TOPOLOGICAL COMPLEXITY OF SURFACES

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The notion of topological complexity of a space has been introduced by M. Farber in [F03] in order to give a topological measure of the complexity of the motion planning problem in robotics. Given a mechanical system, this problem consists of constructing an algorithm telling how to move from any initial state to any final state. If X is the space of all the possible states, which is called the configuration space of the system, then such an algorithm takes as input pairs of configurations $(A, B) \in X \times X$ and produces a continuous path $\gamma\colon I = [0, 1] \to X$ from the initial configuration $A = \gamma(0)$ to the terminal configuration $B = \gamma(1)$. In other words, a motion planning algorithm corresponds to a section $s : X \times X \to X^I$ of the evaluation map

$$\mathrm{ev} : \mathrm{X}^{\mathrm{I}} \to \mathrm{X} \times \mathrm{X}, \quad \gamma \mapsto (\gamma(0), \gamma(1))$$

where X^I is the space of continuous paths $\gamma\colon I = [0, 1] \to X$ (equipped with the compact-open topology). Such a section always exists when X is path-connected but is not continuous in general. For instance, for the circle $X = S^1 \subset \mathbb{R}^2$, one can obtain $s : X \times X \to X^I$ by defining $s(A, B)$ as the shortest path from A to B when A and B are not diametrically opposed ($A \neq -B$) and as the shortest path in the counterclockwise direction when they are diametrically opposed ($A \neq -B$). So defined, the function s is continuous on each piece but is not globally continuous. Actually, given a space X, it is easy to see that there exists a globally continuous section $s\colon X \times X \to X^I$ if and only if X is contractible, i.e., continuously deformable to a point (see Theorem 1 below). This means that we will need at least 2 continuous rules, or more precisely 2 continuous local sections of the evaluation map, to describe a complete motion planning algorithm on a non-contractible space. Roughly speaking, the topological complexity is an invariant which gives a lower bound for the number of continuous rules needed to describe such a complete algorithm. General references on this topic include [F03], [F08].

In this note, after some generalities about this invariant, we will survey the determination of the topological complexity when X is a (connected closed) surface, which has been initiated in [F03] and [FTY03] and recently completed in [D16] and [CV17].

1. Topological complexity

Let X be the configuration space of a mechanical system. In general, such a space (or some of its path components) can be identified to a nice topological space like a manifold, a polyhedron... For instance, the circle $X = S^1$ corresponds to the configuration space of a planar robotic arm revolving about one fixed extremity, the torus $X = S^1 \times S^1$ can be interpreted as the configuration space of an articulated

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arm with two bars, the projective plane $X = \mathbb{R}P^2$ corresponds to all the positions of a bar rotating about its center... Universality theorems (see [JS01], [KM02]) assert that any reasonable topological space (e.g. any smooth compact connected manifold) can be seen as (a path component of) the configuration space of a mechanical system. Examples of mechanisms having some surface as configuration space are for instance given in [JS01] and [H07].

For a general (non-empty) path-connected topological space X, Farber formalized the notion of topological complexity as follows. Note that we here consider the normalized version of this concept in the sense that the topological complexity of a point will be 0 (instead of 1 in the original definition of [F03]).

Definition 1. *The topological complexity of X, $\mathrm{TC}(X)$, is the least integer k such that there exists a cover of $X \times X$ by $k+1$ open sets U_0,..., U_k on each of which the evaluation map*

$$\mathrm{ev} : \mathrm{X^I} \to \mathrm{X} \times \mathrm{X}, \quad \gamma \mapsto (\gamma(0), \gamma(1))$$

admits a local continuous section, that is, for any i, there exists a continuous map $s_i : U_i \to X^I$ satisfying $\mathrm{ev} \circ \mathrm{s_i} = \mathrm{id}$.

