Existence of Positive Periodic Solutions for Scalar Delay Differential Equations with and without Impulses

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To the memory of Professor George R. Sell

Abstract

The paper is concerned with a broad family of scalar periodic delay differential equations with linear impulses, for which the existence of a positive periodic solution is established under very general conditions. The proofs rely on fixed point arguments, employing either the Schauder theorem or Krasnoselskii fixed point theorem in cones. The results are illustrated with applications to an impulsive hematopoiesis model or generalized Nicholson's equations, among other selected examples from mathematical biology. The method presented here turns out to be powerful, in the sense that the derived theorems largely generalize and improve other results in recent literature, even for the situation without impulses.

Keywords: delay differential equation; impulses; positive periodic solution; fixed point theorems; hematopoiesis model; Nicholson equation.

2010 Mathematics Subject Classification: 34K13, 34K45, 92D25.

1 Introduction

Delay differential equations (DDEs) have been extensively used in population dynamics, neural networks, disease modeling, control theory and many other scientific fields, where the delays naturally appear to account for a variety of situations. Among them, periodic DDEs are particularly relevant, since periodic parameters and delays allow incorporating in the models the periodicity of the environment, or some periodic physiological features, as well as other periodic phenomena. Moreover, some evolutionary systems go through abrupt changes, due to predictable or sudden external circumstances, such as weather, resource availability, food supplies, drug treatments, etc. These phenomena are better described by the so-called impulsive differential equations.

This paper is devoted to establishing the existence of at least one positive periodic solution for a large family of periodic scalar DDEs with linear impulses, given in the general form

$$\begin{aligned} &x'(t) + a(t)x(t) = g(t, x_t), \quad t_0 \le t \ne t_k, \\ &\Delta(x(t_k)) := x(t_k^+) - x(t_k) = b_k x(t_k), \quad k \in \mathbb{N}, \end{aligned} \tag{1.1}$$

where the functions a and g are continuous, nonnegative and periodic, with a common period $\omega > 0$, the impulses b_k at times t_k, with $t_0 < t_1 < t_2 < \cdots < t_k < \cdots$ and $t_k \to \infty$, occur with periodicity ω, and the solutions are left continuous, continuous for $t \ne t_k$, with jump discontinuities at $t_k, k \in \mathbb{N}$.

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First, some general terminology is introduced here, namely the concept of *solution*.

For an interval $[\alpha, \beta] \subset \mathbb{R}$, denote by $PC([\alpha, \beta]; \mathbb{R})$ the space of functions $\varphi : [\alpha, \beta] \to \mathbb{R}$ which are piecewise continuous on $[\alpha, \beta]$ and left continuous on $(\alpha, \beta]$, endowed with the supremum norm. For an interval $I \subset \mathbb{R}$, define the space $PC(I; \mathbb{R})$ as the space of bounded functions $\varphi : I \to \mathbb{R}$ whose restriction to each compact interval $[\alpha, \beta] \subset I$ is in the closure of $PC([\alpha, \beta]; \mathbb{R})$ in the space of bounded functions $B([\alpha, \beta]; \mathbb{R})$, with the norm $\|\varphi\| = \sup_{t \in \mathbb{R}} |\varphi(t)|$.

If $\tau > 0$ is the time-delay or 'memory' for (1.1), we take $PC([-\tau, 0]; \mathbb{R})$ as the phase space. In (1.1), we adopt the following standard notations: $x'(t)$ is the left-hand derivative of $x(t)$ and $x_t \in PC([-\tau, 0]; \mathbb{R})$ is defined by

$$x_t(s) = x(t + s), \quad s \in [-\tau, 0].$$

Initial conditions have the form $x_{t_0} = \varphi$ for $\varphi \in PC([-\tau, 0]; \mathbb{R})$. A **solution** $x(t)$ of (1.1) on an interval $[t_0, \beta)$ (with $\beta > t_0$ or $\beta = \infty$) is a piecewise continuous function in $PC([t_0 - \tau, \beta); \mathbb{R})$, with $x' \in PC([t_0 - \tau, \beta); \mathbb{R})$ and $x(t)$ continuous for $t \neq t_k$, such that $x'(t) + a(t)x(t) = g(t, x_t)$ on each interval $(t_k, t_{k+1}]$, $x(t_k^-) = x(t_k)$ and $x(t_k^+) = (1 + b_k)x(t_k)$, for $k \in \mathbb{N}$. Theoretical results about existence, uniqueness and global continuation of solutions are well-known, see e.g. [9, 13, 23]. It is important to emphasize that the family (1.1) in particular includes periodic DDEs without impulses of the form $x'(t) + a(t)x(t) = g(t, x_t)$.

The unifying method developed here demonstrates the existence of (at least) one positive periodic solution for very general classes of periodic impulsive equations given by (1.1), under very mild assumptions on the impulses and with nonlinearities $g(t, \varphi)\,(t \geq 0, \varphi \in PC([-\tau, 0]; \mathbb{R}))$ satisfying some prescribed behavior for $\varphi \geq 0$ either sufficiently small or sufficiently large. Our method is based on finding a fixed point for a convenient operator, by using either Schauder or Krasnoselskii fixed point theorems. Not only is this technique new, in the sense that a fixed point argument is applied to a new operator constructed here, but also our criteria rely on some sufficient conditions which are satisfied by a large number of relevant biological models and whose validity is easy to check. When applied to concrete models, the main theorems proven here give rise to new criteria on existence of a positive periodic solution, which largely generalize, recover or improve other results in recent papers, even for the situation without impulses. As we shall mention in the last section of the paper, the works of Wan et al. [24] and Li et al. [15] were a strong inspiration for the present study, whose main aim was twofold: first, to prove that a positive periodic solution for impulsive equations more general than the ones considered in [24, 15] (where only one discrete delay was allowed) must exist, and, second, to remove the requirement of having always *positive* impulses.

The organization of this paper is now briefly described. The main results, about the existence of a positive periodic solution for (1.1) with some types of nonlinearities $g(t, \varphi)$, are presented in Section 2. The results apply to both equations with and without impulses. Section 3 is devoted to applications to a selected number of celebrated equations used as blood cell production models or in population dynamics, such as generalized Lasota-Wazewska, Mackey-Glass or Nicholson equations. A discussion of the paper and a further review of relevant literature that focuses on the subject are postponed to Section 4, to better explain the main differences, advantages and disadvantages of our approach. Several open problems, as well as a number of future research directions, are also presented.

2 Main results

In this section, we present very general results on the existence of positive periodic solutions for scalar DDEs with impulses. As a consequence, criteria for non-impulsive DDEs will also be derived.

Consider the following scalar impulsive equation with delays:

$$\begin{cases} y'(t) + a(t)y(t) = g(t, y_t), \ 0 \le t \ne t_k, \\ y(t_k^+) - y(t_k) = b_k y(t_k), \quad k \in \mathbb{N}, \end{cases} \tag{2.1}$$

where $a : \mathbb{R} \to \mathbb{R}, g : \mathbb{R} \times PC([-\tau, 0]; \mathbb{R}) \to \mathbb{R}$ and $(t_k)_{k\in\mathbb{N}}$ is an increasing sequence, and

(h1) the functions $a(t), g(t, \varphi)$ are continuous, nonnegative, not identically zero and ω-periodic in $t \in [0, \infty)$, for some constant $\omega > 0$, and g is bounded on bounded sets of $\mathbb{R} \times PC([-\tau, 0]; \mathbb{R})$;

(h2) there is a positive integer p such that $0 < t_1 < t_2 < \cdots t_p \le \omega$ and

$$t_{k+p} = t_k + \omega, \ b_{k+p} = b_k, k \in \mathbb{N};$$

(h3) the constants $b_1, \dots, b_p \in \mathbb{R}$ satisfy $b_k > -1$;

(h4) $\prod_{k=1}^{p}(1 + b_k) < e^{\int_0^\omega a(t)\,dt}$.

Assumption (h3) guarantees that, at the impulsive points t_k, solutions of (2.1) with $y(t_k^-) = y(t_k) > 0$ must satisfy $y(t_k^+) > 0, k \in \mathbb{N}$. Thus, since g transforms bounded sets of $\mathbb{R} \times PC([-\tau, 0]; \mathbb{R})$ into bounded sets of $\mathbb{R}$, solutions with nonnegative initial conditions $\varphi \in PC([-\tau, 0]; \mathbb{R})$ and $\varphi(0) > 0$ are defined and remain positive on $\mathbb{R}^+ = [0, \infty)$.

