Lagrange multipliers and transport densities

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Information

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Abstract

In this paper we consider a stationary variational inequality with non-constant gradient constraint and we prove the existence of solution of a Lagrange multiplier, assuming that the bounded open not necessarily convex set Ω has a smooth boundary.
If the gradient constraint g is sufficiently smooth and satisfies $\Delta g^2 \leq 0$ and the source term belongs to $L^\infty(\Omega)$, we are able to prove that the Lagrange multiplier belongs to $L^q(\Omega)$, for $1 < q < \infty$, even in a very degenerate case. Fixing $q \geq 2$, the result is still true if Δg^2 is bounded from above by a positive sufficiently small constant that depends on Ω, q, $\min\limits_{\overline{\Omega}} g$ and $\max\limits_{\overline{\Omega}} g$.
Without the restriction on the sign of Δg^2 we are still able to find a Lagrange multiplier, now belonging to $L^\infty(\Omega)'$.
We also prove that if we consider the variational inequality with coercivity constant δ and constraint g, then the family of solutions $(\lambda^\delta, u^\delta)_{\delta>0}$ of our problem has a subsequence that converges weakly to (λ^0, u^0), which solves the transport equation.

Résumé

Dans cet article, nous considérons une inégalité variationnelle stationnaire avec une restriction non-constante sur le gradient et nous prouvons l'existence d'un multiplicateur de Lagrange, en supposant que l'ensemble ouvert et borné Ω, pas nécessairement convexe, a une frontière régulière.

Si la restriction du gradient g est suffisamment régulière et satisfait $\Delta g^2 \leq 0$ et le terme source appartient à $L^\infty(\Omega)$, nous pouvons prouver que le multiplicateur de Lagrange appartient à $L^q(\Omega)$, pour $1 < q < \infty$, même dans un cas très dégénéré. Si nous fixons $q \geq 2$, le résultat est aussi vrai si Δg^2 est borné par une constante positive et suffisamment petite qui dépend de Ω, q, $\min\limits_{\overline{\Omega}} g$ et $\max\limits_{\overline{\Omega}} g$.

Sans la restriction sur le signe de Δg^2 nous sommes capables de trouver un multiplicateur de Lagrange, maintenant appartenant à $L^\infty(\Omega)'$.

Nous montrons aussi que si l'on considère l'inégalité variationnelle avec la coercitivité constante δ et la restriction g, alors la famille des solutions $(\lambda^\delta, u^\delta)_{\delta>0}$ de notre problème a une sous-suite qui converge faiblement vers (λ^0, u^0), ce qui résout l'équation de transport.

1 Introduction

Let Ω be a bounded open subset of $\mathbb{R}^d$ with a Lipschitz boundary $\partial\Omega$. It is well-known that the elastic-plastic torsion problem, that consists in finding a function u belonging to the convex set

$$\mathbb{K} = \left\{ v \in H_0^1(\Omega) : |\nabla v| \leq 1 \text{ a.e. in } \Omega \right\}$$

such that

$$\int_\Omega \nabla u \cdot \nabla(v-u) \geq \int_\Omega f(v-u) \quad \forall v \in \mathbb{K} \tag{1}$$

has the following equivalent formulation using a Lagrange multiplier: to find (λ, u) belonging to a suitable functional space such that

$$\begin{aligned} -\nabla \cdot \big(\lambda \nabla u\big) = f &\text{ in } \Omega, \\ u = 0 &\text{ on } \partial\Omega, \\ |\nabla u| \leq 1 &\text{ in } \Omega, \\ \lambda \geq 1 &\text{ in } \Omega, \\ (\lambda - 1)(|\nabla u| - 1) = 0 &\text{ in } \Omega. \end{aligned} \tag{2}$$

The existence and uniqueness of solution of the variational inequality (1) is immediate. Brezis proved, in 1972, in [4] that there exists a unique $\lambda \in L^\infty(\Omega)$ (and, of course, a unique u) that solves problem (2) when f is constant and Ω is simply connected and Gerhardt ([12]) extended this result to multiply connected domains. In this framework, it is easy to show that $u \in W^{2,q}(\Omega)$, for $1 < q < \infty$.

The existence of $\lambda \in M(\Omega)$ was proved in [5] (for more general operators), assuming that Ω is convex and $f \in L^q(\Omega)$, with $q > d$. Here $M(\Omega)$ denotes the set of Radon measures. A recent result in this direction can be found in [1]. The existence of an essentially bounded Lagrange multiplier was proved, in [21], for the parabolic version of problem (2), with nonhomogeneous Dirichlet boundary condition and in [22] for a constraint g such that $\Delta g^2 \leq 0$ and homogeneous Dirichlet boundary condition, in both cases with f depending only on the variable t.

Other problems have variational or quasivariational inequalities with gradient constraint as models: sandpiles, river networks, superconductors in longitudinal domains (see [23, 17, 18, 19, 14, 20, 22, 3]). Some of these are evolutionary models and the gradient constraints are no longer constant. In the quasi-variational problems these constraints are part of the unknown. For, instance, the sandpile model presented in [17] is the following: suppose we have a rigid surface $y = h_0(x)$ were sand is dropped and we wish to determine the pile generated, $h(x,t)$. The function h will satisfy $\partial_t h - \nabla \cdot (\lambda \nabla h) = w$, where w is given and $\lambda \geq 0$ is an unknown scalar function. If the angle of repose of the material is α, then h must satisfy $|\nabla h| \leq \tan(\alpha)$ in the set $\{h(x,t) > h_0(x)\}$ and λ shall be zero in the set where $|\nabla h| < \tan(\alpha)$. If we define, for a given function $u(x,t)$,

$$M(u)(x,t) = \begin{cases} \tan(\alpha) & \text{if } u(x,t) > h_0(x), \\ \max\{\tan(\alpha), |\nabla h_0(x)|\} & \text{if } u(x,t) \leq h_0(x), \end{cases}$$

the sandpile problem is formulated as the evolutionary version of problem (1), with $\lambda \geq 1$ replaced by $\lambda \geq 0$ and the gradient constraint 1 replaced by the nonconstant unknown gradient constraint $M(h)$.

Existence of solution of problem (2) can be proved using duality theory as it is done, for example, in [17, 6, 7]. However, the approximation of the problem by a family of PDEs has the advantage of obtaining the Lagrange multiplier as a limit of a sequence that depends of the solutions of a family of PDEs (see [2]).

In this paper we are interested in the following problem: under suitable assumptions on f and g and for $\delta \geq 0$, to find $(\lambda^\delta, u^\delta) \in L^q(\Omega) \times W^{1,\infty}(\Omega)$, $1 < q < \infty$, such that

$$-\nabla \cdot \left(\lambda^\delta \nabla u^\delta\right) = f \quad \text{in } \Omega, \tag{3a}$$

$$u^\delta = 0 \quad \text{on } \partial\Omega, \tag{3b}$$

$$|\nabla u^\delta| \leq g \quad \text{in } \Omega, \tag{3c}$$

$$\lambda^\delta \geq \delta \quad \text{in } \Omega, \tag{3d}$$

$$(\lambda^\delta - \delta)(|\nabla u^\delta| - g) = 0 \quad \text{in } \Omega \tag{3e}$$

We are considering solution of (3) in the weak sense, more precisely, if $f \in L^q(\Omega)$ and $g \in L^\infty(\Omega)$ is positive, then $(\lambda^\delta, u^\delta)$ satisfies (3a) if

$$\int_\Omega \lambda^\delta \nabla u^\delta \cdot \nabla v = \int_\Omega f v \qquad \forall v \in W_0^{1,q'}(\Omega).$$

In the case $\delta = 0$ and $g \equiv 1$, this problem is an equivalent formulation for the Monge-Kantorovich mass transfer problem (see [11] for details about this problem). The sandpile model can be seen as an evolutionary Monge-Kantorovich problem.

De Pascale and Pratelli studied in [8, 9], and together with Evans in [10], the integrability of λ, assuming that the set Ω is convex, $g \equiv 1$, $f \in L^q(\Omega)$ for $2 \leq q \leq \infty$ and $\displaystyle\int_\Omega f = 0$. In [13], it can be found the version of problem (3), with $\delta = 0$ and variable constraint g, as modelling the optimal mass transport problem in inhomogeneous domains.

In [2], for any bounded and not necessarily convex subset Ω of $\mathbb{R}^d$ with Lipschitz boundary, it was proved that, if $f \in L^\infty(\Omega)$, $g \in W^{2,\infty}(\Omega)$ and $\delta = 1$ then problem (3) has a solution in $L^\infty(\Omega)' \times W^{1,\infty}(\Omega)$.

In this paper, we generalize the result obtained in [10] in two directions: we are able to treat the case were Ω is not convex and the constraint g on the gradient is not constant, as long as $g \in \mathscr{C}^2(\overline{\Omega})$ satisfies $\Delta g^2 \leq 0$. Assuming that $f \in L^\infty(\Omega)$, we prove that $\lambda^\delta \in L^q(\Omega)$, for any $q \in [1, \infty)$ and $\delta \geq 0$. The proof will be done approximating the problem (3), for $\delta > 0$, by a family of quasilinear elliptic problems. We adapt an idea used by Gerhardt in [12] for the case $\delta = 1$ and gradient constraint $g \equiv 1$, where he approximated the maximal monotone graph $j(s) = 1$ if $s < 0$ and $j(s) = [1, \infty)$ if $s = 0$ by a family of smooth increasing functions $j_\varepsilon(s) = 1$ if $s < 0$ and $j_\varepsilon(s) = e^{\frac{s}{\varepsilon}}$ if $s \geq \varepsilon$. An analogous idea is used in [10], concerning the problem with $\delta = 0$ and $g \equiv 1$. In fact, the authors approximate the maximal monotone graph $j(s) = 0$ if $s < 0$ and $j(s) = [0, \infty)$ if $s = 0$ by a sequence $j_k(s) = e^{\frac{ks}{2}}$, where $k \in \mathbb{N}$. The integral type estimates used in [10] are relevant in the proof of one result in this paper. In our case we have additional difficulties as g may be nonconstant and Ω not convex. As a consequence, we need to control $|\nabla u^{\varepsilon\delta}|$ on the boundary of Ω, being $u^{\varepsilon\delta}$ the solution of the approximating problem.

