Universidade do Minho
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UMinho|2019

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Tese de Doutoramento
Programa Doutoral em Física (MAP-fis)

Trabalho efetuado sob a orientação do
Professor Doutor Filipe Carteado Mena
e do
Professor Doutor José Pizarro de Sande e Lemos

outubro de 2019

DIREITOS DE AUTOR E CONDIÇÕES DE UTILIZAÇÃO DO TRABALHO POR TERCEIROS

Acknowledgments

I would like to show my deep gratitude to all who helped me in the course of this doctoral programme: without their guidance, advice and support the work in this thesis would not have been possible.

I thank my supervisor, Professor Filipe Mena, for the constant support during this past 4 years: his insight and positivism, even when faced with seemingly insurmountable problems, taught me to don't give up and tackle all problems with confidence that a solution can be found. To my co-supervisor, Professor José Sande Lemos, for the many hours he spent teach me physics and how to be a researcher, I am deeply grateful for how much he has done for me. I am very thankful for their patience and how much they helped me, without them, whatever good came from this doctoral programme, would not have been possible.

I am also grateful to all the members of CENTRA and CMAT, for the joyful and exceptional working environment. A special thanks to Dr. Amir Ziaie for our fruitful collaboration; to Dr. Vincenzo Vitagliano, for the many discussions on science and the research life whose, markedly shaped many of my decisions in the course of this PhD. I am indebted to Dr. Sante Carloni for all the opportunities and discussions: how much I have learned with him will deeply impact my life.

I would also like to thank my family for the constant cheer that help me go through these arduous years. To my parents, for the unrelenting support since the first day I enrolled in primary school. A very special thanks to my wife and daughter for reminding of the important things, for the innumerable joys and their hugs at the end of the day.

Lastly, I am very grateful to Fundação para a Ciência e a Tecnologia and the IDPASC programme for the scholarship that made it possible for me to enroll in the doctoral program and complete this thesis.

STATEMENT OF INTEGRITY

I hereby declare having conducted this academic work with integrity. I confirm that I have not used plagiarism or any form of undue use of information or falsification of results along the process leading to its elaboration. I further declare that I have fully acknowledged the Code of Ethical Conduct of the University of Minho.

Objetos compactos, colapso gravitacional e singularidades em teorias relativistas da gravitação

Resumo

A natureza e a evolução de objetos astrofísicos massivos providenciam uma janela natural para estudar a interação gravitacional; em particular, entender o estado final do colapso gravitacional e a formação de singularidades é um dos problemas fundamentais da física moderna. Nesta tese, iremos estudar vários modelos para objetos compactos e fluidos de matéria em colapso gravitacional no contexto de teorias afins da gravidade, em particular, iremos analisar como a inclusão de torsão no espaço-tempo afeta a estrutura e a dinâmica de objetos compactos massivos e a formação de singularidades.

Na primeira parte, iremos desenvolver as ferramentas matemáticas que serão utilizadas ao longo da tese. Começamos por generalizar a decomposição covariante 1+3 para teorias afins da gravitação em espaços-tempo dotados de uma conexão afim, compatível com a métrica. Estes resultados mostram claramente como o tensor de torsão afeta a geometria dos espaços-tempo, em particular conclui-se que este influencia diretamente as quantidades cinemáticas de uma congruência, resolvendo muitos dos equívocos presentes na literatura. Investigamos as condições para o mergulho de uma variedade num espaço-tempo de maior dimensão, generalizando as equações de Gauss-Codazzi-Ricci para conexões afins compatíveis com a métrica. De seguida, estudamos a junção suave de dois espaços-tempo numa fronteira comum na presença de torsão, corrigindo e estendendo os resultados na literatura.

A segunda parte desta tese é dedicada a singularidades do espaço-tempo. Expandimos o âmbito do teorema de Raychaudhuri-Komar na teoria da Relatividade Geral a uma larga gama de teorias da gravitação e mostramos como a presença de torsão e aceleração influencia a formação de singularidades. De seguida, consideramos o colapso de um fluido num espaço-tempo de Szekeres no contexto da teoria de Einstein-Cartan-Sciama-Kibble, encontrando um conjunto de condições aos dados iniciais para evitar a formação de singularidades.

Na última parte consideramos soluções exatas para objetos compactos. Ao generalizar as equações de Tolman-Oppenheimer-Volkoff no contexto da teoria de Einstein-Cartan-Sciama-Kibble, derivamos e estudamos as propriedades de soluções exatas para objetos compactos permeados por um fluido perfeito composto por fermiões, suavemente combinados a um vácuo exterior. Também, provamos sob condições gerais que, no contexto de uma torsão do tipo de Weyssenhoff, não existem objetos compactos estáticos, esfericamente simétricos suportados apenas pelo momento angular intrínseco da matéria. Por fim, consideramos a junção dos espaços-tempo Minkowski — Reissner-Nordström com a presença de uma camada fina de matéria. Encontramos todas as soluções e investigamos as propriedades dos espaços-tempo resultantes.

Palavras-chave: Decomposição covariante do espaço-tempo, formalismo de junção de espaços-tempo, objectos compactos, teoremas de singularidades, torsão.

Compact objects, gravitational collapse and singularities in relativistic theories of gravitation

Abstract

The nature and evolution of massive astrophysical bodies provide a natural window to study the gravitational interaction; in particular, understanding the end state of gravitational collapse and the formation of singularities is one of the fundamental problems in modern physics. In this thesis, we will study several models for compact objects and matter fluids undergoing gravitational collapse in the context of affine theories of gravity, in particular, we will analyze how the inclusion of space-time torsion affects the structure and the dynamics of compact objects and the formation of singularities.