A function $y \in P(\mathbb{R}; \mathbb{R})$ is ω-periodic if $y(t + \omega) = y(t), \forall t \in \mathbb{R}$. For $y : \mathbb{R}^+ \to \mathbb{R}$ piecewise continuous and ω-periodic on $\mathbb{R}^+$, we may identify y with its ω-periodic extension to $\mathbb{R}$.

Let X be the Banach space of the ω-periodic functions $y : \mathbb{R}^+ \to \mathbb{R}$ which are continuous on $\mathbb{R}^+ \setminus \{t_k\}$, with $y(t_k^-) = y(t_k)$ and $y(t_k^+) \in \mathbb{R}$ for all $k \in \mathbb{N}$, endowed with the supremum norm $\|y\| = \sup_{t\in[0,\omega]} |y(t)|, \ y \in X$. The open balls in X of center $y \in X$ and radius $r > 0$ will be denoted by $B_r(y)$. Denote by X^+ the cone of non-negative functions in X, i.e., $X^+ = \{y \in X : y(t) \ge 0, t \in [0, \omega]\}$, and consider the partial order in X induced by X^+, $y_1 \le y_2$ if $y_2 - y_1 \in X^+$.

In order to simplify the writing, in what follows we adopt some notations. Define the functions

$$\begin{aligned} &A(t) = \int_0^t a(u)\,du\,, \quad B(t) = \prod_{k:t_k\in[0,t)} (1 + b_k)^{-1}, \\ &B(s, t) = B(s)B(t)^{-1} = \prod_{k:t_k\in[t,s)} (1 + b_k)^{-1}, \qquad t \ge 0, s \in [t, t + \omega]. \end{aligned} \tag{2.2}$$

Clearly, $B(s + \omega, t + \omega) = B(s, t)$ for $0 \le t \le s \le t + \omega$. Hypotheses (h2),(h3) imply that $\overline{B}, \underline{B} \in (0, \infty)$, where

$$\overline{B} = \sup_{0\le t\le s\le t+\omega} B(s, t), \ \underline{B} := \inf_{0\le t\le s\le t+\omega} B(s, t), \tag{2.3}$$

while $B(\omega)e^{A(\omega)} > 1$ as a result of (h4). The situation without impulses, in which case $b_k = 0$ for all k and $\overline{B} = \underline{B} = 1$, is included in our setting.

With the notations in (2.2), define the operator

$$(\Phi y)(t) = \big(B(\omega)e^{A(\omega)} - 1\big)^{-1} \int_t^{t+\omega} B(s,t)g(s,y_s)e^{\int_t^s a(u)\,du}\,ds \tag{2.4}$$

for $y \in X^+, t \geq 0$, and a new cone

$$K = K(\sigma) = \{y \in X^+ : y(t) \geq \sigma \|y\| \text{ for } t \in [0,\omega]\},$$

where $\sigma \in (0,1)$ is a constant to be chosen below.

Lemma 2.1. *Assume (h1)-(h4) and take $\sigma > 0$ with*

$$\sigma \leq \big(\underline{B}/\overline{B}\big)e^{-A(\omega)}.$$

Then $\Phi(K) \subset K$.

Proof. Let $y \in X^+$. From the definition of Φ and (h1)-(h4), it is clear that $\Phi y \geq 0$ and Φy is ω-periodic. Moreover, $t \mapsto (\Phi y)(t)$ is continuous for $t \neq t_k$, with $(\Phi y)(t_k^-) = (\Phi y)(t_k)$ and

$$(\Phi y)(t_k^+) = (1 + b_k)(\Phi y)(t_k) \quad \text{for} \quad k \in \mathbb{N}. \tag{2.5}$$

Thus, $\Phi(X^+) \subset X^+$. For $y \in K$ and $t \geq 0$, we have

$$(\Phi y)(t) \leq \big(B(\omega)e^{A(\omega)} - 1\big)^{-1}\overline{B}e^{A(\omega)} \int_0^\omega g(s,y_s)ds,$$

and

$$(\Phi y)(t) \geq \big(B(\omega)e^{A(\omega)} - 1\big)^{-1}\underline{B} \int_0^\omega g(s,y_s)ds,$$

implying that

$$\|\Phi y\| \leq \big(B(\omega)e^{A(\omega)} - 1\big)^{-1}\overline{B}e^{A(\omega)} \int_0^\omega g(s,y_s)ds$$

and

$$(\Phi y)(t) \geq \underline{B}(\overline{B}e^{A(\omega)})^{-1}\|\Phi y\|.$$

□

In what follows, even if not mentioned, $K = K(\sigma)$ for some fixed σ, with $0 < \sigma \leq \big(\underline{B}/\overline{B}\big)e^{-A(\omega)}$.

Lemma 2.2. *Assume (h1)-(h4). Then $y = y(t)$ is a nonnegative ω-periodic solution of* (2.1) *if and only if y is a fixed point of the operator Φ.*

Proof. Let $y = y(t) \geq 0$ be an ω-periodic solution of (2.1). From [25, Lemma 3.1], the function $x(t) := B(t)y(t)$ is continuous and satisfies

$$x'(t) + a(t)x(t) = B(t)g(t, y_t), \quad t \geq 0,\ t \neq t_k.$$

Fix $t \neq t_k$. Multiplying by $e^{A(t)}$ and integrating over $[t, t+\omega]$, we obtain

$$\int_t^{t+\omega} \frac{d}{ds}\left[x(s)e^{A(s)}\right] ds = \int_t^{t+\omega} B(s)g(s, y_s)e^{A(s)}\, ds,$$

which is equivalent to

$$\big(B(t+\omega)e^{A(\omega)} - B(t)\big)y(t)e^{A(t)} = \int_t^{t+\omega} B(s)g(s, y_s)e^{A(s)}\, ds.$$

Since $B(t+\omega) = B(t+\omega, t)B(t) = B(\omega)B(t)$ and from the definition of $B(s,t) = B(s)B(t)^{-1}$, one obtains

$$\big(B(\omega)e^{A(\omega)} - 1\big)y(t) = \int_t^{t+\omega} B(s,t)g(s, y_s)e^{\int_t^s a(u)\, du}\, ds,$$

and thus $y = \Phi y$.

Conversely, suppose that $y \in X^+$ and $y = \Phi y$. From (2.5), it follows that $\Delta y(t_k) = b_k y(t_k)$ for $k \in \mathbb{N}$. For $t \neq t_k$, differentiation of $\Phi y = y$ leads to

$$y'(t) = -a(t)y(t) + \big(B(\omega)e^{A(\omega)} - 1\big)^{-1}\left[B(t+\omega, t)e^{\int_t^{t+\omega} a(u)\, du} - 1\right] g(t, y_t) = -a(t)y(t) + g(t, y_t).$$

□

The existence of a positive ω-periodic solution of (2.1) will be proven by using either the Schauder's or the Krasnoselskii's fixed point theorems [12]. For the latter case, the version for compressed cones given below is often used and is a simple corollary of Theorem 6.20.1 in [4].

Theorem 2.1. *Let X be a Banach space, K a closed cone in X and $T : K \to K$ a completely continuous operator. Suppose that there exist r, R with $0 < r < R$ such that:*
(i) $\|Ty\| \leq R$ if $y \in K, \|y\| = R$;
(ii) There exists $\psi \in K \setminus \{0\}$ such that $y \neq Ty + \lambda\psi$ for all $y \in K$ with $\|y\| = r$ and $\lambda > 0$.
Then there exists a fixed point y^* of T, with $y^* \in \{y \in K : r \leq \|y\| \leq R\}$.

The next lemma asserts that Φ given by (2.4) is completely continuous on K if more regularity on g is imposed. The restriction $\Phi|_K$ will be still denoted by Φ.

Lemma 2.3. *Assume (h1)-(h4) and that*

(h5) *the function $g_0 : [0, \omega] \times X^+ \to \mathbb{R}$ defined by $g_0(t, y) = g(t, y_t)$ is uniformly continuous on bounded sets of $[0, \omega] \times K$.*

Then the operator $\Phi : K \to K$ is completely continuous.

Proof. (i) We first prove that Φ is continuous. Fix $y_0 \in K$. For any $\varepsilon > 0, t \in \mathbb{R}^+$ and $y \in K \cap B_\varepsilon(y_0)$, we have $\|y_t - y_{0,t}\| < \varepsilon$ and

$$|(\Phi y)(t) - (\Phi y_0)(t)| \leq \big(B(\omega)e^{A(\omega)} - 1\big)^{-1}\overline{B}e^{A(\omega)} \int_0^\omega |g(s, y_s) - g(s, y_{0,s})| \, ds.$$

As the function $g_0(t, y)$ is uniformly continuous on bounded sets of $[0, \omega] \times K$, this shows that Φ is continuous at y_0.