The solution of problem (2) when $\delta = 0$ is obtained as a limit of a subsequence of $\left((\lambda^\delta, u^\delta)\right)_{\delta > 0}$, when $\delta \to 0$. If we only assume that g is a positive function belonging to $W^{2,\infty}(\Omega)$ and drop the assumption $\Delta g^2 \leq 0$, we can prove, for $\delta > 0$, (exactly as in [2]) that there exists a solution $(\lambda^\delta, u^\delta) \in L^\infty(\Omega)' \times W^{1,\infty}(\Omega)$. The case $\delta = 0$ will also be obtained here letting $\delta \to 0$.

We present below the main results of this paper.

Theorem 1.1. *Suppose that the boundary of* Ω *is of class* $\mathscr{C}^2$, $\delta > 0$, $f \in L^\infty(\Omega)$, $g \in \mathscr{C}^2(\overline{\Omega})$ *is such that* $g > 0$ *and* $\triangle g^2 \leq 0$. *Then problem* (3) *has a solution*

$$(\lambda^\delta, u^\delta) \in L^q(\Omega) \times W^{1,\infty}(\Omega), \quad \textit{for any } 1 < q < \infty.$$

Theorem 1.2. *Under the assumptions of Theorem 1.1, there exists a subsequence of the solutions* $\big((\lambda^\delta, u^\delta)\big)_{\delta>0}$ *of problem* (3) *such that*

$$\begin{aligned} \lambda^\delta &\underset{\delta\to 0}{\rightharpoonup} \lambda^0 \quad \textit{in} \quad L^q(\Omega), \\ u^\delta &\underset{\delta\to 0}{\rightharpoonup} u^0 \quad \textit{in} \quad W^{1,\infty}(\Omega). \end{aligned}$$

Besides, (λ^0, u^0) *solves problem* (3) *for* $\delta = 0$.

Our proof that (λ^0, u^0) solves problem (3) for $\delta = 0$, when g is constant and Ω is convex, is an alternative proof of the result obtained in [10] (in this case we only need to assume that $f \in L^q(\Omega)$, for $2 \leq q \leq \infty$). If Ω is not convex, we still prove the same result, imposing $f \in L^\infty(\Omega)$, in order to control a term on the boundary that we cannot neglect.

In Remark 2.8 we will observe that, for fixed $2 \leq q < \infty$, there exists a constant $C = C(\Omega, q, \max\limits_{\overline{\Omega}} g, \min\limits_{\overline{\Omega}} g) > 0$ such that the assumption $\Delta g^2 \leq 0$ can be replace by $\Delta g^2 \leq C$ in the above two theorems.

We are able to prove that $(\lambda^\delta, u^\delta)$ solves a weaker version of problem (3), with no restrictions on Δg^2. We present below this weaker version of problem (3): for $\delta \geq 0$, to find $(\lambda^\delta, u^\delta) \in L^\infty(\Omega)' \times W^{1,\infty}(\Omega)$ such that

$$-\nabla \cdot (\lambda^\delta \nabla u^\delta) = f \quad \text{in } \mathscr{D}'(\Omega), \tag{4a}$$

$$u^\delta = 0 \quad \text{on } \partial\Omega, \tag{4b}$$

$$|\nabla u^\delta| \leq g \quad \text{in } \Omega, \tag{4c}$$

$$\lambda^\delta \geq \delta \quad \text{in } \mathscr{D}'(\Omega), \tag{4d}$$

$$(\lambda^\delta - \delta)(|\nabla u^\delta| - g) = 0 \quad \text{in } \mathscr{D}'(\Omega) \tag{4e}$$

Theorem 1.3. *Assume that* Ω *is a bounded open subset of* $\mathbb{R}^d$ *with Lipschitz boundary. Given* $f \in L^2(\Omega)$ *and* $g \in W^{2,\infty}(\Omega)$, *with* $g > 0$,

a) *if* $\delta > 0$, *problem* (4) *has a solution*

$$(\lambda^\delta, u^\delta) \in L^\infty(\Omega)' \times W^{1,\infty}(\Omega);$$

b) *at least for a subsequence of* $\big((\lambda^\delta, u^\delta)\big)_{\delta>0}$ *of solutions of problem* (4), *we have*

$$\begin{aligned} \lambda^\delta &\underset{\delta\to 0}{\rightharpoonup} \lambda^0 \quad \textit{in} \quad L^\infty(\Omega)', \\ u^\delta &\underset{\delta\to 0}{\rightharpoonup} u^0 \quad \textit{in} \quad W^{1,\infty}(\Omega). \end{aligned}$$

Besides, (λ^0, u^0) *solves problem* (4) *for* $\delta = 0$.

In Section 2 we define a family of approximating quasilinear elliptic equations and we obtain *a priori* estimates that will be important later. Section 3 and 4 are dedicated to the proofs of Theorem 1.1 and Theorem 1.2, respectively. The last section treats the case without restriction on the sign of Δg^2, proving Theorem 1.3.

We would like to remark that the proof in [22] for the existence of Lagrange multiplier belonging to $L^\infty\big(\Omega\times(0,T)\big)$ for the parabolic version of problem (2) with $\delta=1$, uses a completely different approach. It uses the strong maximum principle applied to an equation satisfied by $|\nabla u^{\varepsilon\delta}|^2$ to obtain the estimate

$$|\nabla u^{\varepsilon\delta}|^2\le g^2+\varepsilon C\qquad\text{a.e. in }\Omega\times(0,T).$$

As we assume here that f is not constant, the reasoning used in [22] is no longer valid in Ω, although we can use part of it to control $|\nabla u^{\varepsilon\delta}|$ on the boundary of Ω.

Along the paper we will use C to designate different constants, whenever it is not necessary to be specific and we will use C_0, C_1, C_2, C_* when the value or the dependence of the constants on the data matters. If in a proof several different constants appear, they are also named differently to avoid confusion. We will use D_q to denote that the constant depends on the L^q exponent.

2 The approximating problem

In this paper Ω is a bounded open subset of $\mathbb{R}^d$ with Lipschitz boundary, $\boldsymbol{n}$ denotes the outward unit normal vector to $\partial\Omega$ and δ_0 is a fixed positive number. We will use the summation convention for repeated indexes, and we will denote the partial derivative of a function u with respect to the variable x_i by u_{x_i}. The normal derivative of a function u on $\partial\Omega$ will be represented by $\frac{\partial u}{\partial \boldsymbol{n}}$.

For $0<\varepsilon<1$, $0<\delta\le\delta_0$ and $r>\max\{1,\frac{d}{2}\}$ to be chosen later, independently of ε and δ, consider

$$\begin{array}{rccl} k_{\varepsilon\delta}: & \mathbb{R} & \longrightarrow & \mathbb{R}. \\ & s & \mapsto & \begin{cases}\delta & \text{if } s<0\\ \delta+\left(\frac{s}{\varepsilon}\right)^r & \text{if } s\ge 0\end{cases}\end{array}\tag{5}$$

We observe that the function $k_{\varepsilon\delta}$ is of class $\mathscr{C}^1$ and $k_{\varepsilon\delta}\ge\delta>0$.

Theorem 2.1. *Let $f\in L^2(\Omega)$ and $g\in L^\infty(\Omega)$ be such that $m=\min\limits_{\overline{\Omega}} g>0$. Then problem*

$$\begin{cases} -\nabla\cdot\big(k_{\varepsilon\delta}(|\nabla u^{\varepsilon\delta}|^2-g^2)\nabla u^{\varepsilon\delta}\big)=f^\varepsilon & \textit{on } \Omega\\ u^{\varepsilon\delta}=0 & \textit{in } \partial\Omega,\end{cases}\tag{6}$$

where f^ε is the regularization by convolution of f, has a unique solution $u^{\varepsilon\delta}$ belonging to $\mathscr{C}^2(\Omega)\cap\mathscr{C}(\overline{\Omega})$.

Proof. The proof of this theorem is straightforward. Details can be found in [2]. □

In what follows, whenever there is no confusion, we will write $k_{\varepsilon\delta}$ instead of $k_{\varepsilon\delta}(|\nabla u^{\varepsilon\delta}|^2-g^2)$.

We are going to obtain some *a priori* estimates for the solution $u^{\varepsilon\delta}$, independent of ε and δ. In some estimates we may assume, without mention, that ε is smaller then 1, if necessary. We observe that when $f\in L^\infty(\Omega)$ then $\|f^\varepsilon\|_{L^\infty(\Omega)}\le\|f\|_{L^\infty(\Omega)}$.

Lemma 2.2. *Let $f \in L^{(2r)'}(\Omega)$, $g \in L^\infty(\Omega)$ be such that $m = \min_{\overline{\Omega}} g > 0$. If $u^{\varepsilon\delta}$ solves problem (6) then,*

$$\|\nabla u^{\varepsilon\delta}\|_{\boldsymbol{L}^{2r}(\Omega)} \leq C_1, \tag{7}$$

$$\|u^{\varepsilon\delta}\|_{L^\infty(\Omega)} \leq C_2, \tag{8}$$

where C_1, C_2 are constants depending only on $\|f\|_{L^{(2r)'}(\Omega)}$, $\|g\|_{L^{2r}(\Omega)}$ and m.

