Q-TOPOLOGICAL COMPLEXITY

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Abstract. By analogy with the invariant Q-category defined by Scheerer, Stanley and Tanré, we introduce the notions of Q-sectional category and Q-topological complexity. We establish several properties of these invariants. We also obtain a formula for the behaviour of the sectional category with respect to a fibration which generalizes the classical formulas for Lusternik-Schnirelmann category and topological complexity.

1. Introduction

The *Q-category* of a topological space X, denoted by $Q\mathrm{cat}\,X$, is a lower bound for the Lusternik-Schnirelmann category of X, $\mathrm{cat}\,X$, which has been introduced by H. Scheerer, D. Stanley, and D. Tanré in [17]. This invariant, defined using a fibrewise extension of a functor Q^k equivalent to $\Omega^k\Sigma^k$, has been in particular used in the study of critical points (see [2, Chap. 7], [15]) and in the study of the Ganea conjecture. More precisely, after N. Iwase [13] showed that the Ganea conjecture, which asserted that $\mathrm{cat}\,(X \times S^n) = \mathrm{cat}\,X + 1$ for any $n \geq 1$, was not true in general, although it was known to be true for many classes of spaces (e.g. [19], [14], [12], [16], [22]), one could ask for a complete characterization of the spaces X satisfying the equality above. In [17], H. Scheerer, D. Stanley, and D. Tanré conjectured that a finite CW-complex X satisfies the Ganea conjecture, that is the equality $\mathrm{cat}\,(X \times S^n) = \mathrm{cat}\,X + 1$ holds for any $n \geq 1$, if and only if $Q\mathrm{cat}\,X = \mathrm{cat}\,X$. One direction of this equivalence has been proved in [23] but the complete answer is still unknown.

In this paper we introduce the analogue of Q-category for Farber's topological complexity [5] and establish some properties of this invariant. Since L.-S. category and topological complexity are both special cases of the notion of *sectional category*, introduced by A. Schwarz in [18], we naturally consider and study a notion of Q-sectional category. Our definition, given in Section 2.3, is based on a generalized notion of Ganea fibrations and on a fibrewise extension of the functor Q^k that we respectively recall in Sections 2.1 and 2.2. We next, in Section 3, establish various formulas for Q-sectional category and Q-topological complexity. It is worth noting that our study of the behaviour of the Q-sectional category in a fibration led us to establish a new formula for the sectional category (see Theorem 3.8), which generalizes both the classical formula for the LS-category in a fibration and the formula established by M. Farber and M. Grant for topological complexity [6]. Finally, we use our results to study some examples. This permits us in particular to

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observe that the analogue of the Scheerer-Stanley-Tanré conjecture for topological complexity is not true. We also include a small observation about the original Scheerer-Stanley-Tanré conjecture.

Throughout this text we work in the category of compactly generated Hausdorff spaces having the homotopy type of a CW-complex.

2. Q-sectional category and Q-topological complexity

2.1. **Sectional category and Ganea fibrations.** Let $f : A \to X$ be a map where X is a well-pointed path-connected space with base point $* \in X$. The *sectional category* of f, $\mathrm{secat}(f)$, is the least integer n (or ∞) for which there exists an open cover $U_0, \dots, U_n$ of X such that, for any $0 \le i \le n$, f admits a local homotopy section on U_i (that is, a continuous map $s_i : U_i \to A$ such that $f \circ s_i$ is homotopic to the inclusion $U_i \hookrightarrow X$). When f is a fibration, we can, equivalently, require local strict sections instead of local homotopy sections. As special cases of sectional category, Lusternik-Schnirelmann category and Farber's topological complexity are respectively given by

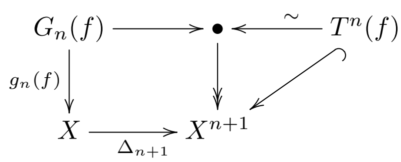
- $\mathrm{cat}(X) = \mathrm{secat}(ev_1 : PX \to X) = \mathrm{secat}(* \to X)$
  where $PX \subset X^I$ is the space of paths beginning at the base point $*$ and ev_1 is the evaluation map at the end of the path,
- $\mathrm{TC}(X) = \mathrm{secat}(ev_{0,1} : X^I \to X \times X) = \mathrm{secat}(\Delta : X \to X \times X)$
  where $ev_{0,1}$ evaluates a path at its extremities and Δ is the diagonal map.

As is well-known, if $f : A \to X$ and $g : B \to Y$ are two maps with homotopy equivalences $A \overset{\sim}{\to} B$ and $X \overset{\sim}{\to} Y$ making the obvious diagram commutative then $\mathrm{secat}(f) = \mathrm{secat}(g)$. Also $\mathrm{secat}(f)$ can be characterized through the existence of a global section for a certain join map which can be, for instance, explicitly constructed through an iterated fibrewise join of f when f is a fibration ([18]). Here we will assume (without loss of generality) that $f : A \to X$ is a (closed) pointed cofibration and consider the following constructions which give us a natural and explicit fibration, the Ganea fibration of f, equivalent to the join map characterizing $\mathrm{secat}(f)$:

- the fatwedge of f ([7], [9]):
$$T^n(f) = \{(x_0, \dots, x_n) \in X^{n+1} \mid \exists j, x_j \in f(A)\}$$
  which generalizes the classical fat-wedge
$$T^n(X) = \{(x_0, \dots, x_n) \in X^{n+1} \mid \exists j, x_j = *\},$$
- the space $\Gamma_n X = \left\{(\gamma_0, \dots, \gamma_n) \in \left(X^I\right)^{n+1} \mid \gamma_0(0) = \dots = \gamma_n(0)\right\}$
  together with the fibration
$$\Gamma_n X \to X, \quad (\gamma_0, \dots, \gamma_n) \mapsto \gamma_0(0) = \dots = \gamma_n(0)$$
  which is a homotopy equivalence,
- the nth Ganea fibration of f, $g_n(f) : G_n(f) \to X$, which is obtained by pull-back along the diagonal map $\Delta_{n+1} : X \to X^{n+1}$ of the fibration associated to the inclusion $T^n(f) \hookrightarrow X^{n+1}$ and explicitly given by:

$$\begin{array}{rcl} G_n(f) = \{(\gamma_0, \dots, \gamma_n) \in \Gamma_n(X) \mid \exists j,\ \gamma_j(1) \in f(A)\} & \to & X \\ (\gamma_0, \dots, \gamma_n) & \mapsto & \gamma_0(0) = \dots = \gamma_n(0). \end{array}$$

All these spaces are considered with the obvious base points $(*, \ldots, *) \in T^n(f)$ and $(\hat{*}, \ldots, \hat{*}) \in G_n(f) \subset \Gamma_n(X)$ (where $\hat{*}$ denotes the constant path). By construction, there exists a commutative diagram in which the square is a (homotopy) pull-back.



By [7], we know that secat(f) is the least integer n such that the diagonal map $\Delta_{n+1} : X \to X^{n+1}$ lifts up to homotopy in the fatwedge of f, $T^n(f)$. By the homotopy pull-back diagram above, this is equivalent to say that secat(f) is the least integer n such that the fibration $g_n(f)$ admits a (homotopy) section. By [9, Th. 8], the fibration $g_n(f)$ is equivalent to the (fibrewise) join of $n+1$ copies of f or of any map weakly equivalent to f. The fibre of $g_n(f)$, denoted by $F_n(f)$, is homotopically equivalent to the (usual) join of $n+1$ copies of the homotopy fibre F of f.