(ii) To prove that Φ is a compact operator in K, first observe that, for any $R > 0$, $\Phi(K \cap B_R(0))$ is bounded, since $g_0(t, y)$ is bounded on bounded sets of $[0, \omega] \times K$.

Secondly, we show that Φ transforms bounded sets of K into relative compact sets of X. We define the auxiliary operator

$$(\mathcal{F}y)(t) = B(t)(\Phi y)(t)$$

and remark that formula (2.5) implies that $(\mathcal{F}y)(t)$ is continuous on $[0, \infty)$. On the other hand, $B(t)$ is bounded on $[0, \omega]$ below and above by positive constants. Take $R > 0$. To prove that $\Phi(K \cap B_R(0))$ is relative compact in X, we show that the family $\mathcal{F}_0 := \{(\mathcal{F}y)|_{[0,\omega]} : y \in K \cap B_R(0)\}$, considered as a subset of $C([0, \omega]; \mathbb{R})$, is equicontinuous.

For $y \in K \cap B_R(0)$, the function $(\mathcal{F}y)(t)$ is piecewise differentiable on $[0, \omega]$, with derivative

$$(\mathcal{F}y)'(t) = B(t)\big[- a(t)(\Phi y)(t) + g(t, y_t)\big], \quad t \in [0, \omega] \setminus \{t_1, \ldots, t_p\}.$$

Clearly, the derivative is uniformly bounded on $K \cap B_R(0)$, i.e., $\|(\mathcal{F}y)'\| \leq M$ for some $M > 0$, because $\Phi(K \cap B_R(0))$ is bounded and

$$|(\mathcal{F}y)'(t)| \leq \overline{B}\left[\max_{t\in[0,\omega]} a(t) \, \|\Phi y\| + \sup_{t\in[0,\omega], y\in B_R(0)} |g(t, y_t)|\right].$$

Therefore, $\mathcal{F}_0$ is equicontinuous, hence relatively compact in $C([0, \omega]; \mathbb{R})$. By multiplying each function $x(t)$ in $\mathcal{F}_0$ by $B(t)^{-1}$ and considering its ω-periodic extension on $\mathbb{R}^+$, we conclude that $\Phi(K \cap B_R(0))$ is relatively compact in K. □

We are finally in the position to prove the main results of this section. We shall consider two distinct situations. First, we address the case where $y(t) \equiv 0$ is not a solution of (2.1) and $g_0(t, y)$ is uniformly bounded on $[0, \omega] \times K$.

Theorem 2.2. *Consider* (2.1) *and assume that (h1)-(h5) hold. In addition, suppose that* $g_0(t, y) := g(t, y_t)$ *is uniformly bounded on* $[0, \omega] \times K$ *and that* $g(t, 0) \not\equiv 0$. *Then there exists at least one positive* ω*-periodic solution of* (2.1).

Proof. Consider $\Phi : K \to K$, where $K = K(\sigma) \subset X$ is the cone defined above, with e.g. σ fixed as $\sigma = \big(\underline{B}/\overline{B}\big)e^{-A(\omega)}$. By Lemma 2.2, it suffices to show that there exists a positive fixed point of Φ.

Let $|g(t, y_t)| \leq M$ for $(t, y) \in [0, \omega] \times X^+$, for some $M > 0$. For $R \geq M\omega\big(B(\omega)e^{A(\omega)} - 1\big)^{-1}\overline{B}e^{A(\omega)}$, we have

$$\Phi\left(K \cap \overline{B_R(0)}\right) \subset K \cap \overline{B_R(0)}.$$

On the other hand, observe that $K \cap \overline{B_R(0)}$ is a closed, bounded, convex subset of X. Since Φ is completely continuous, by Schauder's fixed point theorem there exists a fixed point y^* of Φ in $K \cap \overline{B_R(0)}$. Since $\Phi 0 \neq 0$, $y^* \neq 0$, thus $y^*(t) \geq \sigma\|y^*\| > 0$. □

For a large number of models with real world applications, one however has $g(t, 0) \equiv 0$, and thus $y(t) \equiv 0$ is a solution of (2.1). For (2.1) under hypotheses (h1)-(h5), if a type of monotonicity behavior on $g(t, \varphi)$ is prescribed for $\varphi \geq 0$ in the vicinity of 0^+ and 'at infinity', the use of Krasnoselskii's theorem on a suitable annulus $\{y \in K : r \leq \|y\| \leq R\}$ enables us to conclude that (2.1) has at least a positive ω-periodic solution in K.

Theorem 2.3. *Consider* (2.1) *and assume that (h1)-(h5) hold. In addition, suppose that there are constants* r_0, R_0 *with* $0 < r_0 \leq R_0$ *and continuous functions* $b, h : \mathbb{R}^+ \to \mathbb{R}^+$, *such that:*

(i) $b(t)$ *ω-periodic and* $b(t) \not\equiv 0$,

(ii) $h(u) > 0$ *for* $u > 0$, *with*

$$\lim_{u \to 0^+} \frac{u}{h(u)} < C, \quad \lim_{u \to +\infty} \frac{u}{h(u)} = \infty, \tag{2.6}$$

where

$$C = \big(B(\omega)e^{A(\omega)} - 1\big)^{-1} \underline{B} \inf_{t \geq 0} \int_t^{t+\omega} b(s) e^{\int_t^s a(u)\, du}\, ds, \tag{2.7}$$

(iii) for any $r \geq 0, \varphi \geq 0$ *and* $t \geq 0$,

$$\begin{aligned} g(t, \varphi) &\leq b(t)h(r) \quad \text{if} \quad R_0 \leq \varphi \leq r, \\ g(t, \varphi) &\geq b(t)h(r) \quad \text{if} \quad r \leq \varphi \leq r_0. \end{aligned} \tag{2.8}$$

Then there exists at least one positive ω-periodic solution of (2.1).

Proof. Again, consider $\Phi : K \to K$, where $K = K(\sigma) \subset X$ is the closed cone with $\sigma = \big(\underline{B}/\overline{B}\big)e^{-A(\omega)}$. We shall show that Φ has a fixed point in $\{y \in K : r \leq \|y\| \leq R\}$, for some $0 < r < R$.

Take $R \geq \sigma^{-1} R_0$. For $y \in K$ with $\|y\| = R$, we have $R_0 \leq y(t) \leq R$ and

$$(\Phi y)(t) \leq h(R)\big(B(\omega)e^{A(\omega)} - 1\big)^{-1} \overline{B} e^{A(\omega)} \int_0^\omega b(s) ds, \quad t \in [0, \omega].$$

Since $\lim_{u \to +\infty} \frac{h(u)}{u} = 0$, we can choose $R > 0$ sufficiently large such that

$$(\Phi y)(t) \leq R, \quad t \in [0, \omega].$$

Next, as a result of $\lim_{u \to 0^+} \frac{u}{h(u)} < C$, one can choose $r > 0$ small so that $r \leq r_0$ and $\frac{u}{h(u)} < C$ for $0 < u \leq r$, where C is as in (2.7).

With $\psi \equiv 1$, we claim that $\Phi y + \lambda \psi \neq y$ for all $\lambda > 0, y \in K, \|y\| = r$.

Otherwise, there are $\lambda_0 > 0$ and $y_0 \in K$ with $\|y_0\| = r$, such that $y_0 = \Phi y_0 + \lambda_0 \psi$. Choose $t_* \in [0, \omega]$ such that $\mu := \inf_{t \in [0, \omega]} y_0(t) \geq y_0(t_*) - \lambda_0/2$. Clearly, $0 < \sigma r \leq \mu \leq r$ and $g(t, y_{0,t}) \geq b(t)h(\mu)$, hence

$$\begin{aligned} \mu &\geq y_0(t_*) - \lambda_0/2 = \lambda_0/2 + (\Phi y_0)(t_*) \\ &\geq \lambda_0/2 + \big(B(\omega)e^{A(\omega)} - 1\big)^{-1} \underline{B} h(\mu) \int_{t_*}^{t_* + \omega} b(s) e^{\int_{t_*}^s a(u)\, du}\, ds \\ &\geq \lambda_0/2 + C h(\mu), \end{aligned}$$

implying that $\frac{\mu}{h(\mu)} > C$, which contradicts the choice of r. This proves the claim. By Theorem 2.1, the proof is complete. □

A careful analysis of the arguments above shows that they are valid if the nonlinearity $g(t,\varphi)$ in (2.8), where $\varphi \in PC([-\tau,0];\mathbb{R}^+)$, is replaced by $g_0(t,y) = g(t,y_t)$, where now $y \in K = K(\sigma)$. Therefore we state below a slightly stronger version of Theorem 2.3, which will turn out to be very useful in some applications (see Section 3).

