PORTUGUESE AND BRAZILIAN CHILDREN UNDERSTANDING THE INVERSE RELATION BETWEEN QUANTITIES – THE CASE OF FRACTIONS

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This study compares Portuguese and Brazilian fourth-graders (n=84) understanding of the inverse relation between quantities when fractions are presented in quotient and part-whole interpretations. It addresses three questions: 1) How do children understand this inverse relation in quotient interpretations of fractions? 2) How do children understand this inverse relation in part-whole interpretation of fractions? 3) Are there differences in performance between Brazilian and Portuguese children concerning these issues? A survey by questionnaire was applied and 16 part-whole and quotient problems were analyzed. Results indicate that quotient interpretation promotes more the understanding of this inverse relation; Portuguese and Brazilian children perform differently when solving the fraction problems.

FRAMEWORK

This study focuses on a comparative analysis conducted with Portuguese and Brazilian children's understanding of the inverse relation between quantities, when fractions are involved. To understand rational numbers is one of the greatest conceptual challenges faced by children as they learn mathematics (Behr, Wachsmuth, Post & Lesh, 1984; Singler, Thompson & Schineider, 2011; Hallett, Nunes, Bryant, Thorpe, 2012), since it requires a reorganization of numerical knowledge (Stafylidou & Vosniadou, 2004), as well as an understanding that the properties of integers do not define numbers in general, and thus, require other types of more complex cognitive skills (Jordan, Hansen, Fuchs, Siegler, Gersten & Micklos, 2013).

The understanding of inverse relation between two quantities is an important skill for the conceptual knowledge of rational numbers (Hallett, Nunes, Bryant, Thorpe, 2012). Literature presents several studies focused on the students' understanding of the inverse relationship between quantities. Some are focused on the concept of fraction (see Behr, Wachsmuth, Post & Lesh, 1984; Kornilaki & Nunes, 2005; Mamede, Nunes & Bryant, 2005; Mamede & Cardoso, 2010; Mamede & Vasconcelos, 2014). Recent research (see Nunes, Bryant, Pretzlik, Evans, Wade & Bell, 2004; Mamede, Nunes & Bryant, 2005) consider that the conceptual knowledge of fractions comprises: (1) the invariance principle, that is, the division of a whole into equal parts, while maintaining the initial quantity; (2) the ability of representation, being written as $\frac{a}{b}$, where *a* and *b* are whole numbers (with $b \neq 0$) and the same symbols can represent different quantities (e.g., $\frac{1}{2}$ of 8 and $\frac{1}{2}$ of 12); (3) the understanding of equivalence ($\frac{1}{2}, \frac{2}{4}, \frac{3}{6}$) and ordering ($\frac{1}{2} > \frac{1}{3} > \frac{1}{4}$) of fractions; and (4) the diverse and complex interpretations, meanings or situations of fractions. Children's understanding of the inverse relation between numerator and denominator is crucial for the concept of fractions, and this understanding seems to be affected by the type of interpretation of fractions.

The literature presents different classifications of interpretations or meanings for fractions. Kieren (1993) distinguishes four categories known as "sub-constructs" that are relevant for the concept of rational number: (1) quotient; (2) measure; (3) operator; and (4) ratio. Berh, Lesh, Post and Silver (1984) consider five "sub-constructs" to clarify the concept of rational number, which are: (1) part-whole; (2) quotient; (3) ratio; (4) operator; and (5) measure. More recently, Nunes, Bryant, Pretzlik, Evans, Wade and Bell (2004) presented a classification based on "situations" in which fractions are used, relying on the meaning of the magnitudes assumed in each case, distinguishing: (1) part-whole; (2) quotient; (3) operator; and (4) intensive quantities.

This study adopts Nunes et al. (2004) classification in which in quotient interpretation or situation, $\frac{a}{b}$ can represent the relationship between the number of recipients and items to be distributed (e.g., $\frac{2}{3}$ can represent 2 chocolate bars to be shared fairly by 3 children), but it also represents the quantity of an item received by each recipient (e.g., $\frac{2}{3}$ corresponds to the amount of chocolate received by each child). In the part-whole situation, $\frac{a}{b}$ represents the relationship between the number of equal parts in which the whole is divided and the number of these parts to be taken (e.g., $\frac{2}{3}$ of a chocolate bar means that this was divided into 3 equal parts and 2 of these parts were considered).

