Title: Design and Analysis of Cross Vaults Along History

Angelo Gaetani[a,]*, Giorgio Monti[b], Paulo B. Lourenço[c], Giancarlo Marcari[d]

Running head/short title: Cross Vaults in History

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[a] PhD student, Department of Structural Engineering and Geotechnics, Sapienza University of Rome, Faculty of Engineering, 00184 Rome, Italy. Phone: +39 0644585405; Fax: +39 064884852; E-mail: angelo.gaetani@uniroma1.it

[b] Full Professor, Department of Structural Engineering and Geotechnics, Sapienza University of Rome, Faculty of Architecture, 00197 Rome, Italy. Phone: +39 0649919197; Fax: +39 0649919150; E-mail: giorgio.monti@uniroma1.it

[c] Full Professor, ISISE, Department of Civil Engineering, University of Minho, Campus de Azurem, 4800-058, Guimarães, Portugal. Phone: +351 253510209; Fax: +351 253510217; E-mail: pbl@civil.uminho.pt

[d] Postdoctoral Research Fellow, Department of Structural Engineering and Geotechnics, Sapienza University of Rome, Faculty of Engineering, 00184 Rome, Italy. Phone: +39 0644585399; Fax: +39 064884852; E-mail: giancarlo.marcari@uniroma1.it

* Corresponding author

Abstract: The history of cross vaults began almost two thousand years ago with a widespread use during the Middle Ages and Renaissance, becoming nowadays one of the most diffused and fascinating structural typologies of the European building cultural heritage. However, conversely to the undeniable excellence achieved by the ancient masons, the structural behaviour of these elements is still at the centre of the scientific debate. In this regard, with the aim of reviewing the knowledge on this subject as a concise and valuable support for researchers involved in conservation of historical buildings, with a focus on design rules and structural analysis, the present study firstly introduces the cross vaults from a historical perspective, by describing the evolution of the main geometrical shapes together with basic practical rules used to size them. Then, the paper deals with the subsequent advancements in structural analysis methods of vaults, until the development of modern limit analysis.

Keywords: vaulted structures; cross vault; historical development; practical sizing rules; limit analysis.

1. Introduction

Largely adopted since the Roman Empire Age and exalted by the Gothic Architecture, cross vaults are spread all over the European countries in churches, cloisters and palaces. However, despite the relevance and the long-lasting history, which clearly proves an accurate design process, it is only from the 18th century that scholars have tried to tackle the problem of analysing the structural behaviour of this element. Nowadays, although a certain consensus has been reached in case of only gravity loads, deeper and more comprehensive studies are still dedicated to this topic.

In ancient times, the design process followed what would be presently defined as "a rudimentary scientific approach", i.e. trial-and-error. Considering each building as a scaled specimen of a new one to be built, the ancient builders achieved a proper competence made up by simple geometrical definitions, gathered under the designation of *rules of thumb*. This almost forgotten knowledge, validated by the very existence of those buildings today, represents a valuable source of information about the structural decisions made by the ancient builders for sizing the elements related to vaulted structures. Being an essential aspect for an efficient and respectful conservation of historical monuments, together with a history-based assessment of the building (ICOMOS/ISCARSAH Committee, 2005), many authors have recently grounded their research in these features, e.g. (Brencich and Morbiducci, 2007; Roca et al., 2013).

With this aim – and in order to lay the foundations for a better understanding of the structural behaviour of vaulted structures – this paper addresses the historical development of cross vaults. Without claiming to fully treat this topic, for which specialized literature in the field of architectural history is suggested, the first section of the paper is devoted to the evolution of cross vaults from the geometrical and constructive standpoint, two essential features in structural modelling. In this regard, historical written sources, as manuals and treatises, represent an essential support but, since they were often conceived with no structural purposes, only a critical analysis of the sources can clarify implicit information, e.g. on dimensions of the main elements and constructive phases. Together with the relevant literature, this study allowed to collect the rules of thumb that led to the size ranges of the structural elements present in the vaults (described in the subsequent section). For a detailed review on historical aspects of arches, vaults and domes, the reader is referred to (Huerta, 2004), whereas (Willis, 1842) still represents a valuable reference for the study of gothic vault geometry.

Regarding the analysis of vaulted structures, ancient scholars focused almost exclusively on masonry arches in a way that nowadays can be regarded as an anticipation of the fundamental principles of limit analysis. Having rudimentary mathematical tools available, only few attempted the study of vaults. It is necessary to wait until the 18th century for significant developments, when more advanced approaches allowed to qualitatively describe the complex behaviour of cross vaults. In this scenario, today limit analysis plays again an essential role, lending itself to easy implementation of the intuition of ancient scholars and providing a valuable support for setting up a simplified framework for practitioners in the field of conservation of cultural heritage buildings.

On this matter, the historical developments of cross vault analysis are revised also with reference to modern structural analysis models. The history of continuum mechanics and of arch theory are only briefly outlined for the purpose of the subsequent developments, as they have been treated in depth in other references, such as (Heyman, 1972; Benvenuto, 1991; Di Pasquale, 1996; Kurrer, 2008).

2. Form evolution

Cross vaults appeared in Europe during the Roman Empire Age (1st century BC – 5th century AD) with the construction of thermal baths. The first form was the rounded cross vault composed by the orthogonal intersection of two semi-circular barrel vaults, i.e. two semi-cylindrical shells on a square bay with no ribs (Alberti, 1485), which is generally referred to as groin vault. The Basilica of Maxentius and the Baths of Diocletian (Figure 1), both spanning more than 25 m, are remarkable results of the Roman technical skills and of the unique features of *opus caementicium* (pozzolana concrete). Several authors described its efficiency as a "miracle" (Branca, 1783) while Cavalieri San-Bartolo (1826) stressed the role of its tensile strength in avoiding the thrust on the supports. However, although Romans conceived the vault as a one-piece structure, Tomasoni (2008) stressed how the possible cracks development could have led the builders to strengthen the most stressed parts of the structure by placing brickwork hidden ribs in the concrete mass. For cross vaults this meant building perimeter arches and internal diagonal ribs (Como, 2013).

At the end of the 5th century AD, the decline and subsequent fall of the Roman Empire led to the Middle Ages, characterized by an overall impoverishment of the construction yard, both in terms of techniques and materials, and the consequent disappearance of the pozzolana concrete. It is only since the 10th century that high and wide spanned vaulted structures reappeared in Central Europe reaching the climax two centuries later when more than 350 cathedrals with the outstanding Gothic style were built in less than 30 years. This architectural style was based on a more rational and optimized building approach: each element was assigned to a precise structural role, giving to gothic churches a sense of profound elegance, along with a considerable saving of resources (Alberti, 1485; Frézier, 1737; Viollet-le-Duc, 1854; Huerta, 2004).

From the structural point of view, directing the self-weight of a vault to the four corner pillars allowed lateral walls to become non-structural elements, to be soon replaced by large stained glass windows, thus decreeing the end of the Romanesque massive style. The originally hidden ribs of the Roman vaults became now of fundamental importance: they were made visible at the intrados and, starting from the 11th century, they represented a sort of independent structural frame supporting the thinner webs - in the early stage probably disconnected each other (Willis, 1842). Although, in the last two centuries a great debate arose regarding the structural role of the ribs during and after the construction process - see §4, but also Abbot Surger's description of the church of St. Denis (Frankl, 1960) - studies and experiments suggest that the centring that supported the ribs remained in place until the webs were completed (Wendland, 2007). In this so-called *rib cross vaults*, the preferential force flow path proved to be so efficient that it was possible to build them with 10-15 m span and only 0.20 m thickness, which implied less weight and, thus, less thrust (Como, 2013).

Looking at the construction process, the intersection of two semi-cylinders produces semi-elliptical diagonals, difficult to be built for the masons of that time who started to prefer segmental arcs with circular shape, that is, its centre below the impost level, or semi-four-centred arc ribs (Tosca, 1707; Rondelet, 1802; Willis, 1842). Accordingly, defining the cross arches as autonomous elements, it could be reasonable to adopt centring in-plane arches with an elementary geometry, simply and straightforwardly attainable (Wendland, 2007). On the basis of constructive criteria of rationality and simplification, this process improved leading to design ribs with the same curvature, that is, to carve identical *voussoirs* for different parts of the vault (Willis, 1842; Palacios, 2006).