In the first part, we develop a set of mathematical tools that will be used throughout the thesis. We start by generalizing the 1+3 covariant decomposition for generic affine theories of gravitation in space-times endowed with a metric compatible affine connection. These results, clearly show how the torsion tensor affects the geometry of the space-time, in particular it is found that it directly impacts the kinematical quantities of a congruence, solving many of the ambiguities lingering in the literature. We investigate the conditions for the embedding of a manifold in a higher dimensional space-time, generalizing the Gauss-Codazzi-Ricci equations for metric compatible affine connections. Then, we study the smooth matching of two space-times at a common boundary in the presence of torsion, correcting and extending the results in the literature.

The second part of this thesis is devoted to space-time singularities. We extend the scope of the Raychaudhuri-Komar theorem of General Relativity to a wide class of theories of gravitation and show how the presence of torsion and acceleration influences the formation of singularities. We then consider the case of a collapsing fluid permeating a Szekeres space-time in the context of the Einstein-Cartan-Sciama-Kibble theory, finding a set of conditions on the initial data for the avoidance of singularities.

In the last part we consider exact solutions for compact objects. Generalizing the Tolman-Oppenheimer-Volkoff equations in the context of the Einstein-Cartan-Sciama-Kibble theory, we derive and study the properties of exact solutions for compact objects permeated by a perfect fluid composed of fermions, smoothly matched to an exterior vacuum. Moreover, we prove under generic conditions that, in the context of a Weyssenhoff-like torsion, no static, spherically symmetric compact objects supported only by the intrinsic spin of matter can exist. Lastly, we consider the junction of a Minkwoski — Reissner-Nordström space-times by means of a thin matter shell. We find all the possible solutions and investigate the properties of the resulting space-times.

Keywords: Compact objects, covariant space-time decomposition, singularity theorems, space-time junction formalism, torsion.

Contents

Acronyms

ECSK	Einstein-Cartan-Sciama-Kibble
EFE	Einstein field equations
FLRW	Friedmann-Lemaître-Robertson-Walker
GR	General Relativity
LHS	Left-hand side
LRS	Locally rotationally symmetric
LRS I	Locally rotationally symmetric of type I
LRS II	Locally rotationally symmetric of type II
LTB	Lemaître-Tolman-Bondi
NEC	Null energy condition
RHS	Right-hand side
TOV	Tolman-Oppenheimer-Volkoff
WEC	Weak energy condition

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Preamble

The research presented in this thesis was carried out at Centro de Matemática da Universidade do Minho and Centro de Astrofísica e Gravitação — CENTRA, Instituto Superior Técnico, Universidade de Lisboa and was supported by FCT-Portugal through Grant No. PD/BD/114074/2015, awarded in the framework of the Doctoral Programme IDPASC-Portugal.

The results in Chapter 2 were done in collaboration with Professor José Sande Lemos, Doctor Sante Carloni and Doctor Vincenzo Vitagliano; Chapter 3 with Professor José Sande Lemos and Professor Filipe Mena; Chapter 4 with Professor Filipe Mena; Chapter 5 with Professor Filipe Mena and Doctor Amir Hadi Ziaie; Chapter 6 with Doctor Sante Carloni and Chapter 7 was done in collaboration with Professor José Sande Lemos. The unpublished results are in Chapters 2, 3 and 7.

The works published or to be published presented in this thesis are included bellow:

(1) P. Luz and V. Vitagliano, "Raychaudhuri equation in spacetimes with torsion", Phys. Rev. D **96**, 024021 (2017); arXiv:1709.07261 [gr-qc] (Chapters 2 and 4).

(2) P. Luz and J. P. S Lemos, "1+3 space-time decomposition for affine theories of gravity: the effects of torsion in relativistic cosmology", in preparation (Chapter 2).

(3) P. Luz and J. P. S Lemos, "Junction conditions for Lorentzian manifolds with torsion", in preparation (Chapter 3).

(4) P. Luz, "Generalized Gauss-Codazzi-Ricci embedding equations in the presence of torsion", in preparation (Chapter 3).

(5) P. Luz and F. C. Mena, "Singularity theorems and the inclusion of torsion in affine theories of gravity", submitted; arXiv:1909.00018 [gr-qc] (Chapters 3 and 4).

(6) P. Luz, F. C. Mena and A. H. Ziaie, "Influence of intrinsic spin in the formation of singularities for inhomogeneous effective dust space-times", Class. Quant. Grav. **36**, 015003 (2019); arXiv:1811.12292 [gr-qc] (Chapter 5).

(7) P. Luz and Sante Carloni, "Static compact objects in Einstein-Cartan theory", Phys. Rev. D **100**, 084037 (2019); arXiv:1907.11489 [gr-qc] (Chapters 2, 3 and 6).

(8) P. Luz and J. P. S. Lemos, "Complete classification of static electrically charged thin-shells with a flat core", in preparation. (Chapter 7).

(9) P. Luz and J. P. S. Lemos, "Electrically charged tension shells", in The Fourteenth Marcel Grossmann Meeting (2017), p. 1641 (Chapter 7).

Chapter 1

General introduction

The geometrization of the gravitational interaction is one the most remarkable achievements of humankind. The paradigm shift introduced by Einstein's theory of General Relativity (GR), where the gravitational field is associated with the metric tensor of a manifold, not only reconciled the theory of special relativity with gravity or solved the enigma of the precession of the perihelion of Mercury, but it also predicted a plethora of new phenomena. Notably, the behavior of matter under extreme gravitational fields, where the theory predicts a drastically distinct behavior when compared to the classical Newtonian gravitation, for instance, fully relativistic stars are expected to have, under rather generic conditions, a maximum allowed mass [1], beyond which they will undergo gravitational collapse, possibly forming an event horizon, a surface dividing the space-time in two causally separated regions; even predicting the formation of singularities, not only of the theory, but actual points removed from the space-time manifold.

The existence and formation of space-time singularities has been an area of active discussion since the early days of GR. Not only the first non-trivial solutions: the Schwarzschild [2] and Friedman-Lemaître-Robertson-Walker (FLRW) models [3–9], were known to harbor space-time singularities; after the seminal work of Oppenheimer and Snyder [10], it was realized that these singularities — in the latter, of the stress-energy tensor — could indeed form from physical phenomena.

