

# Endogenous Growth: Analytical Review of its Generating Mechanisms

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## Abstract

This paper consists of an analytical review of the most relevant endogenous growth models. The objective of this literature review is to discuss analytically and understand, in an integrated form, the main mechanisms, identified in the existing literature, that generate endogenous growth.

Endogenous or new growth theory has, so far, produced three main types of mechanisms through which endogenous sustained positive economic growth is made possible.

One strategy brings a theory of innovations or R&D into the growth model. In this type of model, endogenously determined technological progress is the engine of economic growth.

The second mechanism delivers sustained positive growth through the introduction of an endogenously determined accumulation of human capital. In this kind of model, the source of long-run per-capita growth is human capital accumulation.

And a third way to obtain endogenous growth is simply to abandon one of the standard assumptions of the neoclassical model, more precisely the assumption of diminishing returns to capital.

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# 1 Introduction

This paper consists of an analytical review of the most relevant endogenous growth models. The objective of this literature review is to discuss analytically and understand, in an integrated form, the main mechanisms, identified in the existing literature, that generate endogenous growth.

Classical economic growth theory states that sustained positive growth is achieved whenever a non-declining marginal productivity of capital is attained. In this sense, Solow's [1956] neoclassical model demonstrates that, with labour constant, technological progress can overcome the effects of diminishing returns to capital and thus deliver sustained positive per-capita growth in the long-run, with per-capita output growing at the same rate as the rate of technological progress.

The rate of technological progress in Solow's model is exogenous, which means that the neoclassical model fails to explain how the key parameter of a growth model - the economic growth rate - is generated. Consequently, in Solow's model, neither tastes nor policies are able to influence the long-run per-capita growth rate of the economy.

Even though Solow [2000] argues that every area of economic theory has to rest on some exogenous elements, he himself agrees that it is not entirely satisfactory that the theory of economic growth regards economic growth as exogenous.

These results have led to further research on how to endogenise the growth rate. Such research gave rise to endogenous growth theory. Having started with the well known papers of Paul Romer [1986] and Robert Lucas [1988], this new growth theory is already vast and continues to be a very active research field.

As Solow [2000] describes, the endogenous or new growth theory has, so far, produced three main types of mechanisms through which endogenous sustained positive economic growth is made possible.

One strategy, first introduced by Romer [1987,1990], brings a theory of innovations or R&D into the growth model. In this type of model, endogenously determined technological progress is the engine of economic growth.

The second mechanism, owed to Lucas [1988], delivers sustained positive growth through the introduction of an endogenously determined accumulation of human capital. That is, in this kind of model, the source of long-run per-capita growth is human capital accumulation.

And a third, more direct, way to obtain endogenous growth is simply to abandon one of the standard assumptions of the neoclassical model, more precisely the assumption of diminishing returns to capital. This is experimented by Jones and Manuelli [1990].

In this paper, we propose to analyse these three alternative ways of generating endogenous growth.

We will attempt to dissect the above referred models, so that we can clearly expose the roots of endogenously sustained positive long-run economic growth.

With this study, we also aim at providing an integrated, comprehensive and global view over endogenous growth theory.

Hence, in order to better compare the differences between the three main types of endogenous growth models, we adapt the models under our analysis, so that: (1) they all assume a constant population, and (2) they all have a common production function, namely a Cobb-Douglas function with labour-augmenting productivity.

We finalise the paper with a discussion of some limitations that characterise the endogenous growth models analysed in our literature review.

This paper is organised as follows. After this Introduction, Section 2 discusses Solow's [1956] neoclassical model, the starting point of all studies on economic growth. Section 3 analyses Romer's [1990] R&D or idea-based model and the mechanism through which R&D generates endogenously sustained growth. Section 4 discusses Lucas' [1988] model, and investigates the ways in which human capital accumulation leads to endogenous growth. Section 5 analyses Jones and Manuelli's [1990] and Barro and Sala-i-Martin's [1995, Chp.5, page 172] models and the elimination of the diminishing returns to capital assumption as their means to obtain sustained economic growth. The models analysed are compared in Section 6. In Section 7, we analyse the models by Grossman and Helpman [1991] and Aghion and Howitt [1992]. Section 8 is dedicated to a discussion of some limitations of endogenous growth models. We close the analytical literature review with some Final Remarks.

## 2 Solow's Standard Model

Solow's [1956] model is the starting point for almost all studies on growth. Even models that depart fundamentally from Solow's assumptions can be best understood through comparison with the Solow model.

We discuss this exogenous growth model with the purpose of clearly exposing the root of sustained positive per-capita growth.

Such positive sustained growth is achieved in any growth model that is able to obtain a non-declining marginal productivity of capital, for constant labour. In Solow's model, a constant marginal productivity of capital is made possible because of technological progress. Let us analyse how this is obtained:

The neoclassical model is set up for a closed economy with competitive markets, identical rational agents and a production function for the single good  $Y_t$  of the form:

$$Y_t = K_t^\alpha (A_t L_t)^{1-\alpha} \quad , \quad 0 < \alpha < 1 \quad (1)$$

Variable  $A_t$  represents the state of technology,  $K_t$  is the capital stock, and  $L_t$  is the labour force, assumed to be equal to the economy's population.

This production function assumes that technology is labour-augmenting. Barro and Sala-i-Martin [1995, Chp.1] point out that technological progress

must take the labour-augmenting form in the production function if the models are to display a steady-state.

The optimising version of consumers behaviour is adopted here. The optimising version means that the immortal representative consumer is dedicated to planning optimally, that is, he/she wishes to maximise the present discounted value of the utilities of his/her present and future consumption streams. That is, preferences over consumption streams are described by:

$$\text{Max} \int_0^{\infty} U(C_t)e^{-\rho t} dt \quad , \quad U(C_t) = \frac{C_t^{1-\sigma}}{1-\sigma}, \quad (2)$$

where real consumption is a stream  $C_t$  of units of the single good produced, and the discount rate  $\rho$  and the coefficient of risk aversion  $\sigma$  are both positive.

A second branch of growth literature assumes that consumers save a fixed amount of output. Solow [2000] refers to this alternative form of consumption/savings specification as the “behaviouristic” version of savings.

As analysed by Helpman [1992], both forms of saving lead to the same result that sustained positive per-capita long-run growth is obtained if physical capital can be accumulated forever without decreasing its marginal productivity. We will further analyse Solow’s model, with the “behaviouristic” version of savings, in Section 5.

Turning now to the form of the utility function adopted for most growth models:

$$U(C_t) = \frac{C_t^{1-\sigma}}{1-\sigma} \quad , \quad \sigma > 0$$

As Romer [1996, Chp. 2] analyses, this is a constant-relative-risk-aversion (CRRA) utility function. The coefficient of relative risk aversion is:

$$-\frac{C \frac{d^2U(C)}{dC^2}}{\frac{dU(C)}{dC}} = \sigma,$$

which is the reciprocal of the elasticity of intertemporal substitution.

Romer [1996, Chp. 2] further analyses that when  $\sigma$  is close to zero, the utility function is almost linear in  $C_t$ . And when  $\sigma$  is close to one, the utility function approaches  $\ln C_t$ .

Additionally, he explains that if  $\sigma < 1$ , then  $C_t^{1-\sigma}$  is increasing in  $C_t$ . Whereas if  $\sigma > 1$ , then  $C_t^{1-\sigma}$  is decreasing in  $C_t$ . So, dividing  $C_t^{1-\sigma}$  by  $1-\sigma$  ensures that the marginal utility of consumption:

$$\frac{dU(C)}{dC} = C^{-\sigma}$$

is positive regardless of the value of  $\sigma$ .

Most growth models adopt this isoelastic utility function in order to obtain a balanced growth path solution. They do this because, as pointed out by Barro and Sala-i-Martin [1995], the result of a balanced growth path solution agrees

with the empirical observation that many developed countries achieve per-capita growth rates which are positive and trendless for long periods of time. Hence it is believed that a useful growth theory should predict that the per-capita growth rate approaches a constant in the long-run.

Continuing with the description of the standard growth model. Output  $Y_t$  is divided between aggregate consumption  $C_t$  and capital accumulation  $\dot{K}_t$ . Capital depreciation is assumed to be zero, for simplicity. Hence, the closed economy's budget constraint is:

$$Y_t = C_t + \dot{K}_t \quad (3)$$

The resource allocation problem faced by the social planner of this economy consists in maximising utility 2 subject to the budget constraint 3. This translates into maximising the *current-value Hamiltonian*  $H$  defined by:

$$H_t = \frac{C_t^{1-\sigma} - 1}{1-\sigma} + \theta_t [K_t^\alpha (A_t L_t)^{1-\alpha} - C_t]$$

An optimal allocation must maximise the expression  $H$  at each date  $t$ , provided the *current-value* of capital accumulation,  $\theta_t$ , is chosen correctly. The solution is obtained by imposing the following conditions:

First-Order Condition:

$$\frac{dH_t}{dC_t} = 0 \quad (4)$$

Co-State Condition:

$$\frac{dH_t}{dK_t} = \rho\theta_t - \dot{\theta}_t \quad (5)$$

Transversality Condition:

$$\lim_{t \rightarrow \infty} e^{-\rho t} \theta_t K_t = 0 \quad (6)$$

As explained by Lucas [1988], the first-order condition 4 says that at every instant of time the shadow price of investment  $\theta_t$  must be equal to the marginal utility of consumption. It means that at each instant of time, output will be best allocated between consumption and investment when the marginal gain from a unit increase in consumption is just equal to the marginal loss of a unit decrease in investment.

The co-state equation 5 is the Fisher equation that must also hold at every moment of time. It says that the sum of the marginal product of capital and the capital gain per unit of capital must equal the pure rate of time preference.

The transversality condition 6 means that the value of capital must tend to zero. It decides which of the one-parameter family of solutions given by 4 and 5 for one initial condition  $K(0) = K_0$ , is the right one.

Now, conditions 4 and 5 are developed:

$$\begin{aligned}\frac{dH}{dC} &= 0 \\ \Leftrightarrow C^{-\sigma} &= \theta \\ \Leftrightarrow \frac{\dot{C}}{C} &= -\frac{1}{\sigma} \frac{\dot{\theta}}{\theta}\end{aligned}$$

$$\begin{aligned}\frac{dH}{dK} &= \rho\theta - \dot{\theta} \\ \Leftrightarrow \theta\alpha K^{\alpha-1}(AL)^{1-\alpha} &= \rho\theta - \dot{\theta} \\ \Leftrightarrow \frac{\dot{\theta}}{\theta} &= \rho - \alpha K^{\alpha-1}(AL)^{1-\alpha}\end{aligned}$$

The problem is solved for a particular solution to the model - the balanced growth path - which is the solution  $(K_t, \theta_t, C_t)$  such that the growth rates of these three variables are constant. This solution is constructed using equations 3, 4 and 5.

So, equations 4 and 5 together give:

$$g_C = \frac{1}{\sigma}[\alpha K^{\alpha-1}(AL)^{1-\alpha} - \rho] \quad (7)$$

Since  $\sigma$  and  $\rho$  are constants, equation 7 says that a balanced growth path with a positive growth rate requires a constant marginal productivity of capital,  $\alpha K^{\alpha-1}(AL)^{1-\alpha}$ , above the discount rate  $\rho$ . That is:

$$\begin{aligned}g_C = 0 &\Rightarrow [\alpha K^{\alpha-1}(AL)^{1-\alpha}] = 0 \\ \Leftrightarrow (\alpha - 1)g_K + (1 - \alpha)(g_A + g_L) &= 0,\end{aligned} \quad (8)$$

We now consider two cases:

## 2.1 Case 1: Solow's Model Without Technological Progress

Assume first that  $L_t$  and  $A_t$  are constant, that is,  $g_L = 0$  and  $g_A = 0$ . In this case, condition 8 says that:

$$(\alpha - 1)g_K + (1 - \alpha)(g_A + g_L) = 0 \Leftrightarrow g_K = 0,$$

that is, with  $g_A = 0$  and  $g_L = 0$ , the balanced growth path solution has  $g_K = 0$ .

Then, log-differentiation of the production function 1 gives us the growth rate of output:

$$\begin{aligned}Y &= K^\alpha(AL)^{1-\alpha} \\ \Rightarrow \\ g_Y &= \alpha g_K + (1 - \alpha)(g_A + g_L) = 0\end{aligned}$$

Finally, equation 7 can be rewritten to give:

$$\frac{Y}{K} = \frac{1}{\alpha} \frac{dY}{dK} = \frac{C + \dot{K}}{K} = \frac{\sigma g_C + \rho}{\alpha} \quad (9)$$

With  $g_K$  and  $g_L$  equal to zero, equation 9 implies that the growth rate of consumption must also equal zero, because  $\frac{\sigma g_C + \rho}{\alpha}$  is a constant.

For a constant population, per-capita variables grow at the same rate as their aggregate counterparts, so the balanced growth path solution of Solow's model for  $A$  constant and  $L$  constant is:

$$g_C = g_Y = g_K = g_c = g_y = g_k = 0 \quad (10)$$

This economy displays zero growth in the long-run because of diminishing returns to capital. In fact, the marginal productivity of capital is:

$$\frac{dY}{dK} = \alpha K^{\alpha-1} (AL)^{1-\alpha} = \frac{\alpha (AL)^{1-\alpha}}{K^{1-\alpha}},$$

which decreases as  $K$  increases, because the numerator is constant whilst the denominator grows.

Lucas [1988] writes that solving this optimisation problem for its transitional dynamics would show that accumulation of capital  $K$  will eventually drive its marginal productivity down until it equals  $\rho$ , which means that the economy reaches and stays in a steady-state level of  $K$ ,  $K^*$ , for which the economy does not grow.

## 2.2 Case 2: Solow's Model With Technological Progress

Let us now analyse the case in which technology grows at a positive rate  $g_A > 0$ . Population is held constant,  $g_L = 0$ . In this case, condition 8 implies:

$$\begin{aligned} [\alpha K^{\alpha-1} (AL)^{1-\alpha}] &= 0 \\ &\Leftrightarrow \\ (\alpha - 1)g_K + (1 - \alpha)(g_A + g_L) &= 0 \\ &\Leftrightarrow \\ g_K &= g_A, \end{aligned}$$

and log-differentiation of the production function 1 gives:

$$g_Y = \alpha g_K + (1 - \alpha)(g_A + g_L) = g_A$$

Equation 9 then gives the growth rate of consumption:

$$\begin{aligned} \frac{C + \dot{K}}{K} &= \frac{C}{K} + \frac{\dot{K}}{K} = \text{constant} \\ \Rightarrow g_C = g_K &= g_A \end{aligned} \quad (11)$$

As  $L$  is constant, per-capita variables grow at the same rate as the aggregate variables, that is:

$$g_c = g_k = g_y = g_A$$

Another important feature of this solution is that when the economy reaches the balanced growth path, it will be saving and investing a constant fraction of its income, which is:

$$\frac{I}{Y} = \frac{\dot{K}}{Y} = \frac{g_K}{\frac{Y}{K}} = \frac{\alpha g}{\frac{dY}{dK}} = \frac{\alpha g}{\sigma g + \rho} \quad (12)$$

Concluding, when  $g_A$  is positive output per-capita grows at a positive constant rate in the long-run. What is happening here is that, because there is technological progress, capital can grow in the long-run without decreasing its marginal productivity. That is, technological progress overcomes diminishing returns to capital. The marginal productivity of capital can then be permanently sustained (as a constant) above  $\rho$ , allowing for sustained positive per-capita growth. Let us see this:

$$\frac{dY}{dK} = \alpha K^{\alpha-1} (AL)^{1-\alpha} = \frac{\alpha A^{1-\alpha} L^{1-\alpha}}{K^{1-\alpha}}$$

For  $L$  constant, with  $g_K = g_A$ , the marginal productivity of  $K$  is held constant. This delivers sustained positive per-capita growth in the long-run in the neoclassical growth model.