Proof. Let $A_{\varepsilon\delta} = \big\{x \in \Omega : |\nabla u^{\varepsilon\delta}(x)| \geq g(x)\big\}$. Then we have, in $A_{\varepsilon\delta}$,

$$\begin{aligned}|\nabla u^{\varepsilon\delta}(x)|^{2r} &\leq 2^{r-1}\big(|\nabla u^{\varepsilon\delta}(x)|^2 - g^2\big)^r + 2^{r-1}g^{2r}\\ &\leq 2^{r-1}\varepsilon^r k_{\varepsilon\delta}\big(|\nabla u^{\varepsilon\delta}(x)|^2 - g^2\big) + 2^{r-1}g^{2r}.\end{aligned}$$

Then

$$\begin{aligned}\int_{A_{\varepsilon\delta}} |\nabla u^{\varepsilon\delta}(x)|^{2r} &\leq 2^{r-1}\varepsilon^r \int_{A_{\varepsilon\delta}} k_{\varepsilon\delta} + 2^{r-1}\int_\Omega g^{2r}\\ &\leq \frac{2^{r-1}}{m^2}\int_\Omega k_{\varepsilon\delta}|\nabla u^{\varepsilon\delta}|^2 + 2^{r-1}\|g\|^{2r}_{L^{2r}(\Omega)}\\ &= \frac{2^{r-1}}{m^2}\int_\Omega f^\varepsilon u^{\varepsilon\delta} + 2^{r-1}\|g\|^{2r}_{L^{2r}(\Omega)}\\ &\leq C\left(\|f\|^{(2r)'}_{L^{(2r)'}(\Omega)} + 1\right) + \tfrac{1}{2}\int_\Omega |\nabla u^{\varepsilon\delta}|^{2r} + 2^{r-1}\|g\|^{2r}_{L^{2r}(\Omega)},\end{aligned}$$

being the last inequality true by application of Young and Poincaré inequalities.

Noticing that

$$\begin{aligned}\int_\Omega |\nabla u^{\varepsilon\delta}|^{2r} &= \int_{\Omega\setminus A_{\varepsilon\delta}} |\nabla u^{\varepsilon\delta}|^{2r} + \int_{A_{\varepsilon\delta}} |\nabla u^{\varepsilon\delta}|^{2r}\\ &\leq \int_\Omega g^{2r} + C\left(\|f\|^{(2r)'}_{L^{(2r)'}} + 1\right) + \tfrac{1}{2}\int_\Omega |\nabla u^{\varepsilon\delta}|^{2r} + 2^{r-1}\|g\|^{2r}_{L^{2r}(\Omega)},\end{aligned}$$

we get the inequality (7).

The estimate (8) is an immediate consequence of the Sobolev inclusion $W^{1,2r}(\Omega) \hookrightarrow C^{0,1-\frac{d}{2r}}(\overline{\Omega})$. ☐

With the purpose of controlling $|\nabla u^{\varepsilon\delta}|$ on $\partial\Omega$, we introduce some definitions. Let

$$\mathbb{K}^\varepsilon_\nabla = \{v \in H^1_0(\Omega) : |\nabla v|^2 \leq g^2 + \varepsilon \text{ a.e. in } \Omega\}.$$

We define an auxiliary function ψ^ε as follows:

$$\psi^\varepsilon(x) = \bigvee_{v \in \mathbb{K}^\varepsilon_\nabla} v(x).$$

Define, for $x, z \in \overline{\Omega}$,

$$D_\varepsilon(x,z) = \inf\left\{\int_0^{T_0} \sqrt{g(\xi(s))^2 + \varepsilon}\, ds : T_0 > 0,\ \xi : [0,T_0] \to \overline{\Omega} \text{ smooth}, \xi(0) = x,\ \xi(T_0) = z,\ \|\xi'\| \leq 1\right\}.$$

D_ε is a distance in Ω and

$$\psi^\varepsilon = D^\varepsilon(x, \partial\Omega), \quad \psi^\varepsilon_{|\partial\Omega} = 0, \quad |\nabla\psi^\varepsilon|^2 = g^2 + \varepsilon \quad \text{and} \quad \Delta\psi^\varepsilon \leq C_0, \tag{9}$$

where C_0 is a positive constant that depends on $\|g\|_{\mathscr{C}^2(\overline{\Omega})}$ and is independent of ε (for details see [16, Theorem 5.1, Theorem 8.2]).

Lemma 2.3. *Assume that $\partial\Omega$ is of class $\mathscr{C}^2$, $f \in L^\infty(\Omega)$ and $g \in \mathscr{C}^2(\overline{\Omega})$ is such that $g \geq m > 0$. Let $u^{\varepsilon\delta}$ be the solution of problem (6), for sufficiently large r, depending only on the given data. Then there exists a positive constant C_* and $\varepsilon_0 > 0$, depending only on $\|f\|_{L^\infty(\Omega)}$, $\|g\|_{\mathscr{C}^2(\overline{\Omega})}$ and m, such that, for $\varepsilon \in (0, \varepsilon_0)$, we have*

$$\forall \delta > 0 \ \forall x \in \partial\Omega \quad |\nabla u^{\varepsilon\delta}(x)|^2 \leq g^2(x) + C_*\varepsilon.$$

Proof. Consider the operator

$$L(v) = -\nabla \cdot \left(k_{\varepsilon\delta}(|\nabla v|^2 - g^2)\nabla v\right).$$

We are going to construct a supersolution and a subsolution of problem (6). The calculations below are based on related calculations presented in [22, Proposition 3.6], where a different function $k_{\varepsilon\delta}$ is used.

For $\varepsilon > 0$ let ψ^ε be the function defined in (9). In this proof, for simplicity, we will omit the superscripts and subscripts $\varepsilon\delta$ and we will denote $\|\cdot\|_{L^\infty(\Omega)}$ simply by $\|\cdot\|_\infty$.

Consider $\eta_\varepsilon(s) = s + \varepsilon\left(1 - e^{-Bs}\right)$, where B is a positive constant to be chosen later, and let $\phi = \eta_\varepsilon(\psi)$. Notice that $\phi_{|\partial\Omega} = 0$. Then

- $|\nabla\phi|^2 = \eta_\varepsilon'(\psi)^2(g^2 + \varepsilon)$;
- $\frac{|\nabla\phi|^2 - g^2}{\varepsilon} \geq 1$;
- $\eta_\varepsilon'(s) \leq 2$, if $\varepsilon B \leq 1$;
- $\frac{|\nabla\phi|^2 - g^2}{\varepsilon} \leq 3g^2B + 4$, if $\varepsilon B \leq 1$.

We intend to find $B > 0$ and r such that $L(\phi) \geq \|f^\varepsilon\|_\infty \geq f^\varepsilon = L(u^{\varepsilon\delta})$.

Notice that

$$L(\phi) = \left(\frac{|\nabla\phi|^2 - g^2}{\varepsilon}\right)^{r-1}\left(\frac{2r}{\varepsilon}\left(-\phi_{x_i}\phi_{x_j}\phi_{x_ix_j} + gg_{x_i}\phi_{x_i}\right) - \frac{|\nabla\phi|^2 - g^2}{\varepsilon}\triangle\phi\right) - \delta\triangle\phi,$$

and

- $\phi_{x_ix_j} = \eta_\varepsilon''(\psi)\psi_{x_i}\psi_{x_j} + \eta_\varepsilon'(\psi)\psi_{x_ix_j}$;
- $\triangle\phi = \eta_\varepsilon''(\psi)|\nabla\psi|^2 + \eta_\varepsilon'(\psi)\triangle\psi = \eta_\varepsilon''(\psi)(g^2 + \varepsilon) + \eta_\varepsilon'(\psi)\triangle\psi \leq \eta_\varepsilon'(\psi)\triangle\psi$;
- $\psi_{x_i}\psi_{x_j}\psi_{x_ix_j} = gg_{x_i}\psi_{x_i} = g\nabla g \cdot \nabla\psi$;
- $gg_{x_i}\phi_{x_i} = \eta_\varepsilon'(\psi)gg_{x_i}\psi_{x_i} = \eta_\varepsilon'(\psi)g\nabla g \cdot \nabla\psi$;
- $\phi_{x_i}\phi_{x_j}\phi_{x_ix_j} = \eta_\varepsilon'(\psi)^2\eta_\varepsilon''(\psi)(g^2 + \varepsilon)^2 + \eta_\varepsilon'(\psi)^3 g\nabla\psi \cdot \nabla g$.

