A decision model based on expected utility, entropy and variance

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Abstract

A decision model - the expected utility, entropy and variance (EU-EV) model - is proposed, where the decision criterion depends on the uncertainty risk factors entropy and variance and on the subjective expected utility. The EU-EV model is applied to decision problems existing in the literature, and it is shown that they can be well explained with this model.

Keywords: decision analysis, expected utility, entropy, variance, decision model, risk

1. Introduction

Measuring risk and taking decisions under risk is important for many problems in various contexts of science, such as in economics, finance or medicine. Several decision models and risk measures have been proposed in the literature. Some of them privilege decision analysis based on the expected utility model developed by von Neumann and Morgenstern [1] describing how decision makers choose between uncertain prospects. According to that model, there exists a utility function $u(\cdot)$ to appraise different risky outcomes and a decision maker chooses the outcome which maximizes expected utility E[u(·)]. However the empirical validity of expected utility alone was questioned, due to the non-conformity with experimental studies (see e.g.[2],[3]). Therefore, several new models generalizing the expected utility model were proposed. Dyer and Jia [4] presented relative risk-value models, incorporating a relative risk variable into the expected utility model, and generalized those models extending them for non-expected utility preferences, see also [3] and references therein for non-expected utility models of preferences. Measures of perceived risk have been developed satisfying empirically verified properties [5], [6] (e.g. the property that perceived risk increases if there is an increase in range, variance or expected loss). The mean-variance model presented in [6] or the Markowitz mean-variance model for Modern Portfolio Theory [7],[8] is one of those models. The main idea of the mean-variance model was to measure return as expected value and risk as variance. Jia et al. [9] constructed two-attribute models for perceived risk combining the effect of the expected value and a standard measure of risk (for example the expected utility or a combination of variance and skewness). The structure of those models has the intuitive interpretation, that people judge risk considering the effect of the expected value of a risky outcome and also its variation or uncertainty. See [9] for a review of perceived risk measures. Ortobelli [10] emphasized that risk cannot be assessed by measuring only the uncertainty. Popular uncertainty measures, such as the standard deviation or variance, are not always adequate as a proxy for risk [11]. Even so there are cases where the variance can serve as an index for risk as discussed in [12].

Another kind of decision models and risk measures are based on entropy, which has the advantage that it can be computed from nonmetric data and is free from an assumption concerning the underlying distribution. In finance and economic literature one can find entropy models and measures to model an uncertain environment and to obtain optimal economic decisions, e.g to model portfolio (investment) risk [13], [14]. Nawrocki and Harding [15] proposed state-value weighted entropy as a measure of investment risk, because in their opinion, entropy was not a good measure of security risk since the dispersion of security frequency classes was not taken into account. Many other authors constructed models using entropy as risk measure for portfolio selection and investment decisions, e.g the expectation-variance-indeterminacy models [16], the mean-entropy models [17], fuzzy cross-entropy models [18] or the mean-entropy-skewness models [19] and recently a logarithmic expectation entropy model [20].

Models, combining both expected utility and entropy (EU-E models) were proposed by Yang and Qiu [21], [22], who showed that these models can be used to explain some famous decision paradoxes, e.g. the Allais paradox. Luce et al. [23], [24] provided entropy-modified expected utility models, that are similar to the EU-E models. Park et al. [25] suggested Bayesian decision models based on expected utility and the prior distribution of the state variables. In their study, these models, called expected utility and uncertainty risk (EU-UR) models, were compaired with the EU-E models and they were also shown to be compatible with the interpretation of the decision paradoxes.

In this paper a new decision model is proposed, which is based on expected utility, entropy and variance. This model, the expected utility, entropy and variance (EU-EV) model, is an extension of the EU-E model, since it includes the variance as an additional risk factor. The rest of the paper is organized as follows. In section 2, the EU-EV decision model and the associated measure of risk are defined. In Section 3, three different problems, considered in the literature, are studied using the EU-EV model. The results are compared with the results obtained from the EU-E model (problems in Sub-sections 3.1, 3.2 and 3.3) and with the EU-UR model (problems in Sub-sections 3.2 and 3.3). It is shown that the EU-EV model can be used to explain decision problems and that it can serve as a reasonable decision model to analyze uncertain actions and preferences. Section 4 contains the conclusions about this work.