All this practical approach inevitably affected the shape, leading the crown of the vault to be higher than the lateral arches and forcing the webs to be portions of a double-curvature irregular spheroid (Frézier, 1737; Huerta, 2004), providing an higher overall stability both in the construction process and once completed (Wendland, 2007). Besides this first variation, although already largely adopted in Middle East countries, it was during the 12th century that the pointed arch appeared in France and England, representing a geometrical revolution allowing for an easier arrangement of the vault geometry, that is, the height of the lateral arches was no longer constrained and the bay could be rectangular. The same goal could be accomplished also rising the arch upon stilts (“stilted arch”) which are straight prolongations of the arch until meeting the springs (Willis, 1842). The pointed arch had also structural relevance because, as stressed by Viollet-le-Duc (1854), it reveals the ability of the masons of approaching, without any scientific assumption, the closest arch shape to the thrust line (see also Section 4).

The geometrical palette available to the masons paved the way to a wealth of different forms that eventually culminated with the English and Spanish Gothic architecture. In order to provide a more stable support, but also for the sake of innovation or extravagance, a multiplication of ribs appeared. As an example, Figure 2 shows 26 different cross vault plans and the so-called *crazy vaults* of the St. Hugh’s Choir of the Lincoln Cathedral in England (1192 and 1265) that seems to challenge any structural rule. According to the shape of the vault surface, which Willis (1842) already pointed as of capital importance in examining existing vaults, a basic classification of the large variety of quadripartite cross vaults was proposed by Barthel (1993) shown in Figure 3a - for a more detailed investigation on the surface shape according to the traditional vault construction without formwork, the reader is referred to (Wendland, 2007). On the other hand, Figure 3b shows the variation of the overall cross vault shape considering the same diagonal arches and different web profiles (Strommer, 2008).

Finally, for the sake of clarity and completeness, the main elements of a quadripartite cross vault are depicted in Figure 4 (Willis, 1842). In particular, the lateral arches are presented, where *arc doubleau* and *arc formeret* are, respectively, transversal and parallel to the longitudinal axis. Moreover, the possible ribs marking the crown are called *longitudinal* and *transverse ridge rib*, *arc tierceron* is a rib extending between one corner and one ridge, and finally *lierne* is a rib not connected to any corner.

3. Rules of thumb

3.1 Review of main treatises

Until the 15th century, the treatises of architecture did not provide any information about the vaults design. In particular, during almost the entire Gothic period (12th – 16th century), the rules were simply handed over mostly in secrecy, appearing only in Renaissance and Baroque treatises, with a delay of almost four centuries.

One of the most important rules of this time was the so called “Fr. Derand’s rule” (Derand, 1643, p. 2, plate 1), better known as “Blondel’s rule” due to the popularity of the successive author (Blondel, 1675, p. 419). In reality, according to Müller (1990), the rule was already known almost one hundred years before, as already cited in Boccojani’s lost treatise of 1546. As shown in Figure 5a, it consisted in the division of the arch in three equal parts from which it was possible to geometrically obtain the width of the abutment (Heyman, 1982; Benvenuto, 1991; Huerta, 2004). The evident handiness, together with the correct ability of providing wider supports for larger thrust (from pointed to flat arches), made this rule become a standard reference for the next centuries, being still present in Vittone’s treatise (1760) even in case other types of vault were considered. On the other hand, although Derand’s treatise is not a source directly connected to the previous Gothic design tradition, Huerta (2004) showed the relevance of Fr. Derand’s rule for Gothic structures. By way of example, Figure 5b-c display the cross section of the Cathedral of Girona (Spain) and the Sainte Chapelle (Paris, France) whose abutment dimensions are in good agreement with the rule application.

Slightly different from Fr. Derand's rule, in 1560 Hernán Ruiz el Joven introduced the arch thickness into the geometrical construction for the abutment width design, which is possibly the first approach to take into account the weight of the vault (Figure 6a). Moreover, for the first time, the stabilizing importance of the infill was stressed and it was recommended to add it until half of the arch rise, while the thickness of the arch should be not less than 1/10 of the span (Navascués Palacio, 1974).

Whereas the previous two rules concerned only the abutment width, the German gothic builders set up a list of geometrical proportions that, without any structural purpose, starting from the span of chorus, led up to the smallest details, e.g. the vault ribs cross-section (Figure 6b). Regarding the abutment width, it must be stressed that the resulting dimension is not referred to the vault spring (as for the other rules) but to the base of the element, allowing for slight tapering towards the top. The rules reported in Table 2 and Table 3 are provided by Coenen (1990) who collected the sources of the Late Gothic German treatises, of which only *Von des Chores Maß und Gerechtigkeit* (c. 1500) and *Wiener Werkmeisterbuch* (15th century) by unknown authors, and Lechler's "Underweysung" (Coenen, 1990), dated 1516, contain information to size the elements related to cross vaults (Huerta, 2004).

A similar but more pronounced approach was adopted by Cataneo (1567) who, instead of suggesting geometrical proportions, proposed the true dimensions of all the parts of five Latin cross plan churches. The Cataneo's purpose was to make the building resemble the Christ body: although rather forced with the aim of meeting tradition, this reasoning seems to disregard any structural aspect. More in detail, Figure 7 shows the general plan and the longitudinal cross section of a three-nave church. The abutment width is equal to one-third of the clear span of the aisle, which, together with a thick external wall, leads to an overall massive buttressing system able to balance the large thrust of the Renaissance rounded vaults. In this regard, Cataneo (1567) did not define the type of vault in the lateral aisles, even if the square bay may suggest cross or sail vaults.

During the 15th and 16th century, when the Late Gothic gives way to the Renaissance, Rodrigo Gil de Hontañón, who represents one of the most important Spanish architects of the past, wrote a booklet (c. 1544 - 1554, unfortunately lost but partially copied by Simón García in 1681) in which Gothic tradition is merged with new mathematical tools and humanist ideas (Sanabria, 1982; Huerta, 2004). Focusing only on cross vaults, he respectively: a) proposed an unexplained geometrical proportion for the abutment width equal to one fourth of the span; b) approached analytical formulations for the sizing of the pier diameter, the abutment width and the weight of the keystone (Table 1); c) suggested to design the minor elements of the vaults according to a forced proportion with human fingers (see Table 3).

Regarding the use of analytical formulations, whereas on one hand is a proof of new mathematical tools available to masons, on the other hand it reveals the efforts of Rodrigo Gil de Hontañón of considering the design process according to a proper structural intuition rather than the tradition made by simple spatial proportions (Sanabria, 1982). By way of example, although clearly incorrect, the formula for sizing the pier diameter regards the height of the pier and the plan dimensions of the nave bay, meaning that he correctly understood the direct proportion with these geometrical quantities.

Almost one hundred years later, Friar Lorenzo de San Nicolás wrote one of the last works on architecture before the Age of Enlightenment (between 1639 and 1664) and addressed general aspects about cross vaults construction without giving practical rules about their dimensions. Nevertheless, in case of rounded cross vaults, he erroneously pointed out that the structural stability was guaranteed only thanks to the infill weight (until one-third of the rise) with no need of abutments (Huerta, 2004).

The subsequent 18th century brought a new interest for vaulted structures, which were a key topic of modern mechanics. However, the new scientific approach was not close to the autonomy and maturity of the following centuries and, in this context, the rules of thumb still played a fundamental role. Validated by centuries-old history, the traditional rules represented the only support to validate the new theories (Benvenuto, 1991; Kurrer, 2008).

In the early 1700s, de La Hire and Belidor were the most representative figures of this *science after tradition* trend. They tried to rigorously study the arch stability (according to the wedge theory) but they just ended up with another geometrical construction (Figure 8a). Nevertheless, although scientifically incorrect, since it perfectly matched the tradition, this geometrical rule swiftly spread over the Europe, together with the common Fr. Derand's rule.

This trend was still present in the following century when, almost at the beginning of the wrought-iron era, despite the important developments of mechanics, Cavalieri San-Bertolo (1826) and Valadier (1832) still focused their attention on the handiness and supposedly safer tradition. In particular, since Fr. Derand's rule did not consider the thickness of the arch and the height of the abutment, Valadier proposed another graphical method. In this regard, he referenced the essays of *Accademia Reale delle Scienze* of 1712, which is the same year of de La Hire's *Memoir* (Paris), but the comparison between the two methods reveals the apparent difference (Figure 8). Regarding the cross vaults abutment, Valadier applied this method on the two elemental barrel vaults obtaining the perpendicular side lengths (Figure 9).

