3D-MAPPINGS AND THEIR APPROXIMATION BY SERIES OF POWERS OF A SMALL PARAMETER

J. F. Cruz [*1], **M.I. Falcão**[2], **and H.R. Malonek** [3]

[*]*Universidade de Aveiro,*
Portugal
E-mail: josefonsecacruz@sapo.pt

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Abstract. *Conformal mappings in the plane are closely linked with holomorphic functions and their property of complex differentiability. In contrast to the planar case, in higher dimensions the set of conformal mappings consists only of Möbius transformations which are not monogenic and therefore also not hypercomplex differentiable. But due to the equivalence between being hypercomplex differentiable and being monogenic the question arises if from this point of view monogenic functions can still play a special role for other types of 3D-mappings, for instance, for quasi-conformal ones. Our goal is to present a case study of an approach to 3D-mappings which is an extension of ideas of L. V. Kantorovich to the 3-dimensional case by using para-vectors and a suitable series of powers of a small parameter. In the case of the application of Bergman's reproducing kernel approach (BKM) to 3D- mapping problems the recovering of the mapping function itself and its relation to the kernel function is still an open problem. The approach that we present here avoids such difficulties and leads directly to an approximation by monogenic polynomials depending on that small parameter.*

1 INTRODUCTION

In classical complex function theory conformal mappings are closely linked with complex differentiable functions. Indeed, every conformal mapping is realized either by a holomorphic or an anti-holomorphic function f. Moreover, due to Riemanns's mapping theorem any two simply connected regions (except $\mathbb{R}^2$, the Euclidean plane) can be mapped conformally onto each other.

In contrast to the planar case, in $\mathbb{R}^n$, with $n \geq 3$, the set of conformal mappings is only the set of Möbius transformations (due to Liouville's theorem, proved in [10] in 1850 under the condition of f being at least a C^3 homeomorphism). Only in 1958, in [7], P. Hartman succeeded to prove this assertion for C^1 homeomorphisms.

The difficulties in characterizing those Möbius transformations in $\mathbb{R}^4$ by some differentiability property have been studied in detail in [9]. In the case of $\mathbb{R}^4$ the application of quaternions is natural, a fact that has already been noticed in [16] and [4], for instance. But the theory of generalized holomorphic functions (by historical reasons they are also called *monogenic functions*, cf. [3]) as it has been developed on the basis of Clifford algebras (with quaternions as a special case, cf. [4]) does *not* cover the set of Möbius transformations in $\mathbb{R}^n$, since Möbius transformations are *not monogenic* and therefore *monogenic functions* are not directly related to conformal mappings in $\mathbb{R}^n$, $n \geq 3$. Here one can only expect that monogenic functions realize quasi-conformal mappings.

It is evident that such a situation has originated many questions concerning the extension of theoretical and practical conformal mapping methods in $\mathbb{C}$ to the higher dimensional case, particularly in the setting of Clifford Analysis (see [13] for a special approach). Notice that, in this setting, contrary to the case of several complex variables there are no restrictions on the real dimension of being even or odd. This implies that the real 3-dimensional Euclidean space, the most important space for concrete applications, can be subject to a treatment similar to the complex one. Some of those practical mapping methods have been discussed in [1] on the occasion of the 16th IKM in Weimar 2003 . That article was mainly concerned with methods based on the application of Bergman's reproducing kernel approach (BKM) to *numerical* conformal mapping problems. Whereas up to then almost all authors working with BKM in the Clifford setting (cf. the references in [1]) had only been concerned with the general algebraic and functional analytic background which allowed the explicit determination of the kernel in special situations, the main goal of [1] was the numerical experiment by using a Maple software specially developed for that purpose. The article [2] is a continuation of that work.

But since BKM is *only one* of a great variety of concrete numerical methods developed for mapping problems, our goal is to present in this case study a completely different approach. In fact, it is an extension of ideas of L. V. Kantorovich (c.f. [8]) to the 3-dimensional case by using quaternions and a suitable series of powers of a small real parameter. Whereas until now in the Clifford case of BKM the recovering of the mapping function itself and its relation to the monogenic kernel function is still not completely solved, the generalized approach of Kantorovich avoids such difficulties and leads to a monogenic mapping function depending on the small power series parameter.

Therefore, like usual (see [3]), let $\{1, e_1, e_2, e_3\}$ be an orthonormal basis of the Euclidean vector space $\mathbb{R}^4$ with the (quaternionic) product given according to the multiplication rules

$$e_1^2 = e_2^2 = e_3^2 = -1, \quad e_1 e_2 = -e_2 e_1 = e_3.$$

Considering the subset

$$\mathcal{A} := span_{\mathbb{R}}\{1, e_1, e_2\}$$

of the quaternion algebra $\mathbb{H}$ (isomorphic to the special Clifford algebra $\mathcal{C}\ell_{0,2}$), the real vector space $\mathbb{R}^3$ can be embedded in $\mathcal{A}$ by the identification of each element $x = (x_0, x_1, x_2) \in \mathbb{R}^3$ with the *paravector* (sometimes also called *reduced quaternion*)

$$z = x_0 + x_1 e_1 + x_2 e_2 \in \mathcal{A}.$$

For $C^1(\Omega, \mathcal{A})$ define the (reduced) quaternionic Cauchy-Riemann operator

$$D = \frac{\partial}{\partial x_0} + e_1 \frac{\partial}{\partial x_1} + e_2 \frac{\partial}{\partial x_2}.$$

Solutions of the differential equations $Df = 0$ (resp. $fD = 0$) are called left-monogenic (resp. right-monogenic) functions in the domain Ω.