Given such an open cover with $k+1$ local sections, we can construct a global non continuous section s of ev by (for instance) setting $s = s_i$ on $F_i = U_i \setminus (U_0 \cup \cdots \cup U_{i-1})$. This is well-defined since the sets F_i give a partition of $X \times X$. Actually, in [F04], Farber has shown that, if X is a manifold or a polyhedron, we can equivalently define $\mathrm{TC}(X)$ to be the least integer k such that there is a partition of $X \times X$ by $k+1$ sets F_i which are required to be ENR (Euclidian Neighborhood Retract) and equipped with local continuous sections of ev. The extra conditions on X and on the sets F_i are not really restrictive for practical purposes, however, the definition given here in terms of open cover is more convenient for the study of the topological properties of TC such as its invariance.

Recall that two spaces X and Y are homotopically equivalent ($X \simeq Y$) if there exist two (continuous) maps $f : X \to Y$ and $g : Y \to X$ such that the composites $g \circ f$ and $f \circ g$ are homotopic to the identity, that is, there exist homotopies $H : X \times I \to X$, $G : Y \times I \to Y$ satisfying $H(x, 0) = g \circ f(x)$, $H(x, 1) = x$, $G(y, 0) = f \circ g(y)$, and $G(y, 1) = y$. If $\mathrm{TC}(Y) \leq k$ and $V_0, ..., V_k$ is an open cover of $Y \times Y$ with local sections $\sigma_0, ..., \sigma_k$ of ev, then the $(k+1)$ sets $U_i = f^{-1}(V_i)$ form an open cover of $X \times X$ and, for each i, the continuous map s_i given by

$$s_i(A, B)(t) = \begin{cases} H(A, 1-3t) & 0 \leq t \leq 1/3 \\ g(\sigma_i(f(A), f(B))(3t-1)) & 1/3 \leq t \leq 2/3 \\ H(B, 3t-2) & 2/3 \leq t \leq 1 \end{cases}$$

takes $(A, B) \in U_i$ to a path in X from A to B and is hence a local section of ev on U_i. Consequently, $\mathrm{TC}(X) \leq k$. This shows that $\mathrm{TC}(X) \leq \mathrm{TC}(Y)$ whenever $X \simeq Y$. Similarly, we can see that $\mathrm{TC}(Y) \leq \mathrm{TC}(X)$ and conclude to the invariance of TC:

Theorem 1 ([F03])**.** *The topological complexity has the following fundamental properties:*

- TC *is a homotopy invariant: if $X \simeq Y$ then* $\mathrm{TC}(X) = \mathrm{TC}(Y)$.
- $\mathrm{TC}(X) = 0$ *if and only if X is contractible (that is, X is homotopically equivalent to a point).*

In order to understand the second statement, observe that if $\mathrm{TC}(X)=0$, then there is a global continuous section s of ev. Then, fixing a point $*\in X$, the map $(x,t)\mapsto s(x,*)(t)$ gives a homotopy between the identity and the inclusion $*\hookrightarrow X$. In other words, X is contractible. The other direction follows directly from the invariance and from the fact that the topological complexity of a point is 0.

Since a sphere is not contractible we always have $\mathrm{TC}(S^n)\geq 1$. Actually we have

Theorem 2 ([F03])**.** $\mathrm{TC}(S^n)=1$ *if n is odd and* $\mathrm{TC}(S^n)=2$ *if n is even.*

We will see at the end of this section why $\mathrm{TC}(S^n)$ must be greater than 1 when n is even. The fact that $\mathrm{TC}(S^n)\leq 1$ for $n=2k-1$ odd can be seen by adapting the construction given for S^1 in the introduction. Fix a nowhere zero continuous tangent vector field V on $S^{2k-1}\subset\mathbb{R}^{2k}$ (for instance $V(x,y)=(-y,x), x,y\in\mathbb{R}^k$) and consider the open cover of $S^{2k-1}\times S^{2k-1}$ given by $U_0=\{(A,B)\,|\,A\neq -B\}$ and $U_1=\{(A,B)\,|\,A\neq B\}$. Then define $s_0(A,B)$ to be the shortest path from A to B (with constant speed) and $s_1(A,B)$ to be the shortest path from A to $-B$ followed by the meridian from $-B$ to B in the direction of the (non-zero) tangent vector $V(-B)$. This construction can be adapted to the even dimensional case but with an additional open set (and local section) since any continuous tangent vector field vanishes on an even-dimensional sphere. Alternatively, one can deduce the general inequality $\mathrm{TC}(S^n)\leq 2$ from the link between the topological complexity and the classical Lusternik Schnirelmann category (which was introduced in the thirties in order to give a lower bound for the number of critical points of a differential map defined on a manifold, see [CLOT03] as a general reference).