Theorem 2.4. *Consider* (2.1)*, fix* $\sigma = \big(\underline{B}/\overline{B}\big)e^{-A(\omega)}$*, consider the cone* $K = K(\sigma) = \{y \in X^+ : y(t) \geq \sigma\|y\| \text{ for } t \in [0,\omega]\}$ *and assume (h1)-(h5). Suppose also that there are constants* r_0, R_0 *with* $0 < r_0 \leq R_0$ *and continuous functions* $b, h : \mathbb{R}^+ \to \mathbb{R}^+$ *such that the additional requirements (i), (ii) in the statement of Theorem 2.3 are satisfied, and:*

(iii) for any* $r \geq 0, y \in K$ *and* $t \in [0,\omega]$*,*

$$\begin{aligned} g(t,y_t) &\leq b(t)h(r) \quad \text{if} \quad R_0 \leq y(t) \leq r, \\ g(t,y_t) &\geq b(t)h(r) \quad \text{if} \quad r \leq y(t) \leq r_0. \end{aligned} \tag{2.9}$$

Then there exists at least one positive ω*-periodic solution of* (2.1)*.*

Remark 2.1. In the situation of Theorems 2.3 and 2.4, it is useful to note that C as in (2.7) satisfies $C \geq C_0$ where

$$C_0 = \big(B(\omega)e^{A(\omega)} - 1\big)^{-1}\underline{B}\int_0^\omega b(s)\,ds,$$

thus the first condition in (2.6) holds if $\lim_{u\to 0^+}\frac{u}{h(u)} < C_0$. Of course, for concrete functions $a(t), b(t)$ it is often possible to deduce that $C > C_0$, in which case the first condition in (2.6) can be replaced by $\lim_{u\to 0^+}\frac{u}{h(u)} \leq C_0$. On the other hand, additional hypotheses on the coefficients permit writing (2.6) in a simpler form, as shown in the next criterion.

Corollary 2.1. *Consider* (2.1)*, assume (h1)-(h5), and suppose also that there are positive constants* r_0, R_0 *and continuous functions* $b, h : \mathbb{R}^+ \to \mathbb{R}^+$ *with* $b(t)$ ω*-periodic, such that* (2.8) *(or* (2.9)*) is satisfied, with:*
(a) $b(t) > a(t)$ *on* $[0,\omega]$*;*
(b) $h(u)$ *is positive for* $u > 0$, $\lim_{u\to 0^+}\frac{u}{h(u)} \leq C_1, \lim_{u\to +\infty}\frac{u}{h(u)} = \infty$, *where*

$$C_1 = \big(B(\omega)e^{A(\omega)} - 1\big)^{-1}\underline{B}\left(e^{A(\omega)} - 1\right).$$

Then there exists at least one positive ω*-periodic solution of* (2.1)*.*

Proof. Note that

$$\int_t^{t+\omega} a(s)e^{\int_t^s a(u)\,du}\,ds = e^{\int_t^{t+\omega} a(u)\,du} - 1 = e^{A(\omega)} - 1, \quad t \geq 0.$$

From (a), we deduce that

$$\inf_{t\geq 0}\int_t^{t+\omega} b(s)e^{\int_t^s a(u)\,du}\,ds > e^{A(\omega)} - 1$$

and therefore $C > C_1$, where C is defined in (2.7). $\square$

The equation without impulses,

$$y'(t) + a(t)y(t) = g(t, y_t), \quad t \geq 0, \tag{2.10}$$

is simply (2.1) with $b_k = 0, k \in \mathbb{N}$, and is included in the above setting. Below, we denote $C^+(\mathbb{R}^+) = C(\mathbb{R}^+; \mathbb{R}^+)$ and $C_\omega^+ = \{y \in C^+(\mathbb{R}^+) : y \text{ is } \omega\text{-periodic}\}$, and state Theorems 2.2 and 2.3 for the non-impulsive case.

Theorem 2.5. *Consider* (2.10) *where* $a \in C_\omega^+$, $a(t) \not\equiv 0$, $g : \mathbb{R}^+ \times C([-\tau, 0]; \mathbb{R}^+) \to \mathbb{R}^+$ *is continuous,* ω*-periodic in* t*, and* $g_0 : \mathbb{R}^+ \times C_\omega^+ \to \mathbb{R}^+$ *defined by* $g_0(t, y) = g(t, y_t)$ *is uniformly continuous and bounded on bounded sets of* $[0, \omega] \times C_\omega^+$*. If* $g(t, 0) \not\equiv 0$*, then there exists at least one positive* ω*-periodic solution of* (2.10)*.*

Theorem 2.6. *Consider* (2.10) *where* $a \in C_\omega^+$, $a(t) \not\equiv 0$, $g : \mathbb{R}^+ \times C([-\tau, 0]; \mathbb{R}^+) \to \mathbb{R}^+$ *is completely continuous,* ω*-periodic in* t*, and* $g_0(t, y) = g(t, y_t)$ *is uniformly continuous on bounded sets of* $[0, \omega] \times C_\omega^+$*. In addition, suppose that there are constants* r_0, R_0*, with* $0 < r_0 \leq R_0$*, and continuous functions* $b, h : \mathbb{R}^+ \to \mathbb{R}^+$*, where:*

(i) $b(t)$ *is* ω*-periodic and* $\int_0^\omega b(s)\,ds > 0$*;*

(ii) $h(u) > 0$ *for* $u > 0$*, with* $\lim_{u \to 0^+} \frac{u}{h(u)} < C, \lim_{u \to +\infty} \frac{u}{h(u)} = \infty$*, where*

$$C = \left(e^{A(\omega)} - 1\right)^{-1} \inf_{t \geq 0} \int_t^{t+\omega} b(s) e^{\int_t^s a(u)\,du}\,ds\,; \tag{2.11}$$

(iii) for all $r \geq 0, \varphi \geq 0$ *and* $t \geq 0$*,*

$$\begin{aligned} g(t, \varphi) &\leq b(t)h(r) \quad \text{if} \quad R_0 \leq \varphi \leq r, \\ g(t, \varphi) &\geq b(t)h(r) \quad \text{if} \quad r \leq \varphi \leq r_0. \end{aligned}$$

Then there exists at least one positive ω*-periodic solution of* (2.10)*.*

Of course, for $\sigma = e^{-A(\omega)}$, in the above statement one can replace (iii) by (iii*) as in Theorem 2.4.

Corollary 2.2. *Consider the hypotheses of Theorem 2.6, with* $\lim\limits_{u \to 0^+} \dfrac{u}{h(u)} < C$ *replaced by one of the following conditions:*

(a) $\lim\limits_{u \to 0^+} \dfrac{u}{h(u)} < C_0$*, where* $C_0 = \left(e^{A(\omega)} - 1\right)^{-1} \int_0^\omega b(s)\,ds$*;*

(b) $\lim\limits_{u \to 0^+} \dfrac{u}{h(u)} \leq 1$ *and* $b(t) > a(t), t \in [0, \omega]$*.*

Then (2.10) *has a positive* ω*-periodic solution.*

Remark 2.2. As in [8], one could consider impulsive DDEs where the impulses are not linear, but instead have the form $\Delta(y(t_k)) = I_k(y(t_k))$, with $I_k : \mathbb{R}^+ \to \mathbb{R}$ continuous, $u + I_k(u) > 0$ for $u > 0$, and there exist a positive integer p such that $t_{k+p} = t_k + \omega, I_{k+p}(u) = I_k(u)$ $(k \in \mathbb{N}, u > 0)$ and constants $a_1, \ldots, a_p$ and $b_1, \ldots, b_p$, with $b_k > -1$ and such that $b_k \leq \frac{I_k(x) - I_k(y)}{x - y} \leq a_k$, $x, y \geq 0, x \neq y, k = 1, \ldots, p$. In this situation, $B(t)$ defined on (2.2) depends on $y \in X^+$ and has the form $B_y(t) = \prod_{k : t_k \in [0,t)} J_k(y(t_k))$, where $J_k(u) := \frac{u}{u + I_k(u)}$ $(u > 0)$ satisfies $(a_k + 1)^{-1} \leq J_k(u) \leq (b_k + 1)^{-1}$. Clearly, the definition of Φ and the above results have to be rewritten with some care. For further comments, see Section 4.

3 Application to some mathematical biology models with and without impulsive effects

The results in Section 2 apply to a very broad class of periodic scalar DDEs, both with linear impulses and without impulses, which include many relevant models used in population dynamics, physiological systems, ecology, physics, and a variety of other fields. The present method has significant applications, as the hypotheses in the statements of Theorems 2.2 and 2.3 are often fulfilled and their validity easy to check.

