Continuity of the core-EP inverse and its applications

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In this paper, firstly we study the continuity of the core-EP inverse without explicit error bounds by virtue of two methods. One is the rank equality, followed from the classical generalized inverse. The other one is matrix decomposition. The continuity of the core inverse can be derived as a particular case. Secondly, we study perturbation bounds for the core-EP inverse under prescribed conditions. Perturbation bounds for the core inverse can be derived as a particular case. Also, as corollaries, the sufficient (and necessary) conditions for the continuity of the core-EP inverse are obtained. Thirdly, a numerical example is illustrated to compare derived upper bounds. Finally, an application to semistable matrices is provided.

Keywords: core-EP inverse, pseudo core inverse, continuity, perturbation bound

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1 Introduction

It is known that the inverse of a nonsingular matrix is a continuous function. However, in general, the operations of generalized inverses such as Moore-Penrose inverse, Drazin inverse, weighted Drazin inverse, generalized inverse $A^{(2)}_{T,S}$ and core inverse are not continuous [1–5]. It is of interest to know whether the continuity of the core-EP inverse holds. In this note, we will answer this question.

Throughout this paper, $\mathbb{C}^n$ denotes the sets of all n-dimensional column vectors and $\mathbb{C}^{m\times n}$ is used to denote the set of all $m \times n$ complex matrices. For each complex matrix $A \in \mathbb{C}^{m\times n}$, A^* denotes the conjugate transpose of A, $\mathcal{R}(A)$ and $\mathcal{N}(A)$ denote the range (column space) and null space of A respectively. The Moore-Penrose inverse of A, denoted by $A^\dagger$, is the unique solution to

$$AXA = A,\ XAX = X,\ (AX)^* = AX \text{ and } (XA)^* = XA.$$

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Let $A \in \mathbb{C}^{n\times n}$, the index of A, denoted by $\mathrm{ind}(A)$, is the smallest non-negative integer k for which $\mathrm{rank}(A^k) = \mathrm{rank}(A^{k+1})$. The Drazin inverse of A, denoted by A^D, is the unique solution of the system

$$AXA^k = A^k,\ XAX = X \text{ and } AX = XA.$$

Recall that the core-EP inverse was proposed by Manjunatha Prasad and Mohana [6] for a square matrix of arbitrary index, as an extension of the core inverse restricted to a square matrix of index one in [7]. Then, Gao and Chen [8] characterized the core-EP inverse (also known as the pseudo core inverse) in terms of three equations. For $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k$, the core-EP inverse of A, denoted by $A^{\textcircled{\dagger}}$, is the unique solution to equations

$$XA^{k+1} = A^k,\ AX^2 = X,\ (AX)^* = AX. \tag{1.1}$$

The core-EP inverse is an outer inverse ($\{2\}$-inverse), i.e., $A^{\textcircled{\dagger}}AA^{\textcircled{\dagger}} = A^{\textcircled{\dagger}}$, see [8]. If $k = 1$, then the core-EP inverse of A is the core inverse of A. denoted by $A^{\textcircled{\#}}$ (see [7]).

Lemma 1.1. [8] *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k$. Then we have the following facts:*
(1) $A^{\textcircled{\dagger}} = A^DA^k(A^k)^\dagger = A^DA^j(A^j)^\dagger$, *for any $j \geq k$;*
(2) $A^D = (A^{\textcircled{\dagger}})^{k+1}A^k$.

From Lemma 1.1, it follows that $A^k(A^k)^\dagger = AA^{\textcircled{\dagger}} = A^j(A^j)^\dagger$ (for any $j \geq k$) and

$$A^{\textcircled{\dagger}} = A^DA^k(A^k)^\dagger = A^DA^{k+1}(A^{k+1})^\dagger = A^k(A^{k+1})^\dagger. \tag{1.2}$$

Recall that the Euclidean vector norm is defined by

$$\|x\|^2 = x^*x \text{ for any } x \in \mathbb{C}^n,$$

the spectral norm of a matrix $A \in \mathbb{C}^{n\times n}$ is defined by

$$\|A\| = \sup_{x\neq 0} \frac{\|Ax\|}{\|x\|},$$

and

$$\begin{aligned}
&\|Ax\| \leq \|A\|\|x\| \text{ for all } A \in \mathbb{C}^{n\times n} \text{ and all } x \in \mathbb{C}^n,\\
&\|AB\| \leq \|A\|\|B\| \text{ for all } A,\ B \in \mathbb{C}^{n\times n},\\
&\|A^*\| = \|A\| \text{ for all } A \in \mathbb{C}^{n\times n}.
\end{aligned}$$

For a nonsingular matrix A, $\kappa(A) = \|A\|\|A^{-1}\|$ denotes the condition number of A. As usual, this is generalized to the core-EP condition number $\kappa_{\textcircled{\dagger}}(A) = \|A\|\|A^{\textcircled{\dagger}}\|$ if A is singular.

Lemma 1.2. [4] *Let $A \in \mathbb{C}^{n\times n}$ with $\|A\| < 1$. Then $I + A$ is non-singular and*

$$\|(I + A)^{-1}\| \leq (1 - \|A\|)^{-1}.$$

The paper is organized as follows. In Section 2, the continuity of the core-EP inverse without explicit error bounds is investigated by means of a rank equation and a matrix decomposition respectively. The continuity of the core inverse are obtained as corollaries. In Section 3, perturbation bounds for the core-EP inverse are investigated respectively under three reasonable cases:

(1) $\mathcal{R}(E) \subseteq \mathcal{R}(A^k)$ and $\mathcal{N}(A^{k*}) \subseteq \mathcal{N}(E)$, where $k = \mathrm{ind}(A)$;

(2) $AA^{\textcircled{\dagger}} = (A+E)(A+E)^{\textcircled{\dagger}}$ and $A^{\textcircled{\dagger}}A = (A+E)^{\textcircled{\dagger}}(A+E)$;

(3) $\mathrm{rank}\left(A^k\right) = \mathrm{rank}\left((A+E)^k\right)$, where $k = \max\{\mathrm{ind}(A), \mathrm{ind}(A+E)\}$.

The above three cases are motivated by articles [4, 9–11].