Finally, differently from the objective of the previous rules referring to churches, the first rules for porticos are also reported. The only available reference has been found in Palladio (1570) who, according to the weight they were supposed to bear, provided ranges of dimensions for the piers width in both public and private buildings (Figure 10). Considering the weight as an additional parameter made the design process nonlinear, in line with the German Late Gothic builders and Friar Lorenzo de San Nicolás who proposed slight adjustments according to the material type. However, no considerations on the piers height, i.e. slenderness, are given.

3.2 Main elements dimensions

In order to create a more synthetic and comparative view, the rules discussed before are now collected in graphs and tables, giving insight on the possible range of sizes of the main elements related to cross vaults of churches. Due to its importance in the overall stability of the construction, particular attention is paid to the buttressing system: abutment width and pier size. Table 2 reports this information together with a general description and an indication whether the thickness of the *arc doubleau* and the height of the abutment (slenderness) affected the design. In this regard, since the strict approach of German Late Gothic builders and Cataneo, all the parts of the church resulted in a fixed proportion with the module.

The relations between abutment width and span are reported in Figure 11 where the abscissa represents the ratio between the rise of the *arc doubleau* and the span. This is the parameter that better describes the overall shape of the vault, as 0.50 represents a semi-circular arch, while smaller or larger values represent flat or pointed arches, respectively. Fr. Derand's and Hernán Ruiz's rules shows a slight decrease of the abutments width from flat to pointed arch. The former (dash-dot line) seems to be less conservative than the latter (dotted line) with values approximately equal to 0.25 and 0.30 respectively. However, it must be stressed that the Hernán Ruiz's rule refers to the base of the abutment and, through a possible tapering towards the vault spring, it can meet the Fr. Derand's rule.

The German Late Gothic rules (solid lines) provide values at the base of the elements and they are in good agreement with the previous ones. In particular, the chorus and nave abutment widths are a sort of average of the values provided by Hernán Ruiz and Derand's rule. Also the Italian Renaissance Cataneo's rule refers to a particular type of cross vault, i.e.

groin vault (rise/span ratio = 0.5). The rule provides an abutment width equal to one-third of the span, in line with Hernán Ruiz's although their clearly different origin.

Figure 12 shows the relations between the abutment width and the pier diameter versus the span of the vault for Rodrigo Gil's formulation. Since the length of the ribs converging on the abutment are requested (from the springing to their respective keystone), they have been calculated on the base of the same rib scheme of the vaults in the Cathedral of Salamanca (Palacios, 2006). Considering all the ribs with the same curvature, that is, the radius equal to half of the diagonal, and starting from the same proportions of the Cathedral (the nave bay has a span of 13m and a width of almost 10m, thus $w_b = 0.77\ s$, whereas h_p is almost two times the span), the bay width and pier height have been moderately changed.

As it is possible to see, the diameter of the pier is a little more than one half of the abutment width. Comparing the latter with the Fr. Derand's rule (leading approximately to a value equal to $s/4$), Rodrigo Gil's considerably diverges, providing similar results only for a span range between 9 and 13 m, being more conservative for smaller values of the span. Additionally, more noticeable than the previous rules, it is shown that the structural elements become slender as the span increases. Huerta (2006) attributed this trend to the stabilizing effect of the increasing weight with larger dimensions but it is also possible that the rules were used only in a limited range of spans.

Finally, Table 3 reports the range of the dimensions provided by the rules of thumb for the other elements composing the cross vault. Even though not exhaustive, it is a general overview of the presented values whose validation is certainly desirable, both in terms of geometrical survey and structural performance. The complexity of the validation increases with the singularity of historical construction, where the economic possibilities of the cities, and technical skills and expertise of the local masons, could have played a decisive role in the design process (Tomasoni, 2008). However, the survival of the rules over the centuries is an implicit and intuitive validation (Benvenuto, 1991) that can be confirmed by a statistical survey, which at the moment is missing.

4. From historical methods to limit analysis

During the 18th century, the study of masonry vaulted structures led modern mechanics to make great progress, providing outcomes still at the basis of current structural approaches in the framework of limit analysis. Moving from the arch-catenary analogy stated by Robert Hooke's Latin anagram in 1675, then independently extended by Gregory as a stability condition (static theorem), around 1730 Couplet described the assumptions that form the basis of limit analysis (Heyman, 1972; Benvenuto, 1991; Kurrer, 2008). High coefficient of friction (to prevent against sliding failure), infinite compressive strength and null tensile strength still represent the usual hypotheses for analytical and simplified tools for the assessment of masonry structures.

In a scenario in which the masonry arch was the protagonist of the scientific debate, the only scholar who focused on masonry cross vaults was Mascheroni (1785). Starting from Bouguer's lesson about the domes of finite thickness, he criticized the *slicing technique* performed until then, which allowed to disassemble a compound vault in its elementary arches, i.e. a reduction from a three-dimensional problem into a well-known in-plane one. This was the case of the famous Poleni's report on Rome's St. Peter's Basilica in 1748. Although this approach is the easiest way to study compound vaults, it inevitably neglects the interaction between two adjacent slices, e.g. the compressive circumferential stresses of the dome (Benvenuto, 1991).

Mascheroni (1785) dedicated one chapter of his treatise to the study of compound arches and vaults. In spite of his idea about the three-dimensional behaviour of vaults, he approached the study of cross vaults by the usual slicing technique, which includes independent web strips whose resultant action is applied to the diagonal arch. However, regarding the diagonal arches and the webs as the main elements (Figure 13), he proposed a dual problem: given the shape of one arch, calculate the balanced profile of the other arch. He also provided hints in case the generatrix of the webs, i.e. line ML and MT in Figure 13, were not horizontal but inclined or curved. With this aim, he extensively used the concept of catenary, easily visualized through the cross vault analysis of Beranek (1988) in the form of inverted hanging cables (Figure 14) and later at the basis of the 3d catenary net proposed by Andreu et al. (2007) .

With the contribution of Mascheroni, the end of the 18th century marked also the end of the rigid and infinitely resistant *voussoirs* theory, and gave way to new theories, namely beams with curvilinear axis, membranes and shells, gathered all together in the framework of the elastic theory. As a consequence, for masonry arches the goal shifted from stability assessment (or limit analysis) to the solution of the linear elasticity problem, which is a statically indeterminate problem. Whereas the former was partially achieved by the ancient scholars thanks to the intuitive idea of *cracking* the structure to obtain a collapse mechanism (i.e., the kinematic theorem of limit analysis), the latter revealed itself as unsuitable for masonry structures analysis (Kurrer, 2008).

The elastic theory began in the 1820s with the Navier's *Leçons*, introducing stress analysis, comparing the resulting stress values with the material strength. Although in his work Navier considered the arch and the cross vault, there is no evidence whether he used the elastic theory to analyse either of them. According to Huerta (2010), the first elastic analysis of an *encastré* (or fully clamped, built-in) arch was anonymously published by Young in 1817, being the work revealed only in 2005. Unfortunately, another Young's work regarding the first complete theory on the *thrust line*, i.e. the line connecting the resultant forces in each cross section, remained unnoticed. It is only in 1831 when F.J. Gerstner established the theory: as the problem is statically indeterminate, he intuitively realised that the capacity increases with the number of indeterminacies (Kurrer, 2008).

Conversely, other scholars were interested in finding the "true" thrust line, sometimes adding *principles* to the equilibrium equations. Moseley, for example, formulated in 1843 the *principle of minimal resistance*, assuming that the true solution is the one with the minimal capacity (Kurrer, 2008). Culmann (1864), instead, adopted the *principle of minimum loading*, i.e. the true thrust line is the one with the smallest deviation from the centre line, which is one of the assumptions adopted by D'Ayala and Casapulla (2001) in their analysis of hemispherical domes with finite friction.