Then, if $c = \|g\|_\infty\|\nabla g\|_\infty\sqrt{\|g\|_\infty^2 + 1}$, and noticing that $\eta_\varepsilon' > 1$,

$$\begin{aligned}
-\phi_{x_i}\phi_{x_j}\phi_{x_ix_j} + gg_{x_i}\phi_{x_i} &= \eta_\varepsilon'(\psi)\left(g\nabla g \cdot \nabla\psi\left(1 - \eta_\varepsilon'(\psi)^2\right) - \eta_\varepsilon'(\psi)\eta_\varepsilon''(\psi)(g^2 + \varepsilon)^2\right) \\
&\geq \eta_\varepsilon'(\psi)\left(c\left(1 - \eta_\varepsilon'(\psi)^2\right) - \eta_\varepsilon'(\psi)\eta_\varepsilon''(\psi)\, m^4\right) \\
&= \eta_\varepsilon'(\psi)Be^{-B\psi}\varepsilon\left(Bm^4 - 2c + \varepsilon B^2 m^4 e^{-B\psi} - c\varepsilon Be^{-B\psi}\right) \\
&\geq \eta_\varepsilon'(\psi)Be^{-B\|\psi\|_\infty}\varepsilon\left(Bm^4 - 3c\right), \quad \text{if } \varepsilon B \leq 1.
\end{aligned}$$

Recalling that (see (9)) $\Delta\psi \leq C_0$,

$$L(\phi) \geq \eta_\varepsilon'(\psi)\left(2r\eta_\varepsilon'(\psi)Be^{-B\|\psi\|_\infty}\left(Bm^4 - 3c\right) - \left(\delta_0 + 3B\|g\|_\infty^2 + 4\right)C_0\right), \quad \text{if } \varepsilon B \leq 1,$$

and then, choosing $\varepsilon_0=\frac{m^4}{3c+1}$, $B=\frac{3c+1}{m^4}$ and $r\geq\frac{\left(\delta_0+3B\|g\|_\infty^2+4\right)C_0+\|f\|_\infty}{2\,Be^{-B\|\psi\|_\infty}}$ then, for $0<\varepsilon\leq\varepsilon_0$ we have $L(\phi)\geq\|f\|_\infty\geq\|f^\varepsilon\|_\infty\geq L(u^{\varepsilon\delta})$ and, by consequence, $L(-\phi)\leq L(u^{\varepsilon\delta})$. As $\phi_{|\partial\Omega}=0$, we have $\left|u^{\varepsilon\delta}\right|\leq\phi$ in Ω. Since $\partial\Omega$ is of class $\mathscr{C}^2$ then, for all $x\in\partial\Omega$, $|\nabla u^{\varepsilon\delta}(x)|^2\leq|\nabla\phi(x)|^2\leq g^2(x)+(3B\|g\|_\infty^2+4)\,\varepsilon$. □

Lemma 2.4. *Assume that $\partial\Omega$ is of class $\mathscr{C}^2$, $f\in L^\infty(\Omega)$ and $g\in\mathscr{C}^2(\overline{\Omega})$ is such that $m=\min\limits_{\overline{\Omega}} g>0$. Let $u^{\varepsilon\delta}$ be the solution of problem (6), for $0<\varepsilon\leq\varepsilon_0$, being ε_0 and r chosen in Lemma 2.3. Let*

$$I=\frac{\partial u^{\varepsilon\delta}}{\partial\boldsymbol{n}}\Delta u^{\varepsilon\delta}-u^{\varepsilon\delta}_{x_i}\frac{\partial u^{\varepsilon\delta}_{x_i}}{\partial\boldsymbol{n}}. \tag{10}$$

Then $\|I\|_{L^\infty(\partial\Omega)}$ is bounded independently of ε and δ.

Proof. For simplicity, we omit the superscripts $\varepsilon\delta$. Given $x_0\in\partial\Omega$, as I is invariant by rotations, we may assume that the outward unit normal vector at x_0 is $(0,\dots,0,1)$ in a system of coordinates $(y_1,\dots,y_d)$ and the boundary of Ω is, in a neighborhood of x_0, given by the equation

$$y_d=\omega(y_1,\dots,y_{d-1}),$$

being ω a $\mathscr{C}^2$ function. Following the calculations done in [15, pp. 20], we get

$$I=-\left(\tfrac{\partial u}{\partial\boldsymbol{n}}\right)^2\,\Delta\omega.$$

As ω is of class $\mathscr{C}^2$ and $\left|\frac{\partial u}{\partial\boldsymbol{n}}\right|^2=|\nabla u|^2$ on $\partial\Omega$, then the conclusion follows by the preceding lemma. □

Lemma 2.5. *Let $2\leq q<\infty$, $f\in L^q(\Omega)$ and $g\in W^{2,\infty}(\Omega)$ be such that $m=\min\limits_{\overline{\Omega}} g>0$. Then there exist positive constants $C,D>0$, independent of ε and δ, such that*

$$\int_\Omega k_{\varepsilon\delta}^q\leq C\left(\int_\Omega|f|^q+1+\delta_0\right)+D\left(\int_\Omega k_{\varepsilon\delta}^q\Delta g^2-\int_{\partial\Omega}k_{\varepsilon\delta}^q\frac{\partial g^2}{\partial\boldsymbol{n}}-2\int_{\partial\Omega}k_{\varepsilon\delta}^q\,I\right), \tag{11}$$

where I is defined in (10).

Proof. We adapt the proof of [10, Theorem 3.1]. There are two main differences:

1. the function $k_{\varepsilon\delta}(|\nabla u^{\varepsilon\delta}|^2-g^2)$ defined here (for ε small) corresponds to the function $\sigma_k=e^{\frac{k}{2}(|\nabla u_k|^2-1)}$, $k\in\mathbb{N}$ of [10]. In their paper, the authors use, explicitly, that $\partial_{x_j}\sigma_k=\partial_{x_j}u_k\,\partial^2_{x_ix_j}u_k\,\sigma_k$. However, the important property is the monotonicity of the scalar function $k_{\varepsilon\delta}(s)$ or, in their case, the monotonicity of $e^{\frac{k}{2}s}$;

2. the estimate (11) has three additional terms, two involving derivatives of g and one that we cannot neglect when Ω is not convex.

For for the sake of completeness, as there are several differences, we present here all the calculations.

To simplify the notations we will write k and u instead of $k_{\varepsilon\delta}(|\nabla u^{\varepsilon\delta}|^2-g^2)$ and $u^{\varepsilon\delta}$. From (6) we obtain, multiplying by $k^{q-1}u$ and integrating by parts

$$\int_\Omega(k\nabla u)\cdot\nabla\left(k^{q-1}u\right)=\int f^\varepsilon k^{q-1}u.$$

As $(k\nabla u)\cdot\nabla\left(k^{q-1}u\right)=k^q\,|\nabla u|^2+(q-1)k^{q-1}\,u\nabla u\cdot\nabla k$, we get, using (8),

$$\int_\Omega k^q\,|\nabla u|^2+(q-1)\int_\Omega k^{q-1}\,u\nabla u\cdot\nabla k=\int_\Omega f^\varepsilon k^{q-1}u\leq C_2\int_\Omega|f^\varepsilon|\,k^{q-1}\leq C_3\int_\Omega|f^\varepsilon|^q+\tfrac{m^2}{2}\int_\Omega k^q. \tag{12}$$

Notice that $m^2k^q \leq k^q\,|\nabla u|^2 + m^2\delta$, as $k=\delta$ if $|\nabla u|<m$, and then by (12) and using (8),

$$m^2\int_\Omega k^q \leq 2C_3\int_\Omega |f^\varepsilon|^q + 2(q-1)C_2\int_\Omega k^{q-1}|\nabla u\cdot\nabla k| + 2\delta m^2\,|\Omega|. \tag{13}$$

Multiplying (6) by $-\nabla\cdot(k^{q-1}\nabla u)$ we obtain

$$\begin{aligned}\int_\Omega \nabla\cdot(k\nabla u)\nabla\cdot(k^{q-1}\nabla u) &= -\int_\Omega f^\varepsilon\,\nabla\cdot(k^{q-1}\nabla u)\\ &= \int_\Omega f^\varepsilon\left(k^{q-2}\nabla\cdot(-k\nabla u) - (q-2)k^{q-2}\nabla u\cdot\nabla k\right)\\ &= \int_\Omega |f^\varepsilon|^2k^{q-2} - (q-2)\int_\Omega f^\varepsilon\,k^{q-2}\nabla u\cdot\nabla k\\ &\leq \int_\Omega |f^\varepsilon|^2k^{q-2} + (q-2)\int_\Omega |f^\varepsilon|\,k^{q-2}\,|\nabla u\cdot\nabla k|\,. \end{aligned}\tag{14}$$

But

$$\begin{aligned}\int_\Omega (ku_{x_i})_{x_i}\left(k^{q-1}u_{x_j}\right)_{x_j} &= -\int_\Omega ku_{x_i}\left(k^{q-1}u_{x_j}\right)_{x_jx_i} + \int_{\partial\Omega} ku_{x_i}n_i\left(k^{q-1}u_{x_j}\right)_{x_j}\\ &= \int_\Omega (ku_{x_i})_{x_j}\left(k^{q-1}u_{x_j}\right)_{x_i} - \int_{\partial\Omega}\left[ku_{x_i}n_j\left(k^{q-1}u_{x_j}\right)_{x_i} - ku_{x_i}n_i\left(k^{q-1}u_{x_j}\right)_{x_j}\right]\\ &= \int_\Omega (ku_{x_i})_{x_j}\left(k^{q-1}u_{x_j}\right)_{x_i} + \int_{\partial\Omega} k^qu_{x_i}\left[n_iu_{x_jx_j} - n_ju_{x_ix_j}\right]\\ &\quad + \int_{\partial\Omega}(q-1)k^{q-1}u_{x_i}u_{x_j}\left[n_ik_{x_j} - n_jk_{x_i}\right]\\ &= \int_\Omega (ku_{x_i})_{x_j}\left(k^{q-1}u_{x_j}\right)_{x_i} + \int_{\partial\Omega} k^q\,I. \end{aligned}$$