Although the existence of singularities in some solutions of GR was known, these were believed to be a consequence of the symmetries imposed on the space-time manifold, and more realistic scenarios would not suffer from such defects. It was not until the works of Raychaudhuri and Komar [11, 12], and Penrose and Hawking [13–16] that it was understood that the formation of singularities due to gravitational collapse is a generic and unavoidable feature of the theory, challenging its validity on such scenarios.

Not only theoretically but also from an experimental point of view, it has become increasingly clear that the theory of General Relativity, even in the non-quantum regimes, may not correspond to the full theory for the gravitational interaction. In spite of its complexity, GR is a comparatively simple theory given the plethora of possible alternatives. It is in fact serendipitous that GR passed so many experimental tests: not only the theory predicts and describes, so far, all observations of the solar system; the, nowadays regular, detection of gravitational waves have attested to the validity of the theory in the vicinity of supermassive astrophysical objects [17]. On the other hand, the data from rotational curves of galaxies [18] and Supernovae emission spectra [19, 20], show that the predictions of GR are incompatible with the experimental observations without the inclusion of extra fields.

The torsion tensor field and the Einstein-Cartan theory

One of the fundamental assumptions of the theory of General Relativity is the fact that the geometry of the space-time manifold is uniquely described by the metric tensor field. In particular, this imposes that the affine connection is of the Levi-Civita type: metric compatible and symmetric. From a theoretical, other than Occam's razor, and experimental point of view [21] there is no reason for such condition. Then, one possible extension of the theory concerns removing the constraint that the connection must be symmetric, leading to an extra geometrical field on the manifold — the torsion tensor. This is the idea of the so called Einstein-Cartan theory [21, 22] where the geometry of the space-time is described by both the metric and the torsion tensor fields[1].

One particular realization of the Einstein-Cartan theory is the Einstein-Cartan-Sciama-Kibble (ECSK) theory [23, 24] where the field equations are still derived from the Einstein-Hilbert action[2]. The ECSK theory represents one of the simplest generalizations of GR. In this regard, it has been shown [21, 23, 24] that the extra degrees of freedom due to torsion can be related with the matter intrinsic spin, allowing for the inclusion of quantum corrections in a macroscopic limit, in a geometric theory of gravitation. The framework of the ECSK theory has lead to many important results, showing how the extra geometrical structure influences the behavior of the matter fluids that permeate the space-time and, consequently, the geometry of the manifold. Notably, early works in a cosmological setting [25–29] showed that the inclusion of torsion could act as a repulsive force, counteracting the gravitational collapse and possibly prevent the formation of singularities.

Compact objects

A natural window to probe the gravitational interaction and test a theory of gravitation is studying the structure and dynamics of compact astrophysical objects. The various experimental tests on the gravitational interaction confidently assert that vacuum solutions of GR, namely the Schwarzschild and Kerr models [30–32], accurately describe the current experimental data on the corresponding regimes. This, then, places strong bounds on the possible alternative theories of gravitation and the corresponding solutions in vacuum. Alternatively, deviations introduced by distinct theories of gravitation are expected to manifest in the presence of matter. This, on the other hand, places the difficulty of separating the effects due to the extra degrees of freedom in a given theory

[1] In general, it is possible to also remove the imposition that the connection is compatible with the metric. There are however, good physical reasons to impose this, for instance, the norm of a tangent vector field to a geodesic would not be constant along the curve.

[2] In the literature, the nomenclature Einstein-Cartan and ECSK theories are usually used interchangeably. This leads to difficulties when we want to refer to the particular theory following from the Einstein-Hilbert action. Based of the initial works of Sciama and Kibble [22–24], in this work we will adopt the convention that the ECSK theory refers to the case when the field equations follow from the Einstein-Hilbert action.

of gravitation or due to the equation of state of the matter fields that permeate the space-time. It is then important to study solutions that markedly differ from GR so that, either there is no equivalent setup in GR or it would lead to unphysical equations of state. In this thesis, we will then study solutions for compact objects in different theories of gravitation, considering either static setups which, can be used to model massive astrophysical bodies such as stars or Neutron stars, or the gravitational collapse of matter fluids. In particular, we will discuss how the presence of a torsion tensor field affects the solutions and the behavior of this kind of objects.

Given the nature of the torsion tensor field, in the Einstein-Cartan theory it is not possible to relegate its effects as extra matter fields on the space-time [33]: since the geometry of the manifold is described by the metric and the torsion tensors, the way this extra geometrical degrees of freedom will couple with matter, will lead — as we will see — to manifestly distinct behavior when compared to theories of gravitation based on manifolds with a Levi-Civita connection. In this respect, the research on the effects of torsion in compact objects is scarce. Early attempts to find solutions with non-vanishing torsion tensor relied on a mapping between this field and an effective fluid in the theory of General Relativity, disregarding the effects of torsion on the whole geometry of the space-time and considering its effects only on the metric tensor, leading to nonsensical results with solutions with singularities [34, 35] or to the conclusion that stable configurations only exist for extremely small densities and radius [36].

Outline of the thesis

To help the reader navigate through this thesis, we briefly provide its outline and structure. In Part I we derive and present the mathematical results that will be used throughout the thesis. Given the plethora of different conventions in the literature, here we will also set the notation and define the various quantities. In that regard, the treatment is intended to be self-contained, assuming only that the reader has a basic understanding of Lorentzian geometry. Moreover, the results in this part are rather general so that, they can be applied to various setups, even in areas outside the theme of this work.