Definition 2. *The LS category of a space Y,* $\mathrm{cat}(Y)$*, is the least integer m such that Y can be covered by $m+1$ open sets U_i which are contractible in Y - that is, for which the inclusion $U_i\hookrightarrow Y$ is homotopic to a constant map.*

The condition for $U\hookrightarrow Y$ to be homotopic to a constant map means that we have a continuous way to associate with a point y of U a path in Y from y to a fixed point. This is then not difficult to see the following relations between TC and cat (see [F03]):

$$\mathrm{cat}(X)\leq \mathrm{TC}(X)\leq \mathrm{cat}(X\times X).$$

The LS-category is also a homotopy invariant which satisfies $\mathrm{cat}(Y)=0$ if and only if Y is contractible. Independently on the dimension, we have $\mathrm{cat}(S^n)=1$ since any sphere can be covered by 2 contractible open sets (in themselves and therefore in S^n). For manifolds, we have $\mathrm{cat}(Y)\leq\dim Y$ and we also always have $\mathrm{cat}(Y\times Z)\leq\mathrm{cat}(Y)+\mathrm{cat}(Z)$. Then the chain of inequalities above can be completed as

$$\mathrm{cat}(X)\leq\mathrm{TC}(X)\leq\mathrm{cat}(X\times X)\leq 2\mathrm{cat}(X)\leq 2\dim(X).$$

In particular $\mathrm{TC}(S^n)\leq 2\mathrm{cat}(S^n)=2$.

We also note the following interesting case of equality:

Theorem 3 ([F04])**.** *If G is a path-connected topological group then* $\mathrm{TC}(G)=\mathrm{cat}(G)$.

Let us see why $\mathrm{TC}(G)\leq\mathrm{cat}(G)$. Consider the continuous map $\mu: G\times G\to G$ given by $\mu(x,y)=xy^{-1}$ and suppose that $U\hookrightarrow G$ is homotopic to a constant map through a homotopy H. Since G is path-connected, we can suppose that the constant map is $u \mapsto e$ where e is the unit of G. Then, for $(x, y) \in V = \mu^{-1}(U) \subset G \times G$ we can consider the path from x to y given by $t \mapsto H(xy^{-1}, t)y$ and the associated local section of ev.

The direct determination of the LS-category and TC of a space is in general not easy. For instance, there are still Lie groups for which the LS-category is not known and one usually tries to use more calculable lower and upper bounds for estimating these invariants.

A very useful lower bound for the LS-category of a space Y is given by the cup-length of its cohomology. We here consider the cohomology of Y with coefficients in a field $\Bbbk$ and suppose that Y is a path-connected manifold. Recall that $H^*(Y;\Bbbk) = \bigoplus_{m\geq 0} H^m(Y;\Bbbk)$ is a graded $\Bbbk$-algebra satisfying $H^m(Y;\Bbbk) = 0$ for $m > \dim(Y)$ and $H^0(Y;\Bbbk) = \Bbbk \cdot 1 = \langle 1 \rangle$ where 1 is the unit of the algebra, and that the graded multiplication (called the cup-product) $H^p(Y;\Bbbk) \otimes H^q(Y;\Bbbk) \to H^{p+q}(Y;\Bbbk)$ is commutative in the graded meaning: $ab = (-1)^{\deg(a)\deg(b)} ba$. The cup-length of $H^*(Y;\Bbbk)$ is defined by

$$\mathrm{cl}_{\Bbbk}(Y) = \max\{n \mid \exists\, a_1, \dots, a_n \in H^{>0}(Y;\Bbbk)\, \text{s.t.}\, a_1 \cdots a_n \neq 0\}$$