Let us remind that the differential operator D is not only a formal linear combination of the real partial derivatives $\frac{\partial}{\partial x_k}$ but, when applied to a given function $f : \Omega \to \mathbb{H}$, is nothing else than an areolar derivative in the sense of Pompeiu (cf. [15] and, in the hypercomplex case, [14]). The same is true for the conjugate quaternionic Cauchy-Riemann operator

$$\overline{D} = \frac{\partial}{\partial x_0} - e_1 \frac{\partial}{\partial x_1} - e_2 \frac{\partial}{\partial x_2}.$$

But if f is a function monogenic in Ω, its areolar derivative Df is vanishing and this is equivalent with the fact that the areolar derivative $\frac{1}{2}\overline{D}f$ can be considered as the hypercomplex derivative of the function f. This has been discussed to some extend in [6]. In [13] details about the corresponding integral representation of the hypercomplex derivative and related mapping properties are given. Finally, similar to what happens in $\mathbb{C}$ where, for a complex differentiable function $f' = \frac{df}{dz} = \frac{1}{2}(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}) = \frac{\partial f}{\partial x}$, in our case also

$$\frac{1}{2}\overline{D}f = \frac{\partial f}{\partial x_0}. \tag{1}$$

Obviously, formula (1) guarantees that the (hypercomplex) derivative of a monogenic function is again a monogenic function.

In general, due to the algebraic properties of $\mathbb{H}$, we have to assume that a monogenic function f has values in $\mathbb{H}$, i.e., it is of the form

$$f(x) = f_0(x) + f_1(x)e_1 + f_2(x)e_2 + f_3(x)e_3,$$

where $f_k, \ k = 0, 1, 2, 3$ are real valued functions in Ω.

But if we are dealing with mappings from one 3-dimensional domain to another 3-dimensional domain we have to restrict the range of f to a quaternion-valued function with one identically zero component. Of course, this can be done by different choices. Here we consider a function f, defined in Ω and being also a paravector (or reduced quaternion), i.e.

$$f : \Omega \to \mathcal{A}$$

with

$$f(x) = f_0(x) + f_1(x)e_1 + f_2(x)e_2.$$

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In this case a monogenic function is bi-monogenic, i.e. $Df = fD = 0$ and in terms of the real partial derivatives both equations are equivalent with the so called Riesz system

$$
\begin{aligned}
\frac{\partial f_0}{\partial x_0} - \frac{\partial f_1}{\partial x_1} - \frac{\partial f_2}{\partial x_2} &= 0 \\
\frac{\partial f_0}{\partial x_1} + \frac{\partial f_1}{\partial x_0} &= 0 \\
\frac{\partial f_0}{\partial x_2} + \frac{\partial f_2}{\partial x_0} &= 0 \\
\frac{\partial f_1}{\partial x_2} - \frac{\partial f_2}{\partial x_1} &= 0.
\end{aligned}
\tag{2}
$$

After these basic requisites about monogenic functions in domains of real dimension three, the following sections are dealing with their power series development and their application to 3D-mappings.

2 MONOGENIC FUNCTIONS AS MAPPING FUNCTIONS

2.1 Paravector-valued monogenic functions of a paravector in $\mathbb{R}^3$

The description of the series development of monogenic functions will be made here in terms of two hypercomplex monogenic variables $z_k = x_k - x_0 e_k, k = 1, 2$. Indeed, using the general approach for Clifford algebra valued monogenic functions ([12], [5]) restricted to our case of $n = 2$, a second hypercomplex structure of $\mathbb{R}^{2+1}$ different from that given by the set of paravectors $\mathcal{A}$ consists in the following isomorphism:

$$
\mathbb{R}^{2+1} \cong \mathcal{H}^2 = \{\vec{z} : z_k = x_k - x_0 e_k; x_0, x_k \in \mathbb{R}\}.
$$

where $k = 1, 2$. More detailed, this corresponds to taking two copies $\mathbb{C}_k$ of $\mathbb{C}$ and identifying $i \cong e_k, \quad (k = 1, 2); \quad x_0 \cong \Re z; \quad x_k \cong \Im z;$ where $z \in \mathbb{C}$, and let $\mathbb{C}_k := -e_k\mathbb{C}$. Then $\mathcal{H}^2$ is the cartesian product $\mathcal{H}^2 := \mathbb{C}_1 \times \mathbb{C}_2$ and $\mathcal{Cl}_{0,2}$-valued functions of the form $f(z) = f_0(z) + f_1(z)e_1 + f_2(z)e_2 + f_{12}(z)e_1e_2$ are considered as mappings

$$
f : \Omega \subset \mathbb{R}^3 \cong \mathcal{H}^2 \longmapsto \mathcal{Cl}_{0,2}.
$$

