}\operatorname{Lip}(f_k^{(i)})a'_{k,n}\|\overline{\alpha}\|\,|\overline{x}-\overline{y}|_{\mathcal{C}_{n,\overline{\alpha}}}\right\}\\
&\leqslant a'_{m,n}\|\overline{\alpha}\|\lambda\,|\overline{x}-\overline{y}|_{\mathcal{C}_{n,\overline{\alpha}}}
\end{aligned}$$

and thus

$$|J\overline{x} - J\overline{y}|_{\mathcal{C}_{n,\overline{\alpha}}} \leqslant \lambda\,|\overline{x}-\overline{y}|_{\mathcal{C}_{n,\overline{\alpha}}}\,.$$

Since $\lambda < 1$, J is a contraction and by the Banach fixed point theorem J has a unique fixed point $\overline{x}^*$. By (21) it follows that the fixed point $\overline{x}^*$ verifies

$$|\overline{x}^*|_{\mathcal{C}_{n,\overline{\alpha}}} \leqslant \frac{1}{1-\lambda}$$

and this proves the theorem. □

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