In Chapter 2 we will introduce the 1+3 covariant space-time decomposition for generic affine theories of gravitation, namely we will properly define the kinematical quantities that characterize a congruence of curves, the electric and magnetic parts of the Weyl tensor and the covariant components of the stress-energy tensor; then, we find the complete set of evolution and constraint equations for these quantities. Next, in Chapter 3 we will study the geometry of sub-manifolds of a higher dimensional Lorentzian manifold. We will first find the conditions for the embedding of hypersurfaces in a space-time, in particular, we will study in what conditions a space-time admits the existence of hypersurface orthogonal congruences. Then, we study the relation between the geometry of an higher dimensional manifold with that of an embedded sub-manifold: we will introduce the first and second fundamental forms and derive the generalized Gauss-Codazzi-Ricci equations for Lorentzian manifolds endowed with a metric compatible, affine connection. To close this part, we consider the junction of two space-times at a common boundary in the presence of a torsion tensor field, deriving the necessary and sufficient conditions for a smooth junction such that, the resulting space-time has a well defined geometry at the matching surface.

In Part II we will focus on studying space-time singularities, in particular, how the inclusion of space-time torsion affects their formation. In Chapter 4 we will study caustics of congruences of causal curves and generalize the classical focus theorems for time-like and null curves, not necessarily geodesics, in the presence of torsion. These results will then be used to construct singularities theorems. In the time-like case, we will extend the scope of the Raychaudhuri-Komar theorem to affine theories of gravitation for perfect fluids and scalar fields. In Chapter 5 we consider the particular case of a perfect fluid permeating a Szekeres space-time, in the context of the ECSK theory. Relating the degrees of freedom of the torsion tensor with the intrinsic angular momentum of matter, we show how the intrisinc spin influences the possible solutions and the formation of singularities. Then, we consider various particular solutions of the Szekeres model, namely, Lemaître-Tolman-Bondi, Friedman-Lemaître-Robertson-Walker, Senovilla-Vera and locally rotationaly symmetric Bianchi type I space-times, extending the previous results or explicitly verifying them — either analytically or numerically.

The third and last part of this thesis is devoted to solutions for compact objects. In Chapter 6 we start by deriving the 1+1+2 covariant formalism for stationary, locally rotationally symmetric space-times permeated by a Weyssenhoff spin fluid, in the context of the ECSK theory. These results are then used to find the generalized Tolman-Oppenheimer-Volkoff structure equations. The newly derived equations allow us to find and study the properties of exact solutions for compact objects. In particular, we analyze how the intrinsic spin of matter changes the Buchdahl limit for the maximum compactness of stars and, under rather generic conditions, we prove that in the context of a Weyssenhoff like torsion, no static, spherically symmetric compact objects supported only by the intrinsic spin can exist. We also provide a set of algorithms to generate new solutions from previously known ones. In Chapter 7 we consider the junction of a Minkwoski — Reissner-Nordström space-times by means of a thin matter shell. Given the complexity of the Reissner-Nordström space-time, we derive two patches of Kruskal-Szekeres coordinates, covering all possible sub-regions; this allows us to find and clearly relate the properties of the fluid composing the shell with the region it inhabits. We then test the results against the classical energy conditions.

In this thesis, all quantities are expressed in the geometrized unit system where $\mathcal{G} = c = 1$, Greek indices run from 0 to $N-1$ and Latin indices run from 1 to $N-1$, where N denotes the dimension of the manifold. Moreover, we will assume the metric signature $(-+++\ldots)$ and use the Einstein summation convention.

Part I

Covariant manifold threading and geometry of hypersurfaces

Chapter 2

Covariant 1+3 space-time decomposition

2.1 Introduction

Covariant space-time decomposition approaches were devised as powerful tools to analyze the geometry and dynamics of tensor fields on a space-time, aiming to take directly into account the symmetries and preferred directions in a manifold. This kind of formalism of partial decomposition of the space-time manifold has been extensively employed to explore the properties of solutions of selected theories of gravitation, namely: cosmological models, either in the context of GR [37–46] or other $f(R)$ modified theories of gravitation [47, 48]; or space-times of astrophysical interest [49–52]. Moreover, these methods play a pivotal role in the study of formation of space-time singularities [16, 53].

One particular realization of such approaches is the so called 1+3 formalism[1] [38–40]. In this formalism, it is assumed the existence of a congruence of smooth curves, so that any tensor quantity on the space-time is, at each point, separated in its components along the direction of the tangent vector field to the congruence and components along orthogonal surfaces — to the curves of the congruence. In fact, this property of the formalism makes it especially useful from a physical point of view since, usually we are interested in studying the evolution of certain quantities in time. Then, assuming the existence of a time-like congruence, the 1+3 formalism naturally decomposes the structure equations that describe the geometry of the space-time and tensor fields in the manifold along a time and spatial directions. Moreover, this natural splitting of the manifold can greatly simplify the problem of finding solutions when the space-time admits the existence of preferred directions, such as Killing vector fields. Another advantage of the formalism comes from the fact that it is, by construction, independent of a coordinate system. In a geometric theory of gravity, the geometry of the space-time is related with the matter fields that permeates it. Since the evolution and constraint equations found

[1]To not be confused with the "3+1 formalism" (see e.g. Ref. [54]). This change in semantics is due to the fact that in the 1+3 formalism the foliation of the space-time in $N-1$ surfaces, does not have to constraint such surfaces to be integral manifolds. The 3+1 formalism refers to the case when the space-time foliation assumes a family of integral manifolds, orthogonal to a given congruence of curves, placing the focus on the geometry of this hypersurfaces. In the case of the 1+3 formalism the manifold is assumed of be *threaded* by a family of curves, a congruence, so that the geometry of the space-time is encoded in the kinematical quantities that characterize the congruence and the Weyl tensor components.

from the 1+3 formalism are completely covariant, they provide a clearer interpretation of the relations between the kinematics of the congruence and the properties of the matter fields that permeate the space-time.

Although the 1+3 formalism was initially devised to study solutions of the theory of General Relativity and recently it has been employed in the context of other $f(R)$ theories of gravity, the paradigm of covariant space-time decomposition is applicable to an even wider class of geometric theories of gravitation. In this chapter we will focus on the extension of the 1+3 formalism for space-times with a metric compatible affine connection. This will allow us to introduce the general structure equations, the geometric quantities and set the conventions that will be used throughout this thesis.