In this section, we illustrate the results with a few selected examples which, in view of their numerous and important applications, have been widely studied. Although a brief comparison with some related results in recent literature will be given, we of course will not mention all the relevant contributions regarding these selected models. Typically, the equations under consideration are periodic, have the form

$$y'(t) + a(t)y(t) = \sum_{i=1}^{m} f_i(t, y(t - \tau_i(t))),\ 0 \le t \ne t_k,$$

and are subjected to periodic linear impulses at the points $t_k, k \in \mathbb{N}$. In this case, $g(t,\varphi) = \sum_{i=1}^{m} f_i(t, \varphi(-\tau_i(t)))$ is completely continuous and (h5) is satisfied if $f_i(t,x)$ $(1 \le i \le m)$ are continuous on $\mathbb{R}^+ \times \mathbb{R}^+$. As we shall see, dependence on multiple discrete delays in each function f_i can also be considered. Clearly, models with distributed delays can again be included in the form (2.1), see Section 4 for a discussion of this subject. Further comments regarding the novelty of our method, as well as directions for future research, will be postponed to the last section of the paper.

3.1 Models without the trivial solution

The first example concerns a DDE with a bounded nonlinearity, for which $y = 0$ is not a solution.

Consider the following generalized impulsive model of hematopoiesis:

$$\begin{cases} y'(t) + a(t)y(t) = \sum_{i=1}^{m} \dfrac{b_i(t)}{1 + y(t - \tau_i(t))^n},\ 0 \le t \ne t_k, \\ y(t_k^+) - y(t_k) = b_k y(t_k), \quad k \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $m \in \mathbb{N}$, n is a positive constant, $(t_k), (b_k)_{k\in\mathbb{N}}$ are sequences satisfying (h2)-(h4), and the delays and coefficients satisfy the following assumption:

(h1*) $a(t), b_i(t), \tau_i(t) \in C_\omega^+$ $(\omega > 0)$, with $a(t), b_i(t)$ not identically zero, $i = 1, \dots, m$.

Theorem 3.1. *Assume (h1*),(h2)-(h4). Then there exists a positive ω-periodic solution of* (3.1).

Proof. Let $\tau = \max_{1\le i\le m} \max_{t\in[0,\omega]} \tau_i(t)$. The hematopoiesis model (3.1) has the form (2.1), where

$$g(t,\varphi) = \sum_{i=1}^{m} \frac{b_i(t)}{1 + \varphi(-\tau_i(t))^n}, \quad t \ge 0, \varphi \in PC([-\tau, 0]; \mathbb{R}^+).$$

Clearly, $g(t,\varphi)$ is uniformly bounded for $(t,\varphi) \in [0,\omega] \times PC([-\tau,0];\mathbb{R}^+)$. Moreover, $g(t,0) = \sum_{i=1}^m b_i(t) \not\equiv 0$, hence 0 is not a solution of (3.1). The result is an immediate consequence of Theorem 2.2. □

For the situation without impulses, we obtain the corollary below.

Corollary 3.1. *The DDE*

$$y'(t) + a(t)y(t) = \sum_{i=1}^{m} \frac{b_i(t)}{1 + y(t-\tau_i(t))^n}, \quad t \geq 0, \tag{3.2}$$

where $m \in \mathbb{N}$, n is a positive constant and the delays and coefficients satisfy (h1), has at least one positive ω-periodic solution.*

The autonomous equation $y'(t)+ay(t) = b/(1+y^n(t-\tau))$ $(a, b, \tau, n > 0)$ was proposed by Mackey and Glass [18] as an appropriate model to describe the process of production of blood cells. Since then a number of generalizations have been analyzed. In the case of the nonimplusive equation (3.2) with a single delay ($m = 1$), the existence of a positive periodic solution was established in [24], among others. Later, the equation with multiple delays (3.2) was studied by Liu et al. in [17], where the authors showed the existence and uniqueness of a positive ω-periodic solution $y^*(t)$ without further constraints if $0 < n \leq 1$ and with the additional restriction

$$(n-1)\left[e^{A(\omega)}\left(e^{A(\omega} - 1\right)^{-1}\int_0^\omega b(s)\,ds\right]^n \leq 1$$

for the case $n > 1$; sufficient conditions for the global asymptotic stability of $y^*(t)$ were also given.

More recently, Saker and Alzabut [22] investigated the global asymptotic behavior of the impulsive system (3.1) with a single discrete constant delay multiple of the period and n a positive integer:

$$\begin{cases} y'(t) + a(t)y(t) = \dfrac{b(t)}{1 + y(t-m\omega)^n}, & 0 \leq t \neq t_k, \\ y(t_k^+) - y(t_k) = b_k y(t_k), \quad k \in \mathbb{N}, & \end{cases} \tag{3.3}$$

where $m, n \in \mathbb{N}$ and $a(t), b(t)$ are positive periodic functions of period $\omega > 0$. Besides (h2),(h3), the requirement that the impulsive function $t \mapsto \prod_{k:t_k \in [0,t)}(1 + b_k)$ is ω-periodic was further assumed in [22], a condition which has been often imposed in the literature of periodic impulsive equations (see e.g. [5, 10, 27]). However, as noticed by Liu and Takeuchi [16], this condition implies (h2) and

$$\prod_{k=1}^{p}(1 + b_k) = 1\,, \tag{3.4}$$

thus it imposes a very restrictive setting. On the other hand, the equation with a single delay multiple of the period is simpler to treat, since to look for an ω-periodic solution of $y^*(t)$ of (3.3) is equivalent to looking for an ω-periodic solution of the corresponding impulsive equation without delay – which was proven to exist in [22] by using the continuation theorem of degree theory. In summary, our method allows to significantly improve and generalize criteria established in recent literature.

For the Lasota-Wazewska model with linear impulses

$$\begin{cases} y'(t) + a(t)y(t) = \displaystyle\sum_{i=1}^{m} b_i(t) e^{-c_i(t)y(t-\tau_i(t))}, \ 0 \le t \ne t_k, \\ \Delta(y(t_k)) = b_k y(t_k), \quad k \in \mathbb{N}, \end{cases} \tag{3.5}$$

with the assumptions (h1*),(h2)-(h4), in a similar way we deduce that there exists a positive ω-periodic solution.

3.2 Mackey-Glass equation

Next example illustrates the application of Theorem 2.3 for both an impulsive equation with increasing nonlinearities and nonmonotone nonlinearities.

Consider the impulsive Mackey-Glass equation

$$\begin{cases} y'(t) + a(t)y(t) = \displaystyle\sum_{i=1}^{m} \frac{b_i(t) y(t-\tau_i(t))}{1 + y(t-\tau_i(t))^n}, \quad 0 \le t \ne t_k, \\ y(t_k^+) - y(t_k) = b_k y(t_k), \quad k \in \mathbb{N}, \end{cases} \tag{3.6}$$

where all the coefficients, delays and impulses satisfy (h1*),(h2)-(h4), $0 \le \tau_i(t) \le \tau$, $m \in \mathbb{N}$ and $n > 0$. Write (3.6) in the form (2.1), where

$$g(t, \varphi) = \sum_{i=1}^{m} \frac{b_i(t)\varphi(-\tau_i(t))}{1 + \varphi(-\tau_i(t))^n}.$$

We consider separately the cases $0 < n \le 1$ and $n > 1$. If $0 < n \le 1$, take $b(t) = \sum_{i=1}^m b_i(t)$ and $h(u) = \frac{u}{1+u^n}$. We have $h(u)$ increasing on $\mathbb{R}^+$, $\lim_{u \to 0^+} \dfrac{u}{h(u)} = 1$, $\lim_{u \to +\infty} \dfrac{u}{h(u)} = +\infty$. Condition (2.8) holds true for any choice of $r_0, R_0 > 0$. If $n > 1$, the function $\frac{u}{1+u^n}$ attains its absolute maximum at the point $u_0 = (n-1)^{-1/n}$, is increasing on $[0, u_0]$ and decreasing on $[u_0, \infty)$. For this situation, (2.8) is satisfied with $r_0 = R_0 = u_0$, $b(t) = \sum_{i=1}^m b_i(t)$ and

$$h(u) = \begin{cases} \frac{u}{1+u^n} & \text{if } u \in [0, u_0] \\ \frac{u_0}{1+u_0^n} & \text{if } u \in [u_0, \infty) \end{cases}.$$

The nondecreasing function $h(u)$ satisfies $\lim_{u \to 0^+} \dfrac{u}{h(u)} = 1$, $\lim_{u \to +\infty} \dfrac{u}{h(u)} = +\infty$. Hence, for any $n > 0$ we may apply Theorem 2.3 (see also Remark 2.1 and Corollary 2.1) and deduce:

Theorem 3.2. *Consider* (3.6) *under the conditions (h1*),(h2)-(h4). Suppose in addition that* $C > 1$, *where* C *is defined in* (2.7), *for* $b(t) = \sum_{i=1}^m b_i(t)$, $B(\omega) = \prod_{k=1}^p (1 + b_k)^{-1}$, $\underline{B} = \min_{1 \le l, q \le p} \prod_{k=1}^q (1 + b_{l+k})^{-1}$. *Then there is at least one positive* ω*-periodic solution of* (3.6).