Culmann (1864) gave also insight into graphical statics. After the pioneering *Mathematicorum Hypomnemata de Statica* by S. Stevinus in 1608 (Lourenço, 2002), at the end of the 19th century this approach gained new vigour paving the way for vaulted structures analysis (Figure 15). Just to mention a few, Wittmann (1879) was the first to study compound vaults, then Planat (1887) and Mohrmann with the third edition of the Gothic construction manual of Ungewitter (1890). Some years later Körner (1901) and Wolfe (1921) used the same approach, which basically consists in the slicing technique, the only feasible for hand calculation. Recently, thanks to automatic procedures, the concept has been extended to catch the three-dimensional behaviour of vaulted structures (O'Dwyer, 1999; D'Ayala and Casapulla, 2001; Andreu et al., 2007; Block, 2009).

Ungewitter-Mohrmann (1890) presented also an easy method to obtain a good estimate of the thrust resultant and its position with respect to the springs of a cross vault. Figure 16 reports an example and a table for a quick calculation. The method was based on the vault thickness, the rise/span ratio and the crack observation at the crown and springs (Heyman, 1995). Moreover, in case of slicing technique on double-curvature portions of vaults, Ungewitter-Mohrmann suggested to divide the webs in elementary arches following the idea of a ball rolling down the extrados. The same idea was followed by Sabouret (1928) and Abraham (1934) but, since only the latter provided explicative drawings (Figure 17), the entire credit was given to Abraham (Huerta, 2009).

In spite of these last developments in graphical methods and thrust line analysis, with the popularity of wrought-iron structures, starting from 1860s the supremacy of elastic theory was inevitable. Although clearly misleading in case of masonry structures, as stressed by Castigliano's statement "*masonry arches as an imperfectly elastic systems*" in 1879 and the Bavarian Railways engineer Haase in 1885 (Kurrer, 2008), it is only at half of the 20th century that elastic theory definitely lost ground to plastic theory. Thanks to the studies of Drucker, Kooharian and Prager (between 1949 and 1953), later rearranged in the well-known work by Heyman (1966), ultimate load analysis re-emerged together with Couplet's assumptions providing the ground for the three fundamental theorems of plasticity, namely uniqueness, lower bound (or static/safe) and upper bound (or kinematic).

The safe theorem of plasticity scientifically proves what was stated by Hooke and extended by Gregory almost three hundred years before. This theorem also confirms the applicability of the graphical method with the slicing technique: a masonry arch/vault is stable if at least one of the infinite admissible equilibrated thrust lines/surfaces falls entirely into the thickness of the element. Still, it is not easy to discuss the safety of the structure despite the attempts to introduce the so-called *geometrical safety factor* (Heyman, 1982).

Moreover, without entering into the merits of the debate which involved several scholars (Willis, 1835; Viollet-le-Duc, 1854; Sabouret, 1928; Abraham, 1934; Heyman, 1968; Mark, 1982; Huerta, 2009; Tarrío, 2010), the in-service structural role of cross vault ribs can be addressed in the framework of the safe theorem. The hypothesis of ribs as the main structural elements (slicing technique and graphical method) is the simplest of the infinite possible solutions and, although a stress concentration is expected in the junction between two shells surfaces, the ribs are not strictly necessary for the global equilibrium (Heyman, 1977).

Regardless of this idea, which adopted a bi-dimensional response of the vault, it is only in the last two decades that researchers have proposed alternative computational methods to meet this goal, also thanks to more appropriate constitutive laws, failure criteria and plastic flow laws (D'Ayala and Casapulla, 2001; Andreu et al., 2007; Block, 2009; Milani et al., 2014). However, whereas on one hand a significant work has been done on the static behaviour of vaults under gravitational loads, on the other hand, very few has been done in case of seismic action and settlements (Rossi et al., 2014; McInerney and DeJong, 2014).

5. Conclusions

The historical developments of cross vaults reveals the uninterrupted progress of ancient builders in achieving such a high level of complexity and perfection. Conversely, the understanding of cross vaults structural behaviour is nowadays still a challenging task in the conservation of cultural heritage buildings. More efforts are requested and, in order to provide a basic support for deeper structural investigation, this work presented the fundamental knowledge related to the historical developments of cross vaults.

Without any doubt, the shape and the proper geometrical representation of the vault play a fundamental role in its overall stability (Wendland, 2007). Double-curvature webs contribute to reach an higher capacity, i.e. resistant-by-shape structures, and *in situ* geometrical surveys could give valuable insight into the performance of these vaults, e.g. (Theodossopoulos, 2008; Rodriguez et al., 2012; Palacios and Martín Talaverano, 2013; Wendland et al., 2014; Capone et al., 2015).

On the other hand, the study of the rules of thumb provided grounds for a database of the possible dimensions of the elements related to the cross vault. However, according to the available historical sources on this vault typology, the present study focuses more on the structural aspects related to the stability of the building, such as abutment dimensions. Nevertheless, the research has a twofold goal. It provides the basis of a parametric analysis aimed at understanding the influence of each parameter in the overall structural behaviour. At the same time, well aware of the singularities of each historical building, the collected data may represent a practical reference point for practitioners involved in monuments conservation. In this regard, further work is still requested to validate and to expand the overall database or to delimit it to a particular geographical area. In the words of Willis (1842), a catalogue of dimensions following surveys (by researchers and professionals) is rather desirable.

Finally, the study of the historical methods for the analysis of masonry vaulted structures, particularly cross vaults, highlighted the continuous effort of scholars and researchers in studying and explaining the statics of such a complex element. Nowadays, whereas on one hand advanced FE nonlinear analyses are developing as an important branch of structural analysis, on the other hand, several works are focusing on limit analysis as a powerful tool for assessing the collapse failure and the safety of structures composed by macro-blocks, such as vaults. As stressed in the paper, limit analysis has an ancient origin linked to the masonry arch and, with no surprise, old outcomes are still used in modern implementations of the method, for instance the 3d compression only surface as a generalization of the thrust line.

Even though approximate, researchers of the past achieved an appreciable understanding of the stability of cross vaults under gravitational loads, but no considerations seem to have been made in case of seismic action. Looking at the large presence in cultural heritage buildings and the high vulnerability of cross vaults revealed by recent earthquakes, this topic still represents an open issue for both researchers and practitioners, and more research is welcome.

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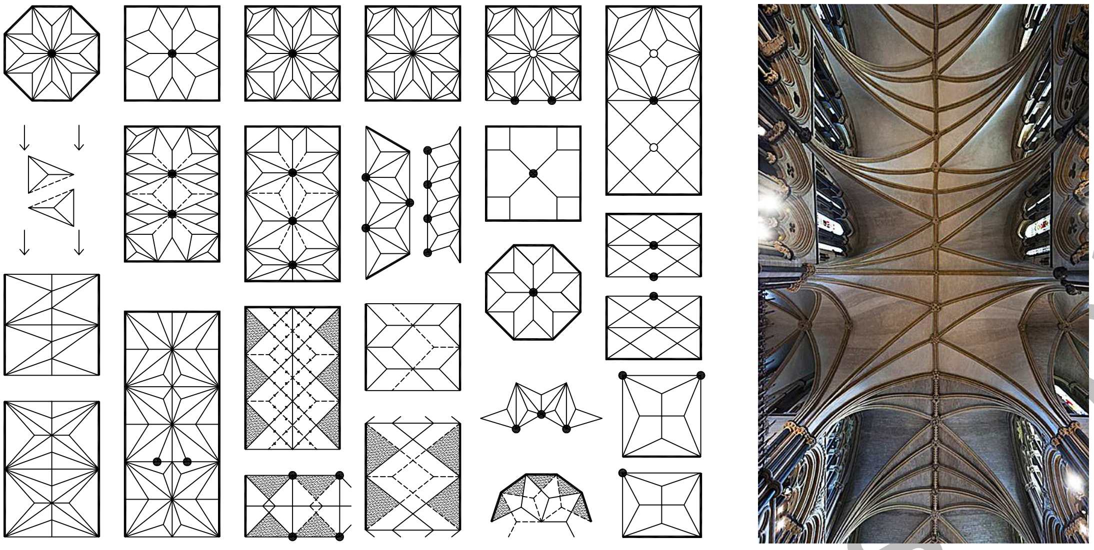