2.2 Geometry of Lorentzian manifolds with torsion

Due to the wide variety of different conventions used in literature, let us start by gently introducing the basic definitions and setting the conventions that will be used throughout this thesis.

Let $(\mathcal{M}, g)$ be an N–dimensional Lorentzian manifold endowed with a metric compatible affine connection, $C^{\gamma}_{\alpha\beta}$, such that, the covariant derivative of a vector field u is given by,

$$\nabla_\alpha u^\gamma = \partial_\alpha u^\gamma + C^{\gamma}_{\alpha\beta} u^\beta \,. \tag{2.1}$$

The anti-symmetric part of the connection, $C^{\gamma}_{\alpha\beta}$, defines a tensor field called torsion

$$S_{\alpha\beta}{}^{\gamma} \equiv C^{\gamma}_{[\alpha\beta]} = \frac{1}{2}\left(C^{\gamma}_{\alpha\beta} - C^{\gamma}_{\beta\alpha}\right) \,. \tag{2.2}$$

Using this definition, it is possible to split the connection into an appropriate combination of the torsion tensor plus the usual metric connection[2], $\Gamma^{\gamma}_{\alpha\beta}$,

$$C^{\gamma}_{\alpha\beta} = \Gamma^{\gamma}_{\alpha\beta} + S_{\alpha\beta}{}^{\gamma} + S^{\gamma}{}_{\alpha\beta} - S_{\beta}{}^{\gamma}{}_{\alpha} \,, \tag{2.3}$$

with

$$\Gamma^{\gamma}_{\alpha\beta} = \frac{1}{2} g^{\gamma\sigma}\left(g_{\alpha\sigma} + g_{\sigma\beta} - g_{\alpha\beta}\right) \,. \tag{2.4}$$

The sum of the three torsion pieces on the RHS of Eq. (2.3) is frequently dubbed in the literature as the contorsion tensor,

$$K_{\alpha\beta}{}^{\gamma} \equiv S_{\alpha\beta}{}^{\gamma} + S^{\gamma}{}_{\alpha\beta} - S_{\beta}{}^{\gamma}{}_{\alpha} \,. \tag{2.5}$$

From the anti-symmetry of the torsion tensor in the first two indices, it is straightforward to verify the following identities for the contorsion tensor,

$$K_{\alpha\beta\gamma} = -K_{\alpha\gamma\beta} \,, \tag{2.6}$$

[2] Also referred as the Levi-Civita connection or Christoffel symbols.

$$K_{[\alpha\beta]}{}^{\gamma} = S_{\alpha\beta}{}^{\gamma} \, . \tag{2.7}$$

Having in mind the general affine connection (2.3), the Lie derivative between two vectors can be expressed in terms of the torsion-full covariant derivative as

$$(\mathcal{L}_u v)^\gamma = [u, v]^\gamma = u^\alpha \nabla_\alpha v^\gamma - v^\alpha \nabla_\alpha u^\gamma - 2S_{\alpha\beta}{}^{\gamma} u^\alpha v^\beta \, . \tag{2.8}$$

This last equation and the definition of the Riemann tensor associated with $C^\gamma_{\alpha\beta}$,

$$R_{\alpha\beta\gamma}{}^{\rho} = \partial_\beta C^\rho_{\alpha\gamma} - \partial_\alpha C^\rho_{\beta\gamma} + C^\rho_{\beta\sigma} C^\sigma_{\alpha\gamma} - C^\rho_{\alpha\sigma} C^\sigma_{\beta\gamma} \, , \tag{2.9}$$

lead to a modified version of the relation between the Riemann curvature tensor and the commutator of two covariant derivatives in the case of non-vanishing torsion:

$$R_{\alpha\beta\gamma}{}^{\delta} \psi_\delta = \nabla_\alpha \nabla_\beta \psi_\gamma - \nabla_\beta \nabla_\alpha \psi_\gamma + 2S_{\alpha\beta}{}^{\delta} \nabla_\delta \psi_\gamma \, , \tag{2.10}$$

where ψ is an arbitrary 1-form. In the case of a Lorentzian manifold with non-null torsion, the Riemann curvature tensor does not possess the same symmetries as in the torsion-free case, namely it verifies

$$\begin{aligned} R_{\alpha\beta\gamma\delta} &= -R_{\beta\alpha\gamma\delta} \, , \\ R_{\alpha\beta\gamma\delta} &= -R_{\alpha\beta\delta\gamma} \, , \end{aligned} \tag{2.11}$$

and the modified Bianchi identities

$$R_{[\alpha\beta\gamma]}{}^{\delta} = -2\nabla_{[\alpha} S_{\beta\gamma]}{}^{\delta} + 4S_{[\alpha\beta|}{}^{\rho} S_{|\gamma]\rho}{}^{\delta} \, , \tag{2.12}$$

$$\nabla_{[\alpha} R_{\beta\gamma]\delta}{}^{\rho} = 2S_{[\alpha\beta|}{}^{\sigma} R_{|\gamma]\sigma\delta\rho} \, . \tag{2.13}$$

From Eq. (2.12) we find that the usual symmetry of exchanging the first and second pair of indices of the Riemann tensor is modified in the presence of torsion, so that

$$2R_{\gamma\delta\alpha\beta} = 2R_{\alpha\beta\gamma\delta} + 3A_{\alpha\gamma\beta\delta} + 3A_{\delta\alpha\beta\gamma} + 3A_{\gamma\delta\alpha\beta} + 3A_{\beta\delta\gamma\alpha} \, , \tag{2.14}$$

where $A_{\alpha\beta\gamma\delta} \equiv -2\nabla_{[\alpha} S_{\beta\gamma]\delta} + 4S_{[\alpha\beta|}{}^{\rho} S_{|\gamma]\rho\delta}$.