2. Expected utility, entropy and variance model

Consider the decision model under risk $G(\Theta, A, X)$, where $\Theta = \{\theta\}$ is the state space, $A = \{a\}$ the action space and $X = X(a, \theta)$ the payoff function. Let $u(X) = u(X(a, \theta))$ be the utility function of the decision maker. Then, the decision model under risk can be written as $G(\Theta, A, u)$. For a finite action space, let $A = \{a_1, a_2, \ldots, a_m\}$ and suppose that the state θ_i corresponding to action a_i, $i = 1, \ldots, m$, has n_i outcomes: $\theta_i = \{\theta_{i1}, \theta_{i2}, \ldots, \theta_{in_i}\}$, the state space being $\Theta = \{\theta_1, \ldots, \theta_m\}$. Let p_{ij}, $i = 1, \ldots, m$, $j = 1, \ldots, n_i$, denote the distribution law of θ_i, where $\sum_{j=1}^{n_i} p_{ij} = 1$ and $p_{ij} \geq 0$. Then $p_{ij} = P(\theta = \theta_{ij} | a = a_i)$ denotes the probability that state θ_{ij} occurs when taking action a_i. The corresponding payoff, when taking action a_i while state θ_{ij} occurs, is given by $X = X(a_i, \theta_{ij}) = x_{ij}$, $i = 1, \ldots, m$, $j = 1, \ldots, n_i$.

In decision analysis the decision maker's choice of an action is determined by the following two main factors: the uncertainty of outcomes, resulting from the uncertainty of state occurrence, and the decision maker's expected utility when taking a certain action. These factors are related to risk as follows. The higher the uncertainty of an outcome is, the higher the risk. The higher the expected utility of an action is, the lower the risk. Since uncertainty is related to the state's entropy and variance, the perception of risk is that risk increases with entropy and with variance. Considering the expected utility of an action, risk decreases, when the expected utility increases.

One can therefore think of a risk measure combining entropy and variance on the one hand and expected utility on the other hand. The expected utility, entropy and variance (EU-EV) risk measure is defined as follows.

Definition 1. *Let $G(\Theta, A, u)$ be a decision analysis model, where the utility function $u = u(X(a, \theta))$ is increasing and $a \in A$. Then, the expected utility, entropy and variance (EU-EV) measure of risk for an action a is defined by*

$$R(a) = \frac{\lambda}{2}\left[H_a(\theta) + \frac{\mathrm{Var[X(a,}\theta)]}{\max\limits_{a \in A}\{\mathrm{Var[X(a,}\theta)]\}}\right] - (1-\lambda)\frac{\mathrm{E[u(X(a,}\theta))]}{\max\limits_{a \in A}\{\mathrm{E[u(X(a,}\theta))]\}}, \qquad (1)$$

where λ is a real constant satisfying $0 \leq \lambda \leq 1$ and $H_a(\theta)$ is the entropy of the state distribution corresponding to action a.

One says that $R(a)$ is the risk of action a. The constant λ is called trade-off coefficient, since it is used in the definition to balance the decision maker's expected utility of an action and the uncertainty reflected by the state's entropy and variance associated with the action. In this risk measure, the entropy and variance are combined as arithmetic mean. If $\lambda = 0$, then the risk measure is based only on the expected utility and if $\lambda = 1$ the risk measure depends only on the uncertainty given by the entropy and variance. If $\lambda \in (0,1)$, then the effect of the expected utility on the risk measure is bigger if λ approaches 0 and if λ approaches 1, the risk measure will be more influenced by the uncertainty than by the expected utility.

Here, the entropy defined by Shannon [26] will be used. When θ is a discrete random variable having n outcomes with corresponding probabilities p_i, $i = 1, \ldots, n$, then the entropy of the state θ corresponding to action a is

$$H_a(\theta) = -\sum_{i=1}^{n} p_i \ln p_i. \tag{2}$$

If a decision maker must choose between two actions a_1 and a_2, then he compares $R(a_1)$ with $R(a_2)$ by taking into account the subjective attitude towards risk given by the expected utility and the objective uncertainty. The decision maker chooses the action with lowest EU-EV risk.

The expected utility, entropy and variance (EU-EV) decision model can then be defined as follows.

Definition 2. *Let $G(\Theta, A, u)$ be a decision analysis model.*

1. *Consider two actions $a_1, a_2 \in A$ with corresponding EU-EV risk measures $R(a_1)$ and $R(a_2)$. Then:*
   (i) a_1 is strictly preferred over a_2, $a_1 \succ a_2$, if $R(a_1) < R(a_2)$.
   (ii) a_1 is weakly preferred over a_2, $a_1 \succeq a_2$, if $R(a_1) \leq R(a_2)$.
   (iii) a_1 is indifferent to a_2, $a_1 \sim a_2$, if $R(a_1) = R(a_2)$.
2. *Consider various actions, $A = \{a_1, a_2, \ldots, a_m\}$, then they can be ordered using the EU-EV measure of risk. The optimal action is the one with minimum EU-EV risk. In that case, one chooses a_i if*

$$R(a_i) = \min_{a_k \in A} R(a_k).$$

Next, some properties of the EU-EV risk measure and the associated decision model will be presented. For simplicity, $\mathrm{E[u(X(a,\theta))]}$ will be denoted by $\mathrm{E[u(a)]}$ and $\mathrm{Var[X(a,\theta)]}$ by $\mathrm{Var[a]}$. First, observe that for $\lambda = 0$, the decision model is reduced to the expected utility model: given two actions a_1 and a_2, $R(a_1) <$