Barro and Sala-i-Martin [1995, Chp. 2, pages 72-80], rewrite Solow's model in terms of consumption, capital and output per effective-labour  $(\frac{C}{AL}, \frac{K}{AL}, \frac{Y}{AL})$ , and analyse the transitional dynamics towards the steady-state of the model, with the use of phase diagrams. They show that the system exhibits saddle-path stability.

This study of Solow's model allows us to conclude that sustained positive per-capita growth in the long-run can be achieved through sustained technological progress.

In Solow's model, the rate of technological progress is exogenously provided, and therefore not explained by the model. Hence this model is called an exogenous growth model. For this reason, in this neoclassical model, the preference parameters  $\sigma$  and  $\rho$  do not influence the equilibrium growth rate. Neither does the technological parameter  $\alpha$ . These parameters do influence the steady-state value of the investment-output ratio, but not the equilibrium growth rate. In fact, the only parameter that affects the growth rate is  $g_A$ , which is exogenous to the model.

In order to better compare Solow's model with Romer's [1990] model, which is analysed next, we represent, in Figure 1, Solow's economy in the space  $(r, g)$ .

In the absence of capital depreciation, the interest rate is equal to the marginal productivity of capital. So curve P (Preferences) represents the equation 7 and describes the consumption side in terms of balanced growth paths



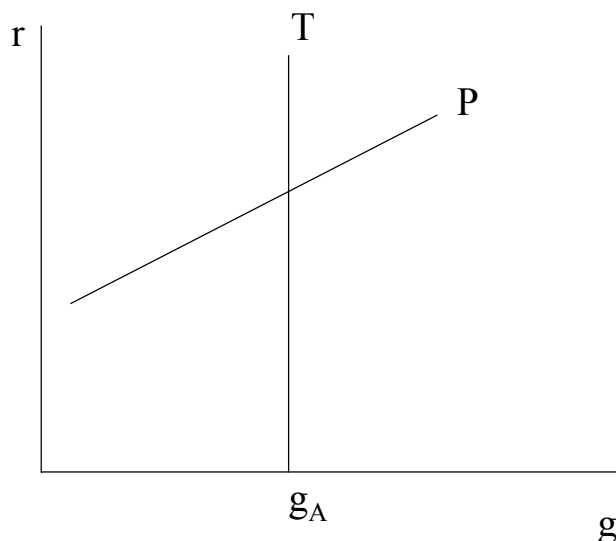


Figure 1:

expressed in terms of pairs of  $(r, g)$ . And curve T (Technology) describes the production side in terms of balanced growth paths given by pairs of  $(r, g)$ <sup>1</sup>. As we can see, the equilibrium growth rate, that results from the crossing of the two curves, is given by the exogenous variable  $g_A$ .

Notice, however, that as Solow [2000] points out, to treat a parameter as exogenous is not the same thing as to treat it as a permanent constant or as something inexplicable.

We can, in fact, advance reasons for an increase in  $g_A$ , especially *a posteriori*. But whatever reasons we come up with, they cannot be explained within Solow's model. That is, the exogenous growth model cannot itself explain whether nor how tastes or policies have an impact on the long-run per-capita growth rate.

This limitation of Solow's neoclassical model has led to further investigation into the fundamental questions of growth. In particular, growth theorists have tried to endogenise the engine of growth, that is, to have the engine of growth determined within the model.

From their attempts to create a systematic and generally acceptable theory, these research activities gave rise to the endogenous or new growth theory.

One group of endogenous growth theorists believe that technological progress is the driving force of growth, and in this sense they agree with Solow's result. But whereas in the neoclassical model technological progress is exogenous to the model, the new growth models have it determined within the model. In this group of endogenous growth models, technological progress arises as a result

<sup>1</sup>The curves are named Preferences and Technology after Rivera-Batiz and Romer [1991].

of research and development (R&D) activities. Romer [1990] provides the first fully-articulated growth model based on R&D, and this branch of the literature also includes the important initial contributions by Grossman and Helpman [1991] and Aghion and Howitt [1992]. The models that fall into this category are called R&D-based or idea-based growth models.

The second main stream of endogenous growth models were built under the belief that the engine of growth is human capital accumulation. This branch of the literature was fathered by Lucas [1988], with further contributions by Becker, Murphy and Tamura [1990] and Stokey [1991]. In our review, we name these kind of models as human capital-based growth models.

Endogenous growth literature also includes a third category of models that produce endogenous growth via the straightforward elimination, from the production function, of the assumption of diminishing returns to physical capital. Jones and Manuelli's [1990] and King and Rebelo's [1990] models are the outstanding examples of such models.

This review continues with the analyses of the models that classically represent these three groups of endogenous growth models.

### 3 R&D-Based Growth Models

Until the second half of the eighties, attempts to explain technological progress were not very successful. The inclusion of a theory of technological progress in the neoclassical framework is not an easy task, because the standard competitive assumptions cannot be maintained. In fact, the returns to scale of the output production function  $Y = K^\alpha (AL)^{1-\alpha}$  tend to be increasing if technology  $A$  is a factor of production.

Previous growth models have avoided this difficulty in many ways. Shell [1967] treats  $A$  as a public input that is provided by the government and therefore receives no compensation. Arrow [1962] and Sheshinski [1967] assume that ideas,  $A$ , are unintended by-products of production or investment, a mechanism described as learning-by-doing. And each persons' discoveries automatically spillover to the whole economy, a process known as knowledge spillovers<sup>2</sup>.

As discussed by Helpman [1992], these models do not capture the deliberate efforts to develop new products and technologies. And the impressive developments in consumer electronics, computers and pharmaceuticals are good examples of the rising importance of deliberate R&D efforts to develop new products and technologies in today's industrial economies.

As the competitive framework breaks down when discoveries depend on intentional R&D effort, and firms can enjoy the exclusivity of their inventions through the use of patent rights, a decentralised theory of technological progress requires basic changes to the neoclassical model in order to incorporate imperfect competition. This was first done by Romer [1987, 1990].

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<sup>2</sup>Romer [1990] notes that in Arrow's model, the nonrival input produced through learning must be completely nonexcludable, otherwise Dasgupta and Stiglitz [1988] show that the decentralised equilibrium with many firms would not be sustainable.

Romer [1987, 1990] proposed to model profit-seeking firms' R&D efforts. He followed Ethier [1982] in reinterpreting the utility function used by Dixit and Stiglitz [1977] as a production function, to capture a preference for variety. In this reinterpretation, final output is an increasing function of the total number of differentiated capital goods used by a final goods producer.

A key feature of Romer's model is the introduction of imperfect competition in the capital goods sector, which makes it possible to model firms' behaviour as engaging in deliberate research activities and thereby being compensated with monopoly rents for a successful innovation.

By introducing profit-seeking research behaviour in the growth model, Romer generates an explanation for technological progress inside the model. That is, sustained positive per-capita growth, possible because of technological progress, can now be explained within the model. This fact makes Romer's model an endogenous growth model. We now analyse Romer's [1990] model, in the version presented by Jones [1995] and Aghion and Howitt [1998, Chp.1].

On the preferences side of Romer's decentralised model, homogeneous consumers maximise, subject to a budget constraint, the discounted value of their utility:

$$\text{Max} \int_0^{\infty} e^{-\rho t} U(C_t) dt \quad , \quad U(C_t) = \frac{C_t^{1-\sigma}}{1-\sigma},$$

where variable  $C_t$  is aggregate consumption in period  $t$ ,  $\rho$  is the rate of time preference, and  $\frac{1}{\sigma}$  is the elasticity of substitution between consumption at two periods of time.

The representative consumer facing a constant interest rate  $r$ , chooses to have consumption growing at the constant rate  $g_C$  given by the Euler equation<sup>3</sup>:

$$g_C = \frac{\dot{C}}{C} = \frac{1}{\sigma}(r - \rho) \tag{13}$$

On the production side, the model can be understood as having three sectors. The final good sector, the capital goods sector and the R&D sector.

The final good  $Y$  is produced using as inputs labour devoted to final output,  $L_Y$  and a number  $A$  of differentiated durable capital goods,  $i$ , each produced in quantity  $x(i)$ . All capital goods have additively separable effects on output<sup>4</sup>. The production function is then:

$$Y_t = L_{Yt}^{1-\alpha} \int_0^{A_t} x_t(i)^\alpha di \tag{14}$$

For  $A$  constant, the production function displays constant returns to scale in  $L_Y$  and  $x(i)$ , and diminishing returns in  $x(i)$ , for  $L_Y$  fixed. But in this model  $A$  is productive as well. Therefore, technological growth, that is, continuous

<sup>3</sup>In Section 8, we derive the Euler Equation.

<sup>4</sup>See Grossman and Helpman [1991, Ch.5] for an alternative description.

increases in  $A$ , avoids the tendency for diminishing returns to increases in  $x(i)$ . This delivers endogenous growth, as will be shown later.

Capital accumulation is given by:

$$\dot{K}_t = Y_t - C_t$$

Assuming that it takes one unit of foregone consumption to produce one unit of any type of capital good,  $K$  is related to the capital goods according to the following rule:

$$K_t = \int_0^{A_t} x_t(i) di$$

The process for accumulation of new designs, that is, the production function of new designs is defined as:

$$\dot{A}_t = \delta L_{At} A_t, \quad (15)$$

where  $L_A$  is total labour employed in research.

Variables  $L_A$  and  $L_Y$  are related by the constraint:

$$L_{At} + L_{Yt} = L_t,$$

so that any person can devote labour either to the final good sector or to the research sector.

According to specification 15, all researchers have access to the total stock of knowledge  $A$  (equal to the total number of designs). In this model, knowledge enters production in two distinct ways. Firstly, a new design corresponds to a new capital good which is used to produce final goods. Additionally, a new design increases the total stock of knowledge and therefore increases the productivity of labour in the research sector.

The owner of a design has property rights over the production of the respective capital good but not over the use of the created design in the research sector. That is, knowledge is a nonrival good that is partially excludable and privately provided.

This equation relies upon important assumptions. Firstly, it assumes that devoting more labour to R&D leads to a higher growth rate of  $A$ .

Secondly, the higher the total stock of  $A$ , the higher the marginal productivity of a researcher.

The third very important assumption is that the output of designs is linear in  $A$ . This is the assumption that makes possible the existence of a balanced growth path, that is, an equilibrium with a constant growth rate for  $A$ ,  $K$ ,  $Y$ , and  $C$ .

Continuing with the description of the model. Being in a perfect competition environment, final good producers rent each capital good according to the profit maximisation rule:

$$\frac{dY_t}{dx_t(i)} = R_t(i),$$

where  $R(i)$  is the rental price of each capital good.

This gives the inverse demand curve faced by each capital good producer:

$$R_t(i) = \alpha L_Y^{1-\alpha} x_t(i)^{\alpha-1} \quad (16)$$

With given values of  $r$  and  $L_Y$ , each capital good producer, who has already incurred the fixed cost investment in a design,  $P_A$ , and has the patent on it, will maximise its revenue minus variable cost at every date:

$$\text{Max} \quad \pi_t(i) = R_t(i)x_t(i) - r_t x_t(i)$$

With a constant marginal cost and a constant elasticity demand curve, this monopolistic competitor solves his problem by charging a monopoly price which is a markup over marginal cost. The markup is determined by the elasticity of demand<sup>5</sup>:

$$\begin{aligned} \text{Max} \quad \pi(i) &= \alpha L_Y^{1-\alpha} x(i)^\alpha - r x(i) \\ &\Rightarrow \\ \frac{d\pi(i)}{dx(i)} &= \alpha^2 L_Y^{1-\alpha} x(i)^{\alpha-1} - r = 0 \\ &\Leftrightarrow \\ R(i) &= \frac{r}{\alpha} \end{aligned}$$

The idea is that a firm incurs a fixed cost when it produces a new capital good. It recovers this cost by selling its good for a price  $R(i)$  that is higher than its constant marginal cost. The fixed cost is the defining characteristic of this technology.

The decision to produce a new capital good depends on the comparison between the discounted stream of net revenues that the patent on this good will bring in the future, and the cost  $P_A$  of the initial investment in a design. The key feature of R&D costs is that they have to be paid up front, before profits can be earned. This time structure introduces natural dynamics in the model.

The market for designs is competitive, so at every date  $t$  the price for designs will be equalised to the present value of the future revenues that a monopolist can extract. This means that capital good producers earn zero profits in a present value sense. The dynamic zero-profit/free-entry condition is then:

$$\begin{aligned} P_{A_t} &= \int_t^{+\infty} e^{-r(\tau-t)} \pi_\tau(i) d\tau \\ &\Leftrightarrow \\ \dot{P}_{A_t} &= r_t P_{A_t} - \pi_t(i), \end{aligned} \quad (17)$$

assuming that there are no bubbles.

Equation 17 can be presented as:

$$r_t P_{A_t} = \pi_t(i) + \dot{P}_{A_t},$$

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<sup>5</sup>The elasticity of demand is  $\alpha - 1$ .

and interpreted in the following way: Firms can choose between putting the monetary value  $P_{At}$  in the bank and earn interest on this deposit,  $r_t P_{At}$ , or buying a patent for the same value and earn the returns of producing the differentiated good,  $\pi_t(i)$ , plus the capital gain/loss of owning that patent,  $P_{At}$ . It is the Fisher equation of this model.

Next, the model is solved for its balanced growth path, the equilibrium for which variables  $A$ ,  $K$ ,  $C$  and  $Y$  grow at constant exponential rates.

According to the Euler equation 13, in a balanced growth path, the interest rate must be constant. Consequently so is  $R(i)$ .

The model is symmetric, that is, all producers have the same technological characteristics and face the same market conditions and consequently will choose the same equilibrium. This implies that  $R(i) = \bar{R} = R$  and  $x(i) = \bar{x} = x$ . Then, we can rewrite the expressions for  $R_t$  and  $x_t$ :

$$R_t = \alpha L_{Yt}^{1-\alpha} x_t^{\alpha-1},$$

and, equivalently:

$$x_t = L_{Yt} \left[ \frac{\alpha^2}{r} \right]^{\frac{1}{1-\alpha}},$$

from which we can observe that in a balanced growth path, with  $L_Y$  constant (required for a balanced growth path, as will be explained below),  $x$  is also constant.

Since all capital good producers produce in the same quantity, total physical capital is equal to:

$$K_t = \int_0^{A_t} x_t(i) di = A_t x_t,$$

and the production function can be rewritten as:

$$Y_t = L_{Yt}^{1-\alpha} A_t x_t^\alpha$$

With  $L_Y$  and  $x$  constant, it is clear from log-differentiation of the two equations above that  $K$  and  $Y$  grow at the same rate as  $A$ .

Now, rewriting the production function so that  $K$  appears distinctively, we have:

$$\begin{aligned} Y &= L_Y^{1-\alpha} A x^\alpha \\ &\Leftrightarrow \\ Y &= L_Y^{1-\alpha} (A x)^\alpha A^{1-\alpha} \\ &\Leftrightarrow \\ Y &= K^\alpha (L_Y A)^{1-\alpha}, \end{aligned}$$

which is similar to Solow's production function.

The marginal productivity of capital is:

$$\frac{dY}{dK} = \frac{\alpha L_Y^{1-\alpha} A^{1-\alpha}}{K^{1-\alpha}}$$

We can observe that, for  $L_Y$  constant, with physical capital  $K$  growing at the same rate as technology  $A$ , the marginal productivity of capital is held constant. Hence this model delivers sustained per-capita growth, in the same way as that predicted by Solow's model. That is, technological progress overcomes diminishing returns to capital. The root of sustained positive growth in Romer's model has thus been identified.