In particular, this is the case if one of the following conditions holds:

(i) $b(t) > a(t)$ *for* $t \in [0, \omega]$ *and*

$$\left(B(\omega) e^{\int_0^\omega a(s)\, ds} - 1\right)^{-1} \underline{B} \left(e^{\int_0^\omega a(s)\, ds} - 1\right) \ge 1\,; \tag{3.7}$$

(ii)

$$\big(B(\omega)e^{\int_0^\omega a(s)\,ds}-1\big)^{-1}\underline{B}\int_0^\omega b(s)\,ds>1. \tag{3.8}$$

For the ω-periodic DDE without impulses, Theorem 2.6 and Corollary 2.2 yield the following criteria:

Corollary 3.2. *Consider the Mackey-Glass equation*

$$y'(t)+a(t)y(t)=\sum_{i=1}^m \frac{b_i(t)y(t-\tau_i(t))}{1+y(t-\tau_i(t))^n},\quad t\geq 0, \tag{3.9}$$

where $a(t), b_i(t), \tau_i(t)\in C_\omega^+$, for some constant $\omega>0$, $i=1,\dots,m$, with $a(t), b(t):=\sum_{i=1}^m b_i(t)$ not identically zero. If $C>1$, for C given by (2.11), then there exists a positive ω-periodic solution of (3.9). In particular, this holds true if either $\int_0^\omega b(s)\,ds>e^{\int_0^\omega a(s)\,ds}-1$ or $b(t)>a(t)$ on $[0,\omega]$.

The asymptotic properties for the original Mackey-Glass equation [18], as well as for generalized models, have been the subject of extensive researches. The case of the periodic Mackey-Glass equation with a single time-varying delay,

$$y'(t)+a(t)y(t)=\frac{b(t)y(t-\tau(t))}{1+y(t-\tau(t))^n}\quad (n>0),$$

with $a(t), b(t), \tau(t)$ continuous, ω-periodic and $a(t), b(t)$ positive, was studied in [24], where the authors proved the existence of a positive ω-periodic solution under the condition $b(t)>a(t)$ for $t\in[0,\omega]$. This criterion is recovered in Corollary 3.2. For other relevant results, see [1, 2, 15] and references therein. Namely, in [1] the authors considered (3.9) with two different time-varying delays in each term on the righ-hand-side. A similar model, but with a Nicholson-type nonlinearity, will be treated in the next subsection.

Remark 3.1. For the impulsive equation (3.6), if all impulses are nonnegative, i.e., $b_k\geq 0$ for all k, then $\underline{B}=B(\omega)$ and (3.7) is always satisfied, whereas (3.8) is equivalent to

$$\int_0^\omega b(s)\,ds>e^{\int_0^\omega a(s)\,ds}-\prod_{k=1}^p(1+b_k).$$

3.3 On generalized Nicholson's blowflies equations with several delays

In this subsection, we analyze some generalizations of the Nicholson's blowflies equation, with impulses and multiple delays.

First, consider the impulsive Nicholson's model

$$\begin{cases} y'(t)+a(t)y(t)=\sum_{i=1}^m b_i(t)y(t-\tau_i(t))e^{-c_i(t)y(t-\tau_i(t))}, & 0\leq t\neq t_k,\\ y(t_k^+)-y(t_k)=b_k y(t_k), & k\in\mathbb{N}, \end{cases} \tag{3.10}$$

where $m\in\mathbb{N}$, $a(t),b_i(t),c_i(t),\tau_i(t)\in C^+_\omega$ $(\omega>0)$, with $a(t),b(t):=\sum_{i=1}^m b_i(t)$ not identically zero, $c_i(t)>0$, for $t\in[0,\omega], i=1,\dots,m$, and the impulsive times (t_k) and impulses (b_k) satisfy (h2)-(h4).

For (3.10), the application of Theorem 2.3 requires some additional care, not only because the nonlinearities $b_i(t)ue^{-c_i(t)u}$ are nonmonotone on u, but also because they do not have the form $b_i(t)h_i(u)$. Nevertheless, as shown below, the framework of Theorem 2.3 is applicable to this equation.

Theorem 3.3. *Consider the impulsive equation* (3.10)*, with the above requirements on coefficients, delays and impulses. With the notation in* (2.2),(2.3)*, assume that $C>1$, where C is defined in* (2.7)*. Then, there is at least one positive ω-periodic solution of* (3.10)*.*

In particular, this is the case if either $b(t)>a(t)$ for $t\in[0,\omega]$ and (3.7) *holds, or if* (3.8) *is satisfied.*

Proof. Eq. (3.10) takes the form (2.1) with $g(t,\varphi)=\displaystyle\sum_{i=1}^m b_i(t)\varphi(-\tau_i(t))e^{-c_i(t)\varphi(-\tau_i(t))}$. Consider the positive numbers

$$\underline{c}=\min_{1\le i\le m}\min_{0\le t\le\omega}c_i(t),\quad \overline{c}=\max_{1\le i\le m}\max_{0\le t\le\omega}c_i(t),\tag{3.11}$$

and the functions $f_i(t,u):=ue^{-c_i(t)u}$, $u\ge 0$. For $i=1,\dots,m$ and $t\in[0,\omega]$ fixed, $f_i(t,\cdot)$ attains its maximum at $u=c_i(t)^{-1}$, with $f_i(t,\cdot)$ increasing on $[0,c_i(t)^{-1}]$, decreasing on $[c_i(t)^{-1},\infty)$ and $f_i(t,c_i(t)^{-1})=c_i(t)^{-1}e^{-1}\le(\underline{c}e)^{-1}$. Choose $r_0=\overline{c}^{-1}$ and $R_0=\underline{c}^{-1}$. As before, we shall use Theorem 2.3 with $b(t)=\sum_{i=1}^m b_i(t)$.

For $i=1,\dots,m$, one has

$$b_i(t)\varphi(-\tau_i(t))e^{-c_i(t)\varphi(-\tau_i(t))}=b_i(t)f_i(t,\varphi(-\tau_i(t)))\le b_i(t)(\underline{c}e)^{-1},\quad \text{for}\quad 0\le t\le\omega,\varphi\in X.$$

For $i=1,\dots,m$, $0<r\le r_0$ and $\varphi\in X$, one obtains

$$b_i(t)\varphi(-\tau_i(t))e^{-c_i(t)\varphi(-\tau_i(t))}\ge b_i(t)f_i(t,r)\ge b_i(t)re^{-\overline{c}r},\quad \text{for}\quad 0\le t\le\omega, r\le\varphi\le r_0.$$

Therefore, condition (2.8) is satisfied with $h:\mathbb{R}^+\to\mathbb{R}^+$ continuous, linear on $[r_0,R_0]$ and such that

$$h(u)=\begin{cases} ue^{-\overline{c}u} & \text{if } u\in[0,r_0]\\ (\underline{c}e)^{-1} & \text{if } u\in[R_0,\infty)\end{cases}.$$

Clearly, $\displaystyle\lim_{u\to0^+}\frac{u}{h(u)}=1$, $\displaystyle\lim_{u\to+\infty}\frac{u}{h(u)}=+\infty$. From Theorem 2.3, the existence of a positive ω-periodic solution is guaranteed if $C>1$. □

Corollary 3.3. *Consider the periodic Nicholson's equation*

$$y'(t)+a(t)y(t)=\sum_{i=1}^m b_i(t)y(t-\tau_i(t))e^{-c_i(t)y(t-\tau_i(t))},\quad t\ge0,\tag{3.12}$$

where $a(t),b_i(t),c_i(t),\tau_i(t)\in C^+_\omega$ $(\omega>0)$, with $a(t)\not\equiv0$ and $c_i(t)$ positive, $i=1,\dots,m$. If either

$$\sum_{i=1}^m\int_0^\omega b_i(s)\,ds>e^{\int_0^\omega a(s)\,ds}-1,\tag{3.13}$$

or

$$\sum_{i=1}^{m} b_i(t) > a(t), \quad t \in [0,\omega], \tag{3.14}$$

(3.12) *has a positive* ω*-periodic solution.*

Remark 3.2. For the periodic Nicholson equation with a single delay multiple of the period and $c(t) \equiv c > 0$, given by $y'(t) + a(t)y(t) = b(t)y(t - n\omega)e^{-cy(t-n\omega)}$ ($n \in \mathbb{N}$), Saker and Agarwal [21] showed the existence of a positive ω-periodic solution if $\min_{t\in[0,\omega]} b(t) > \max_{t\in[0,\omega]} a(t)$. Later, by using a Krasnoselskii fixed point theorem for an operator as in [11, 24], Li and Du [14] considered the more general version (3.12), and proved the existence of a positive ω-periodic solution under the less restrictive sufficient condition (3.14), as in the above corollary.