$R(a_2)$ if and only if $\mathrm{E[u(a_1)]} > \mathrm{E[u(a_2)]}$. In particular, for a linear utility function $u(x) = ax + b$, $a > 0$, decisions are taken according to the expected value principle, i.e. $R(a_1) < R(a_2)$ if and only if $\mathrm{E[a_1]} > \mathrm{E[a_2]}$.

The EU-EV risk measure satisfies the properties in the following two theorems. The result in Theorem 1 states that risk decreases if a positive constant is added to all outcomes of an action. The result in Theorem 2 states that risk increases if all non-negative outcomes of an action with zero mean are multiplied by a positive constant greater than one.

Theorem 1. *Let $G(\Theta, A, u)$ be a decision analysis model with increasing utility function u. Consider the EU-EV risk measure defined in (1). If $A = \{a, a+k\}$, where k is a positive constant, then,*

$$R(a+k) < R(a). \tag{3}$$

Proof 1. *Since $H_a(\theta) = H_{a+k}(\theta)$, $\mathrm{Var[a]} = \mathrm{Var[a{+}k]}$ and $\mathrm{E[u(a)]} < \mathrm{E[u(a{+}k)]}$, one has*

$$R(a) = \frac{\lambda}{2}\left[H_a(\theta) + 1\right] - (1-\lambda)\frac{\mathrm{E[u(a)]}}{\mathrm{E[u(a+k)]}}$$

and

$$R(a+k) = \frac{\lambda}{2}\left[H_a(\theta) + 1\right] - (1-\lambda).$$

From the fact that the utility function is increasing, it follows that $R(a+k) < R(a)$.

Theorem 2. *Let $G(\Theta, A, u)$ be a decision analysis model with non-negative outcomes and increasing utility function u. Consider the EU-EV risk measure defined in (1). If $A = \{a, ka\}$, where $k > 1$ is a constant, and $\mathrm{E[a]} = 0$, then,*

$$R(ka) > R(a) \tag{4}$$

for $\dfrac{1 - \frac{\mathrm{E[u(a)]}}{\mathrm{E[u(ka)]}}}{\frac{3}{2} - \frac{1}{2k^2} - \frac{\mathrm{E[u(a)]}}{\mathrm{E[u(ka)]}}} < \lambda \leq 1.$

Proof 2. *Since $H_a(\theta) = H_{ka}(\theta)$, $\mathrm{Var[ka]} = \mathrm{k^2E[a^2]} > \mathrm{Var[a]} = \mathrm{E[a^2]}$ and $\mathrm{E[u(a)]} < \mathrm{E[u(ka)]}$, one has*

$$R(a) = \frac{\lambda}{2}\left[H_a(\theta) + \frac{1}{k^2}\right] - (1-\lambda)\frac{\mathrm{E[u(a)]}}{\mathrm{E[u(ka)]}}$$

and

$$R(ka) = \frac{\lambda}{2}\left[H_a(\theta) + 1\right] - (1 - \lambda).$$

It follows that $R(ka) > R(a)$ *if and only if*

$$\lambda\left(\frac{3}{2} - \frac{1}{2k^2} - \frac{\mathrm{E[u(a)]}}{\mathrm{E[u(ka)]}}\right) - \left(1 - \frac{\mathrm{E[u(a)]}}{\mathrm{E[u(ka)]}}\right) > 0.$$

Analysing the last inequality and the expression on the left side as function of λ*, one concludes that there exist intervals for* λ *contained in* $[0, 1]$*, which increase with* k*, where the condition is satisfied. Therefore, one has* $R(ka) > R(a)$ *for*

$$\frac{1 - \frac{\mathrm{E[u(a)]}}{\mathrm{E[u(ka)]}}}{\frac{3}{2} - \frac{1}{2k^2} - \frac{\mathrm{E[u(a)]}}{\mathrm{E[u(ka)]}}} < \lambda \leq 1.$$

In particular, the results in Theorem 1 and Theorem 2 show that the EU-EV risk measure satisfies properties concerning the risk perception in lotteries, namely that perceived risk decreases if a constant positive amount is added to all outcomes of a lottery (see [27], [28]), and that perceived risk increases if all outcomes of a lottery with zero mean are multiplied by a positive constant greater than one [29].