and we have

$$\mathrm{cl}_{\Bbbk}(Y) \leq \mathrm{cat}\,(Y) \leq \dim(Y).$$

For instance the cohomology of the torus $T = S^1 \times S^1$ with coefficients in $\mathbb{Q}$ is given by $H^0(T;\mathbb{Q}) = \langle 1 \rangle$, $H^1(T;\mathbb{Q}) = \langle a, b \rangle$ and $H^2(T;\mathbb{Q}) = \langle \omega \rangle$, with the multiplicative structure $a^2 = b^2 = 0$, $ab = -ba = \omega$. We then have $\mathrm{cl}_{\mathbb{Q}}(T) = 2$ and therefore $\mathrm{cat}\,(T) = 2$ since $\dim(T) = 2$. Since T is a topological group, we can also conclude that $\mathrm{TC}\,(T) = 2$.

A similar lower bound for the topological complexity has been introduced by Farber. Call a *zero-divisor* of $H^*(X;\Bbbk)$ an element of the kernel of the cup-product $H^*(X;\Bbbk) \otimes H^*(X;\Bbbk) \to H^*(X;\Bbbk)$. This kernel is an ideal of the tensor algebra whose multiplication satisfies $(a \otimes b)(c \otimes d) = (-1)^{\deg(b)\deg(c)} ac \otimes bd$. The *zero-divisor cup-length* of $H^*(X;\Bbbk)$ is then defined by

$$\mathrm{zcl}_{\Bbbk}(X) = \max\{n \mid \exists\, n\, \text{zero-divisors}\, z_i\, \text{s.t.}\, z_1 \cdots z_n \neq 0\}$$

and we have

Theorem 4 ([F03])**.** $\mathrm{zcl}_{\Bbbk}(X) \leq \mathrm{TC}\,(X)$.

We can now complete the calculation of $\mathrm{TC}\,(S^n)$ for n even. Recall that we already know that $1 \leq \mathrm{TC}\,(S^n) \leq 2$. Taking coefficients in $\mathbb{Q}$, we have only non-zero cohomology in degree 0 and n:

$$H^0(S^n;\mathbb{Q}) = \langle 1 \rangle, \quad H^n(S^n;\mathbb{Q}) = \langle a \rangle.$$

The zero-divisors are given by

$$\mathbb{Q}(a \otimes 1 - 1 \otimes a) \oplus \mathbb{Q}(a \otimes a).$$

Since $(a \otimes 1 - 1 \otimes a)^2 = (-1 - (-1)^n) a \otimes a = -2a \otimes a \neq 0$ if n is even, we can conclude that $\mathrm{zcl}_{\mathbb{Q}}(S^n) = 2$, and therefore $\mathrm{TC}\,(S^n) = 2$, if n is even.

2. Topological complexity of surfaces

We first consider the orientable (closed connected) surfaces. We denote by Σ_g the orientable surface of genus g ($g \geq 0$) so that $\Sigma_0 = S^2$, Σ_1 is the torus $T = S^1 \times S^1$ and, for $g \geq 2$, Σ_g can be described as the connected sum of g tori T. The topological complexity of Σ_g has been determined by Farber in 2003:

Theorem 5 ([F03])**.** *We have*
- *for* $g \leq 1$, $\mathrm{TC}(\Sigma_g) = 2$*;*
- *for* $g \geq 2$, $\mathrm{TC}(\Sigma_g) = 4$.