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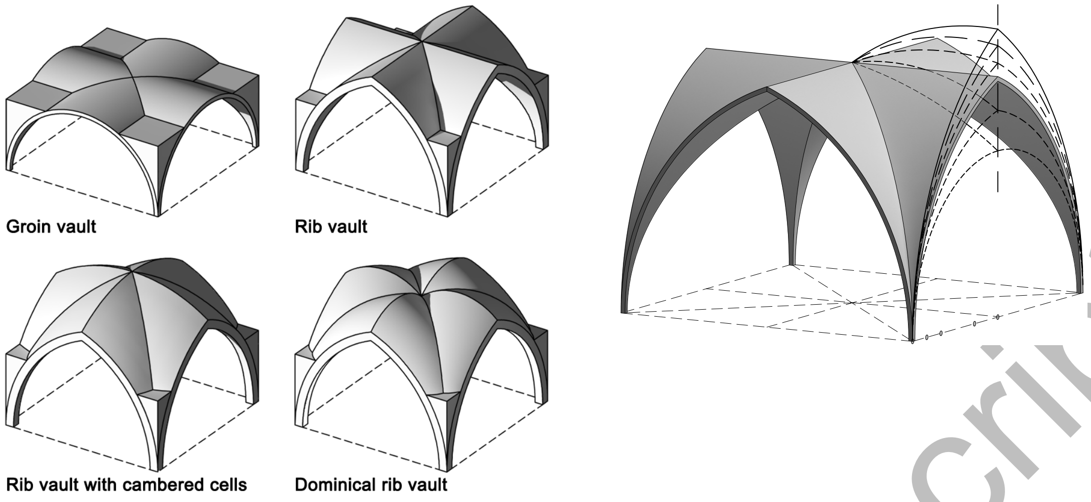

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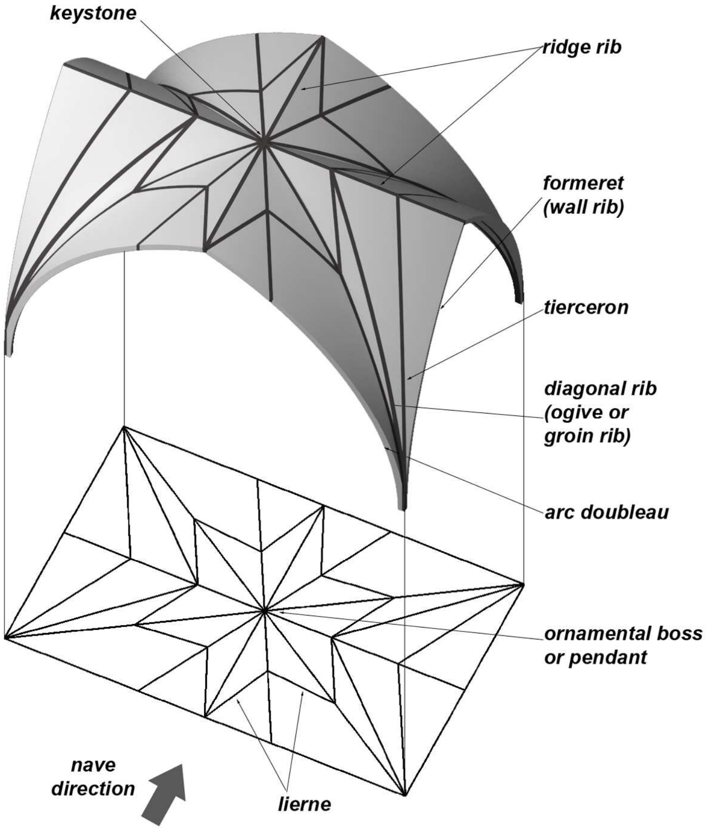

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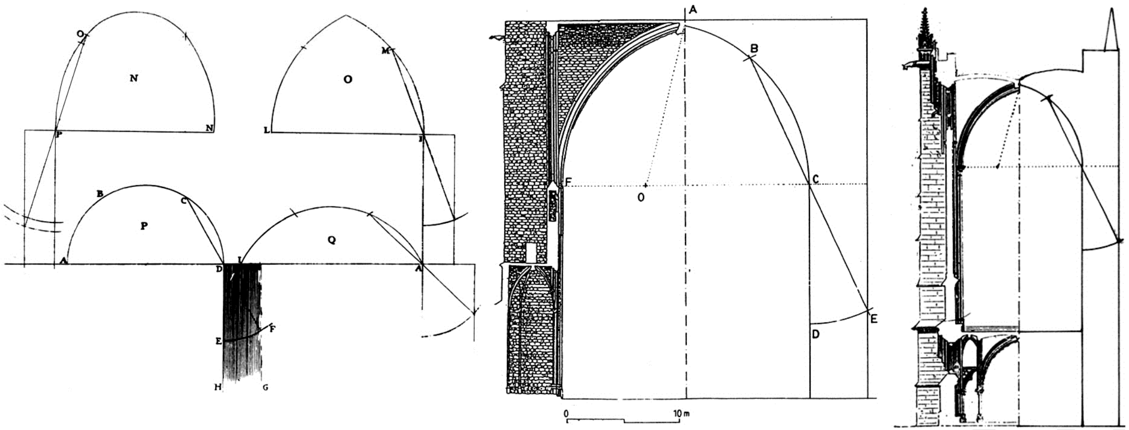

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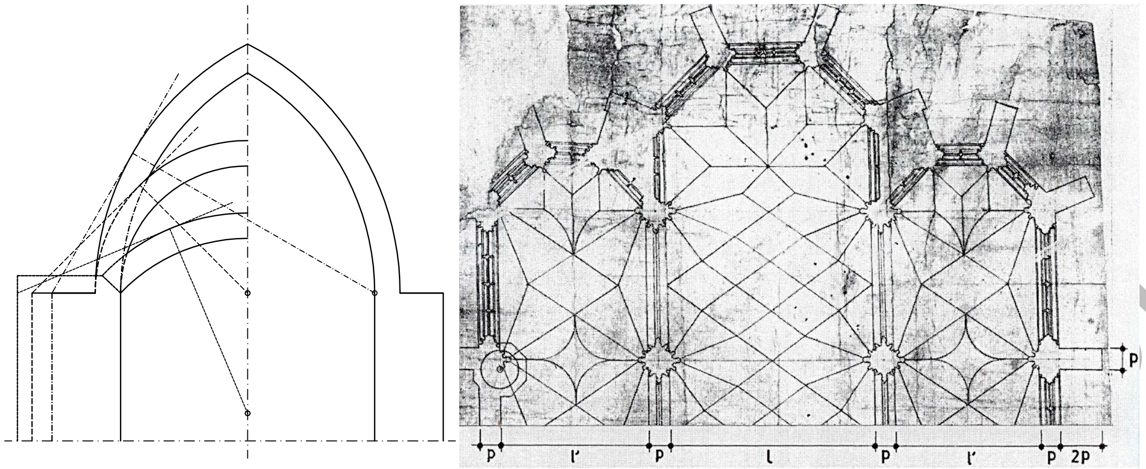

a)

b)

