Filomat xx (yyyy), zzz–zzz
DOI (will be added later)

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A note on clean elements and inverses along an element

Huihui Zhu[a,1], Pedro Patrício[b,1]

[a]*School of Mathematics, Hefei University of Technology, Hefei 230009, China.*
[b]*CMAT-Centro de Matemática, Universidade do Minho, Braga 4710-057, Portugal.*
[c]*Departamento de Matemática e Aplicações, Universidade do Minho, Braga 4710-057, Portugal.*

Abstract. Let R be an associative ring with unity 1 and let $a, d \in R$. An element $a \in R$ is called invertible along d if there exists unique $a^{\parallel d}$ such that $a^{\parallel d}ad = d = daa^{\parallel d}$ and $a^{\parallel d} \in dR \cap Rd$ (see [7, Definition 4]). In this note, we present new characterizations for the existence of $a^{\parallel d}$ by a clean decomposition of ad and da. As applications, existence criteria for the Drazin inverse and the group inverse are given. Also, we show that ad and da are uniquely strongly clean, provided that $a^{\parallel d}$ exists and $ad = da$.

1. Introduction

Throughout this paper, all rings R considered are assumed to be an associative ring with unity 1. We say that $a \in R$ is regular if there exists $x \in R$ such that $a = axa$. Such x is called an inner inverse of a, and is denoted by a^-. An element $a \in R$ is said to be group invertible if there exists $b \in R$ such that $ab = ba$, $aba = a$ and $bab = b$. Such a b is called a group inverse of a, it is unique if it exists, and is denoted by $a^\#$. It is known that $a^\#$ exists if and only if there exist $x, y \in R$ such that $a = a^2x = ya^2$. In this case, $a^\# = yax = y^2a = ax^2$.

Let R be a ring with involution $*$, i.e., $*$ satisfying $(a^*)^* = a$, $(ab)^* = b^*a^*$ and $(a+b)^* = a^* + b^*$ for all $a, b \in R$. An element $a \in R$ (with involution) is Moore-Penrose invertible [14] if there exists $x \in R$ such that $axa = a$, $xax = x$, $(ax)^* = ax$ and $(xa)^* = xa$. Such x is called a Moore-Penrose inverse of a, it is unique if it exists, and is denoted by $a^\dagger$. The standard notions of Drazin inverses can found in mathematical literature [5]. The symbols R^{-1}, $R^\#$, R^D and $R^\dagger$ denote the sets of all invertible, group invertible, Drazin invertible and Moore-Penrose invertible elements in R, respectively.

Green's preorders (see [6]) in a ring R are defined by: (i) $a \leq_{\mathcal{L}} b$ denotes $a \in Rb$; (ii) $a \leq_{\mathcal{R}} b$ denotes $a \in bR$; (iii) $a \leq_{\mathcal{H}} b$ denotes $a \in bR \cap Rb$. Given $a, d \in R$, an element $b \in R$ is called the inverse along d [7] if $bad = d = dab$ and $b \leq_{\mathcal{H}} d$. Such b is unique if it exists, it is called the inverse of a along d, and is denoted by $a^{\parallel d}$. Furthermore, Mary [7] showed that (i) $a \in R^\#$ if and only if it is invertible along a; (ii) $a \in R^D$ if and only if it is invertible along a^m, for some positive integer m; (iii) $a \in R^\dagger$ if and only if it is invertible along a^*. In these cases, $a^\# = a^{\parallel a}$, $a^D = a^{\parallel a^m}$ and $a^\dagger = a^{\parallel a^*}$. More results on the inverse along an element can be referred to [2, 8]. It follows from [7] that if $a^{\parallel d}$ exists then $1 - aa^{\parallel d}$ and $1 - a^{\parallel d}a$ are idempotents. Also, by [8], we know

2010 *Mathematics Subject Classification*. Primary 15A09 (mandatory); Secondary 15A27, 16H99 (optionally)
Keywords. (inverses along an element, one-sided inverses along an element, clean elements, strongly clean decompositions, special clean decompositions, uniquely strongly clean elements)
Received: 01 12 2016; Accepted: dd Month yyyy
Communicated by (Dijana Mosić, mandatory)
This research is supported by the Fundamental Research Funds for the Central Universities, the Natural Science Foundation of Anhui Province (No. 1808085QA16) and the Portuguese Funds through FCT- 'Fundação para a Ciência e a Tecnologia', within the project UID-MAT-00013/2013.
Email addresses: hhhzu@hfut.edu.cn (Huihui Zhu), pedro@math.uminho.pt (Pedro Patrício)

that $a^{\|d}$ exists implies $da + 1 - dd^{-} \in R^{-1}$ and $ad + 1 - d^{-}d \in R^{-1}$, where the regularity of d is ensured by the existence of $a^{\|d}$ ([7, Theorem 7]).