Recall that k denotes $k_{\varepsilon\delta}(|\nabla u^{\varepsilon\delta}|^2-g^2)$, being $k_{\varepsilon\delta}$ the real function defined in (5). Then

$$\begin{aligned}k_{x_j}u_{x_i}u_{x_jx_i} &= \tfrac12\,k'(-)(|\nabla u|^2-g^2)_{x_j}\left(|\nabla u|^2\right)_{x_j}\\ &= \underbrace{\tfrac12\,k'(-)(|\nabla u|^2-g^2)_{x_j}\left(|\nabla u|^2-g^2\right)_{x_j}}_{\geq 0} + \tfrac12\underbrace{k'(-)(|\nabla u|^2-g^2)_{x_j}}_{=k_{x_j}}\left(g^2\right)_{x_j}\\ &\geq \tfrac12 k_{x_j}\left(g^2\right)_{x_j}. \end{aligned}\tag{15}$$

$$\begin{aligned}\int_\Omega (ku_{x_i})_{x_i}\left(k^{q-1}u_{x_j}\right)_{x_j} &= \int_\Omega (ku_{x_i})_{x_j}\left(k^{q-1}u_{x_j}\right)_{x_i} + \int_{\partial\Omega} k^q\,I\\ &= \int_\Omega \left(ku_{x_ix_j} + k_{x_j}u_{x_i}\right)\left(k^{q-1}u_{x_jx_i} + (q-1)k^{q-2}k_{x_i}u_{x_j}\right) + \int_{\partial\Omega} k^q\,I\\ &= \int_\Omega \left(k^q|D^2u|^2 + (q-1)k^{q-2}\,|\nabla u\cdot\nabla k|^2 + qk^{q-1}k_{x_j}u_{x_i}u_{x_jx_i}\right) + \int_{\partial\Omega} k^q\,I\\ &\geq \int_\Omega \left((q-1)k^{q-2}\,|\nabla u\cdot\nabla k|^2 + \tfrac12\nabla k^q\cdot\nabla g^2\right) + \int_{\partial\Omega} k^q\,I, \end{aligned}\tag{16}$$

because, by (15), we have

$$qk^{q-1}k_{x_j}u_{x_i}u_{x_jx_i} \geq \tfrac12 qk^{q-1}k_{x_j}\left(g^2\right)_{x_j} = \tfrac12\nabla k^q\cdot\nabla g^2.$$

From (16), (14) and Young inequality, we obtain

$$\int_\Omega\left((q-1)k^{q-2}\,|\nabla u\cdot\nabla k|^2 + \tfrac12\nabla k^q\cdot\nabla g^2\right) \leq C\int_\Omega |f^\varepsilon|^2k^{q-2} + \tfrac{q-1}{2}\int_\Omega k^{q-2}|\nabla u\cdot\nabla k|^2 - \int_{\partial\Omega} k^q\,I$$

and consequently

$$\begin{aligned}\int_\Omega k^{q-2}\,|\nabla u\cdot\nabla k|^2 &\le \tfrac{2C}{q-1}\int_\Omega |f^\varepsilon|^2 k^{q-2} - \tfrac{1}{q-1}\left(\int_\Omega \nabla k^q\cdot\nabla g^2 + 2\int_{\partial\Omega} k^q\, I\right)\\ &= \tfrac{2C}{q-1}\int_\Omega |f^\varepsilon|^2 k^{q-2} + \tfrac{1}{q-1}\left(\int_\Omega k^q\Delta g^2 - \int_{\partial\Omega} k^q\tfrac{\partial\, g^2}{\partial \boldsymbol{n}} - 2\int_{\partial\Omega} k^q\, I\right).\end{aligned}$$

Returning to (13), we conclude that

$$\int_\Omega k^q \le \tfrac{2C_3}{m^2}\int_\Omega |f^\varepsilon|^q + 2(q-1)\tfrac{C_2}{m^2}\int_\Omega k^{q-1}|\nabla u\cdot\nabla k| + 2\delta\,|\Omega| \tag{17}$$

$$\le \tfrac{2C_3}{m^2}\int_\Omega |f^\varepsilon|^q + \tfrac{1}{3}\int_\Omega k^q + C_4\int_\Omega k^{q-2}|\nabla u\cdot\nabla k|^2 + 2\delta\,|\Omega| \tag{18}$$

$$\le \tfrac{2C_3}{m^2}\int_\Omega |f^\varepsilon|^q + \tfrac{1}{3}\int_\Omega k^q + C_4\left(\tfrac{2C}{q-1}\int_\Omega |f^\varepsilon|^2 k^{q-2} + \tfrac{1}{q-1}\left(\int_\Omega k^q\Delta g^2 - \int_{\partial\Omega} k^q\tfrac{\partial\, g^2}{\partial \boldsymbol{n}} - 2\int_{\partial\Omega} k^q\, I\right)\right) + 2\delta\,|\Omega| \tag{19}$$

$$\le \tfrac{2}{3}\int_\Omega k^q + C_5\int_\Omega |f^\varepsilon|^q + \tfrac{C_4}{q-1}\left(\int_\Omega k^q\Delta g^2 - \int_{\partial\Omega} k^q\tfrac{\partial\, g^2}{\partial \boldsymbol{n}} - 2\int_{\partial\Omega} k^q\, I\right) + 2\delta_0\,|\Omega|. \tag{20}$$

□

Lemma 2.6. *Let Ω, f and g as in Lemma 2.3, with $\Delta g^2 \le 0$. Then, for ε_0 as in Lemma 2.3 and $1 < q < \infty$, there exists a positive constant D_q such that, for all $0 < \varepsilon \le \varepsilon_0$ and $0 < \delta \le \delta_0$*

$$\|k_{\varepsilon\delta}(|\nabla u^{\varepsilon\delta}|^2 - g^2)\|^q_{L^q(\Omega)} \le D_q, \tag{21}$$

Proof. It is enough to prove the result for $q \ge 2$. By Lemma 2.3, there exists a positive constant C_* such that

$$\left|\int_{\partial\Omega} k^q_{\varepsilon\delta}\tfrac{\partial\, g^2}{\partial \boldsymbol{n}}\right| \le (\delta_0 + C_*^r)^q\left|\int_{\partial\Omega}\tfrac{\partial\, g^2}{\partial \boldsymbol{n}}\right|, \qquad \left|\int_{\partial\Omega} k^q_{\varepsilon\delta}\, I\right| \le (\delta_0 + C_*^r)^q\left|\int_{\partial\Omega} I\right|,$$

The conclusion follows by Lemma 2.5 and Lemma 2.4, as $g \in \mathscr{C}^2(\overline{\Omega})$.

□

Lemma 2.7. *Let Ω, f and g as in Lemma 2.3 with $\Delta g^2 \le 0$. Then, for ε_0 as in Lemma 2.3 there exists a positive constant C such that, for all $0 < \varepsilon \le \varepsilon_0$ and $0 < \delta \le \delta_0$*

$$\|k_{\varepsilon\delta}(|\nabla u^{\varepsilon\delta}|^2 - g^2)\nabla u^{\varepsilon\delta}\|_{L^2(\Omega)} \le C. \tag{22}$$

Proof. Using Hölder inequality, we have

$$\int_\Omega k^2_{\varepsilon\delta}|\nabla u^{\varepsilon\delta}|^2 \le \left(\int_\Omega k^{2r'}_{\varepsilon\delta}\right)^{\frac{1}{r'}}\left(\int_\Omega |\nabla u^{\varepsilon\delta}|^{2r}\right)^{\frac{1}{r}},$$

and the result is obtained by applying Lemma 2.6 and Lemma 2.2. □

Remark 2.8. *The assumption $\Delta g^2 \le 0$ in Theorems 1.1 and 1.2 can be weakened if we fix $q \ge 2$, imposing that*

$$\Delta g^2 < C = C(\Omega, q, \max_{\overline{\Omega}} g, \min_{\overline{\Omega}} g),$$

where C is a certain positive constant.

In fact, observing the last inequality of the proof of Lemma 2.5, instead of negleting the term $\frac{C_4}{q-1}\int_\Omega k^q \Delta g^2$ (*which is nonpositive with the sign assumption in* Δg^2), *we can pass this term to the left hand side and obtain the same conclusion, as long as*

$$\tfrac{1}{3} - \tfrac{C_4}{q-1}\Delta g^2 > 0.$$

So, we need to carefully estimate C_4. *To do so, we start by estimating the constants* C_1 *and* C_2 *in Lemma 2.2. Simple calculations show that we can take, for* r *large enough,* $C_1 = \big(2|\Omega|\big)^{\frac{1}{2r}}(\sqrt{2}\|g\|_{L^\infty(\Omega)}+2)$ *and, using the Sobolev imbedding* $W^{1,2r}(\Omega) \subset C^{0,1-\frac{d}{2r}}(\overline{\Omega})$, *we have*

$$\|u^{\varepsilon\delta}\|_{L^\infty(\Omega)} \leq \left(\tfrac{2}{d}\right)^{\frac{1}{2r}} \left(\frac{|\Omega|}{|B(0,1)|}\right)^{\frac{1}{d}} \left(\frac{2r-1}{2r-d}\right)^{\frac{2r-1}{2r}} (\sqrt{2}\|g\|_{L^\infty(\Omega)}+2) := D_r.$$

We need to pay attention to the fact that $u^{\varepsilon\delta}$ *also depends on* r, *but this is not a problem because we fixed the space were we are obtaining the estimate. Since*

$$\lim_{r\to\infty} D_r = \left(\frac{|\Omega|}{|B(0,1)|}\right)^{\frac{1}{d}} (\sqrt{2}\|g\|_{L^\infty(\Omega)}+2),$$

choosing $C_2 > \left(\frac{|\Omega|}{|B(0,1)|}\right)^{\frac{1}{d}}(\sqrt{2}\|g\|_{L^\infty(\Omega)}+2)$, *we are sure that for* r *suficientlty large,*

$$\|u^{\varepsilon\delta}\|_{L^\infty(\Omega)} \leq C_2.$$

The constant C_4 *in* (17) *is obtained by applying the inequality of Young and we have*

$$C_4 = \frac{3(q-1)^2\, C_2^2}{m^4}.$$

Finally, recalling that $\min\limits_{\overline{\Omega}} g = m$ *and* $\max\limits_{\overline{\Omega}} g = \|g\|_{L^\infty(\Omega)}$, *the assumption we are looking for is the following*

$$\Delta g^2 < C := \frac{\big(\min\limits_{\overline{\Omega}} g\big)^4}{9(q-1)\left(\frac{|\Omega|}{|B(0,1)|}\right)^{\frac{2}{d}}(\sqrt{2}\max\limits_{\overline{\Omega}} g+2)^2}.$$

We observe that, if we have the above assumption in Theorems 1.1 and 1.2, we shall impose the same assumption in Lemmas 2.6 and 2.7.