Since the cases $g = 0, 1$ have been discussed in the previous section, we now focus on the case $g \geq 2$. For dimensional reason, we have $\mathrm{TC}(\Sigma_g) \leq 2\dim(\Sigma_g) = 4$. The cohomology of Σ_g with coefficients in $\mathbb{Q}$ is given by $H^0(\Sigma_g;\mathbb{Q}) = \langle 1\rangle$, $H^2(\Sigma_g;\mathbb{Q}) = \langle\omega\rangle$, and

$$H^1(\Sigma_g;\mathbb{Q}) = \langle a_1, b_1, \ldots, a_g, b_g\rangle,$$

with the multiplicative structure

$$a_i^2 = b_i^2 = 0, a_ib_i = -b_ia_i = \omega, a_ib_j = -b_ja_i = 0 \text{ for } i \neq j.$$

Note that we clearly have $\mathrm{cat}(\Sigma_g) = 2$. Considering the zero-divisors $\alpha_i = a_i \otimes 1 - 1 \otimes a_i$, $\beta_j = b_j \otimes 1 - 1 \otimes b_j$ in $H^*(\Sigma_g;\mathbb{Q}) \otimes H^*(\Sigma_g;\mathbb{Q})$, we can check that, for $i \neq j$,

$$\alpha_i\beta_i\alpha_j\beta_j = 2\omega \otimes \omega \neq 0.$$

As a consequence $\mathrm{zcl}_{\mathbb{Q}}(\Sigma_g) \geq 4$ for $g \geq 2$, which permits us to conclude that $\mathrm{zcl}_{\mathbb{Q}}(\Sigma_g) = \mathrm{TC}(\Sigma_g) = 4$.

We now turn to the non-orientable surfaces. For $g \geq 1$, we denote by N_g the non-orientable surface of genus g, which can be described as the connected sum of g copies of the real projective plane $\mathbb{R}P^2$. In particular, $N_1 = \mathbb{R}P^2$ and $N_2 = \mathbb{R}P^2\#\mathbb{R}P^2$ is the Klein bottle.

The cohomology of N_g with coefficients in $\mathbb{Z}_2$ is given by $H^0(N_g;\mathbb{Z}_2) = \langle 1\rangle$, $H^2(N_g;\mathbb{Z}_2) = \langle\omega\rangle$, and

$$H^1(N_g;\mathbb{Q}) = \langle a_1, \ldots, a_g\rangle,$$

with the multiplicative structure

$$a_i^2 = \omega, \quad a_ia_j = a_ja_i = 0 \text{ for } i \neq j.$$

The LS-category is then easy to determine ($\mathrm{cat}(N_g) = 2$) since we have $\mathrm{cl}_{\mathbb{Z}_2}(N_g) = \dim(N_g) = 2$ for any $g \geq 1$. Regarding zero-divisors with coefficients in $\mathbb{Z}_2$ we can check that

$$(a_i \otimes 1 - 1 \otimes a_i)^3 = \omega \otimes a_i + a_i \otimes \omega \neq 0$$

and that any product of 4 zero-divisors vanishes. Consequently, $\mathrm{zcl}_{\mathbb{Z}_2}(N_g) = 3$ and

$$3 = \mathrm{zcl}_{\mathbb{Z}_2}(N_g) \leq \mathrm{TC}(N_g) \leq 2\dim(N_g) = 4.$$

It then follows that $\mathrm{TC}(N_g)$ is either 3 or 4. The topological complexity of $N_1 = \mathbb{R}P^2$ has been determined by Farber, Tabachnikov, and Yuzvinsky in 2003:

Theorem 6 ([FTY03])**.** $\mathrm{TC}(\mathbb{R}P^2) = 3$.

The inequality $\mathrm{TC}(\mathbb{R}P^2) \leq 3$ can be obtained through an explicit open cover of $\mathbb{R}P^2 \times \mathbb{R}P^2$ with 4 local sections of the evaluation map as described in [FTY03]. However, it is worth noting the following more general result on the topological complexity of the n-dimensional real projective space $\mathbb{R}P^n$ which was established in [FTY03]:

Theorem 7 ([FTY03])**.** *For n distinct from 1, 3, 7, $\mathrm{TC}(\mathbb{R}P^n)$ is the least integer k such that there exists an immersion of $\mathbb{R}P^n$ in $\mathbb{R}^k$. For $n \in \{1, 3, 7\}$, $\mathrm{TC}(\mathbb{R}P^n) = \mathrm{cat}(\mathbb{R}P^n) = n$.*