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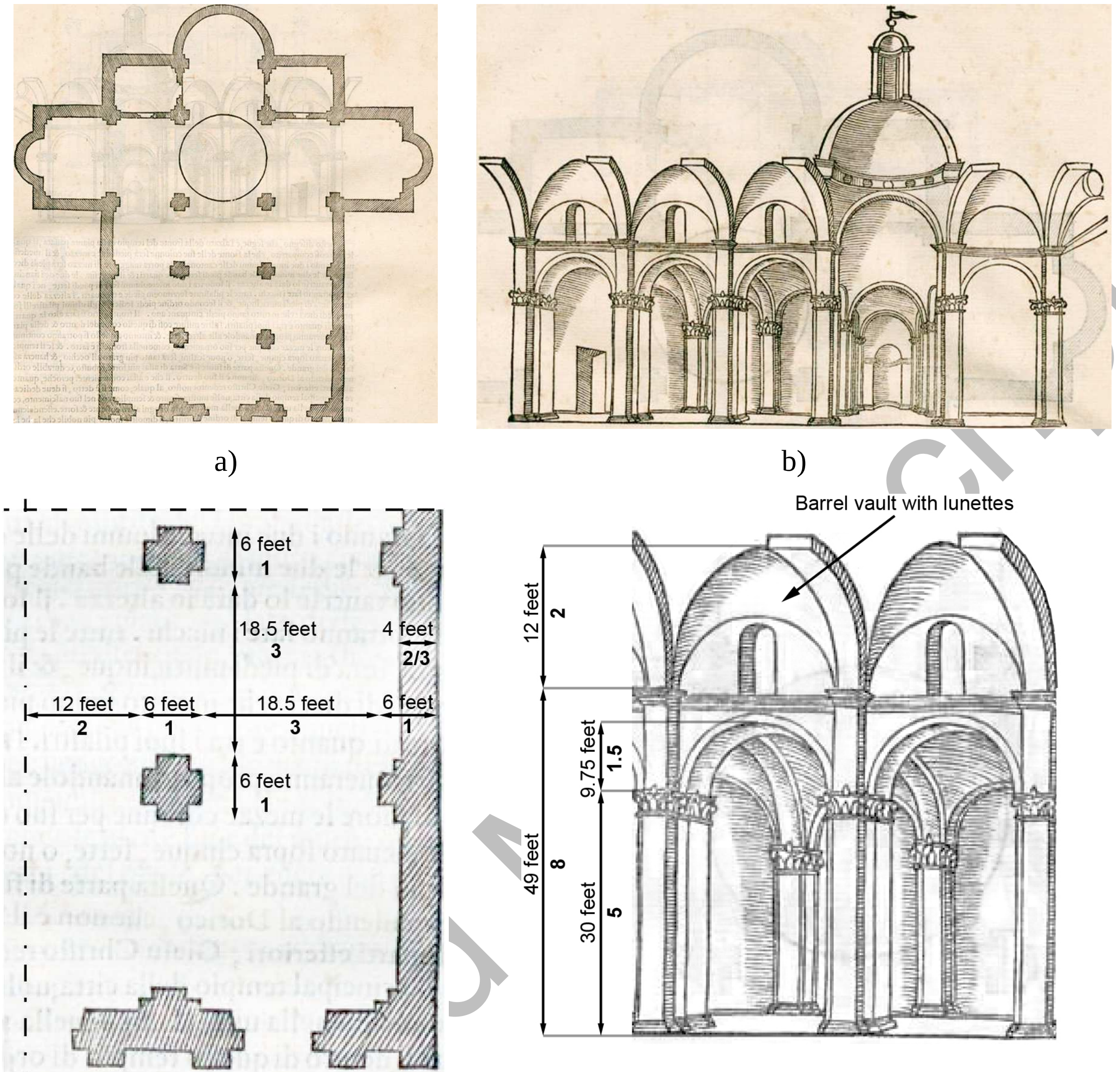

a) b)

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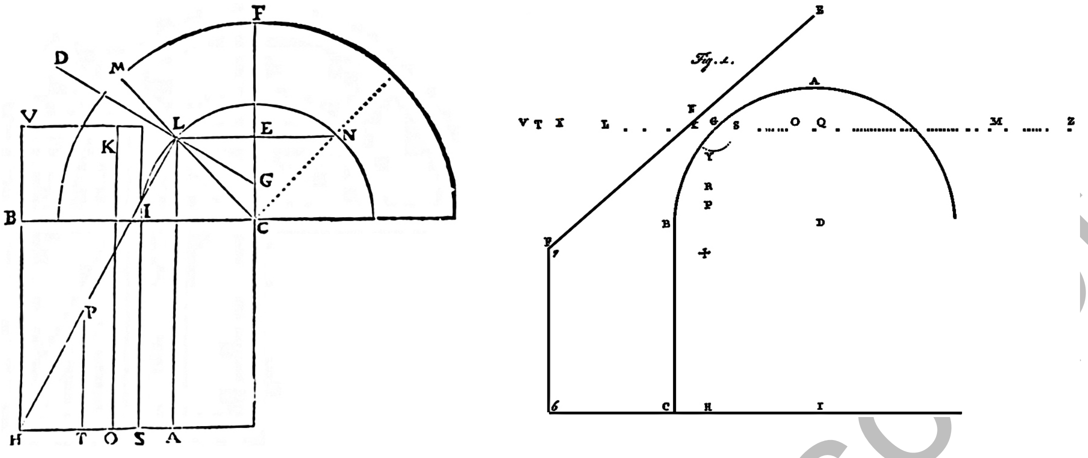

a) b)

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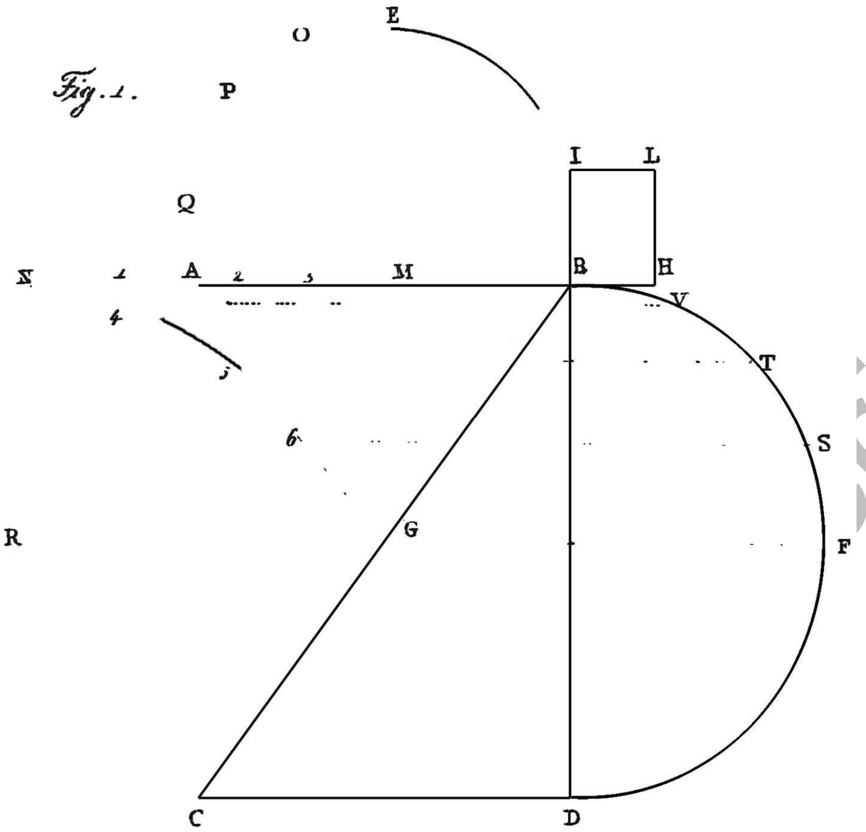

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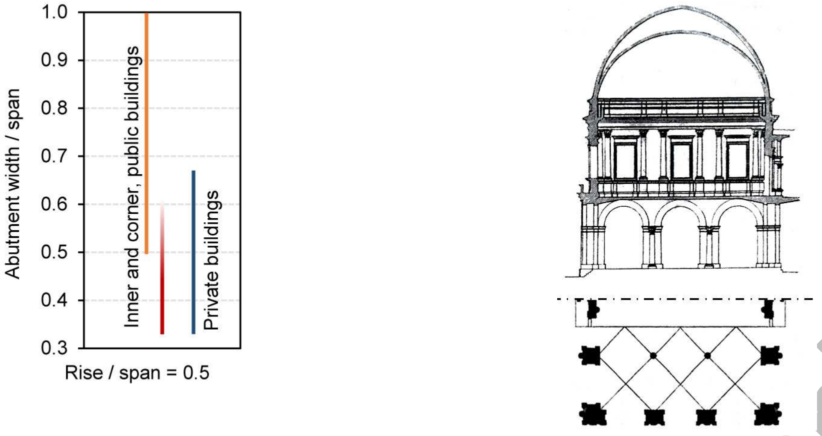

a) b)

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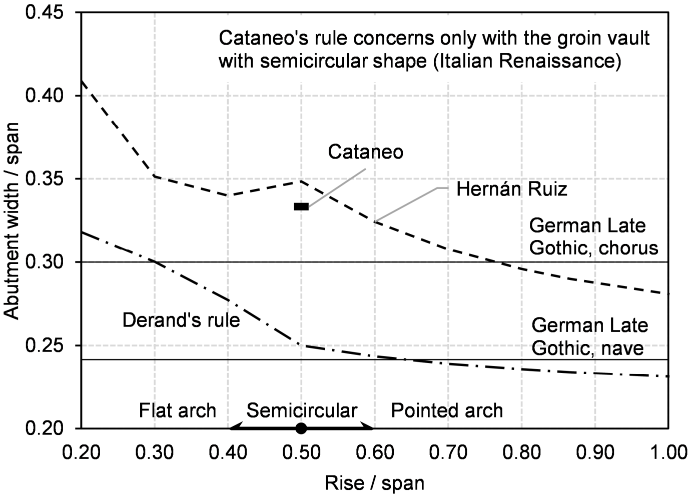