When f is the inclusion $* \to X$ (which is a cofibration since X is well-pointed) we recover a possible description of the classical Ganea fibration of X and we will use, in that case, the classical notation

$$g_n(X) : G_n(X) = \{(\gamma_0, \ldots, \gamma_n) \in \Gamma_n(X) \ : \ \exists j,\ \gamma_j(1) = *\} \to X.$$

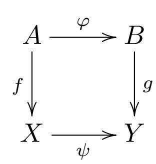
We have

$$\mathrm{cat}(X) \leq n \quad \Leftrightarrow \quad g_n(X) : G_n(X) \to X \text{ admits a (homotopy) section.}$$

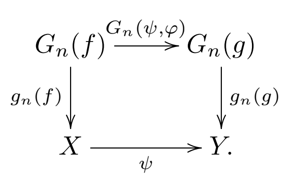
When X is a CW-complex, $f = \Delta : X \to X \times X$ is a cofibration and we have

$$\mathrm{TC}(X) \leq n \quad \Leftrightarrow \quad g_n(\Delta) : G_n(\Delta) \to X \times X \text{ admits a (homotopy) section.}$$

We finally note that, if we have a commutative diagram where f and g are cofibrations



then we have a commutative diagram



Moreover, we have

(a) if φ and ψ are homotopy equivalences, then so is $G_n(\psi, \varphi)$;
(b) if the first diagram is a homotopy pull-back, then so is the second diagram.

Note that (b) follows (for instance) from the equivalence between $g_n(f)$ and the fibrewise join of $n+1$ copies of f together with the Join Theorem ([3]).

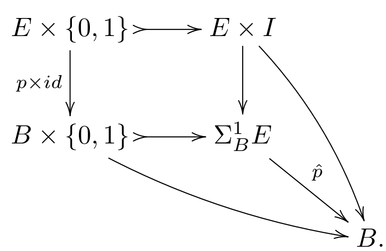
2.2. **Fibrewise Q construction.** The notion of Q-category defined in [17] is based on the Dror-Frajoun fibrewise extension [4] of a functor Q^k equivalent to $\Omega^k\Sigma^k$. Instead of using the Dror-Farjoun construction, we will here use the (equivalent) explicit fibrewise extension of Q^k given in [23]. We describe the case $k = 1$, recall more quickly the general case and refer to [17],[2, Sections 4.5, 4.6, 4.7] and [23] for further details.

We denote by $\Sigma^1 Z$ the unreduced suspension of a space Z, *i.e.* $\Sigma^1 Z = Z \times I/\sim$ with $(z, 0) \sim (z', 0)$ and $(z, 1) \sim (z', 1)$ for $z, z' \in Z$. Denoting by $[z, t] \in \Sigma^1 Z$ the class of $(z, t) \in Z \times I$, we have a canonical map $\alpha^1 = \alpha^1_Z : \{0, 1\} = \partial I \to \Sigma^1 Z$ given by $\alpha^1(0) = [z, 0]$ and $\alpha^1(1) = [z, 1]$ where $z \in Z$. The functor Q^1 is defined by

$$Q^1(Z) = \{\omega : I \to \Sigma^1 Z | \omega_{|\partial I} = \alpha^1_Z\}$$

and is given with a co-augmentation $\eta^1 = \eta^1_Z : Z \to Q^1 Z$ which takes $z \in Z$ to the path $z \to [z, t]$. If Z is pointed and $\widetilde{\Sigma} Z$ denotes the reduced suspension of Z, then there is a natural map $Q^1(Z) \to \Omega\widetilde{\Sigma} Z$, which is a homotopy equivalence induced by the identification map $\Sigma^1 Z \xrightarrow{\sim} \widetilde{\Sigma} Z$ and which makes compatible the coaugmention of Q^1 with the usual coaugmentation $\tilde{\eta} : Z \to \Omega\widetilde{\Sigma} Z$.

We now describe a fibrewise extension of Q^1. Let $p : E \to B$ be a fibration (over a path-connected space) with fibre F. The fibrewise suspension of p, $\Sigma^1_B E \to B$, is defined by the push-out:



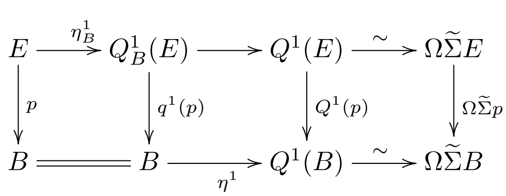
The resulting map $\hat{p} : \Sigma^1_B E \to B$ is a fibration whose fibre over b is the (unreduced) suspension $\Sigma^1 F_b$ of the fibre of $p : E \to B$ over b. By construction, we have a canonical map

$$\mu^1 : B \times \{0, 1\} \to \Sigma^1_B E$$

which is a fibrewise extension of α^1: for any $b \in B$, $\mu^1(b, -) = \alpha^1_{F_b} : \{0, 1\} \to \Sigma^1 F_b$. We define

$$Q^1_B(E) = \{\omega : I \to \Sigma^1_B E \mid \exists b \in B, \hat{p}\omega = b \text{ and } \omega_{|\partial I} = \mu^1(b, -)\}$$

together with the map $q^1(p) : Q^1_B(E) \to B$ given by $\omega \mapsto \hat{p}\omega(0)$. This is a fibration whose fibre over b is $Q^1(F_b)$ ([23, Lemma 8]). We also have a fibrewise coaugmentation $\eta^1_B : E \to Q^1_B(E)$ which extends $\eta^1 : F \to Q^1(F)$ and we have a commutative diagram



where the map $Q^1_B(E) \to Q^1(E)$ is induced by the identification map $\Sigma^1_B E \to \Sigma^1 E$.

In general, for any $k \geq 1$, we consider:

- The k fold unreduced suspension of a space Z, $\Sigma^k Z$, which can be described as $(Z \times I^k)/\sim$ where the relation is given by
$$(z, t_1, \dots, t_k) \sim (z', t'_1, \dots, t'_k)$$
if, for some i, $t_i = t'_i \in \{0, 1\}$ and $t_j = t'_j$ for all $j > i$. We write $[z, t_1, \dots, t_k]$ for the class of an element.
- The kth fibrewise suspension of $p : E \to B$, $\hat{p}^k : \Sigma^k_B E \to B$, whose fibre over b is $\Sigma^k F_b$. As before, $\Sigma^k_B E$ can be described as $(E \times I^k)/\sim$ where the relation is given by $(e, t_1, \dots, t_k) \sim (e', t'_1, \dots, t'_k)$ if $p(e) = p(e')$ and, for some i, $t_i = t'_i \in \{0, 1\}$ and $t_j = t'_j$ for all $j > i$. We write $[e, t_1, \dots, t_k]$ for the class of an element.
- The canonical map $\alpha^k : \partial I^k \to \Sigma^k Z$ given by $\alpha^k(t_1, \dots, t_k) = [z, t_1, \dots, t_k]$ (where $z \in Z$ is any element) and its fibrewise extension $\mu^k : B \times \partial I^k \to \Sigma^k_B E$ satisfying $\mu^k(b, -) = \alpha^k : \partial I^k \to \Sigma^k F_b$ for any $b \in B$.
- The fibration $q^k(p) : Q^k_B(E) \to B$ where $Q^k_B(E)$ is the (closed) subspace of $(\Sigma^k_B E)^{I^k}$ given by
$$Q^k_B(E) = \{\omega : I^k \to \Sigma^k_B E \mid \exists b \in B, \hat{p}^k\omega = b \text{ and } \omega_{|\partial I^k} = \mu^k(b, -)\}$$
and $q^k(p)(\omega) = \hat{p}^k\omega(0)$. The fibre is
$$Q^k(F) = \{\omega : I^k \to \Sigma^k F | \omega_{|\partial I^k} = \alpha^k\} \xrightarrow{\sim} \Omega^k \widetilde{\Sigma}^k F$$
and is given with an obvious coaugmentation $\eta^k : F \to Q^k(F)$, equivalent to the classical augmentation $\tilde{\eta}^k : F \to \Omega^k \widetilde{\Sigma}^k F$. We denote by $\eta^k_B : E \to Q^k_B(E)$ the fibrewise extension of η^k. When it is relevant $Q^k(F)$ and $Q^k_B(E)$ are considered with the base point given by the map $u \mapsto [*, u]$ where $u \in I^k$ and $*$ is the base point of F and E.