The EU-EV risk measure is an extension of the expected utility and entropy (EU-E) risk measure proposed by Yang and Qiu [21], since in addition to the entropy it includes also the variance as uncertainty factor. The EU-E risk measure is defined by

$$R(a) = \lambda H_a(\theta) - (1 - \lambda)\frac{\mathrm{E[u(X(a,\theta))]}}{\max\limits_{a \in A}\{\mathrm{E[u(X(a,\theta))]}\}}, \tag{5}$$

where $0 \leq \lambda \leq 1$.

Another model proposed by Park et al. [25], the expected utility and uncertainty risk (EU-UR) model, uses the prior distribution of the states as uncertainty factor and is expressed by

$$R(a) = \frac{\lambda}{\pi_a(p_{ij})/\max\limits_{a \in A}\{\pi_a(p_{ij})\}} - (1 - \lambda)\frac{\mathrm{E[u(X(a,\theta))]}}{\max\limits_{a \in A}\{\mathrm{E[u(X(a,\theta))]}\}}, \tag{6}$$

where $0 \leq \lambda \leq 1$ and $\pi_a(p_{ij})$ is the prior distribution of action a.

Yang and Qiu [21] and Park et al. [25] studied classical decision problems, namely the problem of Levi and the Allais paradox, using the EU-E model and the EU-UR model, respectively. Now, in the following section, these two problems will be studied with the EU-EV model and the results will be compared with those obtained for the EU-E and EU-UR models. Also another problem analysed by Nawrocki and Harding [15] will be considered as an example for which the EU-EV measure as an extension of the EU-E measure is more appropriate.

3. Numerical examples and results

The following example justifies the use of the EU-EV as risk measure and it is shown that the classical Levy problem and Allais paradox can be explained with the EU-EV model (examples in Subsections 3.2 and 3.3). The examples motivate to derive particular results, which are presented as propositions at the end of the first two subsections.

3.1. Security problem

Nawrocki and Harding [15] explored the use of the entropy as a measure of risk and concluded that it is not adequate for the use in finance and economics, since it ignores the dispersion of frequency classes. They proposed instead a state-value weighted entropy to increase the investment performance of the entropy risk measure. The following example was considered by Nawrocki and Harding to illustrate the problem of using the entropy as a measure of security risk, where two securities have the same entropy, but different levels of risk.

Let a_1 and a_2 be two securities having the same return state values θ_i, $i = 1, \ldots, 5$, with different probabilities p_{1j} and p_{2j}, $j = 1, \ldots, 5$, respectively (see Table 1).

Table 1: Outcomes of a_1 and a_2.

		θ_1	θ_2	θ_3	θ_4	θ_1	Expected value	Variance	Entropy
a_1	x_{1j}	1	2	3	4	5	3	1.2	1.47
	p_{1j}	0.1	0.2	0.4	0.2	0.1			
a_2	x_{2j}	1	2	3	4	5	3	1.8	1.47
	p_{2j}	0.2	0.1	0.4	0.1	0.2			

In this example, a_1 and a_2 have the same entropy: $H_{a_1}(\theta) = H_{a_2}(\theta) = 1.47$, and the same mean: $\mathrm{E[a_1]} = \mathrm{E[a_2]} = 3$, however, different variances: $\mathrm{Var[a_1]} = 1.2$, $\mathrm{Var[a_1]} = 1.8$. Although a_1 and a_2 have the same entropy, a_1 is less risky than a_2.

This example will be analysed using the EU-E and EU-EV risk measures for different utility functions: a risk neutral utility $u(x) = x$ and two risk averse utilities: $u(x) = \log(x)$ and $u(x) = \sqrt{x}$. Table 2 contains the corresponding risk measures for a_1 and a_2.

Table 2: Risk measures EU-E and EU-EV for a_1 and a_2.

Utility function	$R(a_i)$	EU-E	EU-EV
$u(x) = x$	$R(a_1)$	$2.47\lambda - 1$	$2.07\lambda - 1$
	$R(a_2)$	$2.47\lambda - 1$	$2.24\lambda - 1$
$u(x) = \log(x)$	$R(a_1)$	$2.47\lambda - 1$	$2.07\lambda - 1$
	$R(a_2)$	$2.42\lambda - 0.95$	$2.19\lambda - 0.95$
$u(x) = \sqrt{x}$	$R(a_1)$	$2.47\lambda - 1$	$2.07\lambda - 1$
	$R(a_2)$	$2.46\lambda - 0.99$	$2.23\lambda - 0.99$

Risk-neutral utility: $u(x) = x$

Note that for the risk neutral utility function, the expected utility equals the expected value: $\mathrm{E[u(x)]} = \mathrm{E[x]}$. Therefore the EU-E measure depends only on the entropy and expected value and since these are equal for a_1 and a_2, one obtains $R(a_1) = R(a_2)$. In this case, a decision maker is indifferent in choosing between a_1 and a_2, although a_1 is less risky than a_2.