For the next step we apply

Definition 1 *Let $V_{+,\cdot}$ be a commutative or non-commutative ring, $a_k \in V$ $(k = 1, \dots, n)$, then the symmetric "$\times$"-product is defined by*

$$
a_1 \times a_2 \times \cdots \times a_n = \frac{1}{n!} \sum_{\pi(i_1,\dots,i_n)} a_{i_1} a_{i_2} \cdots a_{i_n}
\tag{3}
$$

*where the sum runs over **all** permutations of all $(i_1, \dots, i_n)$*

together with the
Convention:
If the factor a_j occurs μ_j-times in (3), we briefly write

$$
\underbrace{a_1 \times \cdots \times a_1}_{\mu_1} \times \cdots \times \underbrace{a_n \times \cdots \times a_n}_{\mu_n}
\tag{4}
$$

$$
= a_1{}^{\mu_1} \times a_2{}^{\mu_2} \times \cdots \times a_n{}^{\mu_n} = \vec{a}^{\,\mu}
$$

where $\mu = (\mu_1, \dots, \mu_n,)$ and set parentheses if the powers are understood in the ordinary way (see [14]).

Since the symmetric products of μ_1 factors z_1 and μ_2 factors z_2 are monogenic functions of homogeneous degree $\mu_1 + \mu_2$ which form a basis for the Taylor series of a monogenic function in $\mathbb{R}^3$ (see [14]) they are called *generalized powers*. Using $\vec{x} = (x_1, x_2)$ and the multi-indices $\mu = (\mu_1, \mu_2)$ as well as $\nu = (\nu_1, \nu_2)$ it is easy to verify that the partial derivatives of $(\vec{z} - \vec{a})^\mu$ with respect to x_1 and x_2 result in

$$\frac{\partial^{|\nu|}}{\partial \vec{x}^\nu}(\vec{z} - \vec{a})^\mu|_{\vec{z} = \vec{a}} = \begin{cases} \mu! & \text{if } \nu = \mu \\ 0 & \text{if } \nu \neq \mu \end{cases}$$

This implies (cf. [12])

Theorem 1 *Every convergent R-power series (L-power series) generates in the interior of its domain of convergence a monogenic function $f(\vec{z})$ and coincides there with the Taylor series of $f(\vec{z})$, i. e. in a neighborhood of $\vec{z} = \vec{a}$ we have*

$$f(\vec{z}) = \sum_\mu \frac{1}{\mu!} \frac{\partial^{|\mu|} f(\vec{a})}{\partial \vec{x}^\mu}(\vec{z} - \vec{a})^\mu \ \text{ resp. } \ f(\vec{z}) = \sum_\mu \frac{1}{\mu!}(\vec{z} - \vec{a})^\mu \frac{\partial^{|\mu|} f(\vec{a})}{\partial \vec{x}^\mu}.$$

It is usual to consider instead of these general multiple power series the corresponding series ordered by powers of the same homogeneous degree (with a different domain of convergence, in general, cf. [3]). If $|\mu| = n$ then it holds

$$\frac{1}{\mu!} = \frac{1}{n!}\binom{n}{\mu}$$

and by setting $\mu_1 = (n-k), \mu_2 = k; k = 0, 1, \dots n$, the corresponding form of the L - Taylor series can be written as

$$f(z_1, z_2) = \sum_{n=0}^\infty \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} \ (z_1 - a_1)^{n-k} \times (z_2 - a_2)^k \frac{\partial^n f(a_1, a_2)}{\partial x_1^{n-k} \partial x_2^k} \tag{5}$$

(analogously for R-series, with the coefficients on the left side of the powers.) In the following we shall only consider the case of L-monogenic functions.

Obviously, in the neighborhood of the origin the series reduces to

$$f(z_1, z_2) = \sum_{n=0}^\infty \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} \ z_1^{n-k} \times z_2^k \frac{\partial^n f(0,0)}{\partial x_1^{n-k} \partial x_2^k} = \sum_{n=0}^\infty \sum_{k=0}^n \binom{n}{k} \ z_1^{n-k} \times z_2^k \alpha_{(n-k,\,k)}$$

with

$$\alpha_{(n-k,\,k)} := \frac{1}{n!} \frac{\partial^n f(0,0)}{\partial x_1^{n-k} \partial x_2^k}. \tag{6}$$

Being obtained by the partial derivatives of a function $f : \Omega \to \mathcal{A}$ it is evident that the coefficients $\alpha_{(n-k,\,k)}$ should have the form of a reduced quaternion (or paravector):

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$$\alpha_{(n-k,\,k)} = \alpha^0_{(n-k,\,k)} + \alpha^1_{(n-k,\,k)}\,e_1 + \alpha^2_{(n-k,\,k)}\,e_2, \quad n = 0, 1, 2\ldots; \; k = 0, \ldots, n, \qquad (7)$$

if explicitly written with their real and imaginary parts. As we now prove, the last equation of the Riesz system (2) implies a special relationship between adjacent coefficients which guarantees that the sum is a paravector-valued polynomial.