Now, whereas in Solow's model, the growth rate of  $A$  is exogenous to the model, in Romer's model, this growth rate is determined within the model. Let us analyse how this growth rate is endogenously determined:

The engine of growth is given by equation 15, repeated here:

$$\dot{A}_t = \delta L_{At} A_t$$

It implies that:

$$g_A = \delta L_{At},$$

that is, technological progress,  $g_A$ , depends on  $L_A$ , the number of people that choose to work in the research sector. Equation 15 makes it obvious that a balanced growth path solution, that is a solution with a constant growth rate, requires that  $L_A$  remains constant. Thus the existence of a balanced growth equilibrium requires that prices and wages are such that  $L_Y$  and  $L_A$  remain constant as  $A$ ,  $K$ ,  $Y$  and  $C$  grow at a constant exponential rate.

The allocation of workers between the final output and research sectors obeys the labour market equilibrium condition that remuneration of labour must be the same in both sectors.

In the final goods sector, the wage paid to  $L_Y$  is:

$$w_{Yt} = \frac{dY_t}{dL_{Yt}} = (1 - \alpha) L_{Yt}^{-\alpha} A_t x_t^\alpha,$$

and in the research sector, remuneration is:

$$w_{At} = \frac{d\dot{A}_t}{dL_{At}} P_{At} = \delta A_t P_{At}$$

Equality of the two implies that:

$$P_{At} = \frac{(1 - \alpha)}{\delta} L_{Yt}^{-\alpha} x_t^\alpha \tag{18}$$

Log-differentiation of equation 18 shows that in a balanced growth path, as  $L_Y$  and  $x$  are both constant,  $P_A$  is also constant. Hence, with  $\dot{P}_A = 0$ , the

zero-profit condition 17 becomes:

$$\begin{aligned} 0 &= rP_A - \pi \\ &\Leftrightarrow \\ r &= \frac{\pi}{P_A} \end{aligned} \tag{19}$$

Next, recalling equation 16, and the markup rule  $R(i) = \frac{r}{\alpha}$ , we can rewrite the profits expression as:

$$\begin{aligned} \pi &= Rx - rx \\ &= (1 - \alpha)\alpha L_Y^{1-\alpha} x^\alpha, \end{aligned} \tag{20}$$

and, replacing expressions 20 and 18 in equation 19, we get:

$$r = \frac{\pi}{P_A} = \frac{(1 - \alpha)\alpha L_Y^{1-\alpha} x^\alpha}{\frac{(1-\alpha)}{\delta} L_Y^{-\alpha} x^\alpha} \Leftrightarrow r = \delta\alpha L_Y,$$

which is equivalent to:

$$L_Y = \frac{r}{\delta\alpha} \tag{21}$$

Then, it follows that the growth rate of  $A$  is:

$$\begin{aligned} g_A &= \delta L_A \\ &\Leftrightarrow \\ g_A &= \delta(\bar{L} - L_Y) \\ &\Leftrightarrow \\ g_A &= \delta\bar{L} - \frac{r}{\alpha} \end{aligned} \tag{22}$$

As explained before, output and physical capital grow at the same rate as  $A$ . And, as shown below, the capital accumulation equation implies that consumption also grows at the same rate as  $Y$  and  $K$ . That is:

$$\dot{K} = Y - C \Rightarrow \frac{\dot{K}}{K} = \frac{Y}{K} - \frac{C}{K}$$

A constant  $g_K$  implies that:

$$\frac{d\left(\frac{\dot{K}}{K}\right)}{dt} = 0 \Rightarrow \left(\frac{\dot{Y}}{K}\right) = \left(\frac{\dot{C}}{K}\right)$$

which, because  $g_Y = g_K$ , implies that:

$$\left(\frac{\dot{C}}{C}\right) = \left(\frac{\dot{K}}{K}\right)$$



With a constant population, this growth rate is the same as the per-capita growth rate:

$$g_c = g_y = g_k = g_A = g,$$

and it is:

$$g = \delta \bar{L} - \frac{r}{\alpha} \quad (23)$$

We now head for the final step in the resolution of this decentralised economy's problem, that is, the determination of the general equilibrium solution.

Equation 23 represents pairs  $(r, g)$  of balanced growth paths on the production side. Following Rivera-Batiz and Romer [1991], it will be named the Technology curve. It entails a negative relationship between the interest rate and the growth rate. To understand this relationship, note that the return to investing labour in research is a stream of net revenues that a design generates in the future. Its opportunity cost is the wage rate received in the final goods sector. If the interest rate rises, the present discounted value of the stream of revenues will be lower than before and therefore labour will shift from the research sector into the final goods sector, thus decreasing the growth rate.

The Euler equation 13 represents pairs  $(r, g)$  of balanced growth paths on the consumers side. It postulates the familiar positive relationship between the interest rate,  $r$ , and the growth rate,  $g$ , and, following Rivera-Batiz and Romer [1991], it will be called the Preferences curve.

The general equilibrium balanced growth path for this economy is obtained where the two curves intersect, as shown in Figure 2.

Rivera-Batiz and Romer [1991] point out that a parameter restriction is necessary for the growth rate not to be greater than the interest rate. Because, otherwise, present values would not be finite. This restriction is always met if  $\sigma \geq 1$ , which means that the Preferences curve lies on or above the 45° line. If  $\sigma < 1$ , then the Technology curve cannot lie too far up and to the right.

The equilibrium growth rate is the solution to the system of two equations: 23 and 13, and two unknowns:  $r, g$  :

$$\begin{cases} g = \delta \bar{L} - \frac{r}{\alpha} \\ g = \frac{1}{\sigma}(r - \rho) \end{cases}$$

which is:

$$g = \frac{\alpha \delta \bar{L} - \rho}{\alpha + \sigma} \quad (24)$$

Arnold [2000] provides a complete characterisation of the dynamics of Romer's [1990] model in the neighbourhood of its steady-state. He shows that the equilibrium of the model can be analysed in terms of a system of three differential equations in the three variables  $\chi = \frac{C}{K}$ ,  $Z = \frac{Y}{K}$ , and  $L_Y$ . The steady-state of this system corresponds to the balanced growth path of Romer's model. Arnold [2000] shows that there is a unique and monotonic growth path converging to

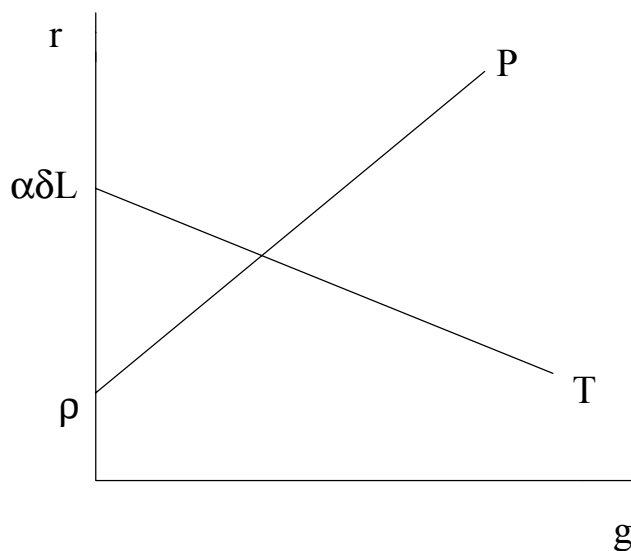


Figure 2:

the steady-state, which is a saddle point. The model does not have local indeterminacy, instability nor cycles. Arnold also demonstrates that the initial value of  $\frac{A}{K}$  uniquely determines the starting point on the saddle-path of the system.

Equation 24 shows that, as opposed to the neoclassical model, in Romer's model the equilibrium growth rate is influenced by the preference parameters  $\sigma$  and  $\rho$ . In a figure such as Figure 2, if either of these two parameters decrease, the Preferences curve shifts to the left, leading to a new equilibrium with a higher growth rate.

The equilibrium growth rate also depends positively on the technology parameter  $\alpha$ , the capital's share in total income.

Additionally, economic growth is proportional to the size of the labour force,  $L$ , equal to total population. In a figure such as Figure 2, a rise in  $L$  shifts the Technology curve to the right, leading to a new balanced growth path with a higher growth rate and a higher interest rate. This proportionality between the size of the population and the growth rate is called the scale-effects property, which characterises virtually all the first-generation of R&D-based models like Romer's [1990], Grossman-Helpman's [1991] and Aghion-Howitt's [1992].

The origin of this scale-effects prediction is the R&D equation 15, which implies that technological growth is proportional to the level of labour resources allocated to research,  $L_A$ . And, with a constant share of labour dedicated to research, this implies that economic growth is proportional to the size of the economy's population.

For Rivera-Batiz and Romer [1991], this dependence on scale is crucial for the analysis of the effects of trade and economic integration on growth. The

effect of integration of two identical economies is straightforward to show with equation 24: Integration doubles the size of the economy, into  $2L$ , increasing the equilibrium growth rate and interest rate.

This scale-effects prediction of the first-generation of R&D-based models is at odds with empirical observation, as was first highlighted by Jones [1995], and a new literature has been developing around the objective of eliminating the scale-effects prediction from R&D-based models<sup>6</sup>.

The equilibrium growth rate expressed in equation 24 is not optimal. There are two sources for this non-optimality: The first source of non-optimality is the fact that capital good producers charge a price that is higher than the marginal cost. Recall the markup rule:

$$R = \frac{r}{\alpha},$$

and recall also expression 16:

$$R = \alpha L_Y^{1-\alpha} x^{\alpha-1}$$

The marginal productivity of capital is:

$$\begin{aligned} \frac{dY}{dK} &= \frac{\alpha L_Y^{1-\alpha} A^{1-\alpha}}{K^{1-\alpha}} = \frac{\alpha L_Y^{1-\alpha} A^{1-\alpha}}{(Ax)^{1-\alpha}} \\ &\Leftrightarrow \\ \frac{dY}{dK} &= \alpha L_Y^{1-\alpha} x^{\alpha-1} \end{aligned}$$

So, it follows that:

$$r = \alpha R = \alpha \frac{dY}{dK},$$

that is, capital is remunerated by less than its marginal productivity. This identifies one source of the non-optimality of Romer's balanced growth path.

The second reason for non-optimality of the decentralised economy is the presence of the externality generated by the fact that the individual decision to do R&D does not take into account the fact that this research will benefit other R&D activities, via the creation of a larger knowledge stock.

The solution to the Social Planner's version of this model, which confirms the non-optimality of Romer's decentralised solution, is presented in the Appendix to this paper.

Two other major references in the literature on R&D-based growth models are the models developed by Grossman and Helpman [1991] and Aghion and Howitt [1992,1998].

Grossman and Helpman's [1991] model differs from Romer's model in two fundamental specifications. Firstly, instead of the presence of differentiated

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<sup>6</sup>Jones [1999] reviews the existing new literature on R&D-based growth models without the scale-effects prediction.

capital goods in the production function, the differentiated goods are final goods that enter the utility function of consumers.

Secondly, production of one unit of each intermediate good requires one unit of labour. Therefore physical capital,  $K$ , is not linked to  $Ax$ , and the solution arises directly from the equilibrium conditions in the labour market. In this model, investment in physical capital is not the source of economic growth. Investment consists, instead, of developing new products.

Aghion and Howitt [1992, 1998], bring into the R&D-based growth theory elements of: (1) chance; (2) Shumpeter’s idea of “creative destruction”, according to which new designs render the existing ones obsolete, giving monopoly profits a temporary character; and (3) endogenous cycles caused by the innovation mechanism.

In Grossman and Helpman’s [1991] and Aghion and Howitt’s [1992, 1998] models, economic growth is sustained at a positive level in the long-run because of investment in R&D activities and the accumulation of knowledge. These models are endogenous growth models because they explain what determines the accumulation of knowledge, and they produce a sustained accumulation of knowledge, at a constant exponential rate.

However, these two models do not contemplate physical capital accumulation. Hence sustained growth does not require a non-declining marginal productivity of capital. The exposition of these two models in this Section would, therefore, break the line of thought adopted for this literature review which aims to analyse the three mechanisms to endogenously obtain a non-declining marginal productivity of capital and thus produce sustained positive long-run growth.

Nevertheless, as they constitute two important contributions to endogenous growth theory, we present their analyses in Section 7, after the discussion of the three mechanisms for generating a constant marginal productivity of capital.

Hence, having completed our analytical study of the first R&D-based growth model, we move on to the second group of endogenous growth models, namely those that produce sustained economic long-run per-capita growth through endogenously determined human capital accumulation.

## 4 Human Capital-Based Growth Models

Human capital-based growth models have human capital accumulation, rather than technological progress, as the source of endogenous growth. We now turn to Lucas’ [1988] model, which is the first model of this kind.

Lucas [1988] builds on Solow’s model and considers human capital accumulation as the engine of growth. He introduces a specification for human capital accumulation, which allows for endogenous growth. Its discussion follows:

In this model there are  $L$  workers in total, with skill level  $h$  ranging from zero to infinity.

Contrary to the original Lucas’ model, in this analysis we assume that population is constant in order to highlight the role of human capital accumulation

in the economic growth process.

A worker with skill level  $h$ , and endowed with one unit of time per unit of time, devotes the fraction  $u(h)$  of his non-leisure time to current production, and the remaining  $(1 - u(h))$  to human capital accumulation. The model implicitly assumes that the amount of leisure is fixed exogenously<sup>7</sup>.

The effective workforce in production is then  $L^e = \int_0^\infty u(h)L(h)h dh$ . Output is  $Y = F(K, L^e)$ .

Assuming that all workers are identical, if they have the skill level  $h$  and choose time allocation  $u$ , then  $L^e = uhL$ .

The production function that we assume for this literature review is:

$$Y_t = K_t^\alpha (AL_t^e)^{1-\alpha}, \quad (25)$$

where the technology parameter  $A$  and population  $L$  are assumed constant<sup>8</sup>.

The capital accumulation equation is, as usual:

$$\dot{K}_t = Y_t - C_t \quad (26)$$

Regarding the specification for the accumulation of human capital, Lucas [1988] adopts Uzawa's [1965] linear function:

$$\dot{h}_t = h_t \delta (1 - u_t), \quad (27)$$

where parameter  $\delta$  represents the efficiency of the learning activities.

This specification assumes that the accumulation of human capital is intensive in human capital. Physical capital is not used as an input in this production function of human capital.

Specification 27 allows for sustained per-capita growth at a constant rate, because it does not display diminishing returns. We can already observe that a balanced growth path solution requires a constant  $u$ :

$$g_h = \frac{\dot{h}_t}{h_t} = \delta(1 - u_t)$$

Now, the representative agent's problem is solved by choosing  $C_t$  and  $u_t$  that maximise utility 2 subject to restrictions 26 and 27. The *current-value Hamiltonian*  $H$  is:

$$H_t = \frac{C_t^{1-\sigma}}{1-\sigma} + \theta_{1t} [K_t^\alpha (A_t u_t h_t L_t)^{1-\alpha} - C_t] + \theta_{2t} [h_t \delta (1 - u_t)]$$

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<sup>7</sup>See Solow [2000] for an analysis of Lucas's model with leisure.

<sup>8</sup>The production function of Lucas's model is  $Y_t = A_t K_t^\alpha L_t^{e^{1-\alpha}} h_{at}^\gamma$ , where the term  $h_{at}^\gamma$  captures the external effects of human capital. We choose to use specification 25 so that we can work with a similar production function as that of Solow and Romer's models, in order to clearly expose the differences in the way each model overcomes diminishing returns to physical capital.