Motivated by the nice paper of Chen [3] (see also [1]), next we consider a more general model. Let us first mention that the equation

$$y'(t) + a(t)y(t) = b(t)y(t - \sigma(t))e^{-c(t)y(t-\tau(t))}, \tag{3.15}$$

with $a(t), b(t), \sigma(t), \tau(t), c(t)$ positive, ω-periodic continuous functions ($\omega > 0$), was studied by Chen [3], who proved the existence of a positive ω-periodic solution of (3.15) under the following sufficient conditions:

$$\begin{aligned} \int_0^\omega b(s)\,ds &> \int_0^\omega a(s)\,ds, \quad \text{if} \quad \sigma(t) = n\omega \quad (\text{for some } n \in \mathbb{N}), \\ \int_0^\omega b(s)\,ds &> e^{2\int_0^\omega a(s)\,ds} \int_0^\omega a(s)\,ds, \quad \text{if} \quad \sigma(t) \not\equiv n\omega \quad \forall n \in \mathbb{N}. \end{aligned} \tag{3.16}$$

It is clear that $xe^{2x} > e^x - 1$ for $x > 0$. Hence, the above Corollary 3.3 strongly generalizes and improves the result in [3] for a general equation (3.15) with $\tau(t) = \sigma(t)$ periodic. But nevertheless it does not recover the results when $\sigma(t) \not\equiv \tau(t)$, nor when $\sigma(t)$ is a multiple of the period. Still, we shall show that the refined version of the existence result given in Theorem 2.4 allows us to deal with equations with *two different periodic delays* $\sigma_i(t), \tau_i(t)$ ($1 \le i \le m$) in each term on the right-hand-side of (3.12).

First, we treat the impulsive case. The next result is not derived directly from Theorem 3.3, but similar arguments are used, where now we invoke Theorem 2.4, rather than Theorem 2.3.

Theorem 3.4. *Consider the impulsive equation*

$$\begin{cases} y'(t) + a(t)y(t) = \sum_{i=1}^{m} b_i(t)y(t - \sigma_i(t))e^{-c_i(t)y(t-\tau_i(t))}, & 0 \le t \ne t_k, \\ y(t_k^+) - y(t_k) = b_k y(t_k), \quad k \in \mathbb{N}, \end{cases} \tag{3.17}$$

where $m \in \mathbb{N}$, $a(t), b_i(t), c_i(t), \sigma_i(t), \tau_i(t) \in C_\omega^+$ $(\omega > 0)$*, with* $a(t), b(t) := \sum_{i=1}^m b_i(t)$ *not identically zero,* $c_i(t) > 0$*, for* $t \in [0,\omega], i = 1, \dots, m$*, and the impulses satisfy (h2)-(h4). Assume that*

$$\inf_{t\ge0} \int_t^{t+\omega} b(s)e^{\int_t^s a(u)\,du}\,ds > e^{A(\omega)}\big(B(\omega)e^{A(\omega)} - 1\big)\underline{B}^{-2}\overline{B}\,. \tag{3.18}$$

Then, there is at least one positive ω*-periodic solution of* (3.17).

Proof. Now, (3.10) has the form (2.1) with $g(t,\varphi) = \sum_{i=1}^{m} b_i(t)\varphi(-\sigma_i(t))e^{-c_i(t)\varphi(-\tau_i(t))}$. Take $\sigma = \big(\underline{B}/\overline{B}\big)e^{-A(\omega)}$ and consider the cone $K = K(\sigma)$. Set $\underline{c}, \overline{c}$ as in (3.11), choose

$$r_0 = \overline{c}^{-1}, \quad R_0 = (\underline{c}\sigma)^{-1} = \underline{c}^{-1}\overline{B}e^{A(\omega)}\underline{B}^{-1},$$

and define the functions $b(t) = \sum_{i=1}^{m} b_i(t)$, $f_i(t,u,v) := ue^{-c_i(t)v}$, $u, v \geq 0$ and $t \in [0,\omega]$, $i = 1, \dots, m$.

For $i = 1, \dots, m$, $R \geq R_0$ and $y \in K$ with $R_0 \leq y(t) \leq R$, it follows that $\sigma R \leq y(t) \leq R$, hence

$$\begin{aligned} y(t-\sigma_i(t))e^{-c_i(t)y(t-\tau_i(t))} &= f_i(t, y(t-\sigma_i(t)), y(t-\tau_i(t))) \\ &\leq f_i(t, R, \sigma R) \leq Re^{-\underline{c}\sigma R} \leq (\underline{c}\sigma e)^{-1}, \quad \text{for} \quad 0 \leq t \leq \omega, \end{aligned} \tag{3.19}$$

because the function $ue^{-\underline{c}u}$ attains its maximum at $u = R_0$. Similarly, for $i = 1, \dots, m$, $0 < r \leq r_0$ and $y \in K$ with $r \leq y(t) \leq r_0$, we have

$$\begin{aligned} y(t-\sigma_i(t))e^{-c_i(t)y(t-\tau_i(t))} &\geq f_i(t, \sigma r_0, r_0) \\ &\geq \sigma r_0 e^{-\overline{c}r_0} \geq \sigma r e^{-\overline{c}r}, \quad \text{for} \quad 0 \leq t \leq \omega, \end{aligned} \tag{3.20}$$

because the function $\sigma ue^{-\overline{c}u}$ is increasing on $[0, r_0]$. From (3.19) and (3.20), condition (2.9) is satisfied with $h : \mathbb{R}^+ \to \mathbb{R}^+$ continuous, linear on $[r_0, R_0]$ and such that

$$h(u) = \begin{cases} \sigma ue^{-\overline{c}u} & \text{if } u \in [0, r_0] \\ (\sigma\underline{c}e)^{-1} & \text{if } u \in [R_0, \infty) \end{cases}.$$

Note that the nondecreasing function $h(u)$ satisfies $\lim_{u\to 0^+} \frac{u}{h(u)} = \sigma^{-1} = \overline{B}e^{A(\omega)}\underline{B}^{-1}$, $\lim_{u\to +\infty} \frac{u}{h(u)} = +\infty$. From Theorem 2.4, the existence of a positive ω-periodic solution is guaranteed if

$$\big(B(\omega)e^{A(\omega)} - 1\big)^{-1}\underline{B} \inf_{t\geq 0} \int_t^{t+\omega} b(s)e^{\int_t^s a(u)\,du}\,ds > \overline{B}e^{A(\omega)}\underline{B}^{-1},$$

which is equivalent to (3.18). □

Combining Theorem 3.4 with Corollary 2.2, for the situation without impulses we obtain the following criterion.

Corollary 3.4. *Consider the generalized Nicholson's equation*

$$y'(t) + a(t)y(t) = \sum_{i=1}^{m} b_i(t)y(t-\sigma_i(t))e^{-c_i(t)y(t-\tau_i(t))}, \quad t \geq 0, \tag{3.21}$$

where $m \in \mathbb{N}$, $a(t), b_i(t), c_i(t), \sigma_i(t), \tau_i(t) \in C_\omega^+$ $(\omega > 0)$, with $a(t), b(t) := \sum_{i=1}^{m} b_i(t)$ not identically zero and $c_i(t) > 0$, for $t \in [0,\omega], i = 1, \dots, m$. Then, there is at least one positive ω-periodic solution of (3.21) *if*

$$\int_0^\omega b(s)\,ds > e^{\int_0^\omega a(s)\,ds}\big(e^{\int_0^\omega a(s)\,ds} - 1\big). \tag{3.22}$$

Remark 3.3. Since $xe^{2x} > e^x(e^x - 1)$ for $x > 0$, even in the case of (3.21) with $m = 1$ as in (3.15), the sufficient condition (3.22) is less restrictive than the one in (3.16) (for $\sigma(t)$ not a constant multiple of the period).

4 Discussion and open problems

In this section, we compare the general criteria for existence of positive periodic solutions obtained in Section 2 with related results in recent literature. Some open problems will also be proposed as a subject for future research.