This remarkable result shows that $\mathrm{TC}(\mathbb{R}P^n)$ coincides with the so-called immersion dimension of $\mathbb{R}P^n$ and, as is well known, $\mathbb{R}P^2$ can be immersed in $\mathbb{R}^3$ but not in $\mathbb{R}^2$. Although many values of this immersion dimension are known, the complete determination of this number as a function of n is still an open problem. It then turns out that, while the LS-category of $\mathbb{R}P^n$ is easy to calculate ($\mathrm{cat}(\mathbb{R}P^n) = \mathrm{cl}_{\mathbb{Z}_2}(\mathbb{R}P^n) = \dim(\mathbb{R}P^n) = n$), the topological complexity of this space can be very difficult to determine. Surprisingly, the determination of $\mathrm{TC}(N_g)$ for $g \geq 2$ has also revealed to be less immediate than that of $\mathrm{cat}(N_g)$. In 2016, Dranishnikov established that $\mathrm{TC}(N_g) = 4$ for $g \geq 4$ and showed that his methods do not extend to the lower genus cases $g \in \{2, 3\}$ ([D16], [D17]). The general case has been solved in 2017:

Theorem 8 ([CV17])**.** *For $g \geq 2$, $\mathrm{TC}(N_g) = 4$.*

We briefly describe the method used in [CV17] where it is proved that the topological complexity of the Klein bottle N_2 is 4 with an argument which permits one to inductively prove that $\mathrm{TC}(N_g) = 4$ for any $g \geq 2$.

As the calculation mentioned above shows, the lower bound given by the zero-divisor cup-length of N_g with coefficients in $\mathbb{Z}_2$ does not permit one to complete the calculation of $\mathrm{TC}(N_g)$ for $g \geq 2$. However, a twisted coefficients version of the zero-divisor cup-length has revealed to be sufficient.

Recall that a system of twisted (or local) coefficients on a space Y is given by a module M over the group ring

$$\mathbb{Z}[\pi] = \{ \sum_{finite} n_i a_i \mid n_i \in \mathbb{Z}, a_i \in \pi \}$$

where $\pi = \pi_1(Y)$ is the fundamental group of Y. In other words, M is an abelian group with an action of π.

Let M be a system of twisted coefficients on $X \times X$ and let $M|X$ be the system induced by the diagonal map $\Delta : X \to X \times X$, $x \mapsto (x, x)$. With such coefficients a cohomology class $u \in H^*(X \times X; M)$ is a zero-divisor if

$$\Delta^*(u) = 0 \in H^*(X; M|X)$$

where Δ^* denotes the morphism induced in cohomology by Δ. This is a generalization of the notion of zero-divisor considered above since, when $M = \Bbbk$ (with the trivial action of $\pi_1(X \times X) \cong \pi_1(X) \times \pi_1(X)$), the kernel of the cup-product $H^*(X; \Bbbk) \otimes H^*(X; \Bbbk) \to H^*(X; \Bbbk)$ can be identified with the kernel of Δ^* through the Künneth isomorphism $H^*(X \times X; \Bbbk) \cong H^*(X; \Bbbk) \otimes H^*(X; \Bbbk)$. Moreover, as shown in [F08], one has $\mathrm{TC}(X) \geq k$ whenever the cup-product of k zero-divisors $u_i \in H^*(X \times X; M_i)$ is non-zero.

In [CF10], Costa and Farber associate with a space X a zero-divisor $\mathfrak{v} = \mathfrak{v}_X \in H^1(X \times X; I(\pi))$ where $\pi = \pi_1(X)$ and $I(\pi) = \{\sum n_i a_i \in \mathbb{Z}[\pi] \mid \sum n_i = 0\}$ is the augmentation ideal, which is a $\mathbb{Z}[\pi \times \pi]$-module through the action given by:

$$(a, b) \cdot \sum n_i a_i = \sum n_i (a a_i b^{-1}).$$

Here $n_i \in \mathbb{Z}$ and $a, b, a_i \in \pi$.