We have, for any $k \geq 1$, a commutative diagram:

$$\begin{array}{ccccccc} E & \xrightarrow{\eta^k_B} & Q^k_B(E) & \longrightarrow & Q^k(E) & \xrightarrow{\sim} & \Omega^k \widetilde{\Sigma}^k E \\ \downarrow p & & \downarrow q^k(p) & & \downarrow Q^k(p) & & \downarrow \Omega^k \widetilde{\Sigma}^k p \\ B & = & B & \xrightarrow{\eta^k} & Q^k(B) & \xrightarrow{\sim} & \Omega^k \widetilde{\Sigma}^k B. \end{array} \tag{1}$$

Setting $Q^0 = Q^0_B = id$ we also have a commutative diagram of the following form:

$$\begin{array}{ccccccccccc} E = Q^0_B E & \longrightarrow & Q^1_B(E) & \longrightarrow & \cdots & \longrightarrow & Q^k_B(E) & \xrightarrow{b^k_B} & Q^{k+1}_B(E) & \longrightarrow & \cdots \\ \downarrow p & & \downarrow q^1(p) & & & & \downarrow q^k(p) & & \downarrow q^{k+1}(p) & & \\ B & = & B & = & \cdots & = & B & = & B & = & \cdots \end{array} \tag{2}$$

where the map b^k_B is a fibrewise extension of the map $b^k : Q^k(F) \to Q^{k+1}(F)$ given by $b^k(\omega)(t_1, \dots, t_{k+1}) = [\omega(t_1, \dots, t_k), t_{k+1}]$. Observe that the map b^k is equivalent to the map $\Omega^k(\tilde{\eta}_{\widetilde{\Sigma}^k F}) : \Omega^k(\widetilde{\Sigma}^k F) \to \Omega^k(\Omega\widetilde{\Sigma}\widetilde{\Sigma}^k F) = \Omega^{k+1}\widetilde{\Sigma}^{k+1} F$. We can interpret the sequence (2) as a fibrewise stabilization of the fibration p since, for any integers

k and i, we have $\pi_i(Q^kF) = \pi_{i+k}(\widetilde{\Sigma}^kF)$.

For any $k \geq 0$, the functors Q^k and Q^k_B preserve homotopy equivalences. We also note that the Q^k_- construction is natural in the sense that, if we have a commutative diagram

$$(\dagger)\qquad \begin{array}{ccc} E' & \xrightarrow{f} & E \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ B' & \xrightarrow[g]{} & B \end{array}$$

where p and p' are fibrations (over path-connected spaces), then, for any k, we obtain a commutative diagram

$$(\ddagger)\qquad \begin{array}{ccc} Q^k_{B'}(E') & \longrightarrow & Q^k_B(E) \\ {\scriptstyle q^k(p')}\downarrow & & \downarrow{\scriptstyle q^k(p)} \\ B' & \xrightarrow[g]{} & B. \end{array}$$

Moreover we have:

Proposition 2.1. *With the notations above, if Diagram* ($\dagger$) *is a homotopy pull-back, then so is Diagram* ($\ddagger$).

Proof. Since the functors Q^k and Q^k_B preserve homotopy equivalences, it is sufficient to establish the statement when Diagram ($\dagger$) is a strict pull-back. In this case, the whisker map $\psi : E' \to E \times_B B'$ is a homeomorphism and its inverse ϕ satisfies $p'\phi(e, b') = b'$ for $(e, b') \in E \times_B B'$. We can then check that the map $\bar{\phi} : \Sigma^k_B E \times_B B' \to \Sigma^k_{B'} E'$ induced by $((e, u), b') \mapsto [\phi(e, b'), u]$ for $((e, u), b') \in (E \times I^k) \times_B B'$ is an inverse of the whisker map $\Sigma^k_{B'} E' \to \Sigma^k_B E \times_B B'$. In other words, the diagram

$$\begin{array}{ccc} \Sigma^k_{B'} E' & \longrightarrow & \Sigma^k_B E \\ {\scriptstyle \widehat{p'}^k}\downarrow & & \downarrow{\scriptstyle \hat{p}^k} \\ B' & \xrightarrow[g]{} & B \end{array}$$

is a strict pull-back (and homotopy pull-back since $\hat{p}^k$ is a fibration). We can next check that the map $Q^k_B(E) \times_B B' \to Q^k_{B'}(E')$ that takes $(\omega, b') \in Q^k_B(E) \times_B B'$ to the element $I^k \to \Sigma^k_{B'} E'$ of $Q^k_{B'}(E')$ given by $u \mapsto \bar{\phi}(\omega(u), b')$ is an inverse of the whisker map $Q^k_{B'}(E') \to Q^k_B(E) \times_B B'$. This means that Diagram ($\ddagger$) is a strict pull-back and therefore a homotopy pull-back since $q^k(p)$ is a fibration. $\square$

Remark 2.2. *Assuming that the fibres F and F' of p and p' are path-connected spaces (having the homotopy type of a CW-complex), we can give the following more conceptual proof of Proposition 2.1. In this case, the spaces $Q^k_B(E)$ and $Q^k_{B'}(E')$ are path-connected spaces having the homotopy type of a CW-complex, since B and $Q^k(F) \simeq \Omega^k\widetilde{\Sigma}^kF$ and $Q^k(F') \simeq \Omega^k\widetilde{\Sigma}^kF$ so are (see* [20]*). Since Diagram* ($\dagger$) *is a homotopy pull-back, the map $e : F' \to F$ induced by this diagram is a homotopy equivalence and so is $Q^k(e)$. Therefore Diagram* ($\ddagger$) *is a homotopy pull-back since the whisker map from $Q^k_{B'}(E')$ to $Q^k_B(E) \times_B B'$ induces $Q^k(e)$ between the fibres and is hence a homotopy equivalence.*