Theorem 2 *A homogeneous monogenic polynomial of degree n given by*

$$u(z_1, z_2) := \sum_{k=0}^{n} \binom{n}{k} \; z_1^{\,n-k} \times z_2^{\,k} \alpha_{(n-k,\,k)} \qquad (8)$$

with arbitrary paravector-valued coefficients is paravector-valued if and only if

$$\alpha^2_{(n-k,\,k)} = \alpha^1_{(n-k-1,\,k+1)} \;\; n = 0, 1, 2\ldots; \; k = 0, \ldots, n. \qquad (9)$$

Sketch of proof: To see that (9) is necessary we start from the last equation of (2) with $f = u$, i.e.

$$\frac{\partial u_2}{\partial x_1} = \frac{\partial u_1}{\partial x_2}. \qquad (10)$$

Since monogenic functions are infinitely hypercomplex differentiable and therefore also real differentiable of arbitrary order we can differentiate both sides of (10) $(n - k - 1)$-times by x_1 and k-times by x_2 to end up with

$$\frac{\partial^n u_2}{\partial x_1^{n-k} \partial x_2^k} = \frac{\partial^n u_1}{\partial x_1^{n-k-1} \partial x_2^{k+1}}. \qquad (11)$$

Formula (11), together with (6), (7), leads to the assertion (9). The fact that (9) is also a sufficient condition can be shown by induction over k. $\square$

Note, that an $L-$monogenic paravector-valued function of a paravector of arbitrary dimension is also $R-$monogenic, i.e. bi-monogenic. This was proved in [11] by simple arguments relying on hypercomplex differentiability.

Taking into account that the main goal of this paper is to realize approximations of 3D-mappings by polynomials generated by partial sums of series of the form (5), we discuss now briefly further general aspects of these series inspired by the corresponding complex approach. We are mainly interested in detecting problems by using the Maple software developed for the work with quaternions by S. Bock (*Quatpackage* see [5]). In this sense it is also an experiment to certify the efficiency of this package. Although this package can have broader applications, here we restrict ourselves only to a situation where the interior of a domain should be transformed into the interior of a sphere.

As it is usual in the case of conformal transformations in the complex plane, the domain $\Omega \subset \mathbb{R}^3$ that we are going to transform should contain the origin. We suppose also, that the domain into which we are mapping should be a ball $\mathcal{B} \subset \mathbb{R}^3$ with the center at the origin.

Since the generalized powers $z_1^{\,n-k} \times z_2^{\,k}, \;$ for $\; n = 1, 2, \ldots; \; k = 0, 1, \ldots, n$, are vanishing at the origin, the requirement that the origin is an invariant point under the considered mapping

$f : \Omega \to \mathcal{B}$ means that in (8) the first coefficient should be equal to zero, i.e. $\alpha_{0,0} = 0$ and each monogenic approximation polynomial begins with an expression of the form

$$u = z_1 \alpha_{(1,0)} + z_2 \alpha_{(0,1)}. \tag{12}$$

Clearly, the invariance of the origin guarantees in our settings that the interior of Ω will be mapped to the interior of the ball.

On the other hand, (12) represents the first order (linear) approximation of the mapping function that we are looking for and, as such, is naturally related to its hypercomplex derivative in the origin.

What is the corresponding situation in $\mathbb{C}$? In the complex case we know that the Riemann mapping theorem still allows to prescribe, for example, the direction in which the real axis should be mapped. Such behavior is simply related to a property of the complex derivative. For instance, often the positivity or a special value of the argument of the derivative in the origin is demanded. Or take for example the requirement that $f'(0) = 1$, where for the moment $f : \Omega \to \mathcal{B} \subset \mathbb{C}$ (cf. [8]). This means that the first (linear) approximation of the mapping function f is given as $w = f(z) = z, z \in \mathbb{C}$. In other words, in the first step of approximation nothing else than the identity function is used. Moreover, this means also that in the first step the unit ball in the image plane $\{w : |w| \leq 1\}$, has as its pre-image the unit ball $\{z : |z| \leq 1\}$. But step by step the approximation by polynomials of higher degree changes the situation. In some sense we could say that in $\mathbb{C}$, from the geometric view point the conditions $f(0) = 0$ as well as $f'(0) = 1$, are *normalizing the first step* in the approximation process of a domain Ω to a circle: the simplest polynomial of degree 1, namely the identity $w = f(z) = z$, is used and therefore in this step a $w-$circle is obtained from the corresponding $z-$circle. This works independently from the considered domain Ω and in so far we do not only have the $w-$circle as the canonical "target" domain, but also the $z-$circle as the canonical "starting" domain.