Recall that an element of a ring R is called clean if it can be written as the sum of an idempotent e and a unit u. A clean ring is one whose each element is clean, which dates back to the paper of Nicholson [9]. Given a clean decomposition $a = e + u$, it is called special clean [1] if $aR \cap eR = 0$, and is strongly clean [10] if it is a clean decomposition and $eu = ue$. Later, Chen et al. [3] called $a \in R$ uniquely strongly clean [3] if it has a uniquely strongly clean decomposition. Several scholars [4, 11, 15] paid attention to the cleanness of elements in rings. Since clean elements can be written as the sum of an idempotent and a unit, they has close relations with the inverse along an element (idempotents and units can be constructed by this types of generalized inverses). However, few articles are presented about the connections between the cleanness of elements and their generalized inverses.

It should be noted that the classical invertibility constructed by generalized inverses is usually the form $ek + 1 - e$, where e is an idempotent. In particular, for the case of the inverse of a along d, taking $e = dd^{-}$ and $k = da$. For the group inverse of regular elements d, setting $e = dd^{-}$ and $k = d$. However, a key issue for investigating the clean decomposition of the inverse along an element is that we need the unit of the form $ek - 1 + e$. So, at the beginning of Section 2, we illustrate that $ek - 1 + e$ is a unit if and only if $ek + 1 - e$ is a unit.

In this note, we give an existence criterion for the inverse along an element, which slightly differs from the result of Mary and Patrício [12]. We then characterize the inverse of the product of triple elements along an element by using one-sided inverse along an element. Moreover, the formulae relating them are given. New characterizations for the inverse of a along d are obtained by clean decompositions of ad and da. As special cases, existence criteria for the group inverse and the Drazin inverse are given. Finally, we show that both da and ad are also uniquely strongly clean, provided that $a^{\|d}$ exists and $ad = da$.

2. The cleanness of elements and the inverse along an element

We first begin with the following lemmas, which play an important role in the sequel.

Lemma 2.1. [8, Theorem 2.1] *Let $a, d \in R$. Then the following conditions are equivalent*:
(i) *$a^{\|d}$ exists.*
(ii) *$d \leq_{\mathcal{R}} da$ and $(da) \in R^{\#}$.*
(iii) *$d \leq_{\mathcal{L}} ad$ and $(ad) \in R^{\#}$.*
In this case, $a^{\|d} = d(ad)^{\#} = (da)^{\#}d$.

Lemma 2.2. (Jacobson's Lemma) *Let $a, b \in R$. Then*
(i) *If $(1 - ab) \in R^{-1}$, then $(1 - ba) \in R^{-1}$ and $(1 - ba)^{-1} = 1 + b(1 - ab)^{-1}a$.*
(ii) *If $(ab - 1) \in R^{-1}$, then $(ba - 1) \in R^{-1}$ and $(ba - 1)^{-1} = b(ab - 1)^{-1}a - 1$.*

Mary and Patrício [8] presented the existence criterion for the inverse along an element by units in a ring, i.e., the equivalences (i) ⇔ (iv) ⇔ (v) in Proposition 2.3 below. We next show that $da + (1 - dd^{-}) \in R^{-1}$ is equivalent to $da - (1 - dd^{-}) \in R^{-1}$.

Let $e, k \in R$ with e idempotent. Assume that $ek + 1 - e$ is a unit in R. Then, by Lemma 2.2, $u = eke + 1 - e$ is also a unit in R. Consequently, $eke = eue$ is a unit in eRe, which gives that $e(-k)e$ is a unit, and then $e(-k)e + 1 - e$ is a unit in R. Therefore, $-eke + 1 - e$ is a unit, which implies that $eke - 1 + e$ is a unit and hence $ek - 1 + e$ is a unit by Lemma 2.2. Dually, if $ek - 1 + e$ is a unit then so is $ek + 1 - e$. More details on corner rings can be referred to [13].