Finally, we also note the following constructions which will be, as in [17], useful in our study of products. If $p : E \to B$ and $p' : E' \to B'$ are two fibrations then, there exist, for any $k \geq 0$, commutative diagrams of the following form

$$\begin{array}{ccc} Q^k_B(E) \times E' & \longrightarrow & Q^k_{B\times B'}(E\times E') \\ & {\scriptstyle q^k(p)\times p'} \searrow & \downarrow {\scriptstyle q^k(p\times p')} \\ & & B \times B' \end{array} \qquad \begin{array}{ccc} E \times Q^k_{B'}(E') & \longrightarrow & Q^k_{B\times B'}(E\times E') \\ & {\scriptstyle p\times q^k(p')} \searrow & \downarrow {\scriptstyle q^k(p\times p')} \\ & & B \times B' \end{array}$$

which are induced by the obvious fibrewise extensions of the maps

$$\begin{array}{rcl} \Sigma^k Z \times Z' & \to & \Sigma^k(Z \times Z') \\ ([z, t_1, \ldots, t_k], z') & \mapsto & [(z, z'), t_1, \ldots, t_k] \end{array}$$

and

$$\begin{array}{rcl} Z \times \Sigma^k Z' & \to & \Sigma^k(Z \times Z') \\ (z, [z', t_1, \ldots, t_k]) & \mapsto & [(z, z'), t_1, \ldots, t_k]. \end{array}$$

By considering the fibrewise extensions of the evaluation

$$\begin{array}{rcl} ev : \Sigma^k Q^k Z & \to & \Sigma^k Z \\ [\omega, t_1, \ldots, t_k] & \mapsto & \omega(t_1, \ldots, t_k) \end{array}$$

and of the map

$$\begin{array}{rcl} Q^k \circ Q^k(Z) & \to & Q^k(Z) \\ \omega : I^k \to \Sigma^k Q^k Z & \mapsto & ev \circ \omega \end{array}$$

we also get a commutative diagram

$$\begin{array}{ccc} Q^k_B \circ Q^k_B(E) & \longrightarrow & Q^k_B(E) \\ & \searrow & \downarrow \\ & & B. \end{array}$$

This permits us to establish the following result:

Proposition 2.3. *Let $p : E \to B$ and $p' : E' \to B'$ be two fibrations. For any $k \geq 0$, there exists a commutative diagram*

$$\begin{array}{ccccc} Q^k_B(E) \times Q^k_{B'}(E') & & \longrightarrow & & Q^k_{B\times B'}(E\times E') \\ & {\scriptstyle q^k(p)\times q^k(p')} \searrow & & \swarrow {\scriptstyle q^k(p\times p')} & \\ & & B \times B'. & & \end{array}$$

Proof. Using the constructions above we obtain:

$$\begin{array}{ccccccc} Q^k_B(E) \times Q^k_{B'}(E') & \longrightarrow & Q^k_{B\times B'}(E \times Q^k_{B'}(E')) & \longrightarrow & Q^k_{B\times B'} \circ Q^k_{B\times B'}(E\times E') & \longrightarrow & Q^k_{B\times B'}(E\times E') \\ {\scriptstyle q^k(p)\times q^k(p')} \downarrow & & \downarrow & & \downarrow & & \downarrow {\scriptstyle q^k(p\times p')} \\ B \times B' & = & B \times B' & = & B \times B' & = & B \times B'. \end{array}$$

□

2.3. **Definition of Qsecat and QTC.** Let $f : A \to X$ a cofibration where X is a well-pointed path-connected space and let $k \geq 0$. By applying the Q^k_X construction to the Ganea fibrations $g_n(f)$ we define:

Definition 2.4. *$Q^k\mathrm{secat}(f)$ is the least integer n (or ∞) such that the fibration*

$$q^k(g_n(f)) : Q^k_X(G_n(f)) \to X$$

admits a (homotopy) section.

By Diagram (2), we have:

$$\cdots \leq Q^{k+1}\mathrm{secat}(f) \leq Q^k\mathrm{secat}(f) \leq \cdots \leq Q^1\mathrm{secat}(f) \leq Q^0\mathrm{secat}(f) = \mathrm{secat}(f)$$

As a limit invariant, we set: $Q\mathrm{secat}(f) := \lim Q^k\mathrm{secat}(f)$.

If f is the inclusion $* \to X$, we recover the notion of $Q^k\mathrm{cat}(X)$ introduced by H. Scheerer, D. Stanley and D. Tanré. If X is a CW-complex and f is the diagonal map $\Delta : X \to X \times X$, we naturally use the notation $Q^k\mathrm{TC}(X)$ and $Q\mathrm{TC}(X)$.

Remark 2.5. *The notion of $Q^k\mathrm{secat}$ can be extended to any map g by applying the Q construction to any fibration equivalent to the join map characterizing $\mathrm{secat}(g)$ or, equivalently, by setting $Q^k\mathrm{secat}(g) := Q^k\mathrm{secat}(f)$ where f is any cofibration weakly equivalent to g.*

3. Some properties of Qsecat and QTC

In all the statements in this section, we consider cofibrations whose target is a well-pointed path-connected space.

3.1. Basic properties.

We start with the following properties which permit us to generalize to QTC and Qcat the well-known relationships [5] between TC and cat (see Corollary 3.2):

Proposition 3.1. *Let $f : A \to X$ and $g : B \to Y$ be two cofibrations together with a commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{\psi} & Y. \end{array}$$

- *(a) If ψ is a homotopy equivalence then, for any $k \geq 0$, $Q^k\mathrm{secat}(f) \geq Q^k\mathrm{secat}(g)$.*
- *(b) If the diagram is a homotopy pull-back then, for any $k \geq 0$, we have $Q^k\mathrm{secat}(f) \leq Q^k\mathrm{secat}(g)$.*

Proof. By considering the Ganea fibrations and applying the Q^k_X construction, we obtain the following commutative diagram

$$\begin{array}{ccc} Q^k_X(G_n(f)) & \longrightarrow & Q^k_Y(G_n(g)) \\ {\scriptstyle q^k(g_n(f))}\downarrow & & \downarrow{\scriptstyle q^k(g_n(g))} \\ X & \xrightarrow{\psi} & Y. \end{array}$$

If ψ is a homotopy equivalence, we deduce from a section of the left-hand fibration a homotopy section of the right-hand fibration, which proves the statement (a).

For (b), the hypothesis implies that the diagram is a homotopy pull-back (see Proposition 2.1) which permits us to obtain a section of the left hand fibration from a section of the right hand one. ☐

Corollary 3.2. *Let X be a path-connected CW-complex. For any $k \geq 0$, we have* $Q^k\mathrm{cat}(X) \leq Q^k\mathrm{TC}(X) \leq Q^k\mathrm{cat}(X \times X)$.

Proof. It suffices to apply the two items of the propostion above to the following two diagrams, respectively. The right hand diagram, where i_2 is the inclusion on the second factor, is a homotopy pull-back:

$$\begin{array}{ccc} * & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle \Delta} \\ X \times X & = & X \times X \end{array} \qquad\qquad \begin{array}{ccc} * & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle \Delta} \\ X & \xrightarrow[i_2]{} & X \times X. \end{array}$$

☐

3.2. **Cohomological lower bound.** Let $f : A \to X$ be a cofibration, and let $f^*\colon H^*(X;\Bbbk) \to H^*(A;\Bbbk)$ be the morphism induced by f in cohomology with coefficients in a field $\Bbbk$. As is well-known ([18]), if we consider the index of nilpotency, nil, of the ideal $\ker f^*$, that is, the least integer n such that any $(n+1)$-fold cup product in $\ker f^*$ is trivial, then we have

$$\mathrm{nil}\ker f^* \leq \mathrm{secat}(f).$$

Actually the proof of [8, Thm 5.2] permits us to see that

$$\mathrm{nil}\ker f^* \leq H\mathrm{secat}(f) \leq \mathrm{secat}(f)$$

where $H\mathrm{secat}(f)$ is the least integer n such that the morphism induced in cohomology by the nth Ganea fibration of f, $H^*(X;\Bbbk) \to H^*(G_n(f);\Bbbk)$, is injective.