In the 3D-case (and, in general, for any real dimension $n > 2$) the situation is different, due to the nature of the used function class. Whereas in $\mathbb{C}$ the identity $f(z) = z$ with $f'(z) \equiv 1$ is a holomorphic function, it is not the case that $f(z) = z = x_0 + x_1 e_1 + x_2 e_2 \in \mathcal{A}$ is a monogenic function. Indeed, for a linear monogenic function f with hypercomplex derivative $\frac{1}{2}\overline{D}f(\mathbf{0}) = 1$ according to formula (12) we must have that

$$\frac{1}{2}\overline{D}f(\mathbf{0}) = -e_1 \alpha_{(1,0)} - e_2 \alpha_{(0,1)} = 1. \tag{13}$$

Applying (7) and (9) the formula (13) is equivalent to

$$\frac{1}{2}\overline{D}f(\mathbf{0}) = -e_1(\alpha^0_{(1,0)} + \alpha^1_{(1,0)} e_1) - e_2(\alpha^0_{(0,1)} + \alpha^2_{(0,1)} e_2) = 1. \tag{14}$$

This condition for a linear monogenic function with hypercomplex derivative equal to 1 is equivalent to

$$\alpha^0_{(1,0)} = \alpha^0_{(0,1)} = 0$$

as well as

$$\alpha^1_{(1,0)} + \alpha^2_{(0,1)} = 1. \tag{15}$$

Thus, together with (9) in the form $\alpha^2_{(1,0)} = \alpha^1_{(0,1)} = c$, the linear approximation is obtained as

$$\begin{aligned} w = f(z) &= z_1(\alpha^1_{(1,0)} e_1 + c e_2) + z_2(c e_1 + \alpha^2_{(0,1)} e_2) \\ &= \alpha^1_{(1,0)} z_1 e_1 + \alpha^2_{(0,1)} z_2 e_2 + c (z_2 e_1 + z_1 e_2) \\ &= x_0 + \alpha^1_{(1,0)} x_1 e_1 + \alpha^2_{(0,1)} x_2 e_2 + c (x_2 e_1 + x_1 e_2) \end{aligned} \tag{16}$$

Whereas in the complex case the demand for a function with derivative $f'(0) = 1$ immediately leads to a well defined linear approximation (the first step approximation with the above explained relationship between two circles), we see that in dimension three one condition concerning the hypercomplex derivative is not enough to normalize in the same sense the first step. In fact, three (real) parameters are still left. To overcome this problem we impose one more well motivated initial condition to the mapping function, namely that

$$\tilde{D}f(\mathbf{0}) := (\frac{\partial}{\partial x_0} + e_1 \frac{\partial}{\partial x_1} - e_2 \frac{\partial}{\partial x_2})f(\mathbf{0}) = 1. \tag{17}$$

Since we already required $\frac{1}{2}\overline{D}f(\mathbf{0}) = \frac{\partial}{\partial x_0}f(\mathbf{0}) = 1$ this implies that

$$(e_1 \frac{\partial}{\partial x_1} - e_2 \frac{\partial}{\partial x_2})f(\mathbf{0}) = 0$$

which, by direct calculation carried out on (16) results in the following relationship between the three until now not fixed real parameters:

$$-\alpha^1_{(1,0)} + \alpha^2_{(0,1)} + 2ce_1e_2 = 0;$$

hence $c = 0$ and by (15)

$$\alpha^1_{(1,0)} = \alpha^2_{(0,1)} = \frac{1}{2}. \tag{18}$$

Using these values in (16) we finally obtain for the initial ("first step") approximation the linear polynomial

$$\begin{aligned} w = f(z) &= \frac{1}{2}(z_1 e_1 + z_2 e_2) \\ &= x_0 + \frac{1}{2}(x_1 \, e_1 + x_2 \, e_2). \end{aligned} \tag{19}$$

From the geometrical point of view the result seems not to be a surprise. Indeed, formula (19) means nothing else than that, in the first step of approximation, the interior of the unit $w-$ball

$$\mathcal{B} = \{w : |w| \le 1\}$$

is obtained from the interior of a unit *oblate ellipsoid* or *oblate spheroid* given by

$$\mathcal{O} = \{(x_0, x_1, x_2) : x_0^2 + \frac{1}{4}x_1^2 + \frac{1}{4}x_2^2 = 1\}. \tag{20}$$

From the analytical point of view the additional condition (17) which led to this situation is also not very surprising. Besides others, we mention only two arguments:

(1) Due to the real dimension three, the use of three $\mathcal{H}^2$-linear hypercomplex differential operators is necessary for describing the three real partial derivatives in terms of hypercomplex differential expressions.

(2) The hypercomplex derivative of a monogenic function given by

$$\frac{1}{2}\overline{D}f = -e_1 \frac{\partial f}{\partial x_1} - e_2 \frac{\partial f}{\partial x_2} = \frac{\partial f}{\partial x_0}$$

reflects several essential qualitative properties of a monogenic function, but from the quantitative point of view does not allow to describe the influence of the partial derivatives with respect

to x_1 and x_2 separately. As we saw, the use of the operator $\tilde{D}$ defined by (17) solved this problem. Needless to note that by (17) the number of initial conditions has been increased by one, exactly the same as the real dimension increased by one compared with the complex case.