Studies focused on different interpretations of rational number have suggested that these affect differently how children understand fractions. Some authors argue that the quotient interpretation favors the understanding of the inverse relationship between numerator and denominator of fractions (see Mamede, Nunes & Bryant, 2005).

Mamede, Nunes and Bryant (2005) investigated whether the quotient and part-whole interpretation of fraction influence the children's performance in problem solving tasks. Eighty children participated in the study aged between 6- and 7-year-olds, who haven't had formal instruction on fractions, but some of them were already familiar with the words "half" and "fourths" in social contexts. The authors analyzed how children understand fractions in part-whole and quotient interpretations, in tasks related to equivalence, ordering, and labelling. Results indicated that children performed better in quotient interpretation than in part-whole regarding ordering and equivalence of fractions; children performed similarly when solving labelling tasks presented in quotient and in part-whole interpretations. Children's success levels in ordering and equivalence of fractions in quotient interpretation suggests that they have some informal knowledge about the logic of fractions, developed in their daily life, without school instruction. These results emphasize the idea that different interpretations of fractions create distinct opportunities for children to understand the inverse relation between quantities.

Nunes et al. (2004) suggest that children's understanding of the inverse relation between quantities is facilitated in quotient interpretation because numerator and denominator are variables of different nature. Previous research on these issues was recently conducted by Mamede and Vasconcelos (2014), with Portuguese 4^{th} graders, to understand how the inverse relation between size and number of parts in division situations is related to the concept of fraction presented in quotient and part-whole interpretations. Among other things, they found out that children's performance in solving ordering and equivalence fraction problems in quotient interpretation were related to each other; their performance in solving ordering problems in quotient and in part-whole interpretations were related, but no significant correlations were observed when solving the equivalence problems in quotient and part-whole interpretations.

Traditionally, in Brazil and in Portugal, fractions are introduced to children using the part-whole interpretation of fractions. If children possess an informal knowledge about fractions, and classroom practices emphasize the introduction of fractions in part-whole interpretation from the 3^{rd} grade, how do children who already received some formal instruction on fractions understand the inverse relation between size of *n* and *n*-parts when problems are presented in quotient and part-whole interpretations? Cross-countries systematic comparisons are relevant, as both countries speak the same language, and are necessaries before making generalizations.

This study analyses Brazilian and Portuguese children's ability to establish the inverse relationship between quantities, for understanding the concept of fractions and the logical invariants of ordering and equivalence. It addresses three questions: 1) How do children understand the inverse relation between quantities when fractions are presented in quotient interpretations? 2) How do children understand this inverse relation when fractions are presented in part-whole interpretation? 3) Are there differences in performance between Brazilian and Portuguese children concerning the understanding of the inverse relation between quantities in these interpretations?

METHODS

A survey by questionnaire was conducted with 9- to 10-year-olds Portuguese (n=42; mean age = 9.69), and Brazilian (n=42; mean age = 9.88) children. The questionnaire included 22 tasks: 8 problems with fractions in part-whole interpretation (4 ordering; 4 equivalence); 8 problems with fractions in quotient interpretation (4 ordering; 4 equivalence); and 6 division problems (3 partitive division, 3 quotitive division). Due to length constrains, the analysis presented here will focuses only on problems of fractions presented in quotient and part-whole interpretations.

All fractions involved in the tasks were less than 1 and were the same for the problems proposed with quotient and part-whole interpretation. Table 1 shows an example of tasks presented for each type of fraction interpretation.

The questionnaire was solved individually and lasted for 40 minutes, and was implemented by the class teacher. Each child received a booklet with one problem per sheet to be solved. In each problem, multiple-choice questions were present, and the judgment for relative value of the quotients by using relations "more than/ less than/ same quantity as" was favored.