In [11], Jiang and Wei successfully employed a Krasnoselskii fixed point theorem to establish a general result on the existence of a positive periodic solution, for a family of periodic DDEs with distributed infinite delay, given by

$$y'(t) + a(t)y(t) = b(t)\int_{-\infty}^{0} K(s)g(t, y(t+s))\, ds,$$

where, $a(t), b(t), g(t, y), K(t)$ are continuous, nonnegative, $a(t), b(t), g(t, y)$ are ω-periodic on t ($\omega > 0$), and $\int_{-\infty}^{0} K(s)\, ds = 1$. Using a similar technique, the existence of positive periodic solutions for periodic DDEs with one nonautonomous discrete delay

$$y'(t) + a(t)y(t) = g(t, y(t - \tau(t))) \tag{4.1}$$

(with $a(t), \tau(t), g(t, y)$ continuous, nonnegative and ω-periodic on t) was later studied by Wan et al. [24]. We emphasize that this family encompasses a large number of biomathematics models. The same basic technique has been largely explored in other related papers, see [10, 15, 17, 26] and references therein. On the other hand, one should stress that the approach in [24] was anticipated by a former work of Nieto [19], who employed fixed point arguments to study the existence of solutions for first order ordinary differential equations with periodic boundary conditions and impulses.

The work by Li et al. [15], where the technique in [11, 24] was refined for equations with *positive* impulses, was in fact a strong motivation for our study. In [15], the impulsive version of (4.1) given by

$$\begin{cases} y'(t) + a(t)y(t) = g(t, y(t - \tau(t))), & 0 \le t \ne t_k, \\ \Delta(y(t_k)) = I_k(y(t_k)), & k \in \mathbb{N}, \end{cases} \tag{4.2}$$

was considered, where the functions $a(t), g(t, u), \tau(t)$ are continuous, positive, ω-periodic on t, the functions $I_k : \mathbb{R}^+ \to \mathbb{R}$ are continuous and $t_k, I_k(u)$ ($k \in \mathbb{N}$) satisfy:

(h2*) there is a positive integer p such that $0 < t_1 < t_2 < \cdots t_p \le \omega$ and $t_{k+p} = t_k + \omega,\ I_{k+p}(u) = I_k(u)$ for $k \in \mathbb{N}, u \ge 0$;

(h3*) $I_k(u) > 0$ for $k \in \mathbb{N}, u > 0$.

For (4.2), criteria for the existence of one or two positive ω-periodic solutions were given in [15]. For instance, when applied to (3.1) with *positive* impulses $\Delta(y(t_k)) = I_k(y(t_k))$, the result in [15] shows that at least one positive ω-periodic solution exists if

$$\limsup_{u\to\infty} \sum_{j=1}^{p} \frac{I_j(u)}{u} = 0. \tag{4.3}$$

In spite of this general framework, where nonlinear impulses are allowed, the results in [15] cannot be applied when the impulses are linear as in (h2), because $I_k(u) = b_k u$ and condition (4.3) is not satisfied, nor to equations with more than one delay. Moreover, the requirement of having always *positive* impulses is very restrictive and not very reasonable in real world applications. See e.g. [16] for a discussion on the use and effect of impulses.

When compared with the approach in [11, 15, 24], the main difference of our method is that it relies on finding a fixed point for the operator Φ defined by (2.4), either on a bounded conical sector $(K\cap \overline{B_R(0)})\setminus\{0\}$, or on a suitable closed "conical annulus" $K\cap(\overline{B_R(0)}\setminus B_r(0))$ of the Banach space X, whereas the approach in [15] uses again fixed point theory on cones, but to find a nonzero fixed point for a different operator Ψ. It turns out however that, for the approach in [15], the restriction of positive impulses in (h3*) must hold, to guarantee that $\Psi(X^+\setminus\{0\})\subset X^+\setminus\{0\}$. Although our setting allows the treatment of (2.1) with impulses whose sign may vary, so far it only deals with *linear* impulses. Of course, in the case of nonlinear impulses given by functions $I_k:\mathbb{R}^+\to\mathbb{R}$ as in (h2*), the analogues to the functions $B(t)$ and $B(s,t)$ defined in (2.2) also depend on $y\in X^+$; as mentioned in Remark 2.2, additional conditions on the impulses should be imposed, so that the operator Φ in (2.4) still applies K into K and is completely continuous.

Typically, the impulsive versions of equations (3.2),(3.9) and (3.12) with $m=1$ (i.e., one non-autonomous periodic delay) are included in the family (4.2). Under (h1*),(h2*),(h3*) and (4.3), the existence of a positive ω-periodic solution for such equations was obtained in [15] without further constraints for the case of the hematopoiesis model, and with the additional requirement of $b(t)>a(t)$ for $t\geq 0$ for the case of the Mackey-Glass and the Nicholson's equations. These criteria are similar to ones included in Theorems 3.1, 3.2 and 3.3.

Let us focus, for the sake of illustration, on the hematopoiesis model (3.2). For (3.2) with $m=1$ (i.e., with a single time-varying periodic delay $\tau(t)$), as well as for the model with distributed infinite delay given by

$$y'(t)+a(t)y(t)=b(t)\int_{-\infty}^{0}\frac{K(s)}{1+y(t+s)^n}\,ds, \tag{4.4}$$

where $n>0$, $a(t),b(t)\in C_\omega^+$ and the kernel $K(t)$ is positive, continuous and normalized so that $\int_{-\infty}^0 K(s)\,ds=1$, the general setting established in [11] led to the existence of at least one positive periodic solution $y^*(t)$. In [2], the global dynamics for both equation (3.2) with $m=1$ and (4.4) were investigated. Several criteria for permanence, oscillation about $y^*(t)$ and global asymptotic stability of $y^*(t)$ were given. The extension of those results to equations with several delays and periodic parameters as in (3.2), as well as to equations with linear and nonlinear impulses, was formulated in [2] as an open problem. Partial answers to the open questions in [2] have been given in Section 3.

Clearly, the method proposed in this paper can be applied to a number of scalar DDEs with *distributed* delays. For instance, Corollary 3.1 applies without changes to (3.2) replaced by

$$y'(t)+a(t)y(t)=b_0(t)\int_{-\tau(t)}^{0}\frac{K(s)}{1+y(t+s)^n}\,ds,\quad t\geq 0.$$

where $K:\mathbb{R}^+\to\mathbb{R}^+$ is continuous, with $b(t):=b_0(t)\int_{-\tau(t)}^0 K(s)\,ds$. However, it does not extend immediately to situations with infinite delay, as in equation (4.4). The generalization of Theorems 2.2 to 2.4 to impulsive DDEs with infinite delays requires further investigation, but the framework used here seems sufficiently flexible to be easily adjusted to this situation.

The question of the global attractivity of a positive periodic solution, most relevant in terms of applications, is not addressed in the present work, but will be investigated for the hematopoiesis model in a forthcoming paper. In fact, previously the authors [7, 8] have studied the global asymptotic stability of the zero solution of a general scalar impulsive delay differential equation $x'(t)+a(t)x(t) = g(t, x_t)$ as in (1.1), but with possible infinite delay and nonlinear impulses $\Delta(x(t_k)) = I_k(x(t_k))$ given by continuous functions $I_k : \mathbb{R} \to \mathbb{R}$. The theory in [7, 8] was applied to derive sufficient conditions for the global attractivity of the positive periodic solution (assuming it existed) of the following Lasota-Wazewska model with impulses:

$$\begin{cases} y'(t) + a(t)y(t) = \sum_{i=1}^{m} b_i(t)e^{-c_i(t)y(t-\tau_i(t))}, \ 0 \le t \ne t_k, \\ \Delta(y(t_k)) = I_k(y(t_k)), \quad k \in \mathbb{N}, \end{cases} \tag{4.5}$$

where all the coefficients and delays are ω-periodic, the impulses satisfy (h2*) and $b_k u \le I_k(u) \le a_k u$, with some prescribed behavior for the sequences $(b_k), (a_k)$.

In contrast with the periodic case, the results about almost periodic solutions are not as frequent. To find almost periodic solutions for almost scalar periodic DDEs, the ideas of Sacker and Sell [20] might be useful. Another challenging problem is to treat multi-dimensional equations, without and with impulses. In [5], an interesting result on existence of a positive periodic solution for a neutral predator-prey planar model with impulsive effects was established by using coincidence degree theory. Recently, the Schauder's fixed point theorem was used in [6], to prove that there exists a positive periodic solution for a broad family of n-dimensional periodic nonmonotone DDEs with distributed delays. It is our belief that the method developed here can be extended to several classes of multi-dimensional DDEs.

Acknowledgements

This work was partially supported by Fundação para a Ciência e a Tecnologia under project UID/MAT/-04561/2013 (Teresa Faria) and UID/MAT/00013/2013 (José J. Oliveira).

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