Summarizing all the results that have been explained in this subsection, by including the initial values of the unknown mapping function at the origin, we conclude that the general form of the series that we have to study is

$$f(z_1, z_2) = \frac{1}{2}(z_1 e_1 + z_2 e_2) + \sum_{n=2}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \, z_1^{n-k} \times z_2^{k} \alpha_{(n-k,\,k)} \tag{21}$$

together with the compatibility condition (9), i.e.,

$$\alpha^2_{(n-k,\,k)} = \alpha^1_{(n-k-1,\,k+1)} \ \ n = 0, 1, 2\ldots; \ k = 0, \ldots, n.$$

2.2 The approximate solution of a special 3D-mapping problem by monogenic polynomials involving a small parameter

In the previous subsection we deduced the form (21) of the series that we shall use for approximating a mapping of $\Omega \subset \mathbb{R}^3$ into a ball $\mathcal{B}_\varrho \subset \mathbb{R}^3$.

We suppose that the boundary of Ω can be embedded with a sufficiently small real parameter λ in a family of surfaces parameterized by s and t of the form

$$z = z(s, t, \lambda).$$

Suppose also that the family of surfaces includes the origin for all λ.

This idea follows Kantorovich's method in the complex plane ([8], Ch. V, §5), where an analogous family of curves $z = z(t, \lambda)$ is considered. The corresponding problem (mapping into a circle) together with the usual standardization of the mapping function leads to a series analogous to (21):

$$\varphi(z, \lambda) = z + \alpha_2(\lambda)z^2 + \alpha_3(\lambda)z^3 + \ldots \tag{22}$$

already written with indeterminate coefficients $\alpha_n(\lambda), n = 2, 3, \ldots$. The determination of those $\alpha_n(\lambda)$ by resolution of a non-linear system of algebraic equations depending on relationships between the boundaries of the considered domains is the core of the method.

From the previous subsection it is clear that we are trying to generalize Kantorovich's method by considering the series

$$\varphi(z_1, z_2, \lambda) = \frac{1}{2}(z_1 e_1 + z_2 e_2) + \sum_{n=2}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \, z_1^{n-k} \times z_2^{k} \alpha_{(n-k,\,k)}(\lambda) \tag{23}$$

where the indeterminate coefficients are paravectors satisfying the compatibility property

$$\alpha^2_{(n-k,\,k)}(\lambda) = \alpha^1_{(n-k-1,\,k+1)}(\lambda) \ \ n = 0, 1, 2\ldots; \ k = 0, \ldots, n.$$

As a concrete example, our case study is concerned with the mapping of the interior of the oblate ellipsoid $\mathcal{E}_\lambda, \ (0 \le \lambda < 1)$, defined by

$$x_0 = (1 + \lambda)\cos s, \ x_1 = 2(1 - \lambda)\sin s \cos t, \ x_2 = 2(1 - \lambda)\sin s \sin t$$

with $0 \le s \le \pi$ and $0 \le t < 2\pi$, into the interior of a ball $\mathcal{B}$.

Remark 1 *We notice that, due to (20), we have $\mathcal{E}_0 = \mathcal{O}$, which means that we are studying a small perturbation of the* canonical oblate spheroid $\mathcal{O}$ *which is mapped into the unit sphere by the linear monogenic function $w = \frac{1}{2}(z_1 e_1 + z_2 e_2) = x_0 + \frac{1}{2}(x_1 \; e_1 + x_2 \; e_2)$ (cf. 19). This can also be seen from the hypercomplex equation of $\mathcal{E}_\lambda$, which, in terms of $w = \frac{1}{2}(z_1 e_1 + z_2 e_2)$, is given by*

$$(1 + \lambda^2) w \, \overline{w} - \lambda (w \, w + \overline{w} \, \overline{w}) = (1 - \lambda^2)^2$$

and where the choice of $\lambda = 0$ leads immediately to $w \, \overline{w} = 1$.

The numerical efficiency of Kantorovich's methods relies also on simplifications in the series (22) by making use of symmetry properties of the considered domain Ω. For instance, the fact that, in some cases, one or both of the coordinate axes (or other symmetry axes) can be considered as invariant under the mapping immediately implies a substantial reduction of the indeterminate coefficients $\alpha_n(\lambda)$, $n = 2, 3, \dots$, and therefore reduces the numerical costs.

Carrying out similar calculations and simplifications in the case of $\partial\Omega = \mathcal{E}_\lambda$, we arrived to the following result, which we present here without the straightforward but rather cumbersome proof.