With the EU-EV risk measure, which contains the variance as additional risk factor, one obtains $R(a_1) < R(a_2)$ for $0 < \lambda \leq 1$, so that a decision maker would choose a_1, which is in fact less riskier than a_2. If the effect of entropy and variance vanishes ($\lambda = 0$), then $R(a_1) = R(a_2)$.

Risk-averse utilities: $u(x) = \log(x)$ **and** $u(x) = \sqrt{x}$

Considering the EU-E measure, one has for both risk-averse utility functions $R(a_1) < R(a_2) \Leftrightarrow 0 \leq \lambda < 1$ and $R(a_1) = R(a_2) \Leftrightarrow \lambda = 1$, so that a decision maker could choose a_2 if he ignores the effect of expected utility ($\lambda = 1$). For other values of the trade-off factors, he would choose a_1.

With the EU-EV measure a decision maker with the risk-averse utility functions would always choose a_1, because $R(a_1) < R(a_2)$ for trade-off factors $0 \leq \lambda \leq 1$.

This problem shows that it is relevant to consider the variance as additional risk factor in the risk measure, since it is possible to have actions with the same entropy and even with the same mean, in which case the expected utility will be equal for risk neutral decision makers. The following result can be stated, which follows from Definition 1.

Proposition 1. *Let $G(\Theta, A, u)$ be a decision analysis model with risk neutral linear utility functions $u(x) = ax + b$, $a > 0$. If $a_1, a_2 \in A$, with EU-EV risk measures $R(a_1)$ and $R(a_2)$, are such that $H_{a_1}(\theta) = H_{a_2}(\theta)$ and $\mathrm{E}[\mathrm{a}_1] = \mathrm{E}[\mathrm{a}_2]$, then*

$$R(a_1) < R(a_2) \Leftrightarrow \mathrm{Var}[\mathrm{a}_1] < \mathrm{Var}[\mathrm{a}_2],$$

for $0 < \lambda \leq 1$.

3.2. Levy problem

Consider the following example, presented in [12], with two actions a_1 and a_2 and two states θ_1 and θ_2 shown in Table 3.

Table 3: Outcomes of actions a_1 and a_2.

		θ_1	θ_2	Expected value	Variance	Entropy
a_1	x_{1j}	1	100	20.8	1568	0.5
	p_{1j}	0.8	0.2			
a_2	x_{2j}	10	1000	19.9	9703	0.06
	p_{2j}	0.99	0.01			

According to the mean variance criterion, since $\mathrm{E[a_1]} > \mathrm{E[a_2]}$ and $\mathrm{Var[a_2]} > \mathrm{Var[a_1]}$, a_2 is riskier than a_1 and one would choose a_1. However, according to the expected utility criterion, since $\mathrm{E[u(a_2)]} > \mathrm{E[u(a_1)]}$ for the risk aversion utility function $u(x) = \ln(x)$, a decision maker would choose a_2.

Using the EU-EV criterion one concludes the following. For a risk neutral decision maker with utility function $u(x) = x$, the risk measures for both actions read: $R(a_1) = 1.33\lambda - 1$ and $R(a_2) = 1.49\lambda - 0.96$, implying $R(a_1) < R(a_2)$ for $0 \leq \lambda \leq 1$ and the decision maker always chooses a_1. Considering the risk-averse utility function $u(x) = \log(x)$, one has $R(a_1) = 0.72\lambda - 0.39$, $R(a_2) = 1.53\lambda - 1$ and, consequently, $R(a_2) < R(a_1) \Leftrightarrow 0 \leq \lambda < 0.75$. In this case, the decision maker would choose a_2 for $0 \leq \lambda \leq 0.75$. Suppose that the decision maker has the risk-averse utility function $u(x) = \sqrt{x}$, then $R(a_1) = 1.14\lambda - 0.81$, $R(a_2) = 1.53\lambda - 1$ and thus $R(a_2) < R(a_1) \Leftrightarrow 0 \leq \lambda < 0.49$. In this case a_2 is preferable to a_1 for trade-off factors $0 \leq \lambda < 0.49$.

Table 4 contains the risk-measures EU-E, EU-UR and EU-EV for the actions a_1 and a_2 for the risk-neutral utility function $u(x) = x$ and for both risk-averse utility functions $u(x) = \log(x)$ and $u(x) = \sqrt{x}$. Using these risk measures, one obtains the intervals for the tradeoff factors corresponding to the decisions $R(a_1) < R(a_2)$ and $R(a_2) < R(a_1)$ indicated in Table 5.

Table 4: Risk measures EU-E, EU-UR and EU-EV for actions a_1 and a_2.