The first-order conditions are:

$$\frac{dH_t}{dC_t} = 0 \quad (28)$$

$$\frac{dH_t}{du_t} = 0 \quad (29)$$

And the co-state equations are:

$$\frac{dH_t}{dK_t} = \rho\theta_{1t} - \dot{\theta}_{1t} \quad (30)$$

$$\frac{dH_t}{dh_t} = \rho\theta_{2t} - \dot{\theta}_{2t} \quad (31)$$

Developing the four equations:

Equation 28:

$$\begin{aligned} \frac{dH}{dC} &= 0 \Leftrightarrow \\ &\Leftrightarrow C^{-\sigma} = \theta_1 \\ &\Leftrightarrow \frac{\dot{C}}{C} = -\frac{1}{\sigma} \frac{\dot{\theta}_1}{\theta_1} \end{aligned}$$

Equation 29:

$$\begin{aligned} \frac{dH}{du} &= 0 \Leftrightarrow \\ &\Leftrightarrow \theta_1(1 - \alpha)AhLK^\alpha(AuhL)^{-\alpha} = \theta_2\delta h \end{aligned}$$

Equation 30:

$$\begin{aligned} \frac{dH}{dK} &= \rho\theta_1 - \dot{\theta}_1 \Leftrightarrow \\ &\Leftrightarrow \frac{\dot{\theta}_1}{\theta_1} = \rho - \alpha K^{\alpha-1}(AuhL)^{1-\alpha} \end{aligned}$$

Equation 31:

$$\begin{aligned} \frac{dH}{dh} &= \rho\theta_2 - \dot{\theta}_2 \Leftrightarrow \\ &\Leftrightarrow \theta_1(1 - \alpha)AuLK^\alpha(AuhL)^{-\alpha} + \theta_2\delta(1 - u) = \rho\theta_2 - \dot{\theta}_2 \end{aligned}$$

The model is solved for its balanced growth path, the solution for which  $C$ ,  $K$  and  $h$  grow at constant rates, the prices  $\theta_1$  and  $\theta_2$  decline at constant rates and the time allocation variable  $u$  is constant:

Firstly, as in the resolution of Solow's model, a constant  $g_c$  requires a constant marginal productivity of physical capital. Equations 28 and 30 give:

$$g_c = -\frac{1}{\sigma} \frac{\dot{\theta}_1}{\theta_1} = \frac{1}{\sigma} [\alpha K^{\alpha-1} (AuhL)^{1-\alpha} - \rho],$$

where the marginal productivity of capital is:

$$\frac{dY}{dK} = \alpha K^{\alpha-1} (AuhL)^{1-\alpha}$$

With  $A, L$  and  $u$  constant, a constant marginal productivity of capital implies that:

$$\begin{aligned} \frac{\left(\frac{dY}{dK}\right)}{\frac{dY}{dK}} &= 0 \Rightarrow (\alpha - 1)g_K + (1 - \alpha)g_h = 0 \Leftrightarrow & (32) \\ &\Leftrightarrow g_K = g_h \end{aligned}$$

Then, log-differentiation of the production function gives the growth rate of output:

$$\begin{aligned} Y &= K^\alpha (AuhL)^{1-\alpha} & (33) \\ &\Rightarrow \\ g_Y &= \alpha g_K + (1 - \alpha)g_h = g_h \end{aligned}$$

Then, and as in Solow's and Romer's models, the physical capital accumulation equation:

$$\dot{K} = Y - C$$

ensures that consumption grows at the same rate as output and physical capital. That is:

$$g_c = g_y = g_k = g_h,$$

where variables have been replaced by their per-capita counterparts, because  $L$  is constant.

Concluding, in this model, sustained per-capita growth is obtained through sustained human capital accumulation. That is, Lucas' model overcomes diminishing returns to physical capital through the accumulation of human capital. Hence, physical capital can be accumulated without decreasing its marginal productivity because human capital is also growing at the same rate as physical capital.

We can see this more clearly by rewriting the expression for the marginal productivity of capital:

$$\frac{dY}{dK} = \alpha K^{\alpha-1} (AuhL)^{1-\alpha} = \frac{\alpha (AuL)^{1-\alpha} h^{1-\alpha}}{K^{1-\alpha}},$$

from which we observe that the marginal productivity of capital remains constant because physical capital  $K$  and human capital  $h$  grow at the same rate.

The root of sustained per-capita economic growth in Lucas' model has thus been identified.

Now we only need to analyse how human capital growth is determined within the model, that is, how this model gains the definition of an endogenous growth model.

The next step is, then, to determine the engine of growth,  $g_h$ :

Equation 27 says that:

$$g_h = \delta(1 - u) \quad (34)$$

Then, log-differentiation of equation 29 gives:

$$\begin{aligned} \theta_1(1 - \alpha)AhLK^\alpha(AuhL)^{-\alpha} &= \theta_2\delta h & (35) \\ &\Leftrightarrow \\ \frac{\dot{\theta}_1}{\theta_1} + \frac{\dot{h}}{h} + \alpha\frac{\dot{K}}{K} - \alpha\frac{\dot{h}}{h} &= \frac{\dot{\theta}_2}{\theta_2} + \frac{\dot{h}}{h} \\ &\Leftrightarrow \\ \frac{\dot{\theta}_1}{\theta_1} &= \frac{\dot{\theta}_2}{\theta_2} \end{aligned}$$

Next look at equation 29, repeated below:

$$\theta_1(1 - \alpha)AhLK^\alpha(AuhL)^{-\alpha} = \theta_2\delta h,$$

and at equation 31, repeated below:

$$\theta_1(1 - \alpha)AuLK^\alpha(AuhL)^{-\alpha} + \theta_2\delta(1 - u) = \rho\theta_2 - \dot{\theta}_2$$

The first terms of these two equations have much in common. In fact we have that:

$$\theta_1(1 - \alpha)AuLK^\alpha(AuhL)^{-\alpha} = \theta_1(1 - \alpha)AhLK^\alpha(AuhL)^{-\alpha} \times \frac{u}{h}$$

So these two equations can be combined together to give:

$$\begin{aligned} \frac{\theta_2\delta hu}{h} + \theta_2\delta - u\theta_2\delta &= \rho\theta_2 - \dot{\theta}_2 & (36) \\ &\Leftrightarrow \\ \frac{\dot{\theta}_2}{\theta_2} &= \rho - \delta \end{aligned}$$

Finally, using equations 35 and 36 to substitute for  $\frac{\dot{\theta}_2}{\theta_2}$  gives the rate of human capital accumulation:

$$g_h = g_k = g_c = g = -\frac{1}{\sigma}\frac{\dot{\theta}_1}{\theta_1} = \frac{1}{\sigma}(\delta - \rho), \quad (37)$$



This endogenous growth model predicts, then, that economic growth increases with the effectiveness of investment in human capital,  $\delta$ . It also predicts that economic growth depends negatively on the preference parameters,  $\rho$  and  $\sigma$ .

Benhabib and Perli [1994] and Xie [1994] investigate the transitional dynamics of Lucas' [1988] model off its steady-state. Barro and Sala-i-Martin [1995, Chp. 5, pages 183-194] also analyse the transitional dynamics of Lucas' model<sup>9</sup>. They show that the equilibrium of the model can be analysed in terms of a system of three differential equations in three variables:  $\chi = \frac{C}{K}$ ,  $W = \frac{K}{H}$ , and  $u$ . The steady-state of this system corresponds to the balanced growth path of Lucas' economy.

In order to study the transitional dynamics, Barro and Sala-i-Martin replace variable  $W$  with variable  $Z$  which is the gross average product of  $K$ . They show, with a phase diagram, that the system:

$$\begin{cases} \dot{\chi} = 0 \\ \dot{Z} = 0 \end{cases}$$

is saddle-path stable, for  $\sigma > \alpha$ . They then analyse the behaviour of  $u$  through an adjacent phase diagram.

Other human capital-based models include Becker, Murphy and Tamura [1990], and Rebelo [1991]. Like Lucas [1988], these models assume different technologies for production of the final good and for human capital accumulation. More precisely, they assume that the production function for human capital is more intensive in human capital than the production function of physical capital. Lucas' model assumes the extreme case that the production of human capital involves no physical capital at all.

Next, we analyse the third group of endogenous growth models, which produce endogenous growth via the elimination of the assumption of diminishing returns to capital.

## 5 Models that Eliminate the Diminishing Returns to Capital Assumption

As discussed in Section 2, Solow's [1956] basic proposition is that without technological progress, the effects of diminishing returns to capital will eventually drive the per-capita growth rate to zero.

The building block of this neoclassical model is an aggregate production function exhibiting constant returns to scale and diminishing marginal productivity in each of the inputs, and satisfying the Inada conditions (described below).

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<sup>9</sup>Barro and Sala-i-Martin [1995] work with Lucas's original model. However, performing the same transitional dynamics analysis to our Luca's model, will give us similar results in terms of the model's dynamics around its steady-state.

Consider the aggregate production function adopted in this literature review:

$$Y = K^\alpha(AL)^{1-\alpha} \quad , \quad 0 < \alpha < 1,$$

where  $A$  and  $L$  are constant.

In this model, because there is no technological change (nor human capital accumulation), the engine of growth is physical capital accumulation.

Then:

$$\frac{dY}{dK} = \frac{\alpha(AL)^{1-\alpha}}{K^{1-\alpha}}$$

and

$$\frac{d^2Y}{dK^2} = \frac{-(1-\alpha)\alpha(AL)^{1-\alpha}}{K^{2-\alpha}}$$

The idea of diminishing returns to capital is formally captured by the following assumptions:

$$\frac{dY}{dK} > 0 \text{ and } \frac{d^2Y}{dK^2} < 0 \text{ for all } K,$$

and by the Inada conditions:

$$\lim_{K \rightarrow \infty} \frac{dY}{dK} = 0 \text{ and } \lim_{K \rightarrow 0} \frac{dY}{dK} = \infty, \quad (38)$$

Instead of assuming the optimising behaviour for consumers, we adopt here the alternative form of saving behaviour, namely that people save a constant proportion  $s$  of gross income  $Y$ . We also assume that capital depreciates at the rate  $\delta$ . Thus the capital accumulation equation is:

$$\dot{K} = sY - \delta K \quad (39)$$

The behaviour of this economy is illustrated in Figure 3. Diminishing returns to capital imply that output will not grow as fast as capital, that is, the production function is concave in the  $(Y, K)$  space. Consequently the savings function,  $sY$ , is also concave which means that saving will not grow as fast as depreciation. So, the economy eventually reaches a steady-state equilibrium,  $K^*$ , where depreciation is equal to saving, and capital growth (and consequently output growth) is zero:

$$\begin{aligned} sY &= \delta K \Leftrightarrow sK^\alpha(AL)^{1-\alpha} = \delta K \\ &\Leftrightarrow \\ K^* &= AL \left[ \frac{s}{\delta} \right]^{\frac{1}{1-\alpha}} \quad \therefore \frac{\dot{K}}{K} = 0 \end{aligned}$$

and

$$Y^* = \left[ AL \left[ \frac{s}{\delta} \right]^{\frac{1}{1-\alpha}} \right]^\alpha (AL)^{1-\alpha} = AL \left[ \frac{s}{\delta} \right]^{\frac{\alpha}{1-\alpha}} \quad \therefore \frac{\dot{Y}}{Y} = 0$$

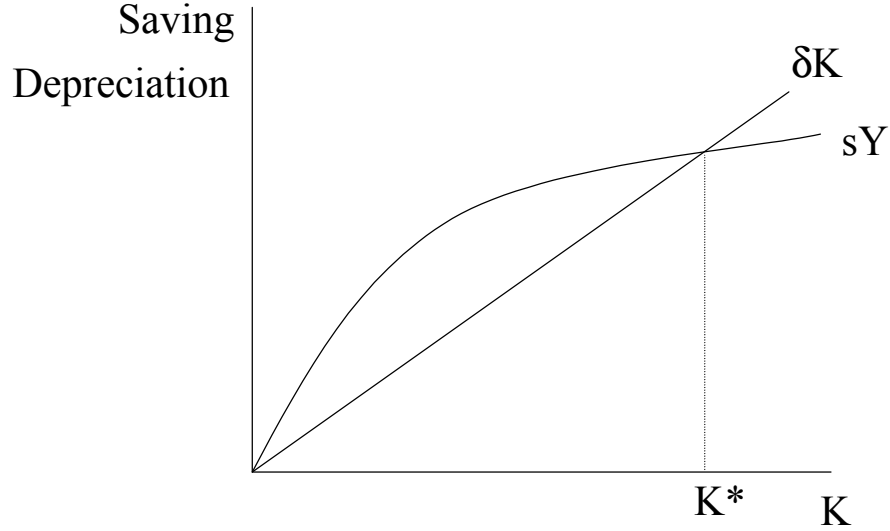


Figure 3:

We can now analyse the third way of obtaining sustained positive economic growth. It entails dropping the standard assumption of the neoclassical model, mentioned above, of diminishing returns to physical capital.

This is done by Jones and Manuelli [1990]. They present a model with a production function that violates the first Inada condition, and thus delivers sustained endogenous growth. Let us analyse their work:

We modify their production function so as to keep similarity with the models discussed previously in this literature review. The production function is, then:

$$Y = vK + K^\alpha(AL)^{1-\alpha}, \quad (40)$$

where  $A$  and  $L$  are constant.

With such production function, the standard conditions:

$$\frac{dY}{dK} = v + \alpha K^{\alpha-1}(AL)^{1-\alpha} > 0,$$

and

$$\frac{d^2Y}{dK^2} = \alpha(\alpha - 1)K^{\alpha-2}(AL)^{1-\alpha} < 0$$

are met.

However, this production function violates the first Inada condition, as can be seen below:

$$\lim_{K \rightarrow \infty} \frac{dY}{dK} = v > 0, \quad (41)$$

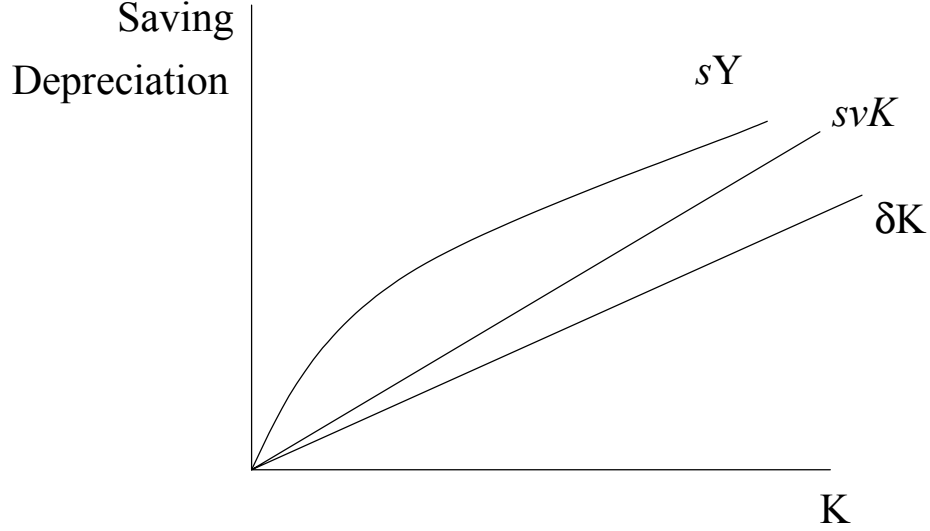


Figure 4:

Let us see how this changes the results of the model.  
The capital accumulation equation is, as before:

$$\dot{K} = sY - \delta K$$

The behaviour of this economy is pictured in Figure 4.