3 Existence of solution for a Lagrange multiplier problem

From now on q will denote any number of the interval $(1,\infty)$.

In this section we will prove, for $\delta > 0$ and convenient assumptions, that a subsequence of $\big((k_{\varepsilon\delta}, u^{\varepsilon\delta})\big)_\varepsilon$ converges, in a suitable space, to a pair of functions $(\lambda^\delta, u^\delta)$ which solve (3).

The estimates obtained in the previous section allows us to obtain a pair of functions $(\lambda^\delta, u^\delta)$, which is the weak limit of a subsequence of $\big((k_{\varepsilon\delta}, u^{\varepsilon\delta})\big)_\varepsilon$, the first component converging in $L^q(\Omega)$, the second one in $W^{1,2r}(\Omega)$. There is a difficulty in the identification of the limit, when $\varepsilon \to 0$, of the product $k_{\varepsilon\delta}\nabla u^{\varepsilon\delta}$, because we have a product of two sequences, both converging weakly. This identification will be done using the approximating problems, following part of the strategy used in [2], where the *a priori* estimates obtained for $k_{\varepsilon\delta}$ were much weaker.

Proof of Theorem 1.1. Let $u^{\varepsilon\delta}$ denote the solution of problem (6) and $k_{\varepsilon\delta}$ denote the function $k_{\varepsilon\delta}(|\nabla u^{\varepsilon\delta}|^2 - g^2)$. By the *a priori* estimates (7), (21) and (22) we conclude that, at least for a subsequence of $(u^{\varepsilon\delta})_\varepsilon$, we have,

$$\begin{aligned} k_{\varepsilon\delta} &\underset{\delta\to 0}{\rightharpoonup} \lambda^\delta \quad \text{in} \quad L^q(\Omega),\\ \nabla u^{\varepsilon\delta} &\underset{\delta\to 0}{\rightharpoonup} \nabla u^\delta \quad \text{in} \quad \boldsymbol{L}^{2r}(\Omega),\\ k_{\varepsilon\delta}\nabla u^{\varepsilon\delta} &\underset{\delta\to 0}{\rightharpoonup} \boldsymbol{\chi}^\delta \quad \text{in} \quad \boldsymbol{L}^2(\Omega). \end{aligned}$$

As $k_{\varepsilon\delta} \geq \delta$ then $\lambda^\delta \geq \delta$ and obviously $u^\delta = 0$ on $\partial\Omega$. Let us prove (3c). Define $A_\varepsilon = \{x \in \Omega : |\nabla u^{\varepsilon\delta}|^2 > g^2 + \sqrt{\varepsilon}\}$ and observe that

$$|A_\varepsilon| = \int_{A_\varepsilon} 1 \leq \int_{A_\varepsilon} \varepsilon^{\frac{r}{2}} \Big(\frac{|\nabla u^{\varepsilon\delta}|^2 - g^2}{\varepsilon}\Big)^r \leq \varepsilon^{\frac{r}{2}} \int_\Omega k_{\varepsilon\delta} \leq \varepsilon^{\frac{r}{2}} D_1,$$

using (21). Then $|A_\varepsilon| \underset{\varepsilon\to 0}{\longrightarrow} 0$ and so

$$\begin{aligned} \int_\Omega \big(|\nabla u^\delta|^2 - g^2\big)^+ &= \int_\Omega \liminf_{\varepsilon\to 0} \big(|\nabla u^{\varepsilon\delta}|^2 - g^2 - \sqrt{\varepsilon}\big)^+\\ &\leq \liminf_{\varepsilon\to 0} \int_\Omega \big(|\nabla u^{\varepsilon\delta}|^2 - g^2 - \sqrt{\varepsilon}\big)^+\\ &\leq \liminf_{\varepsilon\to 0} \||\nabla u^{\varepsilon\delta}|^2 - g^2 - \sqrt{\varepsilon}\|_{L^r(\Omega)} |A_\varepsilon|^{\frac{r-1}{r}} = 0, \end{aligned}$$

by (7). Then we have $|\nabla u^\delta| \leq g$, proving (3c).

Using the first equation of problem (6), we obtain, for any $v \in H^1_0(\Omega)$, that

$$\int_\Omega k_{\varepsilon\delta} \nabla u^{\varepsilon\delta} \cdot \nabla v = \int_\Omega f v \tag{23}$$

and, letting $\varepsilon \to 0$

$$\int_\Omega \boldsymbol{\chi}^\delta \cdot \nabla v = \int_\Omega f v.$$

Taking $v = u^{\varepsilon\delta}$ in (23), we obtain

$$\lim_{\varepsilon\to 0} \int_\Omega k_{\varepsilon\delta} |\nabla u^{\varepsilon\delta}|^2 = \lim_{\varepsilon\to 0} \int_\Omega f u^{\varepsilon\delta} = \int_\Omega f u^\delta = \int_\Omega \boldsymbol{\chi}^\delta \cdot \nabla u^\delta. \tag{24}$$

Noticing that

$$(k_{\varepsilon\delta} - \delta)\big(|\nabla u^{\varepsilon\delta}|^2 - g^2\big) \geq 0$$

we have

$$\int_\Omega k_{\varepsilon\delta} |\nabla u^{\varepsilon\delta}|^2 - \int_\Omega k_{\varepsilon\delta}\, g^2 \geq \delta \int_\Omega \big(|\nabla u^{\varepsilon\delta}|^2 - g^2\big).$$

So, letting $\varepsilon \to 0$ we obtain, recalling that $\liminf\limits_{\varepsilon\to 0} \|\nabla u^{\varepsilon\delta}\|_{\boldsymbol{L}^2(\Omega)} \geq \|\nabla u^\delta\|_{\boldsymbol{L}^2(\Omega)}$,

$$\begin{aligned} \int_\Omega \boldsymbol{\chi}^\delta \cdot \nabla u^\delta &\geq \int_\Omega (\lambda^\delta - \delta)\, g^2 + \delta \int_\Omega |\nabla u^\delta|^2\\ &= \int_\Omega (\lambda^\delta - \delta)\, (g^2 - |\nabla u^\delta|^2) + \int_\Omega \lambda^\delta\, |\nabla u^\delta|^2\\ &\geq \int_\Omega \lambda^\delta\, |\nabla u^\delta|^2, \end{aligned}$$

because $\lambda^\delta \geq \delta$ and $|\nabla u^\delta| \leq g$.

On the other hand, using (23) and (24),

$$0 \leq \int_\Omega k_{\varepsilon\delta}|\nabla(u^{\varepsilon\delta} - u^\delta)|^2 = \int_\Omega k_{\varepsilon\delta}|\nabla u^{\varepsilon\delta}|^2 - 2\int_\Omega k_{\varepsilon\delta}\,\nabla u^{\varepsilon\delta}\cdot\nabla u^\delta + \int_\Omega k_{\varepsilon\delta}|\nabla u^\delta|^2$$
$$\underset{\varepsilon\to 0}{\longrightarrow} -\int_\Omega \boldsymbol{\chi}^\delta\cdot\nabla u^\delta + \int_\Omega \lambda^\delta|\nabla u^\delta|^2.$$

So

$$\int_\Omega \boldsymbol{\chi}^\delta\cdot\nabla u^\delta = \int_\Omega \lambda^\delta|\nabla u^\delta|^2,$$

and then

$$\delta\int_\Omega |\nabla(u^{\varepsilon\delta} - u^\delta)|^2 \leq \int_\Omega k_{\varepsilon\delta}|\nabla(u^{\varepsilon\delta} - u^\delta)|^2 \underset{\varepsilon\to 0}{\longrightarrow} 0, \tag{25}$$

concluding that $u^{\varepsilon\delta} \underset{\varepsilon\to 0}{\longrightarrow} u^\delta$ strongly in $H^1_0(\Omega)$. As $(k_{\varepsilon\delta} - \delta)(g - |\nabla u^{\varepsilon\delta}|)^+ = 0$ then, letting $\varepsilon \to 0$, we obtain

$$0 = (\lambda^\delta - \delta)(g - |\nabla u^\delta|)^+ = (\lambda^\delta - \delta)(g - |\nabla u^\delta|),$$

proving (3e).

Let $v \in \mathscr{D}(\Omega)$. Using the strong convergence of $(\nabla u^{\varepsilon\delta})_\varepsilon$ to ∇u^δ in $\boldsymbol{L}^2(\Omega)$ and the weak convergence of $(k_{\varepsilon\delta})_\varepsilon$ to λ^δ in $L^2(\Omega)$, we conclude from (23) that

$$\int_\Omega fv = \int_\Omega \lambda^\delta\nabla u^\delta\cdot\nabla v.$$

By density, the above equality is also true for any $v \in W^{1,q'}(\Omega)$, for any $1 < q < \infty$, proving (3a). □

4 Existence of solution for a transport densities problem

We will see, in this section, that $\big((\lambda^\delta, u^\delta)\big)_{\delta>0}$ converges, when $\delta \to 0$, in a suitable space, to a pair of functions (λ^0, u^0) which solves problem (3) with $\delta = 0$.