Utility function	$R(a_i)$	EU-E	EU-UR	EU-EV
$u(x) = x$	$R(a_1)$	$1.5\lambda - 1$	$2\lambda - 1$	$1.33\lambda - 1$
	$R(a_2)$	$1.02\lambda - 0.96$	$1.21\lambda - 0.96$	$1.49\lambda - 0.96$
$u(x) = \log(x)$	$R(a_1)$	$0.89\lambda - 0.39$	$1.39\lambda - 0.39$	$0.72\lambda - 0.39$
	$R(a_2)$	$1.06\lambda - 1$	$1.24\lambda - 1$	$1.53\lambda - 1$
$u(x) = \sqrt{x}$	$R(a_1)$	$1.31\lambda - 0.81$	$1.81\lambda - 0.81$	$1.14\lambda - 0.81$
	$R(a_2)$	$1.06\lambda - 1$	$1.24\lambda - 1$	$1.53\lambda - 1$

Table 5: Tradeoff factors for decisions with EU-E, EU-UR and EU-EV.

Utility function	Risk decision	EU-E	EU-UR	EU-EV
$u(x) = x$	$R(a_2) < R(a_1)$	$0.08 < \lambda \leq 1$	$0.05 < \lambda \leq 1$	—
	$R(a_1) < R(a_2)$	$0 \leq \lambda < 0.08$	$0 \leq \lambda < 0.05$	$0 \leq \lambda \leq 1$
$u(x) = \log(x)$	$R(a_2) < R(a_1)$	$0 \leq \lambda \leq 1$	$0 \leq \lambda \leq 1$	$0 \leq \lambda < 0.75$
	$R(a_1) < R(a_2)$	—	—	$0.75 < \lambda \leq 1$
$u(x) = \sqrt{x}$	$R(a_2) < R(a_1)$	$0 \leq \lambda \leq 1$	$0 \leq \lambda \leq 1$	$0 \leq \lambda < 0.49$
	$R(a_1) < R(a_2)$	—	—	$0.49 < \lambda \leq 1$

Analysing these results, one can observe that with the risk-averse utility functions one chooses always a_2 with the EU-E and EU-UR decision model. However with the EU-EV decision model, for the risk-averse utility function $u(x) = \log(x)$, a_2 is chosen for $0 \leq \lambda < 0.75$, and for the risk-averse utility function $u(x) = \sqrt{x}$, a_2 is chosen for a smaller range of λ: $0 \leq \lambda < 0.49$. Considering the risk-neutral utility function $u(x) = x$, a_1 is always chosen with the EU-EV model, whereas with the EU-E and EU-UR models a_2 is preferred over a_1 for a wide range of values of λ and there is the possibility to prefer a_1 over a_2 for values of λ close to zero (see Table 5).

Note that for the risk-neutral utility, the expected utility principle is reduced to the expected value principle, so that a_1 is preferred over a_2 in terms of utility. For this reason and due to the high discrepancy of the variances, which has an higher impact than the entropies on the difference of the EU-EV measures for a_1 and a_2, a_1 is always chosen using the EU-EV measure. This suggests the following result, which follows from Definition 1.

Proposition 2. *Let $G(\Theta, A, u)$ be a decision analysis model with risk neutral linear utility functions $u(x) = ax + b$, $a > 0$. If $a_1, a_2 \in A$, with EU-EV risk measures $R(a_1)$ and $R(a_2)$, are such that*

$$\mathrm{E}[\mathrm{a}_1] > \mathrm{E}[\mathrm{a}_2]$$

and

$$\frac{\mathrm{Var}[\mathrm{a}_1] - \mathrm{Var}[\mathrm{a}_2]}{\max\limits_{a \in A}\{\mathrm{Var}[\mathrm{a}]\}} + H_{a_1}(\theta) - H_{a_2}(\theta) < 0 \tag{7}$$

then $R(a_1) < R(a_2)$ for $0 \leq \lambda \leq 1$.

For the Levy problem the normalized difference of the variances in expression (7) is -0.84 and the difference of the entropies 0.44, leading therefore to the preference of a_1 over a_2.

3.3. Allais paradox

The problem known as Allais paradox [30] shows that expected utility may not describe the behavior of decision makers adequately. It consists of two experiments, in each of which people can choose between two possible capital gains: a_1 or a_2 in the first experiment and a_3 or a_4 in the second experiment, see Table 6.

Table 6: Outcomes of actions a_1 and a_2, a_3 and a_4 of Allais paradox.