Because  $\lim_{K \rightarrow \infty} \frac{dY}{dK} = v$ , the savings function tends asymptotically to the line  $svK$ . If this line is above the line  $\delta K$ , then depreciation never catches up with savings, meaning that this economy can sustain forever a positive growth rate for capital and output, which are the same as the per-capita growth rates, because population is constant.

The growth rate of per-capita capital is:

$$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} = \frac{sY}{K} - \delta = s \frac{vK + K^\alpha (AL)^{1-\alpha}}{K} - \delta = sv + s \frac{(AL)^{1-\alpha}}{K^{1-\alpha}} - \delta \quad (42)$$

As  $K$  goes to infinity,  $g_k = sv - \delta$ . So, positive sustained per-capita growth is possible for  $sv > \delta$ .

If, instead of the fixed saving rate, the model adopts the consumers' optimising behaviour, then per-capita growth is given by:

$$g_y = g_k = g_c = \frac{1}{\sigma} \left[ \frac{dY}{dK} - \delta - \rho \right]$$

And then, as  $\lim_{K \rightarrow \infty} \frac{dY}{dK} = v$ , sustained positive per-capita growth is obtained if  $v > \delta + \rho$ .

Another model that assumes away diminishing returns to capital is that of King and Rebelo [1990]. These authors contemplate human capital and assume that both physical capital and human capital are produced with both physical capital and human capital under two distinct production functions. However, King and Rebelo [1990] keep the result that broad capital (which equals physical capital plus human capital) is produced with non diminishing returns.

Barro and Sala-i-Martin [1995, Chp.5, page 172] also analyse a model that eliminates diminishing returns to capital. They work with a constant returns to scale production function and the one-sector-model assumption that output can be used for consumption, investment in physical capital or investment in human capital<sup>10</sup>.

Here, once again, we modify Barro and Sala-i-Martin's [1995, Chp.5, page 172] model so as to work with a production function common to all the models discussed in this literature review.

The assumed production function is, then:

$$Y_t = K_t^\alpha (A_t L_t^h)^{1-\alpha}, \quad (43)$$

where the technology parameter  $A$  is constant,  $K$  is physical capital, and  $L^h = hL$  is the number of workers  $L$  multiplied by their human capital  $h$ . Only the combination  $hL$  is relevant for output.

Labour  $L$  is constant, so growth of  $L^h$  is only due to the growth of  $h$ .

Let us rename  $L^h = H$ . So the production function becomes:

$$Y_t = K_t^\alpha (A_t H_t)^{1-\alpha}$$

The budget constraint for this economy is:

$$Y_t = C_t + I_{Kt} + I_{Ht}, \quad (44)$$

and the investment equations are:

$$\begin{aligned} \dot{K}_t &= I_{Kt} - \delta K_t \\ \dot{H}_t &= I_{Ht} - \delta H_t \end{aligned} \quad (45)$$

Assuming that households are also producers, the *current-value Hamiltonian* expression for the representative agent's maximisation problem is:

$$J_t = \frac{C_t^{1-\sigma}}{1-\sigma} + u_t [I_{Kt} - \delta K_t] + v_t [I_{Ht} - \delta H_t] + w_t [K_t^\alpha (A_t H_t)^{1-\alpha} - C_t - I_{Kt} - I_{Ht}],$$

where  $u$  and  $v$  are the *current-value* of physical capital and human capital accumulation respectively, and  $w$  is the Lagrangian multiplier associated with the budget constraint.

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<sup>10</sup>Barro and Sala-i-Martin [1995, Chp. 5] also discuss models which assume different technologies for the production of goods and the production of human capital.

The three first-order conditions are:

$$\begin{aligned}\frac{dJ_t}{dC_t} &= 0 \\ &\Leftrightarrow \\ \frac{\dot{C}}{C} &= -\frac{1}{\sigma} \frac{\dot{w}}{w}\end{aligned}\tag{46}$$

$$\begin{aligned}\frac{dJ_t}{dI_{Kt}} &= 0 \\ &\Leftrightarrow \\ u &= w\end{aligned}\tag{47}$$

$$\begin{aligned}\frac{dJ_t}{dI_{Ht}} &= 0 \\ &\Leftrightarrow \\ v &= w\end{aligned}\tag{48}$$

and the two co-state equations are:

$$\begin{aligned}\frac{dJ_t}{dK_t} &= \rho u_t - \dot{u}_t \\ &\Leftrightarrow \\ \frac{\dot{u}}{u} &= \rho + \delta - \alpha K^{\alpha-1} (AH)^{1-\alpha}\end{aligned}\tag{49}$$

$$\begin{aligned}\frac{dJ_t}{dH_t} &= \rho v_t - \dot{v}_t \\ &\Leftrightarrow \\ \frac{\dot{v}}{v} &= \rho + \delta - (1-\alpha)AK^\alpha (AH)^{-\alpha}\end{aligned}\tag{50}$$

As  $u = v = w$ , the solution requires:

$$\begin{aligned}\frac{\dot{u}}{u} &= \frac{\dot{v}}{v} \\ &\Leftrightarrow \\ \alpha K^{\alpha-1} A^{1-\alpha} H^{1-\alpha} &= (1-\alpha)AK^\alpha A^{-\alpha} H^{-\alpha} \\ &\Leftrightarrow \\ \frac{K}{H} &= \frac{\alpha}{1-\alpha},\end{aligned}\tag{51}$$

and the growth rate, equal to the per-capita growth rate due to the constancy of the population, is given by the Euler equation:

$$\begin{aligned}g &= \frac{1}{\sigma} [\alpha K^{\alpha-1} (AH)^{1-\alpha} - \delta - \rho] \\ &= \frac{1}{\sigma} [A^{1-\alpha} \alpha^\alpha (1-\alpha)^{(1-\alpha)} - \delta - \rho]\end{aligned}\tag{52}$$

Output is equal to:

$$\begin{aligned}
Y &= K^\alpha (AH)^{1-\alpha} \\
&= K^{\alpha-1} (AH)^{1-\alpha} K \\
&= A^{1-\alpha} \left( \frac{1-\alpha}{\alpha} \right)^{1-\alpha} K \\
&= BK,
\end{aligned}$$

where  $B$  is a constant. This means that  $\frac{d^2 Y}{dK^2} = 0$ , which is a violation of the standard assumptions for physical capital.

Hence this model, too, generates sustained positive endogenous growth via the elimination of the diminishing returns to physical capital assumption, obtained through a constant  $\frac{K}{H}$  ratio.

This model, like the AK model, has no transitional dynamics. The growth rate is always given by expression 52.

## 6 Discussion

In the previous Sections, we have analysed three alternative ways of achieving a constant marginal productivity of capital. The first mechanism relies on technological progress, the second relies on human capital accumulation and the third consists of eliminating the assumption of diminishing returns to capital from the production function.

In their processes of generating sustained economic growth, all the discussed endogenous growth models start up with a production function like  $Y = K^\alpha (AL)^{1-\alpha}$ , and arrive at a production function of the type  $Y = BK$ , with  $B$  constant, which implies a constant marginal productivity of capital.

There is, however, a fundamental difference between both R&D-based and human capital-based growth models and the third type of endogenous growth models studied. In order to see this difference recall, firstly, Romer's [1990] R&D-based growth model, discussed in Section 3. Its production function is:

$$\begin{aligned}
Y &= L_Y^{1-\alpha} \int_0^A x(i)^\alpha di \\
&= L_Y^{1-\alpha} Ax^\alpha \\
&= L_Y^{1-\alpha} K^\alpha A^{1-\alpha} \\
&= L_Y^{1-\alpha} K^{\alpha-1} A^{1-\alpha} K \\
&= \frac{L_Y^{1-\alpha} A^{1-\alpha}}{K^{1-\alpha}} K \\
&= BK,
\end{aligned}$$

where  $B$  is constant because  $L_Y$  is constant and  $K$  grows at the same rate as  $A$ .

Next recall Lucas' [1988] human capital-based growth model, studied in Section 4. Its production function is:

$$\begin{aligned}
Y &= K^\alpha (AuhL)^{1-\alpha} \\
&= K^\alpha (AuL)^{1-\alpha} h^{1-\alpha} \\
&= K^{\alpha-1} (AuL)^{1-\alpha} h^{1-\alpha} K \\
&= \frac{(AuL)^{1-\alpha} h^{1-\alpha}}{K^{1-\alpha}} K \\
&= BK,
\end{aligned}$$

where  $B$  is constant because  $A$ ,  $L$  and  $u$  are constant and  $g_K = g_h$ .

With these two models, the production function  $Y = BK$  is obtained because diminishing returns to physical capital are overcome by either the progress of technology or the accumulation of human capital.

As opposed to these two types of endogenous growth model, in the third group of endogenous growth models, diminishing returns to physical capital are not overcome. They are eliminated. To see this, recall, for instance, the model of Barro and Sala-i-Martin [1995, Chp. 5, page 172] analysed in Section 5. Its production function is:

$$\begin{aligned}
Y &= K^\alpha (AH)^{1-\alpha} \\
&= K^{\alpha-1} (AH)^{1-\alpha} K \\
&= A^{1-\alpha} \left(\frac{H}{K}\right)^{1-\alpha} K \\
&= A^{1-\alpha} \left(\frac{1-\alpha}{\alpha}\right)^{1-\alpha} K \\
&= BK,
\end{aligned}$$

where  $B$  is constant because  $A$  is constant and the ratio  $\frac{H}{K}$  is equal to a constant  $\frac{1-\alpha}{\alpha}$ .

Notice that in this model, the ratio  $\frac{K}{H}$  is equal to  $\frac{\alpha}{1-\alpha}$ , meaning that it is fixed by the technology parameter, which is a given in the model. That is, the value of this ratio will be the same for all balanced growth paths. Whereas, for instance, in the R&D-based model, the ratio  $\frac{K}{A} = x = L_Y \left[\frac{\alpha^2}{r}\right]^{\frac{1}{1-\alpha}}$  depends on the equilibrium values of  $L_Y$  and the interest rate. That is, for each balanced growth path, there is a different  $\frac{K}{A}$  ratio, which is constant because  $g_k = g_A$ .

Arrow's [1962] model can be placed in this third group of growth models, as it also eliminates diminishing returns to capital. It does so by assuming that knowledge creation is a side product of investment. That is, a firm that invests in physical capital learns simultaneously how to produce more efficiently. This learning-by-doing assumption is also combined with the assumption of knowledge spillovers. Arrow's model possesses a constant marginal productivity of capital, for labour constant, because  $g_A = g_K$  by assumption.

The extreme case of a model that drops the diminishing returns to scale assumption is the so called AK model. This model assumes that production



exhibits exactly constant returns to scale to a broad concept of capital, that is, to the collection of all kinds of capital, for instance physical and human capital. The production function is:

$$Y = AK,$$

and the marginal productivity of capital is  $A$ , a constant. Hence, this production function simply does not display diminishing returns to capital. Solow [2000] mentions his dislike of such kind of model, as it seems to assume exactly what it wants to arrive at.

After this discussion, we next analyse two other important initial contributions to endogenous growth theory, namely those by Grossman and Helpman's [1990] and Aghion and Howitt's [1992]. These models are R&D-based growth models.

## 7 Other R&D-Based Growth Models

### 7.1 Grossman and Helpman's R&D-Based Growth Model

In Grossman and Helpman's [1990] model, growth is obtained through the combination of two mechanisms: (1) production of differentiated consumer goods, which are expanding because (2) there is deliberate accumulation of knowledge.

The authors build on Dixit and Stiglitz [1977] in defining an index  $D$  through a constant elasticity of substitution (CES) function<sup>11</sup>:

$$D = \left[ \int_0^A x_j^\alpha dj \right]^{\frac{1}{\alpha}}, \quad 0 < \alpha < 1, \quad (53)$$

where  $x_j$  is the quantity of the differentiated good  $j$ ,  $A$  is the number of available brands, and  $\alpha$  is a parameter. The elasticity of substitution between every pair of goods is  $\frac{1}{1-\alpha}$ .

Equation 53 yields constant elasticity demand functions for each good. It implies that a doubling of each of the  $x_j$ , for given  $A$ , doubles the index  $D$ . The index increases with each of the  $x_j$  individually, but at a non increasing rate. A higher  $\alpha$  means that the goods are better substitutes in consumption. This specification captures the notion that consumers like variety.

The price index of  $D$ ,  $p_D$  is given by:

$$p_D = \left[ \int_0^A p_j^{-\frac{\alpha}{1-\alpha}} dj \right]^{-\frac{1-\alpha}{\alpha}} \quad (54)$$

Now, assuming that, once invented, all brands require one unit of labour per unit output, marginal cost equals the wage rate  $w$  for all brands.

<sup>11</sup>See Solow [2000, Chp.10] for further insight into this Dixit and Stiglitz representation.

Then, assuming that the wage rate equals unity, the profit maximisation problem of these monopolistic competitors leads to the following markup rule:

$$p_j = p = \frac{w}{\alpha} = \frac{1}{\alpha} \quad (55)$$

With this price, profits are:

$$\begin{aligned} \pi &= px - wx \\ &= (1 - \alpha)p \frac{X}{A} \\ &= \frac{1 - \alpha}{\alpha} \frac{X}{A} \end{aligned} \quad (56)$$

where  $X = Ax$ , represents aggregate output of differentiated goods.

Development of new varieties of goods requires an effort in R&D. The R&D costs have to be paid up front, before profits are realised, and this introduces, as in Romer's model, natural dynamics in the model through the Fisher equation: A typical firm holds the patent on the differentiated good and enjoys indefinite monopoly power on the supply of its good. The value of this firm is then equal to the present discounted value of its profits:

$$\begin{aligned} v(t) &= \int_t^\infty e^{-r(\tau-t)} \pi(\tau) d\tau \\ &\Leftrightarrow \\ \dot{v} &= rv - \pi \\ &\Leftrightarrow \\ r &= \frac{\dot{v}}{v} + \frac{\pi}{v} \end{aligned} \quad (57)$$

The cost of inventing a new product is defined as:

$$\frac{aw}{A} = \frac{a}{A}$$

where  $a$  is a parameter and  $A$  represents the stock of knowledge, equal to the number of already invented goods.

Notice that this specification introduces an externality into the model. When they create new goods, producers are increasing the level of  $A$ , which makes innovation more productive for other producers, as it lowers the cost of innovation.

A firm that invests in the creation of a new good expects a reward of  $v$  on its R&D effort. It will then engage in R&D unless the R&D cost is larger than  $v$ . The dynamic free-entry condition is thus that  $v$  must be less than or equal to the cost of creating a new good. And if  $\dot{A} > 0$  then  $v$  is equal to the innovation cost, expressing the idea that the flow of new inventions equals zero unless the innovation cost is just equal to the value of a firm. The free-entry condition is, then:

$$v = \frac{a}{A} \quad \text{if } \dot{A} > 0 \quad (58)$$

Now, clearing of the labour market requires that employment in R&D plus employment in manufacturing of the goods must equal the total supply of labour,  $L$ , which is assumed constant. That is:

$$\frac{a}{A} \dot{A} + X = L, \quad (59)$$

where  $\frac{a}{A}$  is the amount of labour necessary to make one invention,  $A$  is the number of inventions currently being made, and  $X = Ax$  is the total amount of labour involved in the production of goods already in existence.