Through a calculation at the chain/cochain level using the bar resolution associated with a discrete group, it is shown in [CV17] that the fourth power of $\mathfrak{v}$ is not zero when $X = N_2$ is the Klein bottle and that consequently $\mathrm{TC}(N_2) = 4$. Using next the map $N_g \to N_{g-1}$ which collapses the last summand of $N_g = \mathbb{R}P^2 \# \cdots \# \mathbb{R}P^2$ and the associated morphisms (in cohomology, between the fundamental groups...), an inductive argument permits one to see that the fourth power of the class $\mathfrak{v}$ associated to N_g does not vanish for all $g \geq 2$. Consequently, $\mathrm{TC}(N_g) = 4$ for $g \geq 2$.

As a final remark, we note that except S^2 and $\mathbb{R}P^2$ all the surfaces above are aspherical, which means that their only possibly non-zero homotopy group is their fundamental group. Many interesting spaces are aspherical spaces and many works focus on the study of the topological complexity of such spaces, not only with the aim to calculate this invariant for specific examples but also with the general goal to better understand its properties. The homotopy type of an aspherical space, and therefore its LS-category and its topological complexity, is completely determined by its fundamental group. By a theorem of Eilenberg and Ganea [EG57], the LS-category of such a space is known to equal to the cohomological dimension of its fundamental group. The problem, posed by Farber in [F06], of finding such an expression of the topological complexity of an aspherical space in terms of algebraic invariants of its fundamental group has become a challenging open problem on the invariant TC.

References

[CV17] D. Cohen and L. Vandembroucq, *Topological complexity of the Klein bottle*, J. Appl. and Comput. Topology **1** (2017), 199–213.

[CLOT03] O. Cornea, G. Lupton, J. Oprea, and D. Tanré, *Lusternik-Schnirelmann category*, Mathematical Surveys and Monographs, 103. American Mathematical Society, Providence, RI, 2003.

[CF10] A. Costa and M. Farber, *Motion planning in spaces with small fundamental groups*, Commun. Contemp. Math. **12** (2010), 107–119.

[D16] A. Dranishnikov, *The topological complexity and the homotopy cofiber of the diagonal map for non-orientable surfaces*, Proc. Amer. Math. Soc. **144** (2016), 4999–5014.

[D17] A. Dranishnikov, *On topological complexity of non-orientable surfaces*, Topology Appl. **232** (2017), 61–69.

[EG57] S. Eilenberg and T. Ganea. *On the Lusternik-Schnirelmann category of abstract groups*, Ann. of Math. (2) **65** (1957), 517–518.

[F03] M. Farber, *Topological complexity of motion planning*, Discrete Comput. Geom. **29** (2003), 211–221.

[F04] M. Farber, *Instabilities of robot motion*, Topology Appl. **140** (2004), 245–266.

[F06] M. Farber, *Topology of robot motion planning*, Morse theoretic methods in nonlinear analysis and in symplectic topology, 185-230, NATO Sci. Ser. II Math. phys. Chem., 217, Springer, 2006.

[F08] M. Farber, *Invitation to topological robotics*, Zürich Lectures in Advanced Mathematics, European Mathematical Society (EMS), Zürich, 2008.

[FTY03] M. Farber, S. Tabachnikov, and S. Yuzvinsky, *Topological robotics: motion planning in projective spaces*, Int. Math. Res. Not. **34** (2003), 1853–1870.

[H07] J.-C.Hausmann, *Geometric descriptions of polygon and chain spaces*, Contemp Math. 438 (2007) 47–57.

[JS99] D. Jordan and M. Steiner, *Configuration spaces of mechanical linkages*, Discrete and Comput. Geom. **22** (1999), 297–315.

[JS01] D. Jordan and M. Steiner, *Compact surfaces as configuration spaces of mechanical linkages*, Israel Journal of Math. **122** (2001), 175–187.

[KM02] M. Kapovich and J. Millson, *Universality theorems for configuration spaces of planar linkages*, Topology **41** (2002), 1051–1107.