Problems	Equivalence	Ordering
Part-whole	Marco and Lara have each a pizza with the same size. Marco divided his pizza into 5 equal parts and ate one part. Lara divided her pizza into 10 equal parts and ate two parts. Did Marco eat more pizza than, less pizza than, or the same quantity of pizza as Lara? Explain why.	Ana and Rita have each a chocolate with the same size. Ana ate $\frac{1}{2}$ of her chocolate and Rita ate $\frac{1}{3}$ of her chocolate. Did Ana eat more chocolate than, less chocolate than, or the same quantity of chocolate as Rita? Explain why.
Quotient	Children share two same-sized cakes. Two girls share one cake fairly; three boys share the other cake fairly. Does each girl eat more cake than, less cake than, or the same quantity of cake as each boy? Explain why.	Two girls will share a chocolate bar and each one will eat $\frac{1}{2}$ of the chocolate. Three boys will share a chocolate bar and each one will eat $\frac{1}{3}$ of the chocolate. Does each girl eat more chocolate than, less chocolate than, or the same quantity of chocolate as each boy? Explain why.

Table 1: Examples of tasks presented with fractions in quotient and part-whole interpretations.

Questions were presented to the class and read by the researcher using PowerPoint slides. Each child had to indicate the right answer on the booklet and give a written explanation. The tasks used were adapted from the studies of Mamede, Nunes and Bryant (2005) and Spinillo and Lautert (2011).

Results

Results of the children's performances when solving the proposed tasks were analyzed, by assigning 1 to each right answer and 0 to each wrong answer. Table 2 presents the mean of the correct answers and standard deviations, according to the type of problem, presented in part-whole and quotient interpretations.

	Portugal		Brazil	
	Equivalence	Ordering	Equivalence	Ordering
Part-whole	1.4 (1.31)	1.98 (1.47)	0.55 (0.86)	1.07 (1.14)
Quotient	2.33 (1.14)	3.0 (1.19)	1.67 (1.43)	1.57 (1.40)

Table 2: Mean and (standard deviation) of children's correct responses according to the type of problem presented in part-whole and quotient interpretations, by country.

The results suggest that problems presented in quotient interpretation are easier for children than those presented in part-whole interpretation. Results also suggest that ordering problems are easier for children than equivalence ones. Table 1 also gives the idea that Portuguese children seem to perform better than Brazilian solving fractions problems presented in both interpretations.

Children's performance in each type of fractions problem presented in part-whole and quotient interpretations, by country is given by Figures 1-4.