Here we prove:

Theorem 3.3. *For any $k \geq 0$,* $\mathrm{nil}\ker f^* \leq H\mathrm{secat}(f) \leq Q^k\mathrm{secat}(f)$.

Proof. Suppose that $Q^k\mathrm{secat}(f) \leq n$. By applying Diagram (1) to the nth Ganea fibration of f, we obtain the following commutative diagram:

$$\begin{array}{ccccccc} G_n(f) & \longrightarrow & Q^k_X(G_n(f)) & \longrightarrow & Q^k(G_n(f)) & \xrightarrow{\sim} & \Omega^k\widetilde{\Sigma}^k G_n(f) \\ \downarrow{\scriptstyle g_n(f)} & & \downarrow{\scriptstyle q^k(g_n(f))} & & \downarrow{\scriptstyle Q^k(g_n)} & & \downarrow{\scriptstyle \Omega^k\widetilde{\Sigma}^k g_n(f)} \\ X & = & X & \xrightarrow{\eta^k} & Q^k(X) & \xrightarrow{\sim} & \Omega^k\widetilde{\Sigma}^k X. \end{array}$$

The composite $\tilde{\eta}^k : X \to Q^k(X) \xrightarrow{\sim} \Omega^k\widetilde{\Sigma}^k X$, which is the usual coaugmentation, can be identified to the adjoint of the identity of $\widetilde{\Sigma}^k X$ through the k-fold adjunction. From $Q^k\mathrm{secat}(f) \leq n$ we know that there exists a map $\psi : X \to \Omega^k\widetilde{\Sigma}^k G_n(f)$ such that $\Omega^k\widetilde{\Sigma}^k(g_n(f))\psi \simeq \tilde{\eta}^k$. Through the k fold adjunction, we get a homotopy section of $\widetilde{\Sigma}^k(g_n(f))$ which implies that $H^*(g_n(f))$ is injective. ☐

When $f = \Delta : X \to X \times X$, nil ker Δ^* coincides with Farber's *zero-divisor cup-length*, that is, the nilpotency of the kernel of the cup-product $\cup_X : H^*(X;\Bbbk) \otimes H^*(X;\Bbbk) \to H^*(X;\Bbbk)$ and we have:

Corollary 3.4. *Let X be a path-connected CW-complex. For any $k \geq 0$, we have* $\mathrm{nil}\ker \cup_X \leq Q^k\mathrm{TC}(X)$.

3.3. **Products and fibrations.** We here study the behaviour of $Q\mathrm{secat}$ and $Q\mathrm{TC}$ with respect to products and fibrations and establish generalizations of well-known properties of LS-category and topological complexity. Classically these properties are proven using the open cover definition of these invariants. In order to be able to obtain generalizations to $Q\mathrm{secat}$, $Q\mathrm{TC}$ and $Q\mathrm{cat}$, we first need a proof of the classical property based on the Ganea fibrations.

Theorem 3.5. *Let $f : A \to X$ and $g : B \to Y$ be two cofibrations. Then, for any $k \geq 0$, $Q^k\mathrm{secat}(f \times g) \leq Q^k\mathrm{secat}(f) + Q^k\mathrm{secat}(g)$.*

Proof. From the diagram constructed in [11, Pg. 27] (from the product of fibrewise joins of two fibrations to the fibrewise join of the fibration product) we can deduce the existence of a commutative diagram of the following form:

$$\begin{array}{ccc} G_n(f) \times G_m(g) & \longrightarrow & G_{m+n}(f \times g) \\ & {\scriptstyle g_n(f)\times g_m(g)} \searrow \quad \swarrow {\scriptstyle g_{n+m}(f\times g)} & \\ & X \times Y. & \end{array}$$

By applying the $Q^k_{X\times Y}$ construction and using the map of Proposition 2.3 we obtain:

$$\begin{array}{ccccc} Q^k_X(G_n(f)) \times Q^k_Y(G_m(g)) & \longrightarrow & Q^k_{X\times Y}(G_n(f) \times G_m(g)) & \longrightarrow & Q^k_{X\times Y}(G_{m+n}(f \times g)) \\ & {\scriptstyle q^k(g_n(f))\times q^k(g_m(g))} \searrow & & {\scriptstyle q^k(g_n(f)\times g_m(g))} \searrow & \downarrow {\scriptstyle q^k(g_{n+m}(f\times g))} \\ & & X \times Y & = & X \times Y. \end{array}$$

From this diagram, we can establish that, if $Q^k\mathrm{secat}(f) \leq n$ and $Q^k\mathrm{secat}(g) \leq m$ then $Q^k\mathrm{secat}(f \times g) \leq n + m$. □

Corollary 3.6. *Let X and Y be path-connected CW-complexes. For any $k \geq 0$, we have $Q^k\mathrm{TC}(X \times Y) \leq Q^k\mathrm{TC}(X) + Q^k\mathrm{TC}(Y)$.*

Proof. Since the diagonal map $\Delta_{X\times Y} : X \times Y \to X \times Y \times X \times Y$ coincides, up to the homeomorphism switching the two middle factors, with the product $\Delta_X \times \Delta_Y$, we have $Q^k\mathrm{TC}(X \times Y) = Q^k\mathrm{secat}(\Delta_X \times \Delta_Y)$ and the result follows from the theorem above. □

We now turn to the study of fibrations. We first establish the following result.

Proposition 3.7. *Suppose that there exists a commutative diagram*

$$\begin{array}{ccc} A & \longrightarrow & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{\iota} & Y \end{array}$$

in which f, g and ι are cofibrations. Then, for any $k \geq 0$, we have

$$Q^k\mathrm{secat}(g) \leq (Q^k\mathrm{secat}(\iota) + 1)(\mathrm{secat}(f) + 1) - 1.$$

Proof. Suppose that $\operatorname{secat}(f) \leq p$. Then the fibration $g_p(f)$ admits a section. Using the commutative diagram at the bottom of page 3, we obtain from this section a map $\lambda : X \to G_p(g)$ such that $g_p(g) \circ \lambda = \iota$. Explicitly, for $x \in X$, $\lambda(x)$ is an element $(\gamma_0, \cdots, \gamma_p) \in \Gamma_p(Y)$ such that $\gamma_0(0) = \cdots = \gamma_p(0) = \iota(x)$ and at least one path γ_j satisfies $\gamma_j(1) \in \iota f(A) \subset g(C)$. Since ι is a cofibration and the fibration $\Gamma_p(Y) \to Y$, $(\gamma_0, \cdots, \gamma_p) \mapsto \gamma_0(0)$ is a homotopy equivalence, the relative lifing lemma ([21, Th. 9]) in the diagram

$$\begin{array}{ccc} X & \xrightarrow{\lambda} & \Gamma_p(Y) \\ {\scriptstyle \iota}\big\downarrow & \overset{\hat{\lambda}}{\nearrow} & \big\downarrow{\scriptstyle \sim} \\ Y & = & Y \end{array}$$

permits us to extend λ to a map $\hat{\lambda} : Y \to \Gamma_p(Y)$ which takes a point $y \in Y$ to an element $(\gamma_0, \cdots, \gamma_p) \in \Gamma_p(Y)$ where all the γ_i start in y and at least one path γ_j satisfies $\gamma_j(1) \in \iota f(A)$ whenever $y \in \iota(X)$. The concatenation of a path β_i with the path $\hat{\lambda}(\beta_i(1))$ gives, for any m, a map $\lambda_m : G_m(\iota) \longrightarrow G_{pm+p+m}(g)$ such that $g_{pm+p+m}(g) \circ \lambda_m = g_m(\iota)$. As a consequence, we have for any $m \geq 0$ and any $k \geq 0$, a commutative diagram of the following form:

$$\begin{array}{ccc} Q^k_Y(G_m(\iota)) & \xrightarrow{Q^k_Y(\lambda_m)} & Q^k_Y(G_{pm+p+m}(g)) \\ & {\scriptstyle q^k(g_m(\iota))}\searrow \quad \swarrow{\scriptstyle q^k(g_{pm+p+m}(g))} & \\ & Y. & \end{array}$$