The relation scheme of (1)-(3) states as follows : notice that (1) is equivalent to

$$(4)\ E = A^{\textcircled{\dagger}}AE = EAA^{\textcircled{\dagger}};$$

in general, (1) and (2) are independent, see Example 1.3 and Example 1.4; (1) and (3) are independent, see Examples 1.3 and 1.5; (2) implies (3), but (3) may not imply (2), see Example 1.5.

Example 1.3. *Let* $A = \begin{bmatrix} 1 & 0 \\ 0 & 0.1 \end{bmatrix}$, $E = \begin{bmatrix} 0 & 0 \\ 0 & -0.1 \end{bmatrix}$. *Then* $E = A^{\textcircled{\dagger}}AE = EAA^{\textcircled{\dagger}}$. *However,* $AA^{\textcircled{\dagger}} \neq (A+E)(A+E)^{\textcircled{\dagger}}$, $A^{\textcircled{\dagger}}A \neq (A+E)^{\textcircled{\dagger}}(A+E)$ *and* $\mathrm{rank}(A) \neq \mathrm{rank}(A+E)$.

Example 1.4. *let* $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $E = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0 & 0.1 \\ 0 & 0 & 0 \end{bmatrix}$. *Then* $AA^{\textcircled{\dagger}} = (A+E)(A+E)^{\textcircled{\dagger}}$, $A^{\textcircled{\dagger}}A = (A+E)^{\textcircled{\dagger}}(A+E)$. *However,* $AA^{\textcircled{\dagger}}E \neq E$.

Example 1.5. *let* $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $E = \begin{bmatrix} 0.1 & 0.1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. *Then* $\mathrm{rank}(A) = \mathrm{rank}(A+E)$. *However,* $A^{\textcircled{\dagger}}A \neq (A+E)^{\textcircled{\dagger}}(A+E)$ *and* $E \neq EAA^{\textcircled{\dagger}}$.

Among the above conditions, (3) would be the weakest condition to consider perturbation bounds for the core-EP inverse. Although (2) is stronger than (3), yet (2) in conjunction with other restrictions on A, E would help to acquire a better error bound. Thus (1)-(3) are all worth to be studied. As special cases, perturbation bounds for the core inverse are obtained. Meanwhile, the sufficient (and necessary) conditions for which the operation of the core-EP inverse is continuous are derived as natural outcomes. In Section 4, a numerical example is illustrated to compare upper bounds for $\frac{\|(A+E)^{\textcircled{\dagger}} - A^{\textcircled{\dagger}}\|}{\|A^{\textcircled{\dagger}}\|}$ by using derived results in Section 3. It turns out that upper bounds in case (1) and (2) are slightly better than that in case (3). In Section 5, an application to semistable matrices is provided.

2 Continuity of the core-EP inverse

The following example shows that the core-EP inverse of a square matrix is not continuous in general.

Example 2.1. *Let* $A_j = \begin{bmatrix} 1/j & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ *and* $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. *Then* $A_j \to A$ $(j \to \infty)$. *However,*

$$A_j^{\textcircled{\dagger}} = \begin{bmatrix} j & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \nrightarrow 0 = A^{\textcircled{\dagger}} \ (j \to \infty).$$

In the rest of this section, we consider the necessary and sufficient conditions for which the core-EP inverse has the continuity property.

2.1 Rank equality method

In [3], the continuity of classical generalized inverses are studied by means of rank equalities. Analogously, we consider the continuity of the core-EP inverse.

Lemma 2.2. [3] *Let* $\{A_j\} \subseteq \mathbb{C}^{m\times n}$, $A \in \mathbb{C}^{m\times n}$ *with* $A_j \to A$ $(j \to \infty)$. *Then* $A_j^\dagger \to A^\dagger$ $(j \to \infty)$ *if and only if there exists* j_0 *such that* $\operatorname{rank}(A_j) = \operatorname{rank}(A)$ *for all* $j \geq j_0$.

Lemma 2.3. [3] *Let* $\{A_j\} \subseteq \mathbb{C}^{n\times n}$, $A \in \mathbb{C}^{n\times n}$ *with* $A_j \to A$ $(j \to \infty)$. *Then* $A_j^D \to A^D$ $(j \to \infty)$ *if and only if there exists* j_0 *such that* $\operatorname{rank}(A_j^{\operatorname{ind}(A_j)}) = \operatorname{rank}(A^{\operatorname{ind}(A)})$ *for all* $j \geq j_0$.

Lemma 2.4. [3] *Let* $\{A_j\} \subseteq \mathbb{C}^{n\times n}$, $A \in \mathbb{C}^{n\times n}$ *with* $A_j \to A$ $(j \to \infty)$, $A_j^D \to A^D$ $(j \to \infty)$. *Then there exists* j_0 *such that* $\operatorname{ind}(A) \leq \operatorname{ind}(A_j)$ *for all* $j \geq j_0$.

Analogous to Lemma 2.4, we establish a similar result for the core-EP inverse.

Lemma 2.5. *Let* $\{A_j\} \subseteq \mathbb{C}^{n\times n}$, $A \in \mathbb{C}^{n\times n}$ *with* $A_j \to A$ $(j \to \infty)$, $A_j^{\textcircled{\dagger}} \to A^{\textcircled{\dagger}}$ $(j \to \infty)$. *Then there exists* j_0 *such that* $\operatorname{ind}(A) \leq \operatorname{ind}(A_j)$ *for all* $j \geq j_0$.

Proof. The proof is similar to the Drazin inverse case. For completeness, let us give the proof.

Suppose that $A_j \to A$ and $A_j^{\textcircled{\dagger}} \to A^{\textcircled{\dagger}}$ as $j \to \infty$. Let $\{A_{j_i}\}$ be a subsequence with constant index k of $\{A_j\}$. Then $A_{j_i}^{\textcircled{\dagger}}(A_{j_i})^{k+1} = (A_{j_i})^k$. By taking limits, we derive that

$$A^{\textcircled{\dagger}} A^{k+1} = A^k.$$

Hence $\operatorname{ind}(A) \leq k$. Since the index function takes only finitely many values between 0 and n, we obtain that there exists a j_0 such that

$$\operatorname{ind}(A) \leq \operatorname{ind}(A_j) \text{ for all } j \geq j_0.$$

□

Making an integral application of Lemmas 2.2-2.5, we derive the following result.