The previous results are completely general, in particular, they are valid for space-times of any dimension. Let us now consider the case of a space-time of dimension 4. In this case, a useful quantity to define is the Levi-Civita volume form[3]. Introducing the Levi-Civita symbol, $\eta_{\alpha\beta\gamma\delta}$, as the totally skew tensor density whose

[3]Also referred as covariant Levi-Civita tensor or Levi-Civita 4-form.

components in any local coordinate system verify $\eta_{1234} = +1$, the Levi-Civita volume form is defined as

$$\varepsilon_{\alpha\beta\gamma\delta} \equiv \sqrt{|\det g|}\,\eta_{\alpha\beta\gamma\delta}\,, \tag{2.15}$$

where $|\det g|$ represents the absolute value of the determinant of the metric tensor. The Levi-Civita volume form verifies some very useful relations:

$$\begin{aligned}
\nabla_\rho \varepsilon_{\alpha\beta\gamma\delta} &= 0\,, && (2.16)\\
\varepsilon^{\alpha\beta\gamma\delta} &= \frac{\text{sign}\,(g)}{\sqrt{\det|g|}}\eta^{\alpha\beta\gamma\delta}\,, && (2.17)\\
\varepsilon_{\alpha\beta\gamma\delta}\varepsilon^{\rho\sigma\mu\nu} &= -24\, g^{\rho}{}_{[\alpha}\, g^{\sigma}{}_{\beta}\, g^{\mu}{}_{\gamma}\, g_{\delta]}{}^{\nu}\,, && (2.18)\\
\varepsilon_{\alpha\beta\gamma\delta}\varepsilon^{\alpha\sigma\mu\nu} &= -6\, g^{\sigma}{}_{[\beta}\, g^{\mu}{}_{\gamma}\, g_{\delta]}{}^{\nu}\,, && (2.19)\\
\varepsilon_{\alpha\beta\gamma\delta}\varepsilon^{\alpha\beta\mu\nu} &= -4\, g^{\mu}{}_{[\gamma}\, g_{\delta]}{}^{\nu}\,, && (2.20)\\
\varepsilon_{\alpha\beta\gamma\delta}\varepsilon^{\alpha\beta\gamma\delta} &= -24\,, && (2.21)
\end{aligned}$$

where the first equality follows from the assumption that the connection is metric compatible, the second from the properties of the determinant of a matrix (see e.g. Ref. [55]) and in Eqs. (2.18) – (2.21) only the lower indices are to be anti-symmetrized.

In the case of a manifold of dimension 4, the components of the Riemann curvature tensor, $R_{\alpha\beta\gamma\delta}$, can be written as the following sum [56]

$$R_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} + R_{\alpha[\gamma}\, g_{\delta]\beta} - R_{\beta[\gamma}\, g_{\delta]\alpha} - \frac{1}{3}R\, g_{\alpha[\gamma}\, g_{\delta]\beta}\,, \tag{2.22}$$

where $C_{\alpha\beta\gamma\delta}$ represents the Weyl tensor, $R_{\alpha\beta} \equiv R_{\alpha\mu\beta}{}^{\mu}$ the Ricci tensor and R the Ricci scalar. In the presence of torsion, the Weyl tensor is still defined as the trace-free part of the curvature tensor, but it does not retain all the other usual symmetries, instead:

$$\begin{aligned}
C_{\alpha\beta\gamma\delta} &= -C_{\beta\alpha\gamma\delta}\,,\\
C_{\alpha\beta\gamma\delta} &= -C_{\alpha\beta\delta\gamma}\,,\\
C_{[\alpha\beta\gamma]\delta} &= R_{[\alpha\beta\gamma]\delta} + R_{[\alpha\beta}\, g_{\gamma]\delta}\,.
\end{aligned} \tag{2.23}$$

Also, in the presence of torsion, the relation between the derivative of the Weyl tensor and the Riemann tensor is modified (see Appendix A for the derivation),

$$\begin{aligned}
\nabla_\alpha C^{\gamma\delta\beta\alpha} = &\frac{1}{4}\varepsilon^{\mu\nu\lambda\beta} Q_{\mu\nu\lambda\sigma\rho}\varepsilon^{\sigma\rho\gamma\delta} +\\
&+ \frac{3}{2} g^{\beta[\delta} Q^{\gamma]\mu\nu}{}_{\mu\nu} + \nabla^{[\delta} R^{\gamma]\beta} - \frac{1}{6} g^{\beta[\gamma}\nabla^{\delta]} R\,,
\end{aligned} \tag{2.24}$$

where $Q_{\alpha\beta\gamma\delta\rho} \equiv 2S_{[\alpha\beta|}{}^{\sigma}R_{|\gamma]\sigma\delta\rho}$, generalizing the formula given in Ref. [57] (see also Ref. [45]).

2.3 The separation vector between infinitesimally close curves of a congruence

Having introduced the definitions and properties of the basic geometric quantities, we will now consider the notion of separation (sometimes deviation) vector between infinitesimally close curves of a congruence and relate its evolution with the kinematical quantities that characterize the congruence.

Consider a congruence of curves, not necessarily geodesics, such that each curve of the congruence is parameterized by an affine parameter λ. Consider a second congruence, this time of geodesics, parameterized by an affine parameter t, such that each geodesic intersects a curve of the first congruence at one and only one point of the space-time. Given two curves of the first congruence, c_1 and c_2, and a geodesic of the second congruence, γ, let the two points p and q be the intersection points of γ with, respectively, c_1 and c_2, with $c_1(\lambda_0) = \gamma(t_0) = p$. Assuming that the point q is in a small enough neighbourhood of the point p such that $q = \gamma(t_0 + \delta t) \approx p + \frac{\partial \gamma}{\partial t}\big|_{t_0} \delta t$. If $\boldsymbol{\ell}$ is the tangent vector to the geodesic γ in p, then

$$\boldsymbol{\ell} \equiv \delta t \left.\frac{\partial \gamma}{\partial t}\right|_{t_0} = q - p \,, \tag{2.25}$$

gives a meaningful notion of the separation between the curves c_1 and c_2.