Like Romer's, this model is perfectly symmetric: Firms have the same technology of production, goods enter the utility function in the same way and have the same elasticity of demand and the same price. Hence the quantities of each good produced are the same,  $x_i = x$ . We can then evaluate the consumption index,  $C$ . Variable  $C$  represents consumption in terms of index  $D$ . In equilibrium  $C = D$ . So:

$$C = D = \left[ \int_0^A x_j^\alpha dj \right]^{\frac{1}{\alpha}} = A^{\frac{1}{\alpha}} x = A^{\frac{1-\alpha}{\alpha}} X,$$

from which follows that the growth rate of  $C$  is:

$$\frac{\dot{C}}{C} = \frac{1-\alpha}{\alpha} \frac{\dot{A}}{A} + \frac{\dot{X}}{X}$$

Total labour  $L$  is assumed constant, which means that the growth rate of aggregate variables equals the growth rate of the per-capita variables. Therefore we can write:

$$g_c = \frac{1-\alpha}{\alpha} \frac{\dot{A}}{A} + \frac{\dot{X}}{X}$$

Equation 59 says that, as  $L$  is constant, a balanced growth path solution, that is a constant growth rate  $g$ , requires that  $X$  is constant. And so, we have:

$$ag_A + X = L \Leftrightarrow g_A = \frac{1}{a}(L - X)$$

Also, expression:

$$C = A^{\frac{1-\alpha}{\alpha}} X$$

implies that

$$g_c = \frac{1-\alpha}{\alpha} g_A \quad (60)$$

Notice that  $X$  being constant implies that in equilibrium the quantities of the existing goods will be decreasing at the same rate as new goods are being created:

$$X = Ax \Rightarrow \frac{\dot{X}}{X} = 0 \Leftrightarrow \frac{\dot{x}}{x} = -\frac{\dot{A}}{A}$$

Now follows the determination of the growth rate of  $A$ , the engine of growth in this R&D-based model. For that we first work on the Fisher equation 57, repeated here:

$$r = \frac{\dot{v}}{v} + \frac{\pi}{v}$$

As seen before, in a steady-state with positive  $g_A$ , we have, according to equation 58:

$$v = \frac{a}{A} \Rightarrow \frac{\dot{v}}{v} = -g_A$$

Next, we recall equation 56 for profits:

$$\pi = \frac{1 - \alpha}{\alpha} \frac{X}{A}$$

Then:

$$\frac{\pi}{v} = \frac{1 - \alpha}{\alpha} \frac{X}{a}$$

The Fisher equation 57 can then be rewritten as:

$$\begin{aligned} r &= -g_A + \frac{1 - \alpha}{\alpha} \frac{X}{a} & (61) \\ &\Leftrightarrow \\ r &= -g_A + \frac{1 - \alpha}{\alpha} \frac{(L - ag_A)}{a} \\ &\Leftrightarrow \\ g_c &= \frac{(1 - \alpha)^2}{\alpha} \frac{L}{a} - (1 - \alpha)r \end{aligned}$$

where  $X$  was replaced by its equivalent expression given by the resource constraint 59, and  $g_A$  was replaced by its equivalent in terms of  $g_c$ , given by equation 60.

In this study, we assume the optimising version of consumers behaviour. So investment in R&D has to be financed by savings, which are determined by households' intertemporal preferences. The, by now familiar, preferences structure is used.

So maximising:

$$\int_0^{\infty} e^{-\rho t} U(C_t) dt \quad , \quad U(C) = \frac{C^{1-\sigma}}{1-\sigma}$$

subject to an intertemporal constraint, leads the representative household to allocate consumption according to the following rule:

$$\frac{\dot{C}}{C} = g_c = \frac{1}{\sigma} \left( r - \rho - \frac{\dot{p}_D}{p_D} \right) \quad (62)$$

Now, recall  $p_D$ , given in equation 54, and repeated below:

$$p_D = \left[ \int_0^A p_j^{-\frac{\alpha}{1-\alpha}} dj \right]^{-\frac{1-\alpha}{\alpha}}$$

As  $p_j = p$  for every good, it follows that:

$$p_D = A^{-\frac{1-\alpha}{\alpha}} p,$$

which means that:

$$\frac{\dot{p}_D}{p_D} = -\frac{1-\alpha}{\alpha} \frac{\dot{A}}{A}$$

So equation 62 can be rewritten as:

$$\begin{aligned} g_c &= \frac{1}{\sigma} \left( r - \rho + \frac{1-\alpha}{\alpha} g_A \right) \\ &\Leftrightarrow \\ g_c &= \frac{1}{\sigma-1} (r - \rho) \end{aligned} \quad (63)$$

The equilibrium growth rate is determined by the system composed of the two equations 61 and 63, in two unknowns  $r$  and  $g_c$ . Equation 61 displays a negative relationship between  $r$  and  $g_c$ , and equation 63 displays a positive relationship between  $r$  and  $g_c$ . Therefore, as the two curves are linear, the equilibrium growth rate is uniquely determined.

So, the balanced growth rate solution of Grossman and Helpman's model is:

$$\begin{aligned} g_c &= \frac{1}{\sigma-1} (r - \rho) \\ &\Leftrightarrow \\ (\sigma-1)g_c &= -\frac{1}{1-\alpha} g_c + \frac{1-\alpha}{\alpha} \frac{L}{a} - \rho \\ &\Leftrightarrow \\ g_c &= \frac{1-\alpha}{[\alpha + \sigma(1-\alpha)]} \left[ \frac{1-\alpha}{\alpha} \frac{L}{a} - \rho \right] \end{aligned} \quad (64)$$

The growth rate depends positively on the size of population  $L$ . This fact is the scale-effects prediction, mentioned earlier, that characterises the first generation of R&D-based growth models, but which is at odds with empirical results.

Growth will also be higher the lower the value of  $\alpha$ , that is the higher the degree of monopoly ( $1-\alpha$ ). Growth also depends negatively on the value of  $a$ , as the cost of making an innovation is proportional to  $a$ . Hence the lower the cost of innovation, the higher the rate of innovation. Finally, from the preferences side, there is a negative relationship between  $\rho$  and  $\sigma$  and the steady-state growth rate.

## 7.2 Aghion and Howitt's R&D-Based Growth Model

In Aghion and Howitt's [1992] model, economic growth is generated by a random sequence of quality improving innovations that result from research activities which are themselves uncertain.

This model has two positive externalities. One arises from the fact that monopoly rents are smaller than the consumer surplus. The other positive externality has its origin in the fact that one invention makes possible the next invention.

There is, on the other hand, a negative externality which is due to the fact that a new invention renders the previous one obsolete and replaces it.

We pick up Aghion and Howitt's [1998, Chp. 2] description of their model, and develop it below:

This economy has no capital accumulation and it is populated by a continuous mass of individuals,  $L$ , equal to total labour supply, with linear intertemporal preferences, given by:

$$U(y) = \int_0^{\infty} y_t e^{-rt} dt,$$

where  $r$  is the rate of time preference also equal to the interest rate.

The labour force,  $L$ , produces capital goods,  $x$ , in a one-to-one fashion, and then the capital goods are used to produce the final good,  $y$ , according to the following production function:

$$y = Ax^\alpha \quad , \quad 0 < \alpha < 1, \quad (65)$$

where  $x$  is the quantity of capital goods in existence.

Innovation consists of inventing a new intermediate good that, when successful, renders the old one obsolete and raises the technology parameter,  $A$ , by a constant factor,  $\gamma$ :

$$\frac{A_{i+1}}{A_i} = \gamma > 1,$$

where  $i$  is the number of innovations that have occurred so far.

When the amount  $n$  of labour is assigned to R&D, innovations arrive randomly according to a Poisson process with arrival rate  $\lambda n$ ,  $\lambda > 0$ . Parameter  $\lambda$  indicates the productivity of the research technology. This specification means that the probability of an innovation in a given unit of time is  $\lambda n$ <sup>12</sup>.

<sup>12</sup>See Aghion and Howitt [1998, page 55] for an explanation of Poisson Processes.

The economy's total stock of labour is allocated between R&D and the production of capital goods. So the labour market clearing condition is:

$$L = x + n, \quad (66)$$

where  $x$  is the amount of labour devoted to manufacturing (as goods are produced by labour with a one-to-one technology), and  $n$  is the amount of labour dedicated to research.

The amount of labour allocated to research is determined by the following arbitrage condition:

$$w_i = \lambda V_{i+1}, \quad (67)$$

where  $w$  is the wage rate and  $V_{i+1}$  is the discounted expected payoff to the  $(i + 1)th$  innovation.

This arbitrage condition rules the dynamics of the economy over its successive inventions. It means that in equilibrium a worker must be indifferent between an hour's work in manufacturing,  $w_i$ , and an hour's work in research. The value to a worker of an hour's work in research is equal to the flow probability of an innovation,  $\lambda$ , times the value of that innovation,  $V_{i+1}$ , as an hour's work in research after the  $ith$  innovation results in the  $(i + 1)th$  innovation.

Now, the value  $V_{i+1}$  is determined by the following asset condition:

$$rV_{i+1} = \pi_{i+1} - \lambda n_{i+1} V_{i+1}, \quad (68)$$

which says that the expected income generated by a patent on the  $(i + 1)th$  innovation during a unit time interval, namely  $rV_{i+1}$ , is equal to the profit flow that the producer of the  $(i + 1)th$  innovation obtains,  $\pi_{i+1}$ , minus the expected loss that will occur when the next innovation replaces the  $(i + 1)th$  innovation. This expected loss is equal to  $\lambda$ , the flow probability of the innovation occurring, times  $n_{i+1}$ , the amount of labour dedicated to research after the  $(i + 1)th$  innovation, times  $V_{i+1}$ , the value that will be lost.

In other words, there must be indifference between acquiring a patent of an intermediate good to produce it, and putting the same amount of money in the bank and earning its interest.

Now:

$$\begin{aligned} rV_{i+1} &= \pi_{i+1} - \lambda n_{i+1} V_{i+1} & (69) \\ \Leftrightarrow \\ V_{i+1} &= \frac{\pi_{i+1}}{r + \lambda n_{i+1}} \end{aligned}$$

Equation 69 shows the effects of creative destruction. The higher the number of researchers after the  $(i + 1)th$  innovation,  $n_{i+1}$ , the smaller the payoff to innovating the  $ith$  good.

Let us move on to the specification of the profit flow,  $\pi_i$  and of the flow demand for manufacturing labour,  $x_i$ .

Final good production uses each intermediate goods according to the profit maximisation rule:

$$\frac{dy}{dx_i} = p_i,$$

which, recalling the production function 65:

$$y = Ax^\alpha,$$

is equivalent to:

$$\begin{aligned} \alpha Ax^{\alpha-1} &= p_i & (70) \\ \Leftrightarrow \\ x &= \left( \frac{\alpha}{\frac{p_i}{A_i}} \right)^{\frac{1}{1-\alpha}} \end{aligned}$$

Then follows the profit maximisation problem of the intermediate good producer that uses the  $i$ th innovation. This monopolist can be thought to be either the inventor and producer of the good  $i$  or the producer who buys the patent at the price  $V_i$ . His problem is to:

$$Max \quad \pi_i = p_i x - w_i x$$

And its solution entails the markup rule:

$$p_i = \frac{w_i}{\alpha}$$

So, replacing  $p_i$  in equation 70 gives us the required specification for  $x_i$ :

$$x_i = \left( \frac{\alpha^2}{\frac{w_i}{A_i}} \right)^{\frac{1}{1-\alpha}} \quad (71)$$

Then, we can obtain the expression for  $\pi_i$  :

$$\begin{aligned} \pi_i &= p_i x - w_i x & (72) \\ &= (1 - \alpha) p_i x \\ &= (1 - \alpha) \alpha A_i \left( \frac{\alpha^2}{\frac{w_i}{A_i}} \right)^{\frac{\alpha}{1-\alpha}} \end{aligned}$$

Now, the arbitrage condition 67 can be rewritten in the following way:

$$\begin{aligned} w_i &= \lambda V_{i+1} = \lambda \frac{\pi_{i+1}}{r + \lambda n_{i+1}} \\ &= \lambda \frac{(1 - \alpha) \alpha A_{i+1} \left( \frac{\alpha^2}{\frac{w_{i+1}}{A_{i+1}}} \right)^{\frac{\alpha}{1-\alpha}}}{r + \lambda n_{i+1}} \end{aligned}$$



So, recalling that  $\frac{A_{i+1}}{A_i} = \gamma$ , the productivity-adjusted wage rate,  $\omega_i = \frac{w_i}{A_i}$ , is equal to:

$$\omega_i = \frac{w_i}{A_i} = \lambda \frac{\gamma(1-\alpha)\alpha \left( \frac{\alpha^2}{\frac{w_{i+1}}{A_{i+1}}} \right)^{\frac{\alpha}{1-\alpha}}}{r + \lambda n_{i+1}},$$

The new arbitrage condition is then:

$$\omega_i = \frac{\gamma\lambda\tilde{\pi}(\omega_{i+1})}{r + \lambda n_{i+1}}, \quad (73)$$

where  $\tilde{\pi} = \frac{\pi_{i+1}}{A_{i+1}}$ .

The labour market clearing condition can also be written as:

$$L = n_i + \tilde{x}(\omega_i) \quad (74)$$

The steady-state or balanced growth equilibrium is defined as a stationary solution to the system composed by equations 73 and 74 with  $\omega_i = \omega$  and  $n_i = n$ . This means that both  $\omega$  and  $n$  remain constant over time, so that  $w$ ,  $\pi$  and  $y$  are all scaled up by the same  $\gamma > 1$  each time a new invention occurs.

So, in a steady-state the system to be solved is:

$$\begin{cases} \omega = \frac{\gamma\lambda\tilde{\pi}(\omega)}{r + \lambda n} \\ L = n + \tilde{x}(\omega) \end{cases} \quad (75)$$

In the space  $(\omega, n)$ , the arbitrage equation is downward sloping, as a rise in  $n$  increases the denominator of the ratio. In comparison, the labour market clearing equation is upward sloping, because as  $L$  is constant, if  $n$  increases  $\tilde{x}(\omega_i)$  must fall, which happens if  $\omega$  rises.

As one of the equations is positively sloped and the other is negatively sloped, the balanced growth path solution  $(\omega^*, n^*)$  is unique. Figure 5 illustrates such an equilibrium.

Now, continuing with the calculation of this solution, we must replace  $\tilde{\pi}(\omega)$  by a workable expression:

$$\begin{aligned} \pi &= px - wx \\ &= \left( \frac{1-\alpha}{\alpha} \right) wx \\ \Leftrightarrow \\ \tilde{\pi} &= \frac{\pi}{A} = \left( \frac{1-\alpha}{\alpha} \right) \omega \tilde{x} \\ \Leftrightarrow \\ \tilde{\pi} &= \left( \frac{1-\alpha}{\alpha} \right) \omega (L - n) \end{aligned}$$

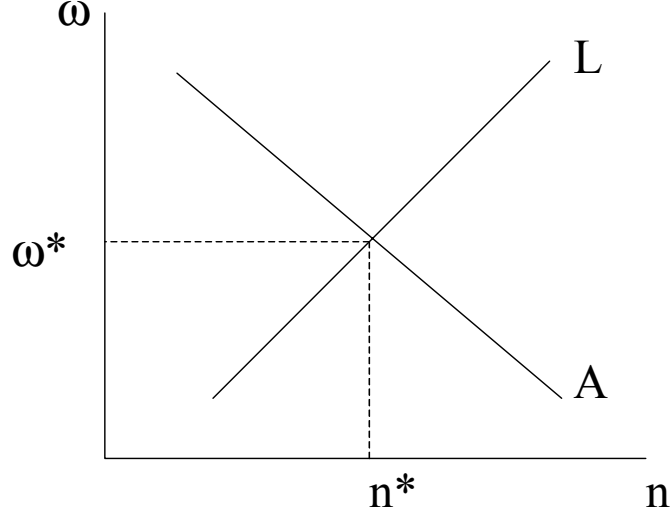


Figure 5:

Then, replacing  $\tilde{\pi}$  in the arbitrage equation 73, we obtain the equilibrium value of  $n$ :

$$\begin{aligned}
 \omega &= \frac{\gamma\lambda \left(\frac{1-\alpha}{\alpha}\right) \omega(L-n)}{r + \lambda n} & (76) \\
 \Leftrightarrow \\
 r + \lambda n &= \gamma\lambda \left(\frac{1-\alpha}{\alpha}\right) (L-n) \\
 \Leftrightarrow \\
 n^* &= \frac{\gamma\lambda \left(\frac{1-\alpha}{\alpha}\right) L - r}{\lambda \left[\frac{\alpha + \gamma(1-\alpha)}{\alpha}\right]}
 \end{aligned}$$

Knowing  $n^*$ , we can then use the labour market clearing condition to implicitly derive the equilibrium value  $\omega^*$ .