Theorem 2.6. *Let* $\{A_j\} \subseteq \mathbb{C}^{n\times n}$, $A \in \mathbb{C}^{n\times n}$ *with* $A_j \to A$ $(j \to \infty)$. *Then the following are equivalent:*

(1) $A_j^{\textcircled{\dagger}} \to A^{\textcircled{\dagger}}$ $(j \to \infty)$;

(2) $A_j^D \to A^D$ $(j \to \infty)$;

(3) *there exists* j_1 *such that* $\mathrm{rank}(A_j^{\mathrm{ind}(A_j)}) = \mathrm{rank}(A^{\mathrm{ind}(A)})$ *for all* $j \geq j_1$;

(4) *there exists* j_2 *such that* $\mathrm{rank}(A_j^{\mathrm{ind}(A_j)}) = \mathrm{rank}(A^{\mathrm{ind}(A_j)}) = \mathrm{rank}(A^{\mathrm{ind}(A)})$ *for all* $j \geq j_2$.

Proof. (1) $\Rightarrow$ (2) From Lemma 1.1, it follows that $A_j^D = (A_j^{\textcircled{\dagger}})^{\mathrm{ind}(A_j)+1}A_j^{\mathrm{ind}(A_j)}$, which converges to $(A^{\textcircled{\dagger}})^{\mathrm{ind}(A_j)+1}A^{\mathrm{ind}(A_j)}$. By Lemma 2.5, there exists j_0 such that $\mathrm{ind}(A_j) \geq \mathrm{ind}(A)$ for all $j \geq j_0$. Then for all $j \geq j_0$,

$$(A^{\textcircled{\dagger}})^{\mathrm{ind}(A_j)+1}A^{\mathrm{ind}(A_j)} = (A^{\textcircled{\dagger}})^{\mathrm{ind}(A)+1}A^{\mathrm{ind}(A)} = A^D.$$

Namely $A_j^D \to A^D$ $(j \to \infty)$.

(2) $\Leftrightarrow$ (3) It is clear by Lemma 2.3.

(3) $\Rightarrow$ (4) Since $\mathrm{rank}(A_j^{\mathrm{ind}(A_j)}) = \mathrm{rank}(A^{\mathrm{ind}(A)})$ for all $j \geq j_1$, then $A_j^D \to A^D$ $(j \to \infty)$ by Lemma 2.3. Thus, there exists j_1' such that $\mathrm{ind}(A_j) \geq \mathrm{ind}(A)$ for all $j \geq j_1'$ in view of Lemma 2.4. Therefore, for all $j \geq j_2 = \max\{j_1, j_1'\}$,

$$A^{\mathrm{ind}(A_j)} = A^{\mathrm{ind}(A)}A^{\mathrm{ind}(A_j)-\mathrm{ind}(A)} \text{ and } A^{\mathrm{ind}(A)} = (A^D)^{\mathrm{ind}(A_j)-\mathrm{ind}(A)}A^{\mathrm{ind}(A_j)}$$

and hence $\mathrm{rank}(A^{\mathrm{ind}(A_j)}) = \mathrm{rank}(A^{\mathrm{ind}(A)})$.

(4) $\Rightarrow$ (1) From the assumption, we derive $A_j^D \to A^D$ $(j \to \infty)$ by applying Lemma 2.3. Then there exists j_3 such that $\mathrm{ind}(A_j) \geq \mathrm{ind}(A)$ for all $j \geq j_3$ by applying Lemma 2.4. Let $j_0 = \max\{j_2, j_3\}$, by Lemma 2.2 and the assumption $\mathrm{rank}(A_j^{\mathrm{ind}(A_j)}) = \mathrm{rank}(A^{\mathrm{ind}(A_j)})$, we derive

$$(A_j^{\mathrm{ind}(A_j)})^\dagger \to (A^{\mathrm{ind}(A_j)})^\dagger \ (j \to \infty).$$

From the above and Lemma 1.1, $A_j^{\textcircled{\dagger}} = A_j^D A_j^{\mathrm{ind}(A_j)}(A_j^{\mathrm{ind}(A_j)})^\dagger \to A^D A^{\mathrm{ind}(A_j)}(A^{\mathrm{ind}(A_j)})^\dagger$ $(j \to \infty)$. Since $\mathrm{ind}(A_j) \geq \mathrm{ind}(A)$ for all $j \geq j_0$, $A^{\mathrm{ind}(A_j)}(A^{\mathrm{ind}(A_j)})^\dagger = A^{\mathrm{ind}(A)}(A^{\mathrm{ind}(A)})^\dagger$ for all $j \geq j_0$. Hence, $A_j^{\textcircled{\dagger}} \to A^D A^{\mathrm{ind}(A)}(A^{\mathrm{ind}(A)})^\dagger = A^{\textcircled{\dagger}}$ $(j \to \infty)$. $\square$

The continuity of the core inverse can be derived as a particular case $\mathrm{ind}(A) = \mathrm{ind}(A_j) = 1$ in Theorem 2.6.

Corollary 2.7. *If* $\{A_j\} \subseteq \mathbb{C}^{n\times n}$, $A \in \mathbb{C}^{n\times n}$ *and* $A_j \to A$ $(j \to \infty)$. *Then the following are equivalent:*

(1) $A_j^{\textcircled{\#}} \to A^{\textcircled{\#}}$ $(j \to \infty)$;

(2) $A_j^{\#} \to A^{\#}$ $(j \to \infty)$;

(3) *there exists* j_0 *such that* $\mathrm{rank}(A_j) = \mathrm{rank}(A)$ *for all* $j \geq j_0$.

2.2 Matrix decomposition method

In [1], Pierce decomposition is used to study the continuity of the Moore-Penrose inverse. However this approach is not suitable for the core-EP inverse since the core-EP inverse is not an inner inverse. As an alternative, we make use of the core-EP decomposition.