Let us now consider a coordinate neighbourhood that contains the points p and q, such that $p = \{x^\alpha\}$, $q = \{x'^\alpha\} = \{x^\alpha + \boldsymbol{\ell}^\alpha\}$, with

$$\boldsymbol{\ell}^\alpha = \delta t \frac{\partial x^\alpha}{\partial t} \,, \tag{2.26}$$

and let u^α be the components of the tangent vector field to the curve c_1 (from here on we will drop the index 1) at p,

$$u^\alpha = \frac{\partial x^\alpha}{\partial \lambda} \,. \tag{2.27}$$

In order to find the general expression for the evolution of the separation vector, $\boldsymbol{\ell}$, we will start by computing the Lie derivative of $\boldsymbol{\ell}$ over the tangent vector u and vice versa. From the definition of the Lie derivative of two vector fields, Eqs. (2.26) and (2.27) we find

$$\mathcal{L}_{\boldsymbol{\ell}}\, u = \mathcal{L}_u\, \boldsymbol{\ell} = 0 \,. \tag{2.28}$$

Using Eqs. (2.8) and (2.28) it is possible to derive an equation for the change of the separation vector along the fiducial curve c,

$$u^\beta \nabla_\beta \boldsymbol{\ell}^\alpha = B_\beta{}^\alpha \boldsymbol{\ell}^\beta \,, \tag{2.29}$$

where

$$B_\beta{}^\alpha = \nabla_\beta u^\alpha + 2S_{\gamma\beta}{}^\alpha u^\gamma \, . \tag{2.30}$$

Then, for infinitesimally close curves, the evolution of the separation vector along the fiducial curve is entirely described by the tensor field B. Let us emphasize that in the derivation of Eqs. (2.29) and (2.30) we have not specified the type of the tangent vector, u, to the fiducial curve, hence, these equations are equally valid for the case of u being time-like, space-like or null, with the fiducial curve being either a geodesic or not.

The second term in the RHS of Eq. (2.30) represents an explicit contribution of the torsion tensor to the evolution of a congruence of curves. In fact, we can verify some geometrical implications of Eqs. (2.29) and (2.30). The derivative along c of the quantity $\boldsymbol{\ell}_\alpha u^\alpha$ reads

$$\frac{D\left(\boldsymbol{\ell}_\alpha u^\alpha\right)}{d\lambda} = \boldsymbol{\ell}_\beta a^\beta + 2S_{\sigma\gamma\alpha} u^\sigma u^\alpha \boldsymbol{\ell}^\gamma \, , \tag{2.31}$$

where we have introduced the acceleration vector a, appearing for non geodesics fiducial curves, with components in a local coordinate system

$$a^\alpha = u^\gamma \nabla_\gamma u^\alpha \, . \tag{2.32}$$

The expression (2.31) represents the failure of the separation vector $\boldsymbol{\ell}$ and the tangent vector u to stay orthogonal to each other, that is, if at a given point, $\boldsymbol{\ell}$ and u are orthogonal to each other, a general non-null torsion, S, or a non-null acceleration, a, will spoil the preservation of such orthogonality along the curve. Note that a torsion field, with no further imposed symmetry, will lead to effects parallel to the direction of u (second term on the right hand side of (2.31)), contributing to a relative acceleration between two initially infinitesimally close particles.

The analysis of Eq. (2.31) leads to the conclusion that the tensor B, describing the behavior of the separation vector will have, for the case of a generic torsion tensor, non-null components tangential and orthogonal to the tangent vector field associated with the fiducial curve c. Without loss of generality, it is then possible to write $B_{\alpha\beta}$ in terms of two components: one completely orthogonal to u and another containing the remaining terms. Given a projector $h_{\alpha\beta}$ onto the surface orthogonal to the curve c at a given point, we can write

$$B_{\alpha\beta} = B_{\perp\alpha\beta} + B_{\|\alpha\beta} \, , \tag{2.33}$$

with

$$B_{\perp\alpha\beta} \equiv h_\alpha{}^\gamma h_\beta{}^\sigma B_{\gamma\sigma} \, , \tag{2.34}$$

$$B_{\|\alpha\beta} \equiv B_{\alpha\beta} - B_{\perp\alpha\beta} \, . \tag{2.35}$$

In analogy to the case of null torsion, we want to define the kinematical quantities identifying expansion, shear, and vorticity, θ, $\sigma_{\alpha\beta}$ and $\omega_{\alpha\beta}$, respectively, of neighboring curves of the congruence. These quantities will only depend on the orthogonal part, $B_{\perp}$, of the tensor B so that, defining the kinematical quantities as

$$\begin{aligned} \theta &= B_{\perp\gamma}{}^{\gamma}\,, \\ \sigma_{\alpha\beta} &= B_{\perp(\alpha\beta)} - \frac{h_{\alpha\beta}}{h^{\gamma}{}_{\gamma}}\theta\,, \\ \omega_{\alpha\beta} &= B_{\perp[\alpha\beta]}\,, \end{aligned} \tag{2.36}$$

without loss of generality $B_{\perp\alpha\beta}$ can be decomposed as

$$B_{\perp\alpha\beta} = \frac{h_{\alpha\beta}}{h^{\gamma}_{\gamma}}\theta + \sigma_{\alpha\beta} + \omega_{\alpha\beta}\,. \tag{2.37}$$

Before we continue let us emphasize that the definitions of the kinematical quantities given by (2.36) are always valid whenever the tensor B is related with the variation of the separation vector between curves of a congruence by an equation such as (2.29) (cf., e.g., Ref. [58]).

2.4 Projector operator and projected covariant Levi-Civita tensor

The set of kinematical quantities that characterize a congruence in a Lorentzian manifold represent one of the building blocks of all covariant space-time decomposition approaches. The results in the previous section are quite general and valid for curves of any kind, however, the procedure that defines the projector operator strictly depends on the specific family of considered curves. Once the projector is assigned, Eq. (2.37) will give an actual expression in terms of the tangent vector, its derivatives and the torsion tensor which, in turn can be related with the Riemann and Weyl tensors. Here we will focus on developing the 1+3 formalism for time-like congruences in a 4–dimensional Lorentzian manifold with torsion[4].