Now, what is left is the determination of the growth rate of the economy. So, in a steady-state, the flow of the final good,  $y$ , produced between innovations  $i$ th and  $(i+1)$ th is:

$$y_i = A_i(x^*)^\alpha = A_i(L - n^*)^\alpha,$$

which implies that:

$$y_{i+1} = A_{i+1}(L - n^*)^\alpha$$

And, therefore:

$$\frac{y_{i+1}}{y_i} = \frac{A_{i+1}}{A_i} = \gamma \quad (77)$$

Equation 77 tells us that  $\ln(y_i)$  increases by an amount equal to  $\ln(\gamma)$  each time an innovation occurs. But, as the real time between two innovations is random, the time path of  $\ln(y)$  is also a random step function, with the size of each step being equal to  $\ln(\gamma) > 0$ . Also the time interval between each step is exponentially distributed with parameter  $\lambda n^*$ . Taking a unit time interval between  $t$  and  $t + 1$ , we have:

$$\ln y(t + 1) = \ln y(t) + (\ln \gamma)\varepsilon(t),$$

where  $\varepsilon(t)$  is the number of innovations between  $t$  and  $t + 1$ .

As  $\varepsilon(t)$  is distributed Poisson with parameter  $\lambda n^*$ , we have:

$$\begin{aligned} E[\ln y(t + 1) - \ln y(t)] &= \lambda n^*(\ln \gamma) \\ &\Leftrightarrow \\ g &= \lambda n^*(\ln \gamma), \end{aligned} \quad (78)$$

where  $g$  is the average growth rate of output.

So, the equilibrium growth rate of this economy has been found. It is proportional to  $n^*$  and so this counts as an endogenously determined growth rate.

Notice that this R&D-based growth model is also characterised by the scale-effects property. In fact, equation 76 says that a rise in  $L$  increases  $n^*$ , and therefore increases  $g$ .

Growth is also influenced positively by  $\lambda$ , the research productivity parameter, as it raises  $n^*$ . On the contrary, a rise in  $r$  or in  $\alpha$  negatively influences  $n^*$ , and so decreases the equilibrium growth rate.

## 8 Some Limitations of the Endogenous Growth Models Studied

In this Section we discuss some limitations found in the endogenous growth models studied in this literature review. We highlight and analyse four such limitations.

The first limitation concerns the function that is responsible for endogenous growth. There seems to be a substantial degree of arbitrariness in the construction of this function in all endogenous growth models.

The second shortcoming arises from the intertemporal preferences structure which these models adopt. This preferences structure generates a positive relationship between the equilibrium interest rate and the equilibrium growth rate. However, such a positive relationship is not empirically supported.

The third limitation of these growth models also concerns the preferences structure. The specification of the preferences structure implies that diversity

of growth rates across countries can only be explained if it is assumed that international capital markets are imperfect. This assumption of imperfect capital markets is required so that countries are able to display different interest rates, and consequently have different growth rates, in equilibrium.

Fourthly, the existing endogenous growth models are not equipped to analyse short-run effects of the aggregate demand on the growth path of the economies.

We complete the Section with a brief exposition of a critical assessment to endogenous growth theory, by Fine [2000], in order to provide an overall insight into important topics within endogenous growth theory.

## 8.1 The Specification of the Function that Generates Endogenous Growth

Solow [2000] considers the endogenous growth models discussed in the previous Sections as very interesting and fertile, and responsible for moving growth theory forward. However, he highlights the arbitrariness of the specifications through which endogenous growth is delivered. This arbitrariness seems to have its roots in the requirement for growth models to achieve a balanced growth path.

Solow [2000] gives three reasons for this tendency of growth theory to focus on steady-states. Firstly, the basic neoclassical model usually has a unique steady-state and all equilibrium paths converge to this steady-state. Moreover, an economy in which the institutional structure has been fixed for a long time is expected to be near a steady-state.

Secondly, it has been thought that a defensible model of economic growth should be able to reproduce the six “stylised facts” about growth asserted by Kaldor. Solow writes that these “stylised facts” are generally a compact description of a steady-state.

His third reason is that some years ago there was no good way to study non-steady-state paths. Even nowadays, policy effects are analysed in terms of steady-state comparisons. The truth is that off the steady-state, the dynamics of endogenous growth models can display local indeterminacies, instability or cycles.

However, to achieve a balanced growth path solution is a very demanding task. A number of specific assumptions is required, so that endogenising the growth rate will not result in explosively fast or decaying growth rates. In this sense, Solow’s [2000] argument is that in the various endogenous growth models, the authors often do not provide detailed justification for these key assumptions which are required to generate a balanced growth path solution.

We now elaborate on this characteristic of arbitrariness of the existing endogenous growth models.

As we discussed in Section 3, in Romer’s [1990] model the specification for the invention of new designs, which is the key specification through which endogenous growth is delivered, is:

$$\dot{A} = \delta L_A A \tag{79}$$

This specification delivers a constant growth rate if labour dedicated to research  $L_A$  is constant:

$$g_A = \frac{\dot{A}}{A} = \delta L_A \quad (80)$$

This balanced growth path solution, that is a solution with a constant growth rate, carries the implication that the growth rate of the economy, equal to the growth rate of designs, is proportional to the number of researchers  $L_A$ . This result is the scale-effects prediction mentioned earlier.

Jones [1995] also wishes to demonstrate that Romer's [1990] specification for R&D can be deemed as arbitrary. With that purpose, Jones presents the following generalised specification for the invention of new designs:

$$\dot{A} = \delta L_A^\lambda A^\phi, \quad (81)$$

that becomes Romer's specification for  $\lambda = \phi = 1$ .

The marginal productivity of researchers is:

$$\frac{d\dot{A}}{dL_A} = \lambda L_A^{\lambda-1} A^\phi,$$

which varies with knowledge according to:

$$\frac{d\left(\frac{d\dot{A}}{dL_A}\right)}{dA} = \phi \lambda L_A^{\lambda-1} A^{\phi-1}$$

In Jones' specification, the assumption  $\phi < 0$  represents negative external returns from the stock of knowledge in the innovation process. Likewise the assumption  $\phi > 0$  implies positive external returns and the assumption  $\phi = 0$  means zero external returns or constant returns to scale.

Jones [1995] argues that the assumption  $\phi = 0$  might seem more natural, as Romer [1990] himself states that whether there are increasing or decreasing returns to R&D is a philosophical question. Jones also states that Romer's assumption of  $\phi = 1$  constitutes a "completely arbitrary degree of increasing returns" and is not empirically supported.

The truth is that, in Romer's model if, for example, instead of equation 79 we had:

$$\dot{A} = \delta L_A A^\eta \quad , \quad \eta \neq 1,$$

then the growth rate of knowledge would be:

$$g = \frac{\dot{A}}{A} = \delta L_A A^{\eta-1} = \frac{\delta L_A}{A^{1-\eta}}$$

The model would not display a balanced growth path, because  $L_A$  is constant (in Romer's model) and  $A$  is growing. If  $\eta < 1$ , the model would display decaying growth and if  $\eta > 1$ , the model would result in explosive growth.

The same kind of arbitrariness can be found in Lucas' [1988] human capital-based model, studied in Section 4. Its human capital accumulation equation, which is the key specification through which endogenous growth is delivered, is:

$$\dot{h} = h\gamma(1 - u) \quad (82)$$

Analogous to Romer's [1990] assumption for the R&D function, Lucas' specification also implies that the accumulation of human capital is proportional to the level of human capital. It delivers a constant growth rate, for  $u$  constant:

$$g = \frac{\dot{h}}{h} = \gamma(1 - u) \quad (83)$$

Solow [2000] states that this specification is based on very powerful assumptions, that of strong increasing returns to scale and constant returns to human capital. However, if, for instance,  $h$  were replaced by  $h^\lambda$ , in Lucas' model:

$$\dot{h} = h^\lambda \gamma(1 - u) \quad , \quad \lambda \neq 1,$$

the growth rate would be:

$$g = \frac{\dot{h}}{h} = h^{\lambda-1} \gamma(1 - u) = \frac{\gamma(1 - u)}{h^{1-\lambda}}$$

The model would not deliver a balanced growth path solution for  $u$  constant. If  $\lambda < 1$ , the model would not have sustained positive long-run growth, as its growth rate would be eroding over time. Likewise, if  $\lambda > 1$ , the model would result in explosive growth.

If we tried an extension to Lucas' model of the same kind as the one that Jones [1995] did with Romer's [1990] model, that is, if we allowed  $u$  to vary with time, we would have:

$$g = \frac{\dot{h}}{h} = h^{\lambda-1} \gamma(1 - u),$$

where a constant growth rate would require that:

$$\begin{aligned} \frac{[h^{\lambda-1} \gamma(1 - u)]}{h^{\lambda-1} \gamma(1 - u)} &= 0 \Leftrightarrow \frac{[h^{\lambda-1} \gamma - u h^{\lambda-1} \gamma]}{h^{\lambda-1} \gamma(1 - u)} = 0 \Leftrightarrow \\ &\Leftrightarrow \frac{(\lambda - 1) h h^{\lambda-2} \gamma - u (\lambda - 1) h h^{\lambda-2} \gamma - \dot{u} h^{\lambda-1} \gamma}{h^{\lambda-1} \gamma(1 - u)} = 0 \\ &\Leftrightarrow \frac{(1 - u) (\lambda - 1) h h^{\lambda-2} \gamma - \dot{u} h^{\lambda-1} \gamma}{h^{\lambda-1} \gamma(1 - u)} = 0 \\ &\Leftrightarrow \frac{\dot{h}}{h} = \frac{\dot{u}}{(1 - u) (\lambda - 1)}, \end{aligned}$$

which is only possible for  $\dot{u} = 0, u \neq 1, \lambda \neq 1$ . Under these conditions there would be no growth, and thus no endogenous growth model.

In Grossman and Helpman's [1991] model, the crucial specification that generates a constant growth rate is that of the cost of inventing a new design:

$$\text{cost} = \frac{a}{A}, \quad (84)$$

where  $A$  is the stock of designs invented so far.

This model obtains a balanced growth path solution because equation 84 fits the labour market equilibrium condition in a way such that it can generate a constant growth rate:

$$\frac{a}{A} \dot{A} + X = L, \quad (85)$$

where  $\frac{a}{A} \dot{A}$  is the amount of labour dedicated to research,  $X$  is the amount of labour employed in manufacturing and  $L$  is total labour. Equation 85 is equivalent to:

$$\begin{aligned} ag &= L - X \\ &\Leftrightarrow \\ g &= \frac{L - X}{a}, \end{aligned} \quad (86)$$

The model displays a balanced growth path solution as  $L$  and  $a$  are constant and  $X$  is required to be constant.

Again, if, instead of 84, the cost specification were:

$$\text{cost} = \frac{a}{A^\phi}, \quad \phi \neq 1,$$

the model would not deliver a balanced growth path solution, as seen below:

$$\begin{aligned} \frac{a}{A^\phi} \dot{A} + X &= L \\ &\Leftrightarrow \\ \frac{ag}{A^{\phi-1}} + X &= L \\ &\Leftrightarrow \\ g &= \frac{A^{\phi-1}(L - X)}{a} \end{aligned}$$

The growth rate is not constant, because  $A$  is growing. If  $\phi > 1$ , which would mean that the cost of innovation would fall with the stock of knowledge (for instance, in the case that one invention makes the following inventions easier, because inventors "stand on the shoulders of giants"), the model would result in explosive growth. If  $\phi < 1$ , the model would result in decaying growth. In particular, if  $\phi < 0$ , the cost of innovation would increase with the stock of knowledge (for instance, if invention of new products becomes increasingly difficult, as the easiest inventions are the first to be made).

Notice that if we tried, once again, a solution of the same kind as Jones' [1995], that is requiring the growth rate of  $X$  to be such as to deliver a balanced growth path:

$$g = \frac{A^{\phi-1}L}{a} - \frac{A^{\phi-1}X}{a},$$

with  $g_X = (1 - \phi)g_A$ , we would still not have a balanced growth path solution, because  $L$  is constant and thus the first term would never be constant.

In Aghion and Howitt's [1992] model, the element of arbitrariness arises in the equation:

$$\frac{A_{i+1}}{A_i} = \gamma > 1, \tag{87}$$

which gives:

$$\frac{Y_{i+1}}{Y_i} = \gamma$$

So:

$$\frac{Y_{t+1}}{Y_t} = \gamma^{e_t}$$

and the growth rate is:

$$g = \ln Y_{t+1} - \ln Y_t = \varepsilon_t \ln \gamma = \lambda n^* \ln \gamma \tag{88}$$

If, instead we had the example given in Solow [2000, page 177]:

$$A_{i+1} = A_i + \gamma,$$

then we would have:

$$Y_{t+1} = Y_t + \lambda n^* \gamma$$

and so the growth rate would be non constant:

$$g = \frac{Y_{t+1} - Y_t}{Y_t} = \frac{\lambda n^* \gamma}{Y},$$

that is the growth rate would tend to zero as output tended to infinity.

Still, Solow [2000] writes that he is not trying to be overly critical and that, although Aghion and Howitt's model is still far from a description of real research, endogenous growth theorists should try and follow their attempt in the sense of creating a theory about the endogenous creation of new technology.



## 8.2 The Interest Rate and the Growth Rate

As Fine [2000] analyses, one example of the microeconomic foundations of endogenous growth theory is the fact that growth models make growth dependent upon the optimising behaviour of representative individuals.

In particular, all the models we have studied and developed in this review adopt the same intertemporal preferences structure for a representative consumer, namely the one for which the intertemporal utility maximisation problem leads to the familiar Euler equation:

$$g = \frac{1}{\sigma}(r - \rho), \quad (89)$$

where  $g$  and  $r$  are the growth rate and the interest rate respectively,  $\frac{1}{\sigma}$  is the elasticity of intertemporal substitution and  $\rho$  is the rate of time preference.

Let us derive this Euler equation, with a simple utility maximisation exercise. Suppose the representative individual maximises the present discounted value of its utility:

$$\text{Max} \int_0^{\infty} U(C_t)e^{-\rho t} dt \quad , \quad U(C_t) = \frac{C_t^{1-\sigma}}{1-\sigma}$$

subject to the following restriction:

$$\dot{A}_t = rA_t + w_t - C_t,$$

where variable  $A$  stands for assets and  $w$  is the wage rate, and it is assumed that households provide one unit of labour per unit of time.