Recall that the core-EP decomposition [12] of A is

$$A = U \begin{bmatrix} T & S \\ 0 & N \end{bmatrix} U^* = U \begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix} U^* + U \begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix} U^* = A_1 + A_2, \tag{2.1}$$

where U is unitary, T is non-singular and N is nilpotent with index k, in which case,

$$A^{\textcircled{\dagger}} = A_1^{\textcircled{\#}} = U \begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*.$$

Fix $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k$ and consider the following equations

$$XA^{k+1} - A^k = E_1,\ AX^2 - X = E_2 \text{ and } AX - (AX)^* = E_3. \tag{2.2}$$

Here X may be thought of as an approximation and the E_i $(i = 1,\ 2,\ 3)$ as error terms. Let $X = A^{\textcircled{\dagger}} + F$. Then (2.2) becomes

$$FA^{k+1} = E_1, \tag{2.3}$$

$$AA^{\textcircled{\dagger}}F + AFA^{\textcircled{\dagger}} + AF^2 - F = E_2, \tag{2.4}$$

$$AF - (AF)^* = E_3. \tag{2.5}$$

Suppose that $F = U \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} U^*$. According to (2.3),

$$U \begin{bmatrix} X_1T^{k+1} & X_1 \sum\limits_{i+j=k} T^iSN^j \\ X_3T^{k+1} & X_3 \sum\limits_{i+j=k} T^iSN^j \end{bmatrix} U^* = E_1,$$

$$\text{i.e., } \begin{bmatrix} X_1 & \Theta_2 \\ X_3 & \Theta_4 \end{bmatrix} = U^*E_1U \begin{bmatrix} (T^{k+1})^{-1} & 0 \\ 0 & I \end{bmatrix}, \tag{2.6}$$

where $\Theta_2 = X_1 \sum\limits_{i+j=k} T^iSN^j$ and $\Theta_4 = X_3 \sum\limits_{i+j=k} T^iSN^j$.

Then according to (2.4),

$$U \begin{bmatrix} \Delta_1 & \Delta_2 \\ \Delta_3 & NX_3X_2 + NX_4^2 - X_4 \end{bmatrix} U^* = E_2, \tag{2.7}$$

where

$$
\begin{aligned}
\Delta_1 &= TX_1T^{-1} + SX_3T^{-1} + TX_1^2 + SX_3X_1 + TX_2X_3 + SX_4X_3,\\
\Delta_2 &= TX_1X_2 + SX_3X_2 + TX_2X_4 + SX_4^2,\\
\Delta_3 &= NX_3T^{-1} - X_3 + NX_3X_1 + NX_4X_3.
\end{aligned}
$$

Finally according to (2.5),

$$
U \begin{bmatrix} \Gamma_1 & TX_2 + SX_4 - (NX_3)^* \\ \Gamma_3 & \Gamma_4 \end{bmatrix} U^* = E_3, \tag{2.8}
$$

where $\Gamma_1 = TX_1 + SX_3 - (TX_1 + SX_3)^*$, $\Gamma_3 = NX_3 - (TX_2 + SX_4)^*$ and $\Gamma_4 = NX_4 - (NX_4)^*$.

If $E_i \to 0$, by applying (2.6)-(2.8), then

$$
X_1 \to 0,\ X_3 \to 0, \tag{2.9}
$$

$$
NX_4^2 - X_4 \to 0, \tag{2.10}
$$

$$
X_2 + T^{-1}SX_4 \to 0. \tag{2.11}
$$

From (2.10), it follows that

$$
X_4 \to NX_4^2 \to N^2X_4^3 \to \cdots \to N^kX_4^{k+1} = 0. \tag{2.12}
$$

Plug $X_4 \to 0$ into (2.11), giving

$$
X_2 \to 0. \tag{2.13}
$$

In view of (2.9), (2.12) and (2.13), $F \to 0$. Hence we have the following result.

Theorem 2.8. *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k$. If $\{X_j\}$ is a sequence of $n \times n$ matrices such that the sequences $\{X_jA^{k+1} - A^k\}$, $\{AX_j^2 - X_j\}$ and $\{AX_j - (AX_j)^*\}$ all converge to zero, then $\{X_j\}$ converges to $A^{\textcircled{\dagger}}$.*

A consequence of Theorem 2.8 is that it makes sense to check a computed $\hat{A}^{\textcircled{\dagger}}$ exactly by using the system (1.1) if A is known.

The case of the core inverse can be derived by letting $k = 1$ in Theorem 2.8.

Corollary 2.9. *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = 1$. If $\{X_j\}$ is a sequence of $n \times n$ matrices such that the sequences $\{X_jA^2 - A\}$, $\{AX_j^2 - X_j\}$ and $\{AX_j - (AX_j)^*\}$ all converge to zero, then $\{X_j\}$ converges to $A^{\textcircled{\#}}$.*

3 Perturbation bounds

In this section, we consider perturbation bounds for the core-EP inverse under reasonable conditions. We refer readers to [4, 9–11, 13–17] for a deep study of perturbation bounds for classical generalized inverses and refer readers to [18] for a study of perturbation bounds for the core inverse.

3.1 The case: $\mathcal{R}(E) \subseteq \mathcal{R}(A^k)$ and $\mathcal{N}(A^{k*}) \subseteq \mathcal{N}(E)$

In this part, we study perturbation bounds for $(A+E)^{\textcircled{\dagger}}$ in the case:

$$\mathcal{R}(E) \subseteq \mathcal{R}(A^k),\ \mathcal{N}(A^{k*}) \subseteq \mathcal{N}(E),\ \text{where } k = \operatorname{ind}(A).$$

This condition is motivated by literatures [4, 9], in which, perturbation bounds for the Moore-Penrose inverse and Drazin inverse were considered respectively.

After which, a sufficient condition for the continuity of the core-EP inverse is derived naturally.