		θ_1	θ_2	θ_3	Expected value	Variance	Entropy
a_1	x_{1j}	1					
	p_{1j}	1			1	0	0
a_2	x_{2j}	1	5	0			
	p_{2j}	0.89	0.1	0.01	1.39	1.46	0.38
a_3	x_{3j}	1		0			
	p_{3j}	0.11		0.89	0.11	0.1	0.35
a_4	x_{4j}		5	0			
	p_{4j}		0.1	0.9	0.5	2.25	0.33

From experimental economy it is known that, if there is the possibility to choose between a_1 and a_2, on the one hand, and a_3 and a_4, on the other hand, then people prefer a_1 over a_2 and a_4 over a_3. However this is not consistent with the expected utility hypotheses, since the preference $\mathrm{E[u(a_1)]} > \mathrm{E[u(a_2)]}$ is equivalent to $0.11u(1) > 0.1u(5) + 0.01u(0)$ and the preference $\mathrm{E[u(a_4)]} > \mathrm{E[u(a_3)]}$ leads to the opposite inequality.

The EU-EV decision model will be applied to this paradox and the results will be compared with those obtained with the EU-E and EU-UR decision models.

Risk-neutral utility: $u(x) = x$

From the risk measures in Table 7 and the tradeoff factors in Table 8, one can observe the following. The risk measure $R(a_1)$ is equal for EU-E, EU-UR and EU-EV, because in this case it depends only on the expected value as risk factor:

$$R(a_1) = -(1-\lambda)\frac{\mathrm{E[a_1]}}{\max\{\mathrm{E[a_1]}, \mathrm{E[a_2]}\}}.$$

The preference of a_1 over a_2 is better explained with the EU-EV model, since this leads to a wider range for the tradeoff factor ($0.29 \leq \lambda \leq 1$), when compared with the other models. As for the second decision, where a_4 is preferred over a_3, it can also be explained with the EU-EV model, however the higher variance of

a_4 reduces the interval for the tradeoff factor ($0 \leq \lambda \leq 0.62$) for this decision.

Table 7: Risk measures EU-E, EU-UR and EU-EV for a risk-neutral utility function $u(x) = x$ for actions a_1, a_2, a_3 and a_4.

	EU-E	EU-UR	EU-EV
$R(a_1)$	$0.72\lambda - 0.72$	$0.72\lambda - 0.72$	$0.72\lambda - 0.72$
$R(a_2)$	$1.38\lambda - 1$	$1.1\lambda - 1$	$1.69\lambda - 1$
$R(a_3)$	$0.43\lambda - 0.08$	$1.08\lambda - 0.08$	$0.42\lambda - 0.22$
$R(a_4)$	$0.69\lambda - 0.36$	$1.32\lambda - 0.36$	$1.67\lambda - 1$

Table 8: Tradeoff factors for decisions with EU-E, EU-UR and EU-EV with a risk-neutral utility function $u(x) = x$.

	EU-E	EU-UR	EU-EV
$R(a_1) < R(a_2)$	$0.42 < \lambda \leq 1$	$0.75 < \lambda \leq 1$	$0.29 \leq \lambda \leq 1$
$R(a_4) < R(a_3)$	$0 \leq \lambda \leq 1$	$0 \leq \lambda \leq 1$	$0 \leq \lambda \leq 0.62$

Risk-averse utility: $u(x) = \sqrt{x}$

Considering the risk-averse utility, one has

$$R(a_1) = -(1-\lambda)\frac{\mathrm{E}\left[\sqrt{\mathrm{a}_1}\right]}{\max\left\{\mathrm{E}\left[\sqrt{\mathrm{a}_1}\right], \mathrm{E}\left[\sqrt{\mathrm{a}_2}\right]\right\}}$$

for the EU-E, EU-UR and EU-EV risk measure, which depends only on the expected utility as risk factor. Therefore, one obtains the same expression $R(a_1) = 0.9\lambda - 0.9$ for the three measures (see Table 9). Again, the choice of a_1 over a_2 is better explained with the EU-EV model, since this leads to a wider range for the tradeoff factor ($0.13 \leq \lambda \leq 1$), when compared with the other models (see Table 10). Concerning the preference of a_4 over a_3, it can be explained with the EU-EV model as well, however the higher variance of a_4 reduces the interval for the tradeoff factor ($0 \leq \lambda \leq 0.5$).

Table 9: Risk measures EU-E, EU-UR and EU-EV for a risk-averse utility function $u(x) = \sqrt{x}$ for actions a_1, a_2, a_3 and a_4.

	EU-E	EU-UR	EU-EV
$R(a_1)$	$0.9\lambda - 0.9$	$0.9\lambda - 0.9$	$0.9\lambda - 0.9$
$R(a_2)$	$1.38\lambda - 1$	$1.1\lambda - 1$	$1.69\lambda - 1$
$R(a_3)$	$0.44\lambda - 0.1$	$1.1\lambda - 0.1$	$0.7\lambda - 0.5$
$R(a_4)$	$0.53\lambda - 0.2$	$1.16\lambda - 0.2$	$1.67\lambda - 1$

Table 10: Tradeoff factors for decisions with EU-E, EU-UR and EU-EV with a risk-averse utility function $u(x) = \sqrt{x}$.