Then the *current-value Hamiltonian* is:

$$H_t = \frac{C_t^{1-\sigma}}{1-\sigma} + \theta_t [rA_t + w - C_t]$$

The first two optimality conditions are:

$$\begin{aligned} \frac{dH}{dC} &= 0 \Leftrightarrow C^{-\sigma} = \theta \\ &\Leftrightarrow g_C = -\frac{1}{\sigma}g_\theta \end{aligned} \quad (90)$$

and

$$\begin{aligned} \frac{dH}{dA} &= \rho\theta - \dot{\theta} \Leftrightarrow r\theta = \rho\theta - \dot{\theta} \\ &\Leftrightarrow g_\theta = \rho - r \end{aligned} \quad (91)$$

Together, they give the Euler equation:

$$g_C = -\frac{1}{\sigma}g_\theta = \frac{1}{\sigma}(r - \rho)$$

which gives us a balanced growth path solution for  $r$  constant.

If, for instance, the utility function were:

$$U(C_t) = \log C_t,$$

the first order condition 90 would be:

$$C^{-1} = \theta$$

which, differentiating, would give:

$$\begin{aligned} -\dot{C}C^{-2} &= \dot{\theta} \\ &\Leftrightarrow \\ \frac{-\dot{C}C^{-2}}{C^{-1}} &= \frac{\dot{\theta}}{\theta} \\ &\Leftrightarrow \\ g_C &= -g_\theta \end{aligned}$$

And the Euler equation would be:

$$g_C = (r - \rho),$$

which would mean that consumption plans do not depend on the elasticity of intertemporal substitution,  $\frac{1}{\sigma}$  nor on the coefficient of relative risk aversion,  $\sigma$ .

Weil [1990] writes that most of the empirical evidence suggests that agents care about intertemporal substitution and also about risk taking, so optimal consumption plans seem to be better reflected by  $U(C_t) = \frac{C_t^{1-\sigma}}{1-\sigma}$ .

Still, the use of equation 89 to describe the consumers' side generates a positive relationship between the general equilibrium interest rate and the general equilibrium growth rate. Although most endogenous growth models obtain this relationship, it is not empirically observed, as pointed out by Helpman [1992]. This constitutes another limitation of the models discussed in this literature review.

However, as Helpman [1992] discusses, this fact does not necessarily reduce the usefulness of endogenous growth models. These models were created to analyse supply-side mechanisms of growth that build on the accumulation of physical capital, the accumulation of human capital and on R&D activities. For this reason, they treat consumption in a simple way.

Weil [1990] also argues that the utility function  $U(C_t) = \frac{C_t^{1-\sigma}}{1-\sigma}$  has the mechanical restriction that the elasticity of intertemporal substitution,  $\frac{1}{\sigma}$  is the reciprocal of the coefficient of relative risk aversion,  $\sigma$ . Such restriction is "devoid of any economic rationale", he says, because it can not capture the empirical evidence of both the consumers' dislike for intertemporal substitution (high  $\sigma$ ) and the consumers' moderate willingness for risk taking (low  $\sigma$ ). Additionally, Weil writes that such a utility function does not allow us to identify between what influences the response of the growth rate to the interest rate - risk aversion

or the intertemporal elasticity of substitution - since such a response depends on the value of  $\sigma$ .

Weil [1990] introduces a utility function for which the constant-relative-risk-aversion coefficient is different from the constant intertemporal elasticity of substitution. By solving an infinite-horizon stochastic consumer problem, he finds that it is the elasticity of intertemporal substitution that governs the response of the growth rate to the interest rate (and not the risk aversion). Still, he obtains the same positive relationship between the growth rate and the interest rate. Epstein and Zin [1987] produce similar results to Weil's [1990] work.

### 8.3 Diversity of Growth Rates Across Countries

The use of equation 89 to describe the consumers' side has one more potential shortcoming, which concerns the explanation of diversity of growth rates across countries. The fact is that if the models that use this equation are to explain diversity of growth rates, they must assume that international capital flows do not lead to the equalisation of interest rates around the world. This is because, as Rebelo [1992] discusses, if international capital markets function perfectly, then the interest rates will equalise and all countries will have their GNPs (but not GDPs) growing at the same rate (assuming all countries share the same values for  $\sigma$  and  $\rho$ ). This rate is given by equation 89.

Rebelo [1992] proposed a utility function that allows for diversity of growth rates in the presence of equalisation of interest rates across countries. His utility function is:

$$U(C_t) = \frac{(C_t - \underline{C})^{1-\sigma}}{1-\sigma} \quad (92)$$

where  $\underline{C}$  is the consumption subsistence level.

Solving the intertemporal maximisation problem of consumers:

$$H_t = \frac{(C_t - \underline{C})^{1-\sigma}}{1-\sigma} + \theta_t [r_t A_t + w_t - C_t],$$

the first-order condition is:

$$\frac{dH}{dC} = 0 \Leftrightarrow (C - \underline{C})^{-\sigma} = \theta$$

Its time differentiation leads to:

$$\begin{aligned} -\sigma \dot{C} (C - \underline{C})^{-\sigma-1} &= \dot{\theta} \\ \Leftrightarrow \\ \frac{\dot{C}}{(C - \underline{C})} &= -\frac{1}{\sigma} \frac{\dot{\theta}}{\theta} \end{aligned}$$

The second-order condition is the same as before:

$$g_{\theta} = \rho - r$$

This implies that the optimal growth rate of consumption is given by:

$$g_c = \frac{\dot{C}}{C} = \frac{1}{\sigma} \frac{C - \underline{C}}{C} (r - \rho) = \frac{1}{\sigma} \left( 1 - \frac{\underline{C}}{C} \right) (r - \rho) \quad (93)$$

This equation implies that for  $r$  constant,  $\underline{C} \neq 0$  and  $C \neq \underline{C}$ , there is no balanced growth path solution. If a country starts with  $C < \underline{C}$ , its growth rate will grow, and if it starts with  $C > \underline{C}$ , its growth rate will decline.

Hence, in order to explain diversity of growth rates across countries with the same interest rate, Rebelo [1992] produced an Euler equation that gives different growth rates for countries with different ratios  $\frac{C}{\underline{C}}$ . Nevertheless, he had to let go of the balanced growth path solution.

This utility function makes sense for developing countries whose growth rates have yet to converge to the higher rates attained by developed economies. On the other hand, empirical evidence for developed countries shows that they tend to grow at a constant rate for long periods of time. Hence Rebelo's result seems not to apply to developed economies, unless their subsistence level of consumption is zero which would result in the Euler equation becoming the familiar one.

## 8.4 The Demand Side

Solow [2000] writes that if the main object of growth theory shifted from researching the steady-state to researching the whole growth path of the economy, the role of aggregate demand and its effects on the long-run path would emerge.

At present, all the existing growth models assume that the economy always achieves its potential output. There is, for example, no distinction between the labour force and employment, nor between the existing stock of capital and its utilisation rate. Some models assume Walrasian equilibrium, with all markets clearing, but Solow argues that this too is a "flat assumption".

Solow agrees that over intervals of thirty to fifty years, growth is clearly dominated by supply-side factors like the increase of the labour force, the accumulation of physical and human capital and the technological progress.

However, he adds that growth paths are not smooth, but instead marked by small or large periods of recession or excess demand. This raises the question of how these macroeconomic fluctuations affect the growth path. The understanding of this requires the linkage of the events of the business cycle to the evolution of the growth path.

Hence Solow argues [2000] that, in order to incorporate the demand side in growth models, short-run and long-run macroeconomics must help each other both analytically and empirically.

Witt [2001] also writes about the demand side as the recently emerging research topic. Researchers in this field believe that more attention must be given to consumer behaviour, which is not totally captured by the theory of utility and the supply-side oriented approach to innovations and growth. There is, for instance, the idea that innovativeness on the part of consumers is as important

as innovativeness on the part of producers. Producers can only sell their innovations if consumers' wants are evolving in that same direction. Another idea is that consumption can be motivated by the attempt to signal social status or distinction. Witt believes that once the insights of what happens on the demand side can be merged with the body of existing research on the production side, a comprehensive picture of the process of economic growth can be obtained.

## 8.5 General Critical Assessment of Endogenous Growth Theory

We finalise our literature review with a reference to Fine's [2000] critical assessment of endogenous growth theory. We summarise below some of his critical thoughts on endogenous growth theory.

Firstly, Fine [2000] writes that as endogenous growth theory is based upon microfoundations, but aims to explain macroeconomic issues, there is always the possibility that it will be jumping between the two. Fine points out that for a partial theory, endogenous growth theory claims too much macroeconomic understanding. He adds that Solow [1991] is also particularly concerned with the long-run in which the "grandly endogenous" elements such as stages of capitalism and shifting social institutions are tied to the simplistic optimising behaviour.

Secondly, endogenous growth theory has reached no policy consensus, nor are its policy implications readily applicable in practice. This is because, in contrast to monetarism or Keynesianism, the abstract, formal and very aggregated content of endogenous growth theory leads to policy ambiguity and imprecision.

Fine [2000] further writes that endogenous growth theory has been rapidly growing and has substantial potential for further expanding its scope. This is because it is based on the microeconomics of market imperfections and technical change, and thus has many resources upon which to draw.

The negative side effect of this is that it makes the content of endogenous growth theory arbitrary, due to the analytical strategy of generating endogeneity, and also subject to methodological individualism. By investing heavily in gradually more sophisticated mathematics and statistics, endogenous growth theorists have departed from assumptions and even from basic descriptive narrative. Fine believes the theory should be developed by going back to methodological first principles and by contemplating the social, historical and other forces within the economy.

Notwithstanding the above mentioned criticisms, Fine [2000] states that endogenous growth theory has the merit of being able to explain the simplest facts about growth (Kaldor's stylised facts, patterns of convergence and divergence) which could not be explained by the orthodox growth theory.

Fine also states that endogenous growth theory has, in fact, shown to be able to accommodate endogenous productivity, monopoly, institutions, money and finance, the patterns of growth and cycles, conflict and inequality. It can also contemplate the political (voting) and the social (stratification). It has been taking over radical political economy and can easily move into the fields

of social sciences as geography and environment. This extraordinary evolution of endogenous growth theory prompts Fine to believe that it will continue to be an important area of the economics discipline.

## 9 Concluding Remarks

In this paper, we have reviewed the prototypical models of endogenous growth in its three branches according to the engine of growth.

The purpose of this paper has been to present a both detailed and panoramic view of the models that constitute the core of endogenous growth literature.

Firstly, we analysed how R&D activities can be modeled in growth models, so as to become the source of sustained positive per-capita growth in the long-run.

Secondly, we studied the modelling of human capital accumulation and its role as the engine of sustained positive economic growth.

In these two kinds of models, the diminishing marginal productivity of capital is overcome, respectively, by technological progress and by human capital accumulation. Sustained positive long-run per-capita growth is thus made possible.

These are endogenous growth models because technological progress and human capital accumulation are determined within the respective models.

The third group of models we discussed produce endogenous sustained growth through the direct elimination of the diminishing returns to capital assumption from the production function.

The numerous models that give body to endogenous growth theory are all based on the models that have been analysed in this literature review. Hence we believe that the comprehension of the mechanisms responsible for endogenous growth in the models analysed here is the first basic and fundamental step towards the ability to work theoretically with endogenous growth models.

Future theoretical work within endogenous growth theory also requires, we believe, awareness of the limitations that characterise existing growth models. Hence our inclusion, in the last part of this literature review, of the discussion of some of such limitations.

## Appendix

### Romer's Model - The Social Planner Solution

In order to confirm the welfare properties of Romer's [1990] model, let us consider the social planner's formulation of this model. The social planner maximises the representative consumer's utility:

$$\max_{c, L_A} \int_0^{\infty} e^{-\rho t} \frac{C_t^{1-\sigma}}{1-\sigma} dt, \quad (94)$$

subject to the following constraints:

$$Y_t = K_t^\alpha L_{Y_t}^{1-\alpha} A_t^{1-\alpha}, \quad (95)$$

$$\dot{K} = Y - C, \quad (96)$$

$$\dot{A} = \delta A L_A, \quad (97)$$

$$L = \bar{L} = L_Y + L_A \quad (98)$$

The *current-value Hamiltonian* is:

$$H = \frac{C^{1-\sigma}}{1-\sigma} + \theta_1 [K^\alpha (L - L_A)^{1-\alpha} A^{1-\alpha} - C] + \theta_2 [\delta A L_A]$$

The two decision variables are  $C_t$  and  $L_{At}$ . So the first-order conditions are:

$$\frac{dH}{dC} = 0 \quad (99)$$

$$\frac{dH}{dL_A} = 0 \quad (100)$$

and the co-state equations are:

$$\frac{dH}{dK} = \rho \theta_1 - \dot{\theta}_1 \quad (101)$$

$$\frac{dH}{dA} = \rho \theta_2 - \dot{\theta}_2 \quad (102)$$

Solving the problem:

Equation 99:

$$\frac{dH}{dC} = 0 \Leftrightarrow C^{-\sigma} = \theta_1 \Leftrightarrow \frac{\dot{C}}{C} = -\frac{1}{\sigma} \frac{\dot{\theta}_1}{\theta_1}$$

Equation 100:

$$\frac{dH}{dL_A} = 0 \Leftrightarrow \theta_1(1 - \alpha)(L - L_A)^{-\alpha} K^\alpha A^{1-\alpha} = \theta_2 \delta A$$

Equation 101:

$$\frac{dH}{dK} = \rho \theta_1 - \dot{\theta}_1 \Leftrightarrow \frac{\dot{\theta}_1}{\theta_1} = \rho - \frac{\alpha(L - L_A)^{1-\alpha} A^{1-\alpha}}{K^{1-\alpha}}$$

Equation 102:

$$\frac{dH}{dA} = \rho \theta_2 - \dot{\theta}_2 \Leftrightarrow \theta_1(1 - \alpha)(L - L_A)^{1-\alpha} K^\alpha A^{-\alpha} + \theta_2 \delta L_A = \rho \theta_2 - \dot{\theta}_2$$

The model is solved for its balanced growth path, the solution for which  $K$ ,  $Y$ , and  $C$  grow at a constant rate (given by the growth rate of  $A$ ), and the current-value prices  $\theta_1$  and  $\theta_2$  decline at constant rates.

The first step is to look at equations 100 and 102 and observe the similarity between their first terms. So, together these two equations give us:

$$\begin{aligned} \frac{\theta_2 \delta A (L - L_A)}{A} + \theta_2 \delta L_A &= \rho \theta_2 - \dot{\theta}_2 & (103) \\ \Leftrightarrow & \\ \frac{\dot{\theta}_2}{\theta_2} &= \rho - \delta L \end{aligned}$$

Next, log-differentiation of equation 100 leads to:

$$\begin{aligned} \frac{\dot{\theta}_1}{\theta_1} + \alpha \frac{\dot{K}}{K} + (1 - \alpha) \frac{\dot{A}}{A} &= \frac{\dot{\theta}_2}{\theta_2} + \frac{\dot{A}}{A} & (104) \\ \Leftrightarrow & \\ \frac{\dot{\theta}_1}{\theta_1} &= \frac{\dot{\theta}_2}{\theta_2} \end{aligned}$$

Now, equations 99, 103 and 104 are used to obtain the equilibrium growth rate of this centralised problem:

$$\begin{aligned} g &= \frac{\dot{C}}{C} = -\frac{1}{\sigma} \frac{\dot{\theta}_1}{\theta_1} = -\frac{1}{\sigma} \frac{\dot{\theta}_2}{\theta_2} & (105) \\ \Leftrightarrow & \\ g_{SP} &= \frac{\delta L - \rho}{\sigma} \end{aligned}$$

The centralised equilibrium growth rate,  $g_{SP}$ , given by equation 105 is higher than the decentralised equilibrium growth rate,  $g_D$ , given by equation 24, repeated here, for better comparison:

$$g_D = \frac{\alpha \delta \bar{L} - \rho}{\alpha + \sigma},$$

confirming that Romer's decentralised model delivers a sub-optimal solution.



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