Theorem 3.1. *Let $A,\ E \in \mathbb{C}^{n\times n}$ and $k = \operatorname{ind}(A)$. If $\mathcal{R}(E) \subseteq \mathcal{R}(A^k)$, $\mathcal{N}(A^{k*}) \subseteq \mathcal{N}(E)$ and $\|A^{\textcircled{\dagger}}E\| < 1$. Then*

$$(A+E)^{\textcircled{\dagger}} = (I + A^{\textcircled{\dagger}}E)^{-1}A^{\textcircled{\dagger}} \tag{3.1}$$

and

$$\frac{\|(A+E)^{\textcircled{\dagger}} - A^{\textcircled{\dagger}}\|}{\|A^{\textcircled{\dagger}}\|} \leq \frac{\|A^{\textcircled{\dagger}}E\|}{1-\|A^{\textcircled{\dagger}}E\|}. \tag{3.2}$$

Proof. In view of (2.5), there exist unitary matrices U such that

$$A = U\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}U^*.$$

Let

$$E = U\begin{bmatrix} E_1 & E_2 \\ E_3 & E_4 \end{bmatrix}U^*.$$

From the assumption $\mathcal{R}(E) \subseteq \mathcal{R}(A^k)$, it follows that $E = A^{\textcircled{\dagger}}AE$, which implies

$$E_3 = 0 \text{ and } E_4 = 0.$$

Then from the assumption $\mathcal{N}(A^{k*}) \subseteq \mathcal{N}(E)$, we have $E = EAA^{\textcircled{\dagger}}$, which deduces that

$$E_2 = 0.$$

Thus,

$$A+E = U\begin{bmatrix} T+E_1 & S \\ 0 & N \end{bmatrix}U^*.$$

Hence,

$$\begin{aligned}(A+E)^{\textcircled{\dagger}} &= U\begin{bmatrix} (T+E_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^* \\ &= (I+A^{\textcircled{\dagger}}E)^{-1}A^{\textcircled{\dagger}}\end{aligned}$$

and

$$(A+E)^{\textcircled{\dagger}} - A^{\textcircled{\dagger}} = -A^{\textcircled{\dagger}}E(A+E)^{\textcircled{\dagger}}.$$

Therefore, by Lemma 1.2,

$$\frac{\|(A+E)^{\textcircled{\dagger}} - A^{\textcircled{\dagger}}\|}{\|A^{\textcircled{\dagger}}\|} \le \frac{\|A^{\textcircled{\dagger}}E\|}{1 - \|A^{\textcircled{\dagger}}E\|}.$$

It completes the proof. □

Corollary 3.2. *Let $A,\ E$ be as in Theorem 3.1 and $\|A^{\textcircled{\dagger}}\|\|E\| < 1$. Then*

$$\frac{\|(A+E)^{\textcircled{\dagger}} - A^{\textcircled{\dagger}}\|}{\|A^{\textcircled{\dagger}}\|} \le \frac{\kappa_{\textcircled{\dagger}}(A)\|E\|\|A\|}{1 - \kappa_{\textcircled{\dagger}}(A)\|E\|\|A\|}. \tag{3.3}$$

The bound (3.3) is perfectly analogous to bounds for the Drazin inverse in [9], the Moore-Penrose inverse and the ordinary inverse in [4].

In the following, a sufficient condition for the continuity of the core-EP inverse is derived as a corollary.

Corollary 3.3. *Let $A \in \mathbb{C}^{n\times n}$ and let $\{E_j\}$ be a sequence of $n \times n$ matrices such that $\|E_j\| \to 0$. If there exists a positive integer j_0 such that $E_j = E_jAA^{\textcircled{\dagger}} = A^{\textcircled{\dagger}}AE_j$ for all $j \ge j_0$, then $(A+E_j)^{\textcircled{\dagger}} \to A^{\textcircled{\dagger}}$.*

Remark 3.4. *If $\operatorname{ind}(A) = 1$, then the condition of Theorem 3.1 is reduced to $E = EAA^{\textcircled{\#}} = A^{\textcircled{\#}}AE$ and $\|A^{\textcircled{\#}}E\| < 1$. Thus, under these assumptions, perturbation bounds for the core inverse are obtained.*

3.2 The case: $AA^{\textcircled{\dagger}} = (A+E)(A+E)^{\textcircled{\dagger}}$ and $A^{\textcircled{\dagger}}A = (A+E)^{\textcircled{\dagger}}(A+E)$

In this part, perturbation bounds for the core-EP inverse are investigated under the assumption that $AA^{\textcircled{\dagger}} = (A+E)(A+E)^{\textcircled{\dagger}}$, $A^{\textcircled{\dagger}}A = (A+E)^{\textcircled{\dagger}}(A+E)$. This condition is motivated by article [10], in which, perturbation bounds for the Drazin inverse were investigated.

A sufficient condition for which the operation of the core-EP inverse is a continuous function is derived as a corollary.

Theorem 3.5. *Let $A,\ E \in \mathbb{C}^{n\times n}$ such that $AA^{\textcircled{\dagger}} = (A+E)(A+E)^{\textcircled{\dagger}},\ A^{\textcircled{\dagger}}A = (A+E)^{\textcircled{\dagger}}(A+E)$ and $\|A^{\textcircled{\dagger}}E\| < 1$. Then*

$$\|(A+E)^{\textcircled{\dagger}}\| \le \frac{\|A^{\textcircled{\dagger}}\|}{1 - \|A^{\textcircled{\dagger}}E\|} \tag{3.4}$$

and

$$\frac{\|(A+E)^{\textcircled{\dagger}} - A^{\textcircled{\dagger}}\|}{\|A^{\textcircled{\dagger}}\|} \le \frac{\|A^{\textcircled{\dagger}}E\|}{1 - \|A^{\textcircled{\dagger}}E\|}. \tag{3.5}$$

Proof. Since $AA^{\textcircled{\dagger}} = (A+E)(A+E)^{\textcircled{\dagger}}$ and $A^{\textcircled{\dagger}}A = (A+E)^{\textcircled{\dagger}}(A+E)$, then

$$(A+E)^{\textcircled{\dagger}} - A^{\textcircled{\dagger}} = A^{\textcircled{\dagger}}[A - (A+E)](A+E)^{\textcircled{\dagger}}.$$

Thus, $(A+E)^{\textcircled{\dagger}} = A^{\textcircled{\dagger}} - A^{\textcircled{\dagger}}E(A+E)^{\textcircled{\dagger}}$. Applying the norm $\|\cdot\|$,

$$\|(A+E)^{\textcircled{\dagger}}\| \leq \|A^{\textcircled{\dagger}}\| + \|A^{\textcircled{\dagger}}E\|\|(A+E)^{\textcircled{\dagger}}\|.$$

Hence (3.4) is obtained since $\|A^{\textcircled{\dagger}}E\| < 1$.