	EU-E	EU-UR	EU-EV
$R(a_1) < R(a_2)$	$0.21 < \lambda \leq 1$	$0.52 < \lambda \leq 1$	$0.13 \leq \lambda \leq 1$
$R(a_4) < R(a_3)$	$0 \leq \lambda \leq 1$	$0 \leq \lambda \leq 1$	$0 \leq \lambda \leq 0.5$

Risk-seeking utility: $u(x) = x^2$

With the risk-seeking utility function one can observe again, as expected, an equal risk measure $R(a_1) = 0.29\lambda - 0.29$ for EU-E, EU-UR and EU-EV (see Table 11). In this case, the expression for $R(a_1)$ reads

$$R(a_1) = -(1-\lambda)\frac{\mathrm{E}\left[\mathrm{a}_1^2\right]}{\max\left\{\mathrm{E}\left[\mathrm{a}_1^2\right], \mathrm{E}\left[\mathrm{a}_2^2\right]\right\}}.$$

The same behaviour as in the risk-neutral case and risk-averse case can be observed for the EU-EV model in the risk-seeking case, when comparing the choice of a_1 over a_2 and of a_4 over a_3. The range of the tradeoff factor is wider for the first decision with EU-EV ($0.51 \leq \lambda \leq 1$) and it is reduced for the second decision ($0 \leq \lambda \leq 0.67$), see Table 12. One concludes that also in the risk-seeking case, the results are consistent with the Allais paradox.

Table 11: Risk measures EU-E, EU-UR and EU-EV for a risk-seeking utility function $u(x) = x^2$ for actions a_1, a_2, a_3 and a_4.

	EU-E	EU-UR	EU-EV
$R(a_1)$	$0.29\lambda - 0.29$	$0.29\lambda - 0.29$	$0.29\lambda - 0.29$
$R(a_2)$	$1.38\lambda - 1$	$1.1\lambda - 1$	$1.69\lambda - 1$
$R(a_3)$	$0.38\lambda - 0.03$	$1.03\lambda - 0.03$	$0.24\lambda - 0.04$
$R(a_4)$	$1.06\lambda - 0.74$	$1.7\lambda - 0.74$	$1.67\lambda - 1$

Table 12: Tradeoff factors for decisions with EU-E, EU-UR and EU-EV with a risk-seeking utility function $u(x) = x^2$.

	EU-E	EU-UR	EU-EV
$R(a_1) < R(a_2)$	$0.65 < \lambda \leq 1$	$0.88 < \lambda \leq 1$	$0.51 \leq \lambda \leq 1$
$R(a_4) < R(a_3)$	$0 \leq \lambda \leq 1$	$0 \leq \lambda \leq 1$	$0 \leq \lambda \leq 0.67$

One can observe further interesting properties from the results presented in Tables 7-12. Comparing the intervals of the tradeoff factors for the decision $R(a_1) < R(a_2)$ with EU-E, EU-UR and EU-EV, the intervals increase with the increase of risk-aversion (cf. Tables 8,10,12). This is consistent with the decision maker's preference of the certain gain a_1 over a_2. Risk-averse decision makers are more likely to choose a_1 than risk-neutral ones, and both are more likely to choose a_1 than risk seeking decision makers. This behavior is expressed well with all three risk measures.

As for the decision $R(a_4) < R(a_3)$, with the EU-E and EU-UR models the intervals of the tradeoff factors are equal, $0 \leq \lambda \leq 1$, independently of the utility function. However, with the EU-EV model, the increasing in the degree of risk aversion is reflected in the intervals' decreasing range. Note that a risk-seeking person chooses a_4 for tradeoff factors $0 \leq \lambda \leq 0.67$, a risk-neutral person, for tradeoff factors $0 \leq \lambda \leq 0.62$ and a risk-averse person, for tradeoff factors $0 \leq \lambda \leq 0.5$.

One can conclude that the Allais paradox can be described by the EU-EV decision model.

4. Conclusions

Decision models based on von Neumann and Morgenstern utility theory alone cannot characterize all types of human behavior, as recognized in the literature. However, risk can also not be assessed measuring only uncertainty, e.g. reflected in entropy or variance. It is reasonable to have decision models taking into account both, the preference based on expected utility and the perception of risk given by uncertainty, whose associated risk measures are consistent with the decision maker's preference ordering. The EU-EV model proposed in this paper is one of these decision models. It is shown that this model can be used to analyze and explain many decision problems. Its corresponding risk measure is defined using expected utility, entropy and variance. Since this model also includes the variance, when compared with the EU-E model, it performs better in certain cases (e.g. it is possible to have different actions with the same entropy, the same expected utility and different variances).

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