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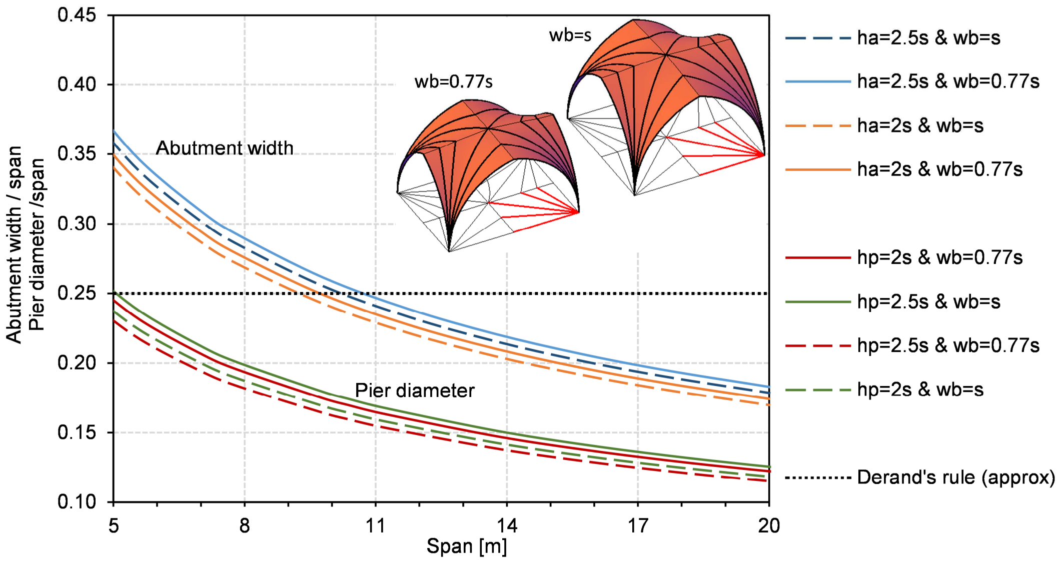

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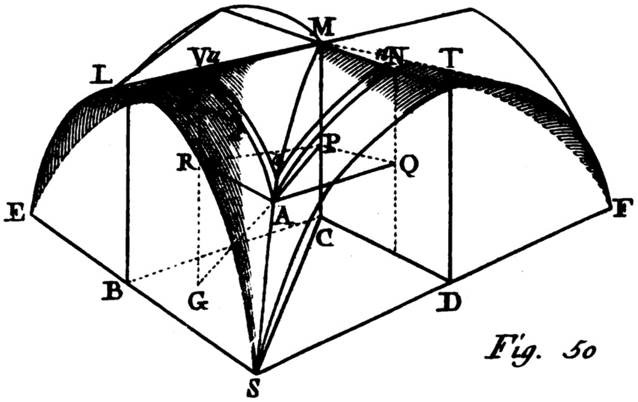

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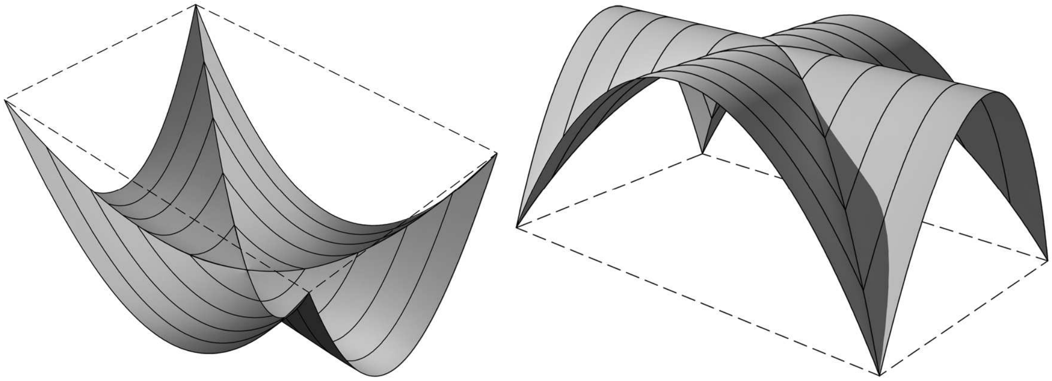

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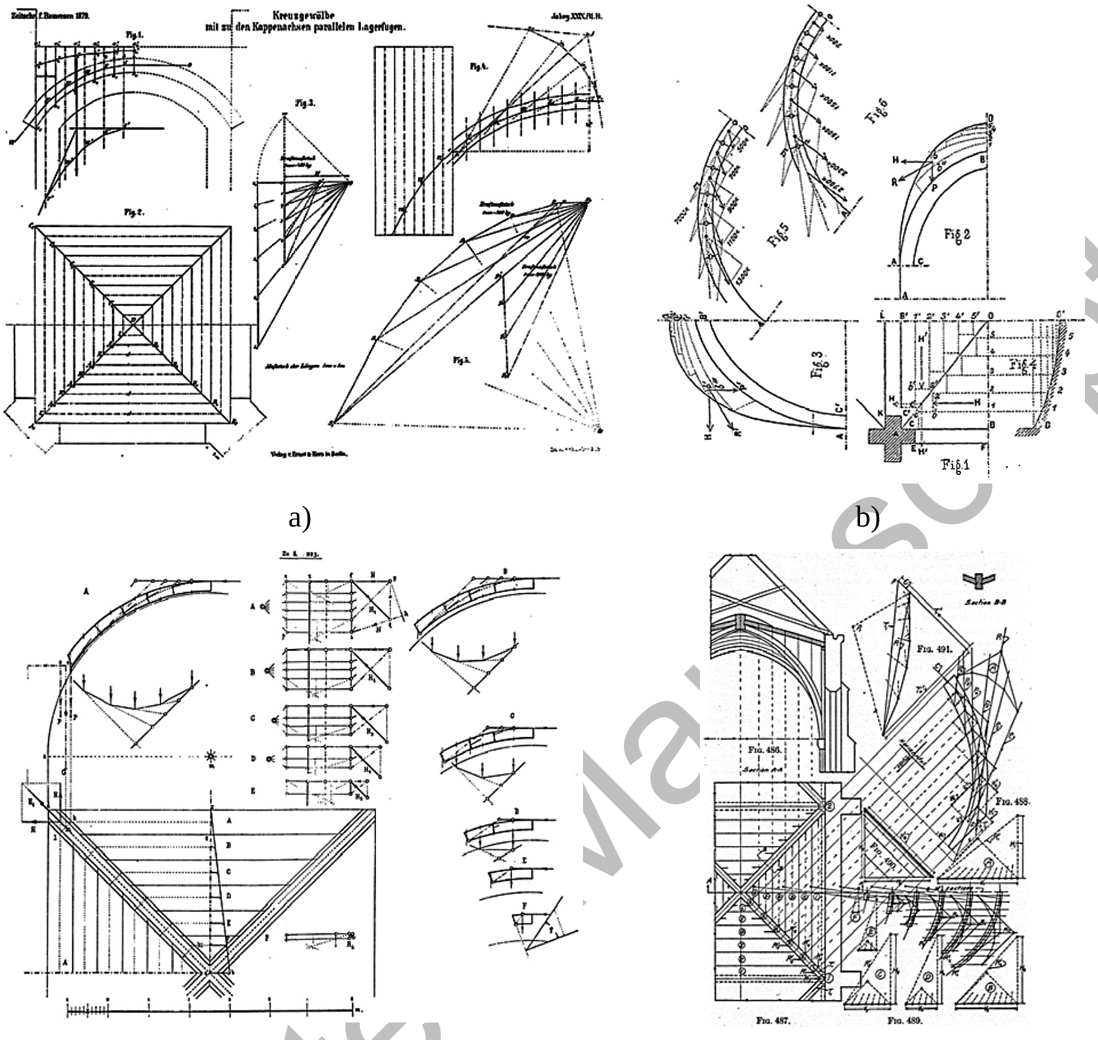

a) b)

c) d)