Again from $(A+E)^{\textcircled{\dagger}} - A^{\textcircled{\dagger}} = A^{\textcircled{\dagger}}[A-(A+E)](A+E)^{\textcircled{\dagger}}$, it follows that

$$(A+E)^{\textcircled{\dagger}} - A^{\textcircled{\dagger}} = -A^{\textcircled{\dagger}}E[A^{\textcircled{\dagger}} + (A+E)^{\textcircled{\dagger}} - A^{\textcircled{\dagger}}].$$

Applying the norm $\|\cdot\|$,

$$\|(A+E)^{\textcircled{\dagger}} - A^{\textcircled{\dagger}}\| \leq \|A^{\textcircled{\dagger}}E\|[\|A^{\textcircled{\dagger}}\| + \|(A+E)^{\textcircled{\dagger}} - A^{\textcircled{\dagger}}\|].$$

Since $\|A^{\textcircled{\dagger}}E\| < 1$, then (3.5) is derived.

□

Corollary 3.6. *Let A, E be as in Theorem 3.5 and $\|A^{\textcircled{\dagger}}\|\|E\| < 1$. Then*

$$\frac{\|(A+E)^{\textcircled{\dagger}} - A^{\textcircled{\dagger}}\|}{\|A^{\textcircled{\dagger}}\|} \leq \frac{\kappa_{\textcircled{\dagger}}(A)\|E\|/\|A\|}{1-\kappa_{\textcircled{\dagger}}(A)\|E\|/\|A\|}. \tag{3.6}$$

The bound (3.6) is perfectly analogous to a bound for the Drazin inverse in [10].

From Theorem 3.5, we derive a sufficient condition for the continuity of the core-EP inverse, as follows.

Corollary 3.7. *Let $A \in \mathbb{C}^{n\times n}$ and let $\{E_j\}$ be a sequence of $n\times n$ matrices such that $\|E_j\| \to 0$. If there exists a positive integer j_0 such that $AA^{\textcircled{\dagger}} = (A+E_j)(A+E_j)^{\textcircled{\dagger}}$ and $A^{\textcircled{\dagger}}A = (A+E_j)^{\textcircled{\dagger}}(A+E_j)$ for all $j \geq j_0$, then $(A+E_j)^{\textcircled{\dagger}} \to A^{\textcircled{\dagger}}$.*

Remark 3.8. *If $\mathrm{ind}(A) = \mathrm{ind}(A+E) = 1$, then the condition of Theorem 3.5 is reduced to $AA^{\textcircled{\#}} = (A+E)(A+E)^{\textcircled{\#}}$, $A^{\textcircled{\#}}A = (A+E)^{\textcircled{\#}}(A+E)$ and $\|A^{\textcircled{\#}}E\| < 1$. Thus, under these assumptions, a perturbation bound for the core inverse is obtained.*

3.3 The case: $\mathrm{rank}\left(A^k\right) = \mathrm{rank}\left((A+E)^k\right)$

It is known from [11] that if A and $\{E_j\}$ are $n\times n$ matrices such that $\|E_j\| \to 0$, then there exists a positive integer j_0 such that

$$\mathrm{rank}\left((A+E_j)^{k_j}\right) \geq \mathrm{rank}\left(A^{k_j}\right)$$

for all $j \geq j_0$, where $k_j = \max\{\mathrm{ind}(A), \mathrm{ind}(A+E_j)\}$.

Let $A,\ E \in \mathbb{C}^{n\times n}$. For an arbitrary positive integer h, define $E(A^h)$ by $E(A^h) = (A+E)^h - A^h$. Then $\|(A+E)^h\| \leq \|A^h\| + \varepsilon(A^h)$, where

$$\varepsilon(A^h) = \sum_{i=0}^{h-1} \mathrm{C}_h^i \|A\|^i \|E\|^{h-i} \geq \|E(A^h)\|$$

and C_h^i is the binomial coefficient.

Lemma 3.9. *Let $k = \max\{\text{ind}(A), \text{ind}(A+E)\}$. If* $\text{rank}\left((A+E)^k\right) > \text{rank}\left(A^k\right)$, *then*

$$\|(A+E)^{\textcircled{\dagger}}\| \geq \frac{1}{[\varepsilon(A^k)]^{1/k}}.$$

Proof. The proof is analogous to the proof of [11, Theorem 3]. For completeness and convenience, we give a proof.

Since $\mathcal{R}(A^k) \oplus \mathcal{N}(A^k) = \mathbb{C}^n$, $\text{rank}\left((A+E)^k\right) > \text{rank}\left(A^k\right)$, then there exists $x \neq 0$ such that $x \in \mathcal{R}((A+E)^k) \cap \mathcal{N}(A^k)$ by [3, Lemma 1]. Without loss of generality, we can assume $\|x\| = 1$. Then $x = [(A+E)^{\textcircled{\dagger}}]^k (A+E)^k x$ and

$$1 = x^* x = x^*[(A+E)^{\textcircled{\dagger}}]^k (A+E)^k x = x^*[(A+E)^{\textcircled{\dagger}}]^k E(A^k) x \leq \|(A+E)^{\textcircled{\dagger}}\|^k \varepsilon(A^k).$$

Hence $\|(A+E)^{\textcircled{\dagger}}\| \geq \frac{1}{[\varepsilon(A^k)]^{1/k}}$. □

Lemma 3.9 declares that $\|(A+E)^{\textcircled{\dagger}}\| \to \infty$ as $\|E\| \to 0$ provided

$$\text{rank}\left((A+E)^k\right) > \text{rank}\left(A^k\right).$$

Also, from Lemma 3.9 we immediately obtain the following result.

Corollary 3.10. *Let $\{E_j\}$ be a sequence of $n \times n$ matrices such that $\|E_j\| \to 0$, $k_j = \max\{\text{ind}(A), \text{ind}(A+E_j)\}$. If $(A+E_j)^{\textcircled{\dagger}} \to A^{\textcircled{\dagger}}$, then there exists j_0 such that* $\text{rank}\left((A+E_j)^{k_j}\right) = \text{rank}\left(A^{k_j}\right)$ *for all $j \geq j_0$.*

Proof. Proof by contradiction. □

Thus, in this section, to consider perturbation bounds for the core-EP inverse, it sufficies to consider the case: $\text{rank}\left(A^k\right) = \text{rank}\left((A+E)^k\right)$.

Lemma 3.11. [11] *Suppose* $\text{rank}\left((A+E)^h\right) = \text{rank}\left(A^h\right)$ *and $\|(A^h)^\dagger\|\varepsilon(A^h) < 1$. Then*

$$\|[(A+E)^h]^\dagger\| \leq \frac{\|(A^h)^\dagger\|}{1 - \|(A^h)^\dagger\|\varepsilon(A^h)}.$$

Combine (1.2) with Lemma 3.11, then we have the following result.