Proof of Theorem 1.2. Let $(\lambda^\delta, u^\delta)$ be the subsequence of $(k_{\varepsilon\delta}, u^{\varepsilon\delta})_\varepsilon$ in the proof of Theorem 1.1. Then

$$\|\lambda^\delta\|_{L^q(\Omega)} \leq \liminf_{\varepsilon\to 0}\|k_{\varepsilon\delta}\|_{L^q(\Omega)}, \qquad \|\nabla u^\delta\|_{L^\infty(\Omega)} \leq \|g\|_{L^\infty(\Omega)}, \qquad \|\lambda^\delta\nabla u^\delta\|_{L^q(\Omega)} \leq \|\lambda^\delta\|_{L^q(\Omega)}\|\nabla u^\delta\|_{\boldsymbol{L}^\infty(\Omega)}$$

and so, at least for a subsequence of $(\lambda^\delta, u^\delta)_{\delta>0}$, we have

$$\lambda^\delta \underset{\delta\to 0}{\rightharpoonup} \lambda^0 \quad \text{in} \quad L^q(\Omega),$$
$$\nabla u^\delta \underset{\delta\to 0}{\rightharpoonup} \nabla u^0 \quad \text{in} \quad \boldsymbol{L}^\infty(\Omega) \text{ weak-*},$$
$$\lambda^\delta\nabla u^\delta \underset{\delta\to 0}{\rightharpoonup} \boldsymbol{\chi}^0 \quad \text{in} \quad \boldsymbol{L}^q(\Omega).$$

Observing that, for any $v \in W^{1,q'}_0(\Omega)$

$$\int_\Omega \lambda^\delta\nabla u^\delta\cdot\nabla v = \int_\Omega fv,$$

we get, letting $\delta \to 0$,

$$\int_\Omega \boldsymbol{\chi}^0\cdot\nabla v = \int_\Omega fv.$$

Since $u^\delta=0$ on $\partial\Omega$, $|\nabla u^\delta|\le g$ and $\lambda^\delta\ge 0$ then $u^0=0$ on $\partial\Omega$, $|\nabla u^0|\le g$ and $\lambda^0\ge 0$, proving (3b), (3c) and (3d). Recalling that $(\lambda^\delta-\delta)(g^2-|\nabla u^\delta|^2)=0$, we have

$$\lim_{\delta\to 0}\int_\Omega \lambda^\delta g^2-\int_\Omega\lambda^\delta|\nabla u^\delta|^2=\lim_{\delta\to 0}\delta\int_\Omega(g^2-|\nabla u^\delta|^2)=0, \tag{26}$$

because $g^2-|\nabla u^\delta|^2$ is bounded in $L^1(\Omega)$. Observing that

$$\int_\Omega\lambda^\delta|\nabla u^\delta|^2=\int_\Omega fu^\delta\underset{\delta\to 0}{\longrightarrow}\int_\Omega fu^0=\int_\Omega\boldsymbol{\chi}^0\cdot\nabla u^0,$$

we conclude from (26) that

$$\int_\Omega\lambda^0 g^2=\int_\Omega\boldsymbol{\chi}^0\cdot\nabla u^0 \tag{27}$$

and because $|\nabla u^0|\le g$ and $\lambda^0\ge 0$ then

$$\int_\Omega\lambda^0|\nabla u^0|^2\le\int_\Omega\lambda^0 g^2,$$

concluding that

$$\int_\Omega\lambda^0|\nabla u^0|^2\le\int_\Omega\boldsymbol{\chi}^0\cdot\nabla u^0. \tag{28}$$

But

$$\begin{aligned}0\le\lim_{\delta\to 0}\int_\Omega\lambda^\delta|\nabla(u^\delta-u^0)|^2&=\lim_{\delta\to 0}\int_\Omega\left(\lambda^\delta|\nabla u^\delta|^2-2\lambda^\delta\nabla u^\delta\cdot\nabla u^0+\lambda^\delta|\nabla u^0|^2\right)\\&=-\int_\Omega\boldsymbol{\chi}^0\cdot\nabla u^0+\int_\Omega\lambda^0|\nabla u^0|^2\end{aligned} \tag{29}$$

and so, using (28) and (29) we get

$$\int_\Omega\lambda^0|\nabla u^0|^2=\int_\Omega\boldsymbol{\chi}^0\cdot\nabla u^0 \tag{30}$$

and as a consequence, we also obtain

$$\lim_{\delta\to 0}\int_\Omega\lambda^\delta|\nabla(u^\delta-u^0)|^2=0. \tag{31}$$

Using (27) and (30), we conclude that

$$\int_\Omega\lambda^0(g^2-|\nabla u^0|)=0$$

and so, as $\lambda^0\ge 0$ and $g^2-|\nabla u^0|\ge 0$, we must have $\lambda^0(g^2-|\nabla u^0|)=0$ a.e. in Ω, concluding (3e).

Given $v\in\mathscr{D}(\Omega)$,

$$\begin{aligned}\left|\int_\Omega\lambda^\delta\nabla(u^\delta-u^0)\cdot\nabla v\right|&\le\int_\Omega\lambda^\delta|\nabla(u^\delta-u^0)||\nabla v|\\&\le\left(\int_\Omega\lambda^\delta|\nabla(u^\delta-u^0)|^2\right)^{\frac12}\left(\int_\Omega\lambda^\delta|\nabla v|^2\right)^{\frac12}\underset{\delta\to 0}{\longrightarrow}0,\end{aligned} \tag{32}$$

because $(\lambda^\delta)_{\delta>0}$ is bounded in $L^1(\Omega)$ and using (31). Then we have

$$\int_\Omega\lambda^\delta\nabla u^\delta\cdot\nabla v=\int_\Omega\lambda^\delta\nabla(u^\delta-u^0)\cdot\nabla v+\int_\Omega\lambda^\delta\nabla u^0\cdot\nabla v=\int_\Omega fv$$

and, letting $\delta\to 0$ and using (32), we conclude (3a), i.e.,

$$\int_\Omega\lambda^0\nabla u^0\cdot\nabla v=\int_\Omega fv$$

and, by density, the same is true for all $v\in W_0^{1,p'}(\Omega)$. □

5 The case when $\Delta g^2 \not\leq 0$

In this section we will prove Theorem 1.3 which is a generalization of Theorem 4.1 of [2]. The new result, here, is the existence of solution $(\lambda^0, u^0) \in L^\infty(\Omega)' \times W^{1,\infty}(\Omega)$ of problem (3), since the proof for the case $\delta > 0$ is identical to the case $\delta = 1$ done in the referred paper. For the sake of completeness and because we are able to simplify the proof presented in [2], we will show here a brief sketch of it when $\delta > 0$. By the inequality (25) we have strong convergence in $H_0^1(\Omega)$ of $u^{\varepsilon\delta}$ to u^δ, when ε tends to zero. So, the uniform boundedeness of $u^{\varepsilon\delta}$ in $H^2_{loc}(\Omega)$, that was used in the proof of Theorem 4.1 of [2] is not necessary. On the other hand, to be able to treat the case $\delta = 0$, we need to prove some *a priori* estimates.

In this section we will also use again $k_{\varepsilon\delta}$ to represent $k_{\varepsilon\delta}(|\nabla u^{\varepsilon\delta}|^2 - g^2)$, whenever there is no confusion.

Proposition 5.1. *Suppose that $\delta \in (0, \delta_0)$, $f \in L^2(\Omega)$ and $g \in L^\infty(\Omega)$ is such that $m = \min_{\overline{\Omega}} g > 0$. Then there exists a positive constant C, independent of ε and δ, such that the solution $u^{\varepsilon\delta}$ of the approximated problem (4) verifies*

$$\|k_{\varepsilon\delta}(|\nabla u^{\varepsilon\delta}|^2 - g^2)|\nabla u^{\varepsilon\delta}|^2\|_{L^1(\Omega)} \leq C,$$
$$\|k_{\varepsilon\delta}(|\nabla u^{\varepsilon\delta}|^2 - g^2)\|_{L^\infty(\Omega)'} \leq C,$$
$$\|k_{\varepsilon\delta}(|\nabla u^{\varepsilon\delta}|^2 - g^2)\nabla u^{\varepsilon\delta}\|_{\boldsymbol{L}^\infty(\Omega)'} \leq C.$$

Proof. Using $u^{\varepsilon\delta}$ as test function in (6), applying Hölder inequality and (8), we get

$$\int_\Omega k_{\varepsilon\delta}|\nabla u^{\varepsilon\delta}|^2 = \int_\Omega f u^{\varepsilon\delta} \leq \|f\|_{L^1(\Omega)}\|u^{\varepsilon\delta}\|_{L^\infty(\Omega)} \leq C_2\|f\|_{L^1(\Omega)}.$$

Since $k_{\varepsilon\delta}(|\nabla u^{\varepsilon\delta}|^2 - g^2) \leq \delta + \frac{1}{m^2}k_{\varepsilon\delta}(|\nabla u^{\varepsilon\delta}|^2 - g^2)|\nabla u^{\varepsilon\delta}|^2$, then

$$\int_\Omega k_{\varepsilon\delta} \leq \delta_0|\Omega| + \frac{1}{m^2}C_2\|f\|_{L^1(\Omega)}.$$

So,

$$\|k_{\varepsilon\delta}(|\nabla u^{\varepsilon\delta}|^2 - g^2)\|_{L^\infty(\Omega)'} = \sup_{\|v\|_{L^\infty(\Omega)}\leq 1}\int_\Omega k_{\varepsilon\delta}v \leq \sup_{\|v\|_{L^\infty(\Omega)}\leq 1}\|k_{\varepsilon\delta}\|_{L^1(\Omega)}\|v\|_{L^\infty(\Omega)} = \|k_{\varepsilon\delta}\|_{L^1(\Omega)}.$$