We remind the reader that there must be a connection with k and e in the first summand x, that is, e must be somehow in x. If it contains no e, then $x + (1 - e) \in R^{-1}$ does not imply $x - (1 - e) \in R^{-1}$ in general. Such as, let $R = \mathbb{R}_{2\times 2}$ be the ring of all 2 by 2 real matrices. Suppose $x = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $e = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then $x + (1 - e) = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \in R^{-1}$. But $x - (1 - e) = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \notin R^{-1}$.

Let $d \in R$ be regular. If $e = dd^-$ and $k = da$, then $ek + 1 - e = da + 1 - dd^- \in R^{-1}$ if and only if $ek - 1 + e = da - 1 + dd^- \in R^{-1}$. Dually, setting $f = d^-d$ and $h = ad$, then $hf + 1 - f = ad + 1 - d^-d \in R^{-1}$ if and only if $hf - 1 + f = ad - 1 + d^-d \in R^{-1}$. We hence add two characterizations for the inverse along an element.

Proposition 2.3. *Let $a, d \in R$ with d regular. Then the following conditions are equivalent*:
(i) $a^{\|d}$ *exists.*
(ii) $u = da - 1 + dd^- \in R^{-1}$.
(iii) $v = ad - 1 + d^-d \in R^{-1}$.
(iv) $u' = da + 1 - dd^- \in R^{-1}$.
(v) $v' = ad + 1 - d^-d \in R^{-1}$.
In this case, $a^{\|d} = u^{-1}d = dv^{-1} = (u')^{-1}d = d(v')^{-1}$.

Proof. By a direct check, we know that $u^{-1}a = av^{-1}$ is the inverse of a along d. $\square$

It follows from [18, p.168] that $a \in R^{\#}$ if and only if 1 is invertible along a. Hence, it follows an existence criterion for the group inverse of a regular element.

Corollary 2.4. *Let $a \in R$ be regular. Then the following conditions are equivalent*:
(i) $a \in R^{\#}$.
(ii) $u = a - 1 + aa^- \in R^{-1}$.
(iii) $v = a - 1 + a^-a \in R^{-1}$.
(iv) $u' = a + 1 - aa^- \in R^{-1}$.
(v) $v' = a + 1 - a^-a \in R^{-1}$.
In this case, $a^{\#} = u^{-1}a = av^{-1} = (u')^{-1}a = a(v')^{-1}$.

Let us recall that [18] $a \in R^{\dagger}$ if and only if a^* is invertible along a. Hence, we get the following result concerning the existence criterion of the Moore-Penrose inverse.

Corollary 2.5. *Let R be a ring with involution and let $a \in R$ be regular. Then the following conditions are equivalent*:
(i) $a \in R^{\dagger}$.
(ii) $u = aa^* - 1 + aa^- \in R^{-1}$.
(iii) $v = a^*a - 1 + a^-a \in R^{-1}$.
(iv) $u' = aa^* + 1 - aa^- \in R^{-1}$.
(v) $v' = a^*a + 1 - a^-a \in R^{-1}$.
In this case, $a^{\dagger} = (u^{-1}a)^* = (av^{-1})^* = ((u')^{-1}a)^* = (a(v')^{-1})^*$.

Let R be a ring with involution and let $a, x \in R$. If x satisfies $axa = a$ and $(ax)^* = ax$, then x is a {1,3}-inverse of a, and is denoted by $a^{(1,3)}$. If x satisfies $axa = a$ and $(xa)^* = xa$, then x is a {1,4}-inverse of a, and is denoted by $a^{(1,4)}$. It is well known that a is Moore-Penrose invertible if and only if it is both {1,3}-invertible and {1,4}-invertible. Moreover, $a^{\dagger} = a^{(1,4)}aa^{(1,3)}$.

Suppose we are given any p, a, q in involutory rings R, the present author Patrício [12] illustrated that paq is Moore-Penrose invertible if and only if pa is {1,3}-invertible and aq is {1,4}-invertible, provided that $p'pa = a = aqq'$ for some $p', q' \in R$. In this case, $(paq)^{\dagger} = (aq)^{(1,4)}a(pa)^{(1,3)}$.

Inspired by the above mentioned author's work, we consider the relations between the inverse of paq along d and left inverses of aq and right inverse of pa along certain element.