Theorem 3.12. *Suppose* $\text{ind}(A+E) = k$, $\text{rank}\left((A+E)^k\right) = \text{rank}\left(A^k\right)$ *and $\|(A^{k+1})^\dagger\|\varepsilon(A^{k+1}) < 1$. Then*

$$\|(A+E)^{\textcircled{\dagger}}\| \leq \frac{(\|A^k\| + \varepsilon(A^k))\|(A^{k+1})^\dagger\|}{1 - \|(A^{k+1})^\dagger\|\varepsilon(A^{k+1})}. \tag{3.7}$$

Theorem 3.12 states that $(A+E)^{\textcircled{\dagger}}$ is bounded provided

$$\text{rank}\left((A+E)^{\text{ind}(A+E)}\right) = \text{rank}\left(A^{\text{ind}(A+E)}\right).$$

This is one of the bases for obtaining a perturbation bound for the core-EP inverse. The other one is contained in the asymptotic expansion of $(A+E)^{\textcircled{\dagger}} - A^{\textcircled{\dagger}}$.

Let $k = \max\{\mathrm{ind}(A), \mathrm{ind}(A+E)\}$. Then

$$\begin{aligned}(A+E)^{\textcircled{\dagger}} - A^{\textcircled{\dagger}} &= -(A+E)^{\textcircled{\dagger}}EA^{\textcircled{\dagger}} + (A+E)^{\textcircled{\dagger}} - A^{\textcircled{\dagger}} + (A+E)^{\textcircled{\dagger}}(A+E-A)A^{\textcircled{\dagger}} \\ &= -(A+E)^{\textcircled{\dagger}}EA^{\textcircled{\dagger}} + (A+E)^{\textcircled{\dagger}}(I - AA^{\textcircled{\dagger}}) - [I - (A+E)^{\textcircled{\dagger}}(A+E)]A^{\textcircled{\dagger}} \\ &= -(A+E)^{\textcircled{\dagger}}EA^{\textcircled{\dagger}} + (A+E)^{\textcircled{\dagger}}[(A+E)^{\textcircled{\dagger}}]^{k*}[E(A^k)]^*(I - AA^{\textcircled{\dagger}}) \\ &\quad + [I - (A+E)^{\textcircled{\dagger}}(A+E)]E(A^k)(A^{\textcircled{\dagger}})^{k+1}.\end{aligned} \tag{3.8}$$

Take $\|\cdot\|$ on (3.8), then

$$\begin{aligned}\|(A+E)^{\textcircled{\dagger}} - A^{\textcircled{\dagger}}\| \leq\ & \|(A+E)^{\textcircled{\dagger}}\|\|A^{\textcircled{\dagger}}\|\|E\| + \\ & \|(A+E)^{\textcircled{\dagger}}\|^{k+1}(I + \|A\|\|A^{\textcircled{\dagger}}\|)\varepsilon(A^k) + \\ & [I + \|(A+E)^{\textcircled{\dagger}}\|\|A\| + \|(A+E)^{\textcircled{\dagger}}\|\|E\|]\|A^{\textcircled{\dagger}}\|^{k+1}\varepsilon(A^k).\end{aligned} \tag{3.9}$$

Now suppose $\|(A^{k+1})^{\dagger}\|\varepsilon(A^{k+1}) < 1$ and $\mathrm{rank}\left((A+E)^k\right) = \mathrm{rank}\left(A^k\right)$, then by Theorem 3.12, $(A+E)^{\textcircled{\dagger}}$ is bounded. Thus, from Equality (3.9), $\|(A+E)^{\textcircled{\dagger}} - A^{\textcircled{\dagger}}\| \to 0$ as $\|E\| \to 0$, that is to say,

$$(A+E)^{\textcircled{\dagger}} = A^{\textcircled{\dagger}} + O(\|E\|). \tag{3.10}$$

In order to derive a perturbation bound, we plug (3.10) into the right side of (3.8). Then

$$\begin{aligned}(A+E)^{\textcircled{\dagger}} - A^{\textcircled{\dagger}} &= -A^{\textcircled{\dagger}}EA^{\textcircled{\dagger}} + A^{\textcircled{\dagger}}[(A^{\textcircled{\dagger}})^k]^*(\sum_{i=0}^{k-1} A^iEA^{k-1-i})^*(I - AA^{\textcircled{\dagger}}) \\ &\quad + (I - A^{\textcircled{\dagger}}A)\sum_{i=0}^{k-1} A^iEA^{k-1-i}(A^{\textcircled{\dagger}})^{k+1} + O(\|E\|^2) \\ &= -A^{\textcircled{\dagger}}EA^{\textcircled{\dagger}} + A^{\textcircled{\dagger}}[\sum_{i=0}^{k-1} A^iE(A^{\textcircled{\dagger}})^{i+1}]^*(I - AA^{\textcircled{\dagger}}) \\ &\quad + (I - A^{\textcircled{\dagger}}A)\sum_{i=0}^{k-1} A^iE(A^{\textcircled{\dagger}})^{i+2} + O(\|E\|^2).\end{aligned} \tag{3.11}$$

Take $\|\cdot\|$ on (3.11), then we obtain the following result.

Theorem 3.13. *Let $k = \max\{\mathrm{ind}(A), \mathrm{ind}(A+E)\}$, $\mathrm{rank}\left((A+E)^k\right) = \mathrm{rank}\left(A^k\right)$ and $\|(A^{k+1})^{\dagger}\|\varepsilon(A^{k+1}) < 1$. Then*

$$\frac{\|(A+E)^{\textcircled{\dagger}} - A^{\textcircled{\dagger}}\|}{\|A^{\textcircled{\dagger}}\|} \leq C(A)\frac{\|E\|}{\|A\|} + o(\|E\|^2), \tag{3.12}$$

where $C(A) = [2\sum_{i=0}^{k-1} \|A\|^i\|A^{\textcircled{\dagger}}\|^{i+1}(1 + \|A\|\|A^{\textcircled{\dagger}}\|) + \|A^{\textcircled{\dagger}}\|]\|A\|$.