Therefore, if $Q^k\operatorname{secat}(\iota) \leq m$, then there exists a section of the right-hand fibration and we can conclude that $Q^k\operatorname{secat}(g) \leq pm + p + m = (m+1)(p+1) - 1$. □

As a consequence we obtain the following property of $Q^k\operatorname{secat}$ (and secat when $k = 0$):

Theorem 3.8. *Let* $F \xrightarrow{\iota} E \xrightarrow{\pi} B$ *be a fibration over a well-pointed space. Suppose that there exists a commutative diagram*

$$\begin{array}{ccccc} A & \longrightarrow & C & & \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} & & \\ F & \xrightarrow{\iota} & E & \xrightarrow{\pi} & B \end{array}$$

in which f and g are cofibrations. Then, for any $k \geq 0$, we have

$$Q^k\operatorname{secat}(g) \leq (Q^k\operatorname{cat}(B) + 1)(\operatorname{secat}(f) + 1) - 1.$$

Proof. Since B is well-pointed, ι is a cofibration ([21, Th. 12]). By Proposition 3.7, we have $Q^k\operatorname{secat}(g) \leq (Q^k\operatorname{secat}(\iota)+1)(\operatorname{secat}(f)+1)-1$. On the other hand, since the following diagram

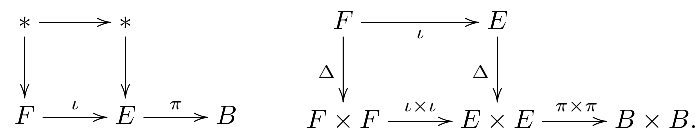
$$\begin{array}{ccc} Q^k_E(G_m(\iota)) & \longrightarrow & Q^k_B(G_n(B)) \\ {\scriptstyle q^k(g_m(\iota))}\big\downarrow & & \big\downarrow{\scriptstyle q^k(g_m(B))} \\ E & \xrightarrow[\pi]{} & B \end{array}$$

is a homotopy pull-back, we have $Q^k\operatorname{secat}(\iota) \leq Q^k\operatorname{cat}(B)$ and the result follows. □

Corollary 3.9. *Let* $F \xrightarrow{\iota} E \xrightarrow{\pi} B$ *be a fibration where all the spaces are well-pointed path-connected CW-complexes. For any* $k \geq 0$*, we have*

(a) $Q^k\mathrm{cat}(E) \leq (Q^k\mathrm{cat}(B)+1)(\mathrm{cat}(F)+1)-1.$
(b) $Q^k\mathrm{TC}(E) \leq (Q^k\mathrm{cat}(B\times B)+1)(\mathrm{TC}(F)+1)-1.$

Proof. We apply the theorem above to the following diagrams, respectively:



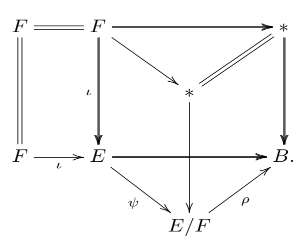
□

Remark 3.10. *If we consider Corollary 3.9 for* $k=0$*, the formula given in* (a) *is the classical formula for LS-category while the formula given in* (b) *corresponds to the formula given in* [6]*. It seems that these two special cases are the only special cases of the formulas established in Proposition 3.7 and Theorem 3.8 which were already known. In particular, Corollary 3.9(a) was not known for* $k \geq 1$ *and we did not find in the literature formulas corresponding to the case* $k=0$ *of Theorem 3.8.*

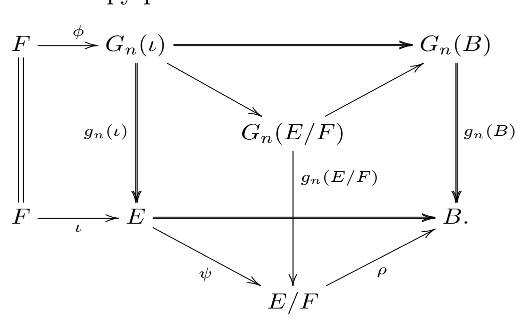
We finally also establish a generalization to Qcat of a property of the LS-category due to O. Cornea [1]. We will next use this result in our final remark on the Scheerer-Stanley-Tanré conjecture.

Theorem 3.11. *Let* $F \xrightarrow{\iota} E \xrightarrow{\pi} B$ *be a fibration where all the spaces are well-pointed path-connected CW-complexes and let* $k \geq 0$*. If* $Q^k\mathrm{cat}(B) \geq 1$ *or* B *is simply-connected, then* $Q^k\mathrm{cat}(E/F) \leq Q^k\mathrm{cat}(B)$*.*

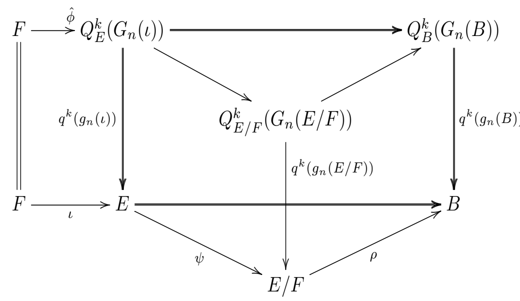
Proof. Consider the following commutative diagram where all the vertical maps are cofibrations, the bold square is a homotopy pull-back, ψ is the identification map and ρ is the map induced by π:



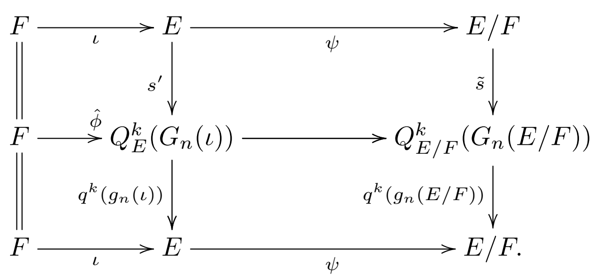
By taking the Ganea fibrations, we obtain the following commutative diagram where the bold square is a homotopy pull-back:



The map ϕ takes a point $f \in F$ to the constant $(n+1)$-tuple of paths $(\hat{f}, \dots, \hat{f}) \in G_n(\iota)$. It is clear that the composite of ϕ with the map $G_n(\iota) \to G_n(E/F)$ is trivial. We now apply the Q^k_- construction to obtain the following diagram



where $\hat{\phi}(f)$ is the map $I^k \to \Sigma^k_E G_n(\iota)$ given by $u \mapsto [(\hat{f}, \dots, \hat{f}), u]$. Once again the bold square is a homotopy pull-back and the composite of $\hat{\phi}$ with the map $Q^k_E G_n(\iota) \to Q^k_{E/F} G_n(E/F)$ is trivial (since its sends f to the base point of the space $Q^k_{E/F}(G_n(E/F)))$. Suppose that $Q^k\mathrm{cat}(B) \leq n$. Then the fibration $q^k(g_n(B))$ admits a section s that we can suppose to be pointed (because, under our hypothesis $n \geq 1$ or B simply-connected, the fibre of $q^k(g_n(B))$ is path-connected). By the unicity (up to homotopy) of the whisker map in the bold homotopy pull-back, we then obtain a homotopy section s' of $q^k(g_n(\iota))$ which satisfies $s' \circ \iota \simeq \hat{\phi}$. Therefore the composition of $s' \circ \iota$ with the map $Q^k_E(G_n(\iota)) \to Q^k_{E/F}(G_n(E/F))$ is homotopically trivial and s' induces a map $\tilde{s} : E/F \to Q^k_{E/F}(G_n(E/F))$ giving the following diagram:



Since $F \to E \to E/F$ is a cofibre sequence and the left and middle vertical composites are homotopy equivalences we obtain that $q^k(g_n(E/F)) \circ \tilde{s}$ is a homotopy equivalence. We can thus conclude that $q^k(g_n(E/F))$ admits a homotopy section and that $Q^k\mathrm{cat}(E/F) \leq n$. □

Remark 3.12. *We note that any non acyclic space satisfies the hypothesis "$Q^k\mathrm{cat}(B) \geq 1$ or B simply-connected" required in Theorem 3.11. Indeed, if $Q^k\mathrm{cat}(B) = 0$ then we can see, using Diagram (1) and induction, that $\widetilde{\Sigma} B$ is contractible, which implies that B is acyclic. We also note that it is possible to show that $Q^1\mathrm{cat}(B) = 0$ implies $B \simeq *$ but we do not know whether there exists a non contractible acyclic space B satisfying $Q^k\mathrm{cat}(B) \geq 1$ for $k \geq 2$.*

3.4. **Condition for** $Q\mathrm{secat} = \mathrm{secat}$**.** Similarly to [23, Theorem 15] we have

Theorem 3.13. *Let $f : A \to X$ be a cofibration which is an $(r-1)$-equivalence with $r\mathrm{secat}(f) \geq 3$. If $\dim(X) \leq 2r\mathrm{secat}(f) - 3$ then $Q\mathrm{secat}(f) = \mathrm{secat}(f)$.*

This result follows by induction from the following proposition which is a generalization of [23, Prop. 16]:

Proposition 3.14. *Let $f : A \to X$ be a cofibration which is an $(r-1)$-equivalence and n be an integer such that $rn \geq 3$. If for $k \geq 0$ one has $Q^k\mathrm{secat}(f) = n$ and $\dim(X) \leq 2rn - 3 + k$ then $Q^{k+1}\mathrm{secat}(f) = n$.*

Proof. Since $Q^{k+1}\mathrm{secat}(f) \leq Q^k\mathrm{secat}(f) = n$, we only have to prove that $Q^{k+1}\mathrm{secat}(f) \geq n$. Suppose that $Q^{k+1}\mathrm{secat}(f) \leq n-1$ and let $\sigma : X \to Q_X^{k+1}(G_{n-1}(f))$ be a (homotopy) section of $q^{k+1}(g_{n-1}(f))$. Consider the following diagram:

$$\begin{array}{ccc}
Q^k(F_{n-1}(f)) & \xrightarrow{\;b^k\;} & Q^{k+1}(F_{n-1}(f)) \\
\downarrow & & \downarrow \\
Q_X^k(G_{n-1}(f)) & \xrightarrow{\;b_X^k\;} & Q_X^{k+1}(G_{n-1}(f)) \\
& {\scriptstyle q^k(g_{n-1}(f))} \searrow \quad \swarrow {\scriptstyle q^{k+1}(g_{n-1}(f))} & \\
& X. &
\end{array}$$

Since f is an $(r-1)$-equivalence, the fibre of f is $(r-2)$-connected and the space $F_{n-1}(f)$ is $(rn-2)$-connected. The condition $rn \geq 3$ ensures that $F_{n-1}(f)$ is at least 1-connected. Using the fact that, for a $(l-1)$-connected space Z with $l \geq 2$, the coaugmentation $\tilde{\eta}_Z : Z \to \Omega\widetilde{\Sigma}Z$ is a $(2l-1)$-equivalence, we can see that the map $b^k : Q^k(F_{n-1}(f)) \to Q^{k+1}(F_{n-1}(f))$, which is equivalent to $\Omega^k\tilde{\eta}_{\widetilde{\Sigma}^k F_{n-1}(f)}$, is a $(2rn-3+k)$-equivalence for all $k \geq 0$. As a consequence the map $b_X^k : Q_X^k(G_{n-1}(f)) \to Q_X^{k+1}(G_{n-1}(f))$ is also a $(2rn-3+k)$-equivalence for all $k \geq 0$. Since $\dim(X) \leq 2rn + k - 3$, we obtain a map $\sigma' : X \to Q_X^k(G_{n-1}(f))$ such that $b_X^k \circ \sigma' \simeq \sigma$. This map is a homotopy section of $q^k(g_{n-1}(f))$ which contradicts the hypothesis $Q^k\mathrm{secat}(f) = n$. Therefore $Q^{k+1}\mathrm{secat}(f) \geq n$. ☐

Corollary 3.15. *Let X be an $(r-1)$-connected CW-complex such that $r\mathrm{TC}(X) \geq 3$. If $\dim(X) \leq \dfrac{2r\mathrm{TC}(X)-3}{2}$ then $Q\mathrm{TC}(X) = \mathrm{TC}(X)$.*

Proof. If X is $(r-1)$-connected then the diagonal map $\Delta : X \to X \times X$ is an $(r-1)$-equivalence and we can apply Theorem 3.13. ☐

3.5. **Examples and observations.**

(1) Let $Y = \mathbb{R}P^6/\mathbb{R}P^2$ and $X = Y \vee Y$. In [11], it is proved that $\mathrm{TC}(X) = \mathrm{nil}\,\ker \cup_X = 4$ and that X is a counter-example for the analogue of the Ganea conjecture for TC since, for any even n, $\mathrm{TC}(X \times S^n) = 5 < \mathrm{TC}(X) + \mathrm{TC}(S^n)$. By Theorem 3.3, it is clear that a space Z satisfying $\mathrm{nil}\,\ker \cup_Z = \mathrm{TC}(Z)$ also satisfies $Q\mathrm{TC}(Z) = \mathrm{TC}(Z)$. Therefore the space $X = Y \vee Y$ satisfies $Q\mathrm{TC}(X) = \mathrm{TC}(X) = 4$. The interest of this example is that it shows that the analogue for TC of the Scheerer-Stanley-Tanré conjecture mentioned in the introduction (which would be $\mathrm{TC}(X \times S^n) = \mathrm{TC}(X) + \mathrm{TC}(S^n)$ if and only if $Q\mathrm{TC}(X) = \mathrm{TC}(X)$) does not hold in general. Actually the analogue of [23, Thm 11] for topological complexity is not true: we cannot have $Q\mathrm{TC}(X \times S^n) = Q\mathrm{TC}(X) + Q\mathrm{TC}(S^n)$ because, in that case, it would be true that $Q\mathrm{TC}(X) = \mathrm{TC}(X)$ implies that

$\mathrm{TC}(X \times S^n) = \mathrm{TC}(X) + \mathrm{TC}(S^n)$. Note that $Q\mathrm{TC}(S^n) = \mathrm{TC}(S^n)$ since $\mathrm{nil}\ker \cup_{S^n} = \mathrm{TC}(S^n)$.