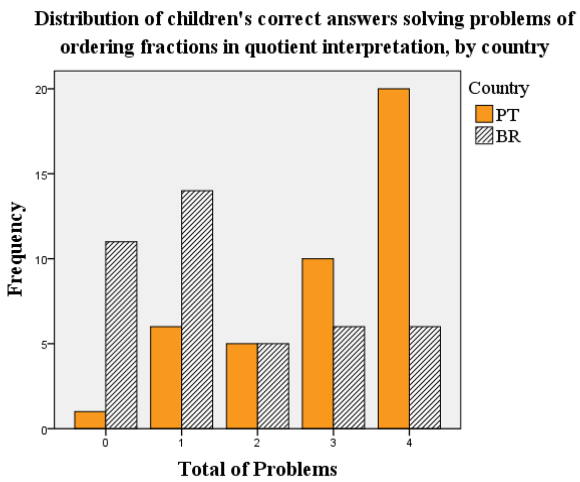


Figure 1

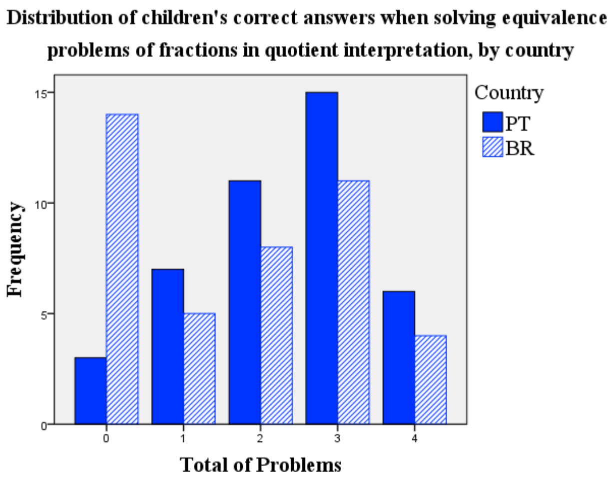
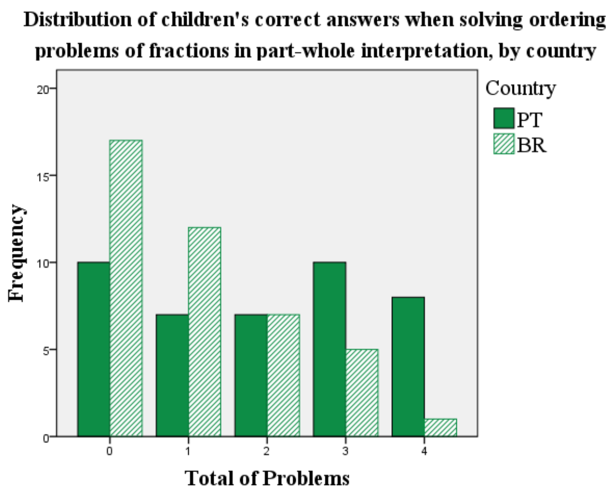


Figure 2



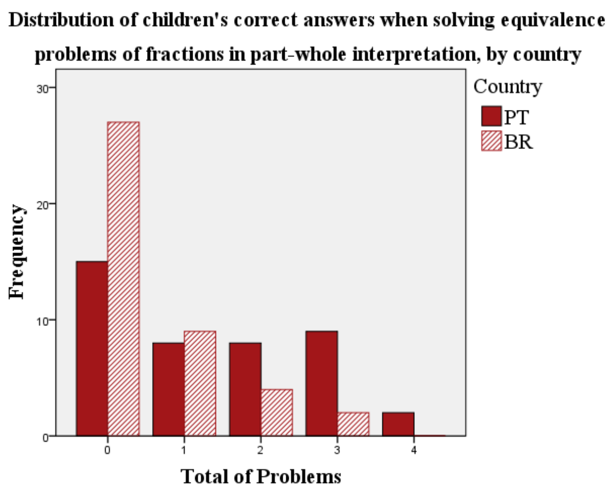


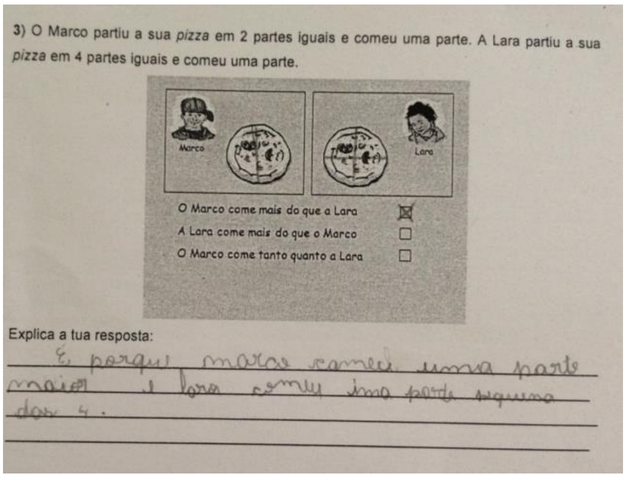
Figure 3 Figure 4

In quotient interpretation, when solving ordering problems, 83.3% of Portuguese and 40.5% of Brazilian children got at least 2 problems correctly solved; all of these problems were correctly solved by 47.6% and by 14.3% of Portuguese and Brazilian children, respectively. Regarding the equivalence problems, 76.2% of Portuguese and 54.7% of Brazilian children got at least half of the problems correctly solved; and 14.3% and 9.5% of Portuguese and Brazilian children, respectively, got all the problems correctly solved.

In part-whole interpretation, when solving ordering problems, 59.5% of Portuguese and 31% of Brazilian children got at least 2 problems correctly solved; all problems were correctly solved by 19% of Portuguese and by 2.4% of Brazilian children. Regarding the equivalence problems, 45.2% of Portuguese and 14.3% of Brazilian children got at least half of the problems correctly solved; and 4.8% of Portuguese children got all problems correctly solved, but none of the Brazilian children did it.