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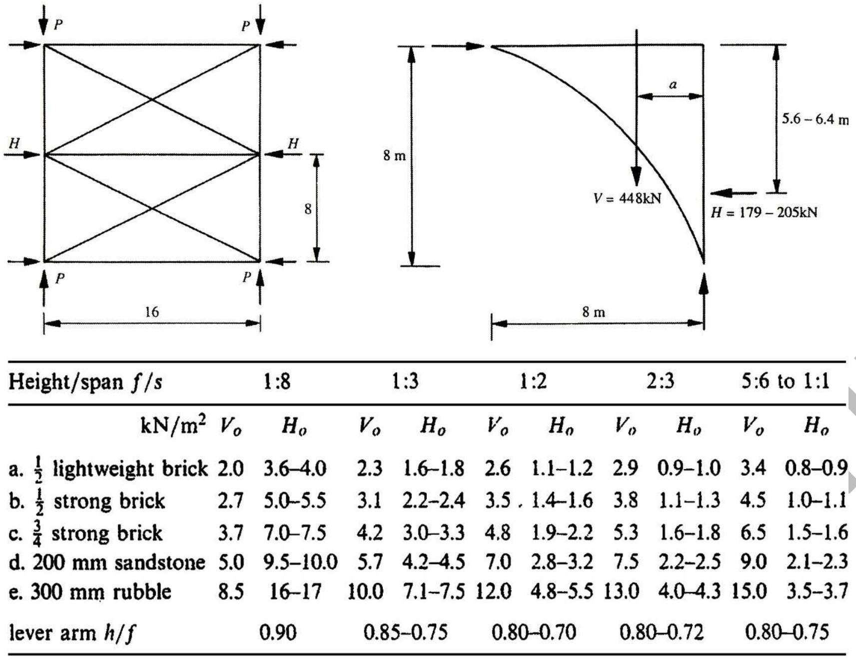
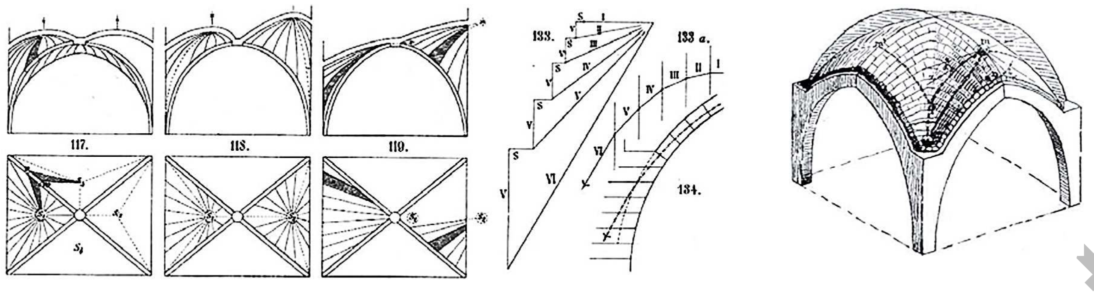


Height/span f/s	1:8		1:3		1:2		2:3		5:6 to 1:1	
kN/m^2	V_o	H_o	V_o	H_o	V_o	H_o	V_o	H_o	V_o	H_o
a. $\frac{1}{2}$ lightweight brick	2.0	3.6–4.0	2.3	1.6–1.8	2.6	1.1–1.2	2.9	0.9–1.0	3.4	0.8–0.9
b. $\frac{1}{2}$ strong brick	2.7	5.0–5.5	3.1	2.2–2.4	3.5	1.4–1.6	3.8	1.1–1.3	4.5	1.0–1.1
c. $\frac{3}{4}$ strong brick	3.7	7.0–7.5	4.2	3.0–3.3	4.8	1.9–2.2	5.3	1.6–1.8	6.5	1.5–1.6
d. 200 mm sandstone	5.0	9.5–10.0	5.7	4.2–4.5	7.0	2.8–3.2	7.5	2.2–2.5	9.0	2.1–2.3
e. 300 mm rubble	8.5	16–17	10.0	7.1–7.5	12.0	4.8–5.5	13.0	4.0–4.3	15.0	3.5–3.7
lever arm h/f	0.90		0.85–0.75		0.80–0.70		0.80–0.72		0.80–0.75	

Figure 16. Example according to the approximated method by Ungewitter- Mohrmann for a cross vault, in case of a 20cm thick sandstone vault and a ratio *f/s* = 1:2 (Heyman, 1995), where *f* is the height and *s* is the span



a) b)

Figure 17. Slicing technique: a) patterns of slicing (Ungewitter and Mohrmann, 1890) and b) “ball principle”(Abraham, 1934)

Pier diameter (at the base) [feet]	$d_p = \frac{1}{2}\sqrt{h_p + w_b + s}$	h_p	height of the pier at the springing of the vault [feet]
		w_b	central nave bay width [feet]
		s	central nave span [feet]
Abutment width (at the spring level, wall included) [feet]	$w_a = \frac{2}{3}\sqrt{h_a + \frac{2}{3}\sum r_i}$	h_a	height of the abutment at the springing of the vault [feet]
		r_i	semi-length of all the ribs connected to the abutment (except for the *arc formeret*) [feet]
			The author suggested the abutment breadth equal to half of w_a
Keystone weight [quintal]	$Q = p_r\sqrt{\sum l_s - \sum l_{ns}}$	p_r	weight of ribs per unit length [quintals per feet]
		l_s	length of the structural elements [feet]
		l_{ns}	length of the non-structural elements [feet]

Table 1. Rodrigo Gil de Hontañón’s rules for dimensions of piers, abutments and keystones, where one Castellano foot is approximately equal to 0.28m and one quintal is about 46 kg (100 old Spanish pounds)

References	**Description**		**Abutment height**	***Arc doubleau* thickness**
Fr. Derand's rule (before 1546)	Graphical method	See Fig. 6	✗	✗
Hernán Ruiz el Joven (1560)	Graphical method	See Fig. 7	✗	✓
German Late Gothic	Direct proportion (chorus w_a^c and nave w_a^n)	$w_a^c > \frac{s}{3.33}$ $w_a^n > \frac{s}{4.14}$	Fixed	Fixed
Cataneo (three nave church) (1567)	Real dimensions	$w_a = \frac{s}{3}$	Fixed	✗
Rodrigo Gil de Hontañón (1550)	Analytical formulation	$d_p = \frac{1}{2}\sqrt{h_p + w_b + s}$ $w_a = \frac{2}{3}\sqrt{h_a + \frac{2}{3}\sum r_i}$	✓	✗
De La Hire (1712) Belidor (1729)	Graphical method	Wedge theory (see Fig. 11a)	✓	✓
Valadier (1832)	Graphical method	See Fig. 11b and Fig. 12	✓	✓

Table 2. Overall description of rules applicable to cross vaults. In particular, w_a is the abutment width, d_p is the pier diameter and s is the span of the vault (for Rodrigo Gil's see Table 1)

Elements	References	Dimensions
Arc doubleau	German Late Gothic	$s_n/22.5$ (central nave) $s_n/30$ (aisle)
	Rodrigo Gil de Hontañón (1550)	$s/20$
	Hernán Ruiz el Joven (1560)	Min $s/10$
Diagonal rib (Central nave)	German Late Gothic	Height: $s_n/30$ Thickness: $s_n/60$
	Rodrigo Gil de Hontañón (1550)	Height: $s/24$
Arc tierceron	Rodrigo Gil de Hontañón (1550)	Height: $s/28$
Arc formeret	Rodrigo Gil de Hontañón (1550)	Height: $s/30$
Web thickness	Como (2013)	$s/50 \div s/75$ (Gothic vaults)
Infill	Friar Lorenzo de San Nicolás (1639-64)	Up to one third of the vault height (rounded cross vault)
	Hernán Ruiz el Joven (1560)	Up to half of the height of *arc doubleau*
Wall thickness (Chorus)	German Late Gothic	$s_c/10$
Wall and pier thickness (Central nave)	German Late Gothic	$s_c/10$ or $0.125 \div 0.141\ s_c$
	Cataneo (1567) (three nave church)	Pier: 1/4 clear nave span Wall: ≈ 1/6 clear nave span
Wall thickness (Aisle)	German Late Gothic	$s_c/10$ or $0.133\ s_c$
	Cataneo (1567) (three nave church)	≈ 2/9 clear aisle span

Table 3. Rules of thumb for the main elements of the church related to the cross vault: s_n and s_c are the span of the central nave and of chorus respectively, whereas s is the span of the element considered.