Let us now recall some definitions and properties of one-sided inverse along an element.

Lemma 2.6. [17] *Let $a, d \in R$.*
(i) *An element $b \in R$ is called a left inverse of a along d if $bad = d$ and $b \leq_{\mathcal{L}} d$. Moreover, a is left invertible along d if and only if $d \leq_{\mathcal{L}} dad$.*
(ii) *An element $b \in R$ is called a right inverse of a along d if $dab = d$ and $b \leq_{\mathcal{R}} d$. Moreover, a is right invertible along d if and only if $d \leq_{\mathcal{R}} dad$.*

Our notation follows [17, 18]. For instance, the symbol $a_l^{\|d}$ (resp., $a_r^{\|d}$) denotes a left (resp., right) inverse of a along d.

It follows from [17, Corollary 2.5] that a is invertible along d if and only if it is both left and right invertible along d. In particular, by Lemma 2.6, a is invertible along d if and only if $d \leq_{\mathcal{H}} dad$. However, the present authors [17] did not present the formula between the inverse along an element and one-sided inverse along an element.

Theorem 2.7. *Let $p, a, q, d \in R$. If there exist $p', q' \in R$ such that $dpp' = d = q'qd$, then the following conditions are equivalent*:

(i) *paq is invertible along d.*

(ii) *aq is left invertible along dp and pa is right invertible along qd.*

In this case, $(paq)^{\|d} = (aq)_l^{\|dp} a(pa)_r^{\|qd}$.

Proof. (i) ⇒ (ii) Since paq is invertible along d, by Lemma 2.6, we have $d \leq_{\mathcal{L}} dpaqd$ and $d \leq_{\mathcal{R}} dpaqd$, and consequently $dp \leq_{\mathcal{L}} dpaqdp$ and $qd \leq_{\mathcal{R}} qdpaqd$. Again, from Lemma 2.6, it follows that aq is left invertible along dp and pa is right invertible along qd.

(ii) ⇒ (i) As aq is left invertible along dp, then there is $x \in R$ such that $(aq)_l^{\|dp} = xdp$. Hence, $(aq)_l^{\|dp} a(pa)_r^{\|qd} = xdpa(pa)_r^{\|qd} = xq'qdpa(pa)_r^{\|qd} = xq'qd = xd$, which implies $(aq)_l^{\|dp} a(pa)_r^{\|qd} paqd = xdpaqd = (aq)_l^{\|dp} aqd = (aq)_l^{\|dp} aqdpp' = dpp' = d$.

Since pa is right invertible along qd, we have $(pa)_r^{\|qd} = qdy$ for some $y \in R$. A straightforward calculation gives $(aq)_l^{\|dp} a(pa)_r^{\|qd} = dy$, we hence get $dpaq(aq)_l^{\|dp} a(pa)_r^{\|qd} = dpaqdy = dpa(pa)_r^{\|qd} = d$.

Finally, $(aq)_l^{\|dp} a(pa)_r^{\|qd} = xd = dy \leq_{\mathcal{H}} d$.

Therefore, paq is invertible along d and $(paq)^{\|d} = (aq)_l^{\|dp} a(pa)_r^{\|qd}$. □

Taking $p = q = 1$ in Theorem 2.7, it follows that

Corollary 2.8. [19, Proposition 2.3] *Let $a, d \in R$. Then the following conditions are equivalent*:

(i) *a is invertible along d.*

(ii) *a is both left and right invertible along d.*

In this case, $a^{\|d} = a_l^{\|d} = a_r^{\|d} = a_l^{\|d} a a_r^{\|d}$.

Next, we give another existence criterion of the inverse along an element by the cleanness of elements.

Theorem 2.9. *Let $a, d \in R$ with $ad = da$. Then the following conditions are equivalent*:

(i) *$a^{\|d}$ exists.*

(ii) *$d \leq_{\mathcal{R}} da$, there exist $e^2 = e \in R$ and $u \in R^{-1}$ such that $da = e + u$ is both a strongly clean decomposition and a special clean decomposition.*

(iii) *$d \leq_{\mathcal{L}} ad$, there exist $f^2 = f \in R$ and $v \in R^{-1}$ such that $ad = f + v$ is both a strongly clean decomposition and a special clean decomposition.*

In this case, $a^{\|d} = u^{-2}dad = dadv^{-2}$.