A non-parametric Wilcoxon-Mann-Whitney test was conducted to compare Portuguese and Brazilian children's performance when solving the fractions problems (equivalence and ordering in quotient and part-whole interpretations). Children's performance when solving problems in quotient interpretation is significantly better in the group of Portuguese than in Brazilian children (($U=403$; $W=1305$; $p<.001$) for ordering problems and ($U=647$; $W=1550$; $p<.05$) for equivalence problems). Portuguese children's performance was significantly better than Brazilian also when solving problems presented in part-whole interpretation, ($U=573$; $W=1476.5$; $p<.05$) for ordering problems and ($U=552.5$; $W=1455.5$; $p<.001$) for the equivalence ones.

The discrepancy of performance between Portuguese and Brazilian children when solving the tasks might be explain by the differences in the mathematics instruction of fourth graders in these countries.



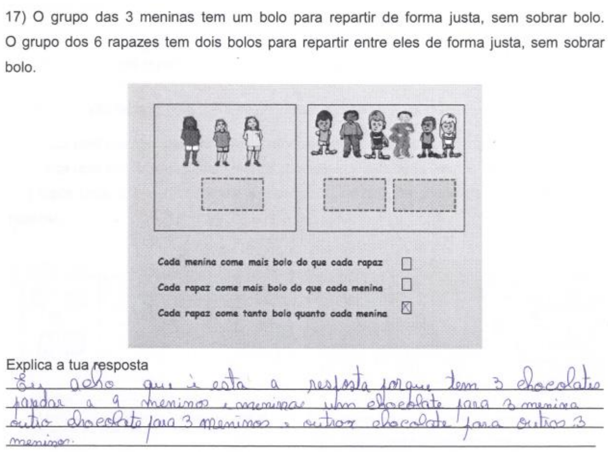


Figure 5 Figure 6

In Figure 5 an ordering problem was presented in part-whole interpretation to compare fractions ½ and ¼. The child justifies that Marco eats more pizza than Lara because

"Marco ate a bigger part and Lara ate a smaller part from the 4.". In Figure 6, an equivalence problem was presented in quotient interpretation to compare 1/3 and 2/6. The child justifies that "I think this is the answer because there are 3 chocolate bars for 9 boys and girls, a chocolate bar for 3 girls another chocolate for 3 boys e another for other 3 boys."

DISCUSSION AND CONCLUSIONS

This study suggests that Portuguese and Brazilian children understand the inverse relation between quantities in quotient interpretation. In spite of some differences in performance across countries, most of the children of each country who participated in this study, succeeded in solving simple ordering and equivalence fractions problems presented to them in the quotient interpretation. In the part-whole interpretation, children of both countries found more difficult to succeed either in ordering or equivalence simple problems.

The results suggest that quotient and part-whole interpretations contribute differently to the children's understanding of the inverse relation between quantities. This idea is in agreement with previous research carried out with 6- and 7-year-olds children (see Mamede, Nunes & Bryant, 2005), who had not received any formal instruction about fractions in school, but could succeed in solving simple fraction problems in quotient interpretation, revealing that children possess some type of informal knowledge on the logic of fractions (ordering and equivalence), developed in their daily life. If different interpretations of fractions involve distinct levels of understanding of the inverse relation between quantities for children, caution should be made when exploring these interpretations in the mathematics classes. Teachers should be aware that an absence of exploration of an interpretation of fractions may compromise children's understanding of rational numbers.

The discrepancy of performance between Portuguese and Brazilian children when solving the tasks might be explained by the differences in the mathematics instruction of fourth graders in these countries. In Portugal, children contact more informally with fractions in 3rd grade and more formally in 4th grade, according to the official curricular guidance; In Brazil, in spite of curricular guidance related to these issues, frequently teachers avoid to explore fractions in mathematics classes, compromising the development of children's understanding on the inverse relation between quantities. Possibly, this happens because teachers do not believe that their children can understand such relations or are unsure about how to explore fractions with their students. This study gives evidence that fourth-graders can understand the inverse relation between quantities, and interesting discussion moments around these subjects could take place in their classes. More research needs to be developed concerning such important topic in order to stimulate properly the children's understanding of the inverse relation between quantities, at primary school levels.

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