Corollary 3.14. *Let $\{E_j\}$ be a sequence of $n \times n$ matrices such that $\|E_j\| \to 0$ and let $k_j = \max\{\mathrm{ind}(A), \mathrm{ind}(A+E_j)\}$. If there exists j_0 such that $\mathrm{rank}\left((A+E_j)^{k_j}\right) = \mathrm{rank}\left(A^{k_j}\right)$ for all $j \geq j_0$, then $(A+E_j)^{\oplus} \to A^{\oplus}$.*

"Let $k = \max\{\mathrm{ind}(A), \mathrm{ind}(A+E)\}$, $\mathrm{rank}\left((A+E)^k\right) = \mathrm{rank}\left(A^k\right)$" is the same meaning as "$\mathrm{rank}\left((A+E)^{\mathrm{ind}(A+E)}\right) = \mathrm{rank}\left(A^{\mathrm{ind}(A)}\right)$". Thus, Corollary 3.14 in conjunction with Corollary 3.10 gives another proof for the equivalence of (1) and (3) in Theorem 2.6, which means that $\mathrm{rank}\left((A+E)^k\right) = \mathrm{rank}\left(A^k\right)$ is the weakest condition for the continuity of the core-EP inverse.

Remark 3.15. *If $k = 1$, then the condition of Theorem 3.13 becomes $\mathrm{rank}(A+E) = \mathrm{rank}(A)$ and $\|(A^2)^{\dagger}\|\varepsilon(A^2) < 1$. Thus, under these assumptions, we derive a perturbation bound for the core inverse.*

4 Numerical examples

In this section, we shall establish a numerical example to compare upper bounds for $\frac{\|(A+E)^{\oplus} - A^{\oplus}\|}{\|A^{\oplus}\|}$ derived in (3.5) and (3.12). Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ E = \begin{bmatrix} \varepsilon & \varepsilon & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then $\mathrm{ind}(A) = \mathrm{ind}(A+E) = 1$, $\mathrm{rank}(A)$ $=\mathrm{rank}(A+E) = 1$ and $AA^{\oplus} = (A+E)(A+E)^{\oplus}$, $A^{\oplus}A = (A+E)^{\oplus}(A+E)$. Thus A and E satisfy the conditions in Theorems 3.5 and 3.13. Table 1 shows that our bound (3.5) is slightly better than (3.12).

Table 1: Comparison of upper bounds of $\|(A+E)^{\oplus} - A^{\oplus}\|/\|A^{\oplus}\|$

	$\varepsilon = 0.1000$	$\varepsilon = 0.0100$	$\varepsilon = 0.0010$	$\varepsilon = 0.0001$
Exact	0.0909	0.0099	0.0010	1.0000e-04
(3.5)	0.1647	0.0143	0.0014	1.4143e-04
(3.12)	0.7070+$o(\|E\|^2)$	0.0705+$o(\|E\|^2)$	0.0070+$o(\|E\|^2)$	7.0710e-04+$o(\|E\|^2)$

5 Applications to semistable matrices

Following [19], a matrix $A \in \mathbb{C}^{n\times n}$ is called semistable if $\mathrm{ind}(A) \leq 1$ and the nonzero eigenvalues λ of A satisfy $\mathrm{Re}\ \lambda < 0$; a semistable matrix with $\mathrm{ind}(A) = 0$ is stable. It is known that we have an integral representation for the inverse of A if A is stable (for example, see [19]):

$$A^{-1} = -\int_0^{\infty} \exp(tA)\mathrm{d}t. \tag{5.1}$$

In this section, an integral representation for the core-EP inverse of a perturbed matrix $A+E$ is discussed under the condition $E = EAA^{\textcircled{\dagger}} = A^{\textcircled{\dagger}}AE$, where A is a semistable matrix.

Theorem 5.1. *Let $A \in \mathbb{C}^{n\times n}$ be semistable and let $E \in \mathbb{C}^{n\times n}$ such that $E = EAA^{\textcircled{\dagger}} = A^{\textcircled{\dagger}}AE$. Then there exists $\delta(A) > 0$ such that for $\|E\| < \delta(A)$,*

$$(A+E)^{\textcircled{\dagger}} = -\int_0^{\infty} \exp(t(A+E))AA^{\textcircled{\dagger}}\mathrm{d}t. \tag{5.2}$$

Proof. For $A \in \mathbb{C}^{n\times n}$, we have

$$A = U\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}U^*$$

as in (2.5), where U is unitary, T is nonsingular and N is nilpotent.

From the assumption $E = EAA^{\textcircled{\dagger}} = A^{\textcircled{\dagger}}AE$, it follows that $E = U\begin{bmatrix} E_1 & 0 \\ 0 & 0 \end{bmatrix}U^*$ according to the proof of Theorem 3.1. Then $\|E_1\| = \|U^*EU\| \leq \|U^{-1}\|\|E\|\|U\| = \kappa(U)\|E\|$. Observe that

$$\begin{aligned} \exp(t(A+E))AA^{\textcircled{\dagger}} &= U\begin{bmatrix} \exp(t(T+E_1)) & \Delta \\ 0 & \exp(tN) \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}U^* \\ &= U\begin{bmatrix} \exp(t(T+E_1)) & 0 \\ 0 & 0 \end{bmatrix}U^*. \end{aligned}$$

Since A is semistable, T is stable. By the continuity of eigenvalues ([20, Section IV.1.1], [19]), there exists $\eta > 0$ such that $\|E_1\| < \eta$ implies that $T+E_1$ is also stable. Set $\delta(A) = \frac{\eta}{\kappa(U)}$, if $\|E\| < \delta(A)$, then $\|E_1\| < \eta$, thus $T+E_1$ is stable. Therefore $T+E_1$ is integrable on the interval $[0,\infty)$. In view of (5.1),

$$\begin{aligned} -\int_0^{\infty} \exp(t(A+E))AA^{\textcircled{\dagger}}\mathrm{d}t &= -U\begin{bmatrix} \int_0^{\infty} \exp(t(T+E_1))\mathrm{d}t & 0 \\ 0 & 0 \end{bmatrix}U^* \\ &= U\begin{bmatrix} (T+E_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^* = (A+E)^{\textcircled{\dagger}}. \end{aligned}$$

It completes the proof. $\square$

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