Proof. (i) ⇒ (ii) Suppose that $a^{\|d}$ exists. Then $d \leq_{\mathcal{R}} da$ by Lemma 2.1. As $a^{\|d}$ exists and $ad = da$, then $d \in R^{\#}$ (see [18, p.170]). Set $e = 1 - dd^{\#}$ and $u = da - 1 + dd^{\#}$. Then $e^2 = e$, and $u \in R^{-1}$ from Proposition 2.3. We compute $eu = -e = ue$ and hence $da = e + u$ is a strongly clean decomposition. Let $b \in daR \cap eR$. Then there are $x, y \in R$ such that $b = dax = ey = edax = (1 - dd^{\#})dax = 0$. So, $da = e + u$ is a special clean decomposition.

(ii) ⇒ (i) Since $da = e + u$ is both a strongly clean decomposition and a special clean decomposition, it follows $dae = eda \in daR \cap eR = 0$. Multiplying by da on both left and right sides gives $(da)^2 = uda = dau$, hence $da = u^{-1}(da)^2 = (da)^2 u^{-1}$. So, $da \in R^{\#}$ and $(da)^{\#} = u^{-2}da$, which together with Lemma 2.1 ensures that $a^{\|d}$ exists since $d \leq_{\mathcal{R}} da$. Moreover, $a^{\|d} = (da)^{\#}d = u^{-2}dad$.

(i) ⇔ (iii) is similar to the proof of (i) ⇔ (ii). □

It is known that $a \in R^D$ if and only if a is invertible along a^m for some positive integer m. Hence, in the characterization of Drazin inverses, the condition $d \leq_{\mathcal{R}} da$ of Theorem 2.9 can be reduced to $a^{n-1} \leq_{\mathcal{R}} a^n$ for some positive integer n. We claim that the condition $a^{n-1} \leq_{\mathcal{R}} a^n$ can be dropped. Indeed, if $a^n = e + u$ is both a strongly clean decomposition and a special clean decomposition for some positive integer n, then $ea^n = a^n e \in eR \cap a^n R = 0$. Hence, $a^n = u^{-1}a^{2n} = a^{2n}u^{-1}$, which implies $a^n \in a^{n+1}R \cap Ra^{n+1}$.

The following result characterizes the existence criterion of the Drazin inverse by clean decompositions of certain element.

Corollary 2.10. *Let $a \in R$. Then the following conditions are equivalent*:

(i) $a \in R^D$.

(ii) *There exist $e^2 = e \in R$ and $u \in R^{-1}$ such that $a^n = e + u$ is both a strongly clean decomposition and a special clean decomposition, for some positive integer n.*

In this case, $a^D = u^{-2}a^{2n-1} = a^{2n-1}u^{-2}$.

By substituting "a projection ($p^2 = p = p^*$)" for "an idempotent" in the appropriate concepts, it follows notions of the strongly $*$-clean decomposition and special $*$-clean decomposition in $*$-ring (see [16]).

It follows from (see e.g. [20, Lemma 2.2]) that $a^{(1,3)}$ exists if and only if $a \leq_{\mathcal{L}} a^*a$, and $a^{(1,4)}$ exists if and only if $a \leq_{\mathcal{R}} aa^*$. We next give characterizations for the Moore-Penrose inverse by $*$-clean properties.

Corollary 2.11. *Let R be a ring with involution and let $a \in R$ with $aa^* = a^*a$. Then the following conditions are equivalent*:

(i) $a \in R^\dagger$.

(ii) *$a^{(1,3)}$ exists, there exist $p^2 = p = p^* \in R$ and $u \in R^{-1}$ such that $a^*a = p + u$ is both a strongly $*$-clean decomposition and a special $*$-clean decomposition.*

(iii) *$a^{(1,4)}$ exists, there exist $q^2 = q = q^* \in R$ and $v \in R^{-1}$ such that $aa^* = q + v$ is both a strongly $*$-clean decomposition and a special $*$-clean decomposition.*

In this case, $a^\dagger = u^{-2}a^*aa^* = a^*aa^*v^{-2}$.

It follows from Theorem 2.9 that da and ad have strongly clean decompositions, under certain assumptions. It is of interest to consider whether such strongly clean decomposition is unique? Finally, we illustrate this fact.