Theorem 3 *Let the series $\varphi(z_1, z_2, \lambda)$ be given by formula (23) with the compatibility condition*

$$\alpha^2_{(n-k, \, k)}(\lambda) = \alpha^1_{(n-k-1, \, k+1)}(\lambda) \;\; n = 2, 3, \dots; \; k = 0, \dots, n,$$

being fulfilled.   (i) The hyperplane $x_0 = 0$ is invariant (i.e., $x_0 = 0$ implies that $\varphi(z_1, z_2, \lambda)$ admits only pure imaginary values), if

$$\alpha^0_{(n-k,k)}(\lambda) := 0, \;\; for \;\; k = 0, \dots, n, \;\; n\text{=}2,3,\dots. \tag{24}$$

(ii) The hyperplane $x_0 = 0$ and the real axis $x_1 = x_2 = 0$ are invariant, if

$$z_1^{n-k} \times z_2^{k} = 0, \;\; for \; every \; even \; n, \; and \;\; k = 0, \dots, n.$$

and

$$\begin{cases} \alpha^2_{(n-k,k)} = 0, & for \; every \; odd \; n \;\; and \; even \; k, \\ \alpha^1_{(n-k,k)} = 0, & for \; every \; odd \; n \;\; and \; odd \;\; k, k = 0, 1, \dots, n. \end{cases}$$

These invariance properties immediately lead to the following

Corrolary 1 *The mapping of domains $\Omega \subset \mathbb{R}^3$, which are axially symmetric with respect to the real axis and admit a planar symmetry with respect to the imaginary hyperplane $x_0 = 0$, into a ball $\mathcal{B} \subset \mathbb{R}^3$ centered in the origin can be realized by a monogenic series of the form*

$$\begin{aligned}
\varphi(z_1, z_2) \;\; = \;\; & \frac{1}{2}(z_1 e_1 + z_2 e_2) + \\
& + (\alpha^1_{(3,0)} \, z_1^3 e_1 + 3\alpha^2_{(2,1)} \, z_1^2 \times z_2^1 e_2 + 3\alpha^1_{(1,2)} \, z_1^1 \times z_2^2 e_1 + \alpha^2_{(0,3)} \, z_2^3 e_2) + \\
& + (\alpha^1_{(5,0)} \, z_1^5 \, e_1 + 5\alpha^2_{(4,1)} \, z_1^4 \times z_2 \, e_2 + 10\alpha^1_{(3,2)} \, z_1^3 \times z_2^2 \, e_1 + 10\alpha^2_{(2,3)} \, z_1^2 \times z_2^3 \, e_2 + \\
& + 5\alpha^1_{(1,4)} \, z_1 \times z_2^4 \, e_1 + \alpha^2_{(0,5)} \, z_2^5 \, e_2) + \\
& + \cdots
\end{aligned} \tag{25}$$

with real coefficients $\alpha^l_{(n-k,k)}; n = 3, 5, 7, \dots; k = 0, 1, \dots, n; l = 1$ for even $k, l = 2$ for odd k.

Implying further in (25) the compatibility conditions

$$\alpha^2_{(n-k,\,k)}(\lambda) = \alpha^1_{(n-k-1,\,k+1)}(\lambda) \ \ n = 2, 3 \ldots; \ k = 0, \ldots, n,$$

writing as abbreviation for the inner coinciding coefficients

$$\beta_{31} := \alpha^2_{(2,1)} = \alpha^1_{(1,2)},$$

$$\beta_{51} := \alpha^2_{(4,1)} = \alpha^1_{(3,2)},$$

$$\beta_{53} := \alpha^2_{(2,3)} = \alpha^1_{(1,4)},$$

and so on for β_{nm} in all the following polynomials of higher homogeneous degree $n = 7, 9, \ldots; m = 1, 3, \ldots, n-2$, the formula (25) reduces to

$$
\begin{aligned}
\varphi(z_1, z_2) \ = \ & \frac{1}{2}(z_1 e_1 + z_2 e_2) + \\
& + (\alpha_{(3,0)} \ z_1^3 e_1 + 3\beta_{(3,1)} \ z_1^2 \times z_2^1 e_2 + 3\beta_{(3,1)} \ z_1^1 \times z_2^2 e_1 + \alpha_{(0,3)} \ z_2^3 e_2) + \\
& + (\alpha_{(5,0)} \ z_1^5 \ e_1 + 5\beta_{(5,1)} \ z_1^4 \times z_2 \ e_2 + 10\beta_{(5,1)} \ z_1^3 \times z_2^2 \ e_1 + 10\beta_{(5,3)} \ z_1^2 \times z_2^3 \ e_2 + \\
& + 5\beta_{(5,3)} \ z_1 \times z_2^4 \ e_1 + \alpha_{(0,5)} \ z_2^5 \ e_2) + \\
& + \cdots .
\end{aligned}
\tag{26}
$$

Here also the upper indices on the outer term coefficients in every homogeneous degree are omited since they are no longer relevant.

Remark 2 *It is obvious that, under the mentioned geometric conditions of Theorem 3, the total degree of freedom d in the choice of real coefficients in (26) corresponding to the homogeneous degree $n = 3, 5, \ldots$ is $d = \frac{1}{2}(n + 3)$. Similar calculations allow to estimate and compare the number of numerical procedures which are needed with and without additional information about symmetries of the considered domain Ω.*

3 A NUMERICAL EXPERIMENT

We now apply Corollary 1 to the example mentioned before, i.e., to the mapping of the interior of the oblate ellipsoid $\mathcal{E}_\lambda$, $(0 \le \lambda < 1)$, defined by

$$x_0 = (1 + \lambda)\cos s, \ x_1 = 2(1 - \lambda)\sin s \cos t, \ x_2 = 2(1 - \lambda)\sin s \sin t \tag{27}$$

with $0 \le s \le \pi$ and $0 \le t < 2\pi$, into the interior of a ball $\mathcal{B}$.