(2) Let $X = S^3 \cup_\alpha e^7$ where $\alpha : S^6 \to S^3$ is the Blakers-Massey element. We know by [10] (or [11]) that $\mathrm{TC}(X) = 3$. Therefore the condition $\dim(X) \leq \frac{2r\mathrm{TC}(X) - 3}{2}$ (with $r = 3$) is satisfied and $Q\mathrm{TC}(X) = 3$ by Corollary 3.15. This is an example for which $\mathrm{nil}\ker \cup_X < Q\mathrm{TC}(X)$. Actually, using [11, Theorem 5.5], we know that for any two-cell complex $X = S^p \cup_\alpha e^{q+1}$ such that
 (a) $2p - 1 < q \leq 3p - 3$
 (b) the Hopf invariant of α, $H_0(\alpha) \in \pi_q(S^{2p-1})$, satisfies the condition $(2 + (-1)^p)H_0(\alpha) \neq 0$

we have $\mathrm{TC}(X) \geq 3$. Corollary 3.15 permits us to see that for these spaces we have $Q\mathrm{TC}(X) = \mathrm{TC}(X)$.

(3) Let $X = S^2 \cup_\alpha e^{10}$ be the space considered by Iwase in [13] to disprove the Ganea conjecture. It is easy to see that $\mathrm{nil}\ker \cup_X = 2$ and, using Hopf invariants, it has been proved in [11] that $\mathrm{TC}(X) \leq 3$. Here we can compute that $Q\mathrm{TC}(X) = 2$. Indeed, by Corollaries 3.2 and 3.4, we have

$$\mathrm{nil}\ker \cup_X \leq Q\mathrm{TC}(X) \leq Q\mathrm{cat}(X \times X).$$

On the other hand, we know from [17] (or Theorem 3.5) that $Q\mathrm{cat}(X\times X) \leq 2Q\mathrm{cat}(X)$. We also know that, for this particular space $X = S^2 \cup_\alpha e^{10}$, we have $Q\mathrm{cat}(X) = 1$ ([17, Ex. 5]). We therefore can conclude that $Q\mathrm{TC}(X) = 2$. More generally, any finite CW-complex X which is a counter-example to the Ganea conjecture satisfies $Q\mathrm{cat}(X) < \mathrm{cat}(X)$ (see below) and therefore $Q\mathrm{TC}(X) \leq 2\mathrm{cat}(X) - 2$.

(4) As mentioned in [15, Remark 7], using [23, Theorem 15] (or Theorem 3.13 above for the cofibration $* \to X$), we can see that, for any path-connected finite dimensional CW-complex X, the product $X \times T^p$, where T^p is the p-fold product of p circles S^1, satisfies $Q\mathrm{cat}(X \times T^p) = \mathrm{cat}(X \times T^p)$ as soon as $p \geq \dim(X) + 3$. In a similar way, considering p-fold products of the sphere S^2, we have:

- If X is a 1-connected CW-complex then, for any $p \geq \frac{2\dim(X) + 3}{4}$, the space $X \times (S^2)^p$ satisfies $Q\mathrm{TC}(X \times (S^2)^p) = \mathrm{TC}(X \times (S^2)^p)$.

Indeed the space $X \times (S^2)^p$ is 1-connected so that we can consider $r = 2$ in Corollary 3.15. On the other hand we have $\dim(X \times (S^2)^p) = \dim(X) + 2p$ and $\mathrm{TC}(X\times(S^2)^p) \geq 2p$ since $X\times(S^2)^p$ dominates $(S^2)^p$ whose topological complexity is $2p$. Then if $p \geq \frac{2\dim(X) + 3}{4}$ we get $2(\dim(X)+2p) \leq 8p-3$ and Corollary 3.15 permits us to conclude.

We finish this text with a small remark on the Scheerer-Stanley-Tanré conjecture:

Conjecture. ([17]) Let X a finite CW-complex. We have $\text{cat}\,(X\times S^n) = \text{cat}\,(X)+1$ for all $n \geq 1$ (X satisfies the Ganea conjecture) if and only if $Q\text{cat}\,(X) = \text{cat}\,(X)$.

As mentioned in the introduction, one direction of the Scheerer-Stanley-Tanré conjecture has been proved in [23]. More precisely, using the fact that the invariant $Q\text{cat}$ satisfies $Q\text{cat}\,(X \times S^n) = Q\text{cat}\,(X) + 1$ ([23, Corollary 12]), we have that $\text{cat}\,(X \times S^n) = \text{cat}\,(X) + 1$ holds for all $n \geq 1$ as soon as $Q\text{cat}\,(X) = \text{cat}\,(X)$.

In the last item above we recalled that for any path-connected finite dimensional CW-complex X, the product $X \times T^p$, where T^p is the p-fold product of p circles S^1, satisfies $Q\text{cat}\,(X \times T^p) = \text{cat}\,(X \times T^p)$ and therefore the Ganea conjecture as soon as $p \geq \dim(X) + 3$ (compare with [16]). We also notice the following consequence of [23, Corollary 12]:

- If we have $Q\text{cat}\,(X\times T^p) = \text{cat}\,(X\times T^p)$ for some p then we have $Q\text{cat}\,(X \times T^q) = \text{cat}\,(X \times T^q)$ for all $q \geq p$.

Indeed, suppose that $Q\text{cat}\,(X \times T^q) = \text{cat}\,(X \times T^q)$. Then $X \times T^q$ satisfies the Ganea conjecture and, since $X \times T^{q+1} = X \times T^q \times S^1$, we have

$$\text{cat}\,(X \times T^{q+1}) = \text{cat}\,(X \times T^q) + 1 = Q\text{cat}\,(X \times T^q) + 1 = Q\text{cat}\,(X \times T^{q+1}).$$

The assertion follows by induction.

We now use Theorem 3.11 to ask a question related to the Scheerer-Stanley-Tanré conjecture:

Suppose that X is a counter-example to the Ganea Conjecture and let p be the least integer such that $X \times T^p$ satisfies the Ganea Conjecture. Consider the space $Y = (X \times T^p)/S^1$ and the following diagram where the dotted map is induced by the projection $X \times T^p \to X \times T^{p-1}$:

$$\begin{array}{ccccc} S^1 & \longrightarrow & X \times T^p & \xrightarrow{\text{pr}} & X \times T^{p-1} \\ & & \downarrow & \nearrow & \\ & & Y = (X \times T^p)/S^1. & & \end{array}$$

Since $X \times T^{p-1}$ does not satisfy the Ganea conjecture, we have, by Theorem 3.11,

$$Q\text{cat}\,(Y) \leq Q\text{cat}\,(X \times T^{p-1}) < \text{cat}\,(X \times T^{p-1}).$$

On the other hand, since Y dominates $X \times T^{p-1}$ we also have $\text{cat}\,(Y) \geq \text{cat}\,(X \times T^{p-1})$. Therefore $Q\text{cat}\,(Y) < \text{cat}\,(Y)$.

Question. Does Y satisfy the Ganea conjecture?

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