Theorem 2.12. *Let $a, d \in R$ with $ad = da$. If $a^{\parallel d}$ exists, then*

(i) *da is uniquely strongly clean.*

(ii) *ad is uniquely strongly clean.*

Proof. (i) It follows from Theorem 2.9 that da has a strongly clean decomposition $da = e + u$, where $e^2 = e$, $u \in R^{-1}$ and $dae = eda = 0$.

Suppose that $da = f + v$ is another strongly clean decomposition, where $f^2 = f, v \in R^{-1}$ and $fda = daf = 0$. To prove da is uniquely strongly clean, it is sufficient to prove $e = f$.

As $eda = 0$, then $da = (1-e)da = (1-e)(e+u) = (1-e)u$, and consequently $1 - e = dau^{-1}$. Multiplying the equality above by f on the left concludes $f(1-e) = fdau^{-1} = 0$. Hence, $f = fe$.

Also, from $daf = 0$, it follows $da = v(1-f)$ and $1 - f = v^{-1}da$. Multiplying by e on the right yields $(1-f)e = v^{-1}dae = 0$. Thus, $e = fe$.

Therefore, $e = f$ and da is uniquely strongly clean.

(ii) can be proved by a similar way of (i). □

As special results of Theorem 2.12, we have

Corollary 2.13. *Let $a \in R^D$. Then a^n is uniquely strongly clean, for some positive integer n.*

Corollary 2.14. *Let $a \in R^\#$. Then a is uniquely strongly clean.*

References

[1] E. Akalan, L. Vaš, Classes of almost clean rings, Algebr. Represent. Theory 16 (2013) 843-857.
[2] J. Benitez, E. Boasso, The inverse along an element in rings with an involution, Banach algebras and C^*-algebras, Linear Multilinear Algebra 65 (2017) 284-299.
[3] J.L. Chen, Z. Wang, Y.Q. Zhou, Rings in which elements are uniquely the sum of an idempotent and a unit that commute, J. Pure Appl. Algebra 213 (2009) 215-223.
[4] A.J. Diesl, Nil clean rings, J. Algebra 383 (2013) 197-211.
[5] M.P. Drazin, Pseudo-inverses in associative rings and semigroups, Amer. Math. Monthly 65 (1958) 506-514.
[6] J.A. Green, On the structure of semigroups, Ann. Math. 54 (1951) 163-172.
[7] X. Mary, On generalized inverses and Green's relations, Linear Algebra Appl. 434(2011) 1836-1844.
[8] X. Mary, P. Patrício, Generalized inverses modulo $\mathcal{H}$ in semigroups and rings, Linear Multilinear Algebra 61 (2013) 1130-1135.
[9] W.K. Nicholson, Lifting idempotents and exchange rings, Trans. Amer. Math. Soc. 229 (1977) 269-278.
[10] W.K. Nicholson, Strongly clean rings and fittings' lemma, Comm. Algebra 27 (1999) 3583-3592.
[11] W.K. Nicholson, Y.Q. Zhou, Clean general rings, J. Algebra 291 (2005) 297-311.
[12] P. Patrício, The Moore-Penrose inverse of a factorization, Linear Algebra Appl. 370 (2003) 227-235.
[13] P. Patrício, R. Puystjens, Generalized invertibility in two semigroups of a ring, Linear Algebra Appl. 377 (2004) 125-139.
[14] R. Penrose, A generalized inverse for matrices, Proc. Camb. Phil. Soc. 51 (1955) 406-413.
[15] J. Šter, Lifting units in clean rings, J. Algebra 381 (2013) 200-208.
[16] L. Vaš, $*$-clean rings; cleanness of some Baer $*$-rings and von Neumann algebras, J. Algebra 324 (2010) 3388-3400.
[17] H.H. Zhu, J.L. Chen, P. Patrício, Further resulst on the inverse along an element in semigroups and rings, Linear Multilinear Algebra 64 (2016) 393-403.
[18] H.H. Zhu, J.L. Chen, P. Patrício, Reverse order law for the inverse along an element, Linear Multilinear Algebra 65 (2017) 166-177.
[19] H.H. Zhu, J.L. Chen, P. Patrício, X. Mary, Centralizer's applications to the inverse along an element, Appl. Math. Comput. 315 (2017) 27-33.
[20] H.H. Zhu, X.X. Zhang, J.L. Chen, Generalized inverses of a factorization in a ring with involution, Linear Algebra Appl. 472 (2015) 142-150.