In this case the coefficients in (25) depend on the (sufficiently small) parameter λ, i.e. $a^l_{(n-k,k)} = a^l_{(n-k,k)}(\lambda)$. Kantorovich's method makes use of the approximation of the mapping function f through its approximation on the boundary surfaces, i.e., we define the coefficients $a^l_{(n-k,k)}(\lambda)$ of $f = \varphi(z_1, z_2, \lambda)$ up to a certain degree n by considering the mapping of $\mathcal{E}_\lambda$ to $\partial \mathbf{B}_\varrho$ with some radius ϱ.

Since we ask for the transformation into a ball, we have to look for the value of $|\varphi(z_1, z_2, \lambda))|^2$ on the surface $z(s, t, \lambda)$ as described in the beginning of subsection 2.2.

It is evident that the multiplication of $\varphi(z_1, z_2, \lambda)$ by its conjugate, together with (27), leads to an expression of the general form:

$$|\varphi(z(s, t, \lambda))|^2 = c_0(\lambda) + \sum_{i,j,k,l} c_{(i,j,k,l)}(\lambda) \ \sin^i s \cos^j s \sin^k t \cos^l t.$$

Due to the fact that on the sphere (i.e. on the boundary of $\mathbf{B}$), the value of $|\varphi(z_1, z_2, \lambda))|^2$ should be constant and equal to ϱ^2, we see that this development of $|\varphi(z(s, t, \lambda))|^2$ results in a nonlinear system of algebraic equations, with $a^l_{(n-k,k)}(\lambda)$ as unknowns, given by

$$c_{(i,j,k,l)}(\lambda) = 0 \tag{28}$$

and, consequently, we have

$$c_0(\lambda) = \varrho^2. \tag{29}$$

We illustrate now the considered approximation by a simple polynomial of degree 3, i.e.

$$
\begin{aligned}
\varphi_3(z_1, z_2, \lambda)) &= \frac{1}{2}(z_1 e_1 + z_2 e_2) + \\
&+ (\alpha_{(3,0)} \, z_1^3 e_1 + 3\beta_{(3,1)} \, z_1^2 \times z_2^1 e_2 + 3\beta_{(3,1)} \, z_1^1 \times z_2^2 e_1 + \alpha_{(0,3)} \, z_2^3 e_2).
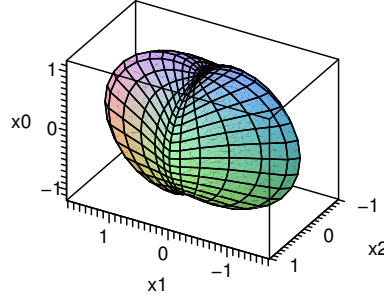\end{aligned} \tag{30}
$$

Solving the corresponding system (28) by using the Maple-Quatpackage from [5], we get

$$
\begin{cases}
\alpha_{(3,0)}(\lambda) = \frac{(59\lambda^4 - 180\lambda^3 + 290\lambda^2 - 180\lambda + 59)\lambda}{12(\lambda^3 - 3\lambda^2 + 3\lambda - 1)(\lambda - 3)(1+\lambda)^3} \\
\beta_{(3,1)}(\lambda) = -\frac{(7 - 2\lambda + 7\lambda^2)\lambda}{12(1+\lambda)(\lambda^4 - 6\lambda^3 + 12\lambda^2 - 10\lambda + 3)} \\
\alpha_{(0,3)}(\lambda) = \frac{(1+\lambda)\lambda}{4(\lambda^4 - 6\lambda^3 + 12\lambda^2 - 10\lambda + 3)}
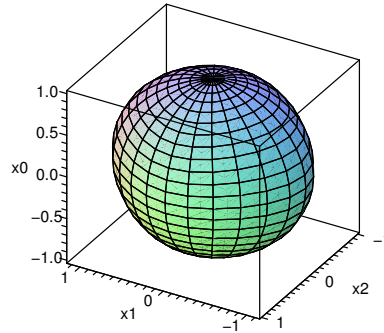\end{cases}
$$

and the radius ϱ of the ball $\mathbf{B}_\varrho$ is obtained as

$$\varrho = \sqrt{\frac{9\lambda^4 - 12\lambda^3 + 22\lambda^2 - 12\lambda + 9}{(\lambda - 3)^2}}.$$
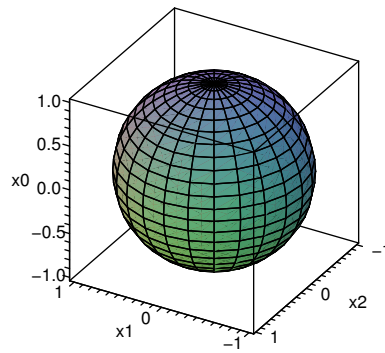
The approximations for different choices of λ can be seen in the following figures

Figure 1: Image in the case of $\lambda = 0.1$

Figure 2: Image in the case of $\lambda = 0.01$

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Figure 3: Image in the case of $\lambda = 0.001$

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