Finally, since

$$\begin{aligned}
\|k_{\varepsilon\delta}(|\nabla u^{\varepsilon\delta}|^2 - g^2)\nabla u^{\varepsilon\delta}\|_{\boldsymbol{L}^\infty(\Omega)'} &= \sup_{\|\boldsymbol{v}\|_{\boldsymbol{L}^\infty(\Omega)}\leq 1}\int_\Omega k_{\varepsilon\delta}\nabla u^{\varepsilon\delta}\cdot\boldsymbol{v} \\
&\leq \sup_{\|\boldsymbol{v}\|_{\boldsymbol{L}^\infty(\Omega)}\leq 1}\|k_{\varepsilon\delta}|\nabla u^{\varepsilon\delta}|^2\|^{\frac12}_{L^1(\Omega)}\|k_{\varepsilon\delta}\|^{\frac12}_{L^1(\Omega)}\|\boldsymbol{v}\|_{\boldsymbol{L}^\infty(\Omega)} \\
&\leq \|k_{\varepsilon\delta}|\nabla u^{\varepsilon\delta}|^2\|^{\frac12}_{L^1(\Omega)}\|k_{\varepsilon\delta}\|^{\frac12}_{L^1(\Omega)},
\end{aligned}$$

the conclusion follows. □

Proof of Theorem 1.3. From Proposition 5.1, applying the Banach-Alaoglu-Bourbaki Theorem, we have, at least for a subsequence,

$$k_{\varepsilon\delta} \underset{\varepsilon\to 0}{\longrightarrow} \lambda^\delta \quad \text{in } L^\infty(\Omega)'$$
$$k_{\varepsilon\delta}\nabla u^{\varepsilon\delta} \underset{\varepsilon\to 0}{\longrightarrow} \boldsymbol{\chi}^\delta \quad \text{in } \boldsymbol{L}^\infty(\Omega)'.$$

We observe that, using arguments as in the proof of Theorem 1.1, replacing the terms $\int_\Omega \boldsymbol{\chi}^\delta \cdot \boldsymbol{v}$ by the duality pairing $\langle \boldsymbol{\chi}^\delta, \boldsymbol{v}\rangle_{\boldsymbol{L}^\infty(\Omega)'\times \boldsymbol{L}^\infty(\Omega)}$ and $\int_\Omega \lambda^\delta v$ by the duality pairing $\langle \lambda^\delta, v\rangle_{L^\infty(\Omega)'\times L^\infty(\Omega)}$, we have

$$\langle \boldsymbol{\chi}^\delta, \nabla u^\delta\rangle_{\boldsymbol{L}^\infty(\Omega)'\times \boldsymbol{L}^\infty(\Omega)} = \langle \lambda^\delta, |\nabla u^\delta|^2\rangle_{L^\infty(\Omega)'\times L^\infty(\Omega)}$$

and so also that

$$\nabla u^{\varepsilon\delta} \underset{\varepsilon\to 0}{\longrightarrow} \nabla u^\delta \quad \text{in } \boldsymbol{L}^2(\Omega).$$

Using the duality pairings instead of the corresponding integrals, all the steps of the proof of this theorem follows as in the proof of Theorem 1.1.

The proof of the case $\delta = 0$ is more delicate. Using Proposition 5.1, there exists a positive constant C such that

$$\|\lambda^\delta\|_{L^\infty(\Omega)'} \leq \liminf_{\varepsilon\to 0} \|k_{\varepsilon\delta}\|_{L^\infty(\Omega)'} \leq C$$
$$\|\boldsymbol{\chi}^\delta\|_{\boldsymbol{L}^\infty(\Omega)'} \leq \liminf_{\varepsilon\to 0} \|k_{\varepsilon\delta}\nabla u^{\varepsilon\delta}\|_{\boldsymbol{L}^\infty(\Omega)'} \leq C$$

and so, again by Banach-Alaoglu-Bourbaki Theorem, at least for a subsequence, we have

$$\lambda^\delta \underset{\delta\to 0}{\longrightarrow} \lambda^0 \quad \text{in } L^\infty(\Omega)',$$
$$\boldsymbol{\chi}^\delta \underset{\delta\to 0}{\longrightarrow} \boldsymbol{\chi}^0 \quad \text{in } \boldsymbol{L}^\infty(\Omega)'.$$

Since, for all $v \in W^{1,\infty}_0(\Omega)$ we have

$$\langle \boldsymbol{\chi}^\delta, \nabla v\rangle_{\boldsymbol{L}^\infty(\Omega)'\times \boldsymbol{L}^\infty(\Omega)} = \int_\Omega f v,$$

letting $\delta \to 0$, we obtain

$$\langle \boldsymbol{\chi}^0, \nabla v\rangle_{\boldsymbol{L}^\infty(\Omega)'\times \boldsymbol{L}^\infty(\Omega)} = \int_\Omega f v.$$

As $\|\nabla u^\delta\|_{\boldsymbol{L}^\infty(\Omega)} \leq \|g\|_{L^\infty(\Omega)}$ we also have

$$\nabla u^\delta \underset{\delta\to 0}{\rightharpoonup} \nabla u^0 \quad \text{in } \boldsymbol{L}^2(\Omega).$$

We are no longer able to prove the strong convergence in $\boldsymbol{L}^2(\Omega)$ of $(\nabla u^\delta)_{\delta>0}$ to ∇u^0, when $\delta \to 0$. However, we can still prove that (λ^0, u^0) solves problem (4).

Denoting below $\langle \cdot, \cdot\rangle_{L^\infty(\Omega)'\times L^\infty(\Omega)}$ simply by $\langle \cdot, \cdot\rangle$, we have

$$\begin{aligned} 0 = \langle \lambda^\delta - \delta, g^2 - |\nabla u^\delta|^2\rangle &= \lim_{\delta\to 0}\langle \lambda^\delta - \delta, g^2 - |\nabla u^\delta|^2\rangle \\ &= \lim_{\delta\to 0}\langle \lambda^\delta, g^2\rangle - \int_\Omega f u^\delta - \delta\int_\Omega (g^2 - |\nabla u^\delta|^2) \\ &= \langle \lambda^0, g^2\rangle - \int_\Omega f u^0, \end{aligned} \tag{33}$$

since the function $g^2 - |\nabla u^\delta|^2$ is uniformly bounded in $L^1(\Omega)$.

Observing that, for any $\delta \in (0, \delta_0)$, $u^\delta = 0$ on $\partial\Omega$, $|\nabla u^{\varepsilon\delta}| \leq g$ and $\lambda^\delta \geq \delta$ in $\mathscr{D}'(\Omega)$ then the same is true for u^0, proving (4b), (4c) and (4d) for $\delta = 0$.

By (4c) we have

$$\langle\lambda^0,|\nabla u^0|^2\rangle \le \langle\lambda^0,g^2\rangle = \int_\Omega fu^0 = \langle\boldsymbol{\chi}^0,\nabla u^0\rangle_{\boldsymbol{L}^\infty(\Omega)'\times\boldsymbol{L}^\infty(\Omega)}. \tag{34}$$

But, arguing as in the proof of Theorem 1.2,

$$\begin{aligned}0 \le \lim_{\delta\to0}\lim_{\varepsilon\to0}\int_\Omega k_{\varepsilon\delta}|\nabla(u^{\varepsilon\delta}-u^0)|^2 &= -\int_\Omega fu^0 + \langle\lambda^0,|\nabla u^0|^2\rangle_{L^\infty(\Omega)'\times L^\infty(\Omega)}\\ &= -\langle\boldsymbol{\chi}^0,\nabla u^0\rangle_{\boldsymbol{L}^\infty(\Omega)'\times\boldsymbol{L}^\infty(\Omega)} + \langle\lambda^0,|\nabla u^0|^2\rangle_{L^\infty(\Omega)'\times L^\infty(\Omega)},\end{aligned} \tag{35}$$

From (34) and (35), we obtain

$$\langle\boldsymbol{\chi}^0,\nabla u^0\rangle_{\boldsymbol{L}^\infty(\Omega)'\times\boldsymbol{L}^\infty(\Omega)} = \langle\lambda^0,|\nabla u^0|^2\rangle_{L^\infty(\Omega)'\times L^\infty(\Omega)} \tag{36}$$

and, as a consequence,

$$\lim_{\delta\to0}\lim_{\varepsilon\to0}\int_\Omega k_{\varepsilon\delta}|\nabla(u^{\varepsilon\delta}-u^0)|^2 = 0. \tag{37}$$

Going back to (33) and using (36), we conclude that

$$\langle\lambda^0, g^2-|\nabla u^0|^2\rangle = 0,$$

proving (4e), since λ^0 and $g^2-|\nabla u^0|^2$ are nonnegative.

Given $v\in\mathscr{D}(\Omega)$, we have

$$\int_\Omega k_{\varepsilon\delta}\nabla u^{\varepsilon\delta}\cdot\nabla v = \int_\Omega fv$$

and so

$$\int_\Omega k_{\varepsilon\delta}\nabla(u^{\varepsilon\delta}-u^0)\cdot\nabla v + \int_\Omega k_{\varepsilon\delta}\nabla u^0\cdot\nabla v = \int_\Omega fv. \tag{38}$$

But we know that

$$0\le\lim_{\delta\to0}\lim_{\varepsilon\to0}\left|\int_\Omega k_{\varepsilon\delta}\nabla(u^{\varepsilon\delta}-u^0)\cdot\nabla v\right| \le \lim_{\delta\to0}\lim_{\varepsilon\to0}\left(\int_\Omega k_{\varepsilon\delta}|\nabla(u^{\varepsilon\delta}-u^0)|^2\right)^{\frac12}\left(\int_\Omega k_{\varepsilon\delta}|\nabla v|^2\right)^{\frac12} = 0,$$

applying (37). So, letting in (38) $\varepsilon\to0$ and after $\delta\to0$ we prove that

$$\langle\lambda^0\nabla u^0,\nabla v\rangle = \int_\Omega fv, \qquad \forall v\in\mathscr{D}(\Omega),$$

concluding (4a) and the proof. □

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