Consider a Lorentzian manifold of dimension 4, $(\mathcal{M}, g, S)$, admitting, in some open neighborhood, the existence of a congruence of time-like curves with tangent vector field u. Without loss of generality, we can foliate the manifold in 3-surfaces, $\mathcal{V}$, orthogonal, at each point, to the curves of the congruence, such that all tensor quantities are defined by their behavior along the direction of u and in $\mathcal{V}$. This procedure is usually called "1+3 space-time decomposition". Such decomposition of the space-time manifold relies on the existence of a projector to the hypersurface $\mathcal{V}$. Assuming each curve of the congruence to be affinely parameterized, so that $u_\alpha u^\alpha = -1$, the projector onto $\mathcal{V}$, at each point can be defined as

$$h_{\alpha\beta} \equiv g_{\alpha\beta} + u_\alpha u_\beta\,, \tag{2.38}$$

[4]Many of the results that we will introduce are valid or easily extended for manifolds of dimension $N \geq 2$. In fact, this will be used in Chapter 4, to study singularities theorems. Nonetheless, quantities and identities that rely on the covariant Levi-Civita tensor, Eq. (2.15), notably the 1+3 decomposition of the Weyl tensor, are dimension dependent, hence, as we will see, the general set of structure equations will depend on the dimension of the manifold.

verifying

$$
\begin{aligned}
h_{\alpha\beta}u^{\alpha} &= 0\,, \\
h_{\alpha\beta} &= h_{\beta\alpha}\,, \\
h_{\alpha}{}^{\gamma}h_{\gamma\beta} &= h_{\alpha\beta}\,, \\
h^{\sigma}_{\sigma} &= 3\,.
\end{aligned}
\tag{2.39}
$$

Another useful operator is the projected covariant Levi-Civita tensor

$$
\varepsilon_{\alpha\beta\gamma} = \varepsilon_{\alpha\beta\gamma\sigma}u^{\sigma}\,, \tag{2.40}
$$

derived from the Levi-Civita volume form, defined in Eq. (2.15), with the following properties

$$
\begin{aligned}
\varepsilon_{\alpha\beta\gamma} &= \varepsilon_{[\alpha\beta\gamma]}\,, \\
\varepsilon_{\alpha\beta\gamma}u^{\gamma} &= 0\,, \\
\varepsilon_{\alpha\beta\gamma}\varepsilon^{\mu\nu\sigma} &= 6h^{\mu}{}_{[\alpha}h^{\nu}{}_{\beta}h_{\gamma]}{}^{\sigma}\,, \\
\varepsilon_{\alpha\beta\gamma}\varepsilon^{\mu\nu\gamma} &= 2h^{\mu}{}_{[\alpha}h_{\beta]}{}^{\nu}\,, \\
\varepsilon_{\mu\nu\alpha}\varepsilon^{\mu\nu\beta} &= 2h_{\alpha}{}^{\beta}\,.
\end{aligned}
\tag{2.41}
$$

Moreover, using Eq. (2.38) and the properties of the Levi-Civita volume form, we find the useful identities

$$
\begin{aligned}
h_{\mu}{}^{\alpha}h_{\nu}{}^{\beta}h_{\rho}{}^{\gamma}h_{\lambda}{}^{\sigma}\varepsilon_{\alpha\beta\gamma\sigma} &= 0\,, \\
\varepsilon_{\alpha\beta\gamma\sigma} &= h_{\alpha}{}^{\mu}\varepsilon_{\mu\beta\gamma\sigma} + u_{\alpha}\varepsilon_{\beta\gamma\sigma}\,.
\end{aligned}
\tag{2.42}
$$

2.5 Torsion, Weyl and stress-energy tensors decomposition

Defined the projector operator $h_{\alpha\beta}$, we will now write the decomposition of the torsion tensor, Eq. (2.2), and the Weyl tensor, (2.22), in terms of their components along the direction of the tangent vector field u and on $\mathcal{V}$. For the torsion tensor we find

$$
S_{\alpha\beta\gamma} = \varepsilon_{\alpha\beta}{}^{\mu}\bar{S}_{\mu\gamma} + W_{[\alpha|\gamma}u_{|\beta]} + S_{\alpha\beta}u_{\gamma} + u_{[\alpha}X_{\beta]}u_{\gamma}\,, \tag{2.43}
$$

with

$$
\begin{aligned}
\bar{S}_{\alpha\beta} &= \frac{1}{2}\varepsilon_{\alpha\mu\nu}h^{\sigma}_{\beta}S^{\mu\nu}{}_{\sigma}\,, \\
W_{\alpha\beta} &= 2u^{\mu}h^{\nu}_{\alpha}h^{\sigma}_{\beta}S_{\mu\nu\sigma}\,, \\
S_{\alpha\beta} &= -h^{\mu}_{\alpha}h^{\nu}_{\beta}u^{\sigma}S_{\mu\nu\sigma}\,, \\
X_{\alpha} &= 2u^{\mu}h^{\nu}_{\alpha}u^{\sigma}S_{\mu\nu\sigma}\,.
\end{aligned}
\tag{2.44}
$$

For the Weyl tensor,

$$C_{\alpha\beta\gamma\delta} = -\varepsilon_{\alpha\beta\mu}\varepsilon_{\gamma\delta\nu}E^{\nu\mu} - 2u_\alpha E_{\beta[\gamma}\,u_{\delta]} + 2u_\beta E_{\alpha[\gamma}\,u_{\delta]} - 2\varepsilon_{\alpha\beta\mu}H^{\mu}{}_{[\gamma}\,u_{\delta]} - 2\varepsilon_{\mu\gamma\delta}\bar{H}^{\mu}{}_{[\alpha}\,u_{\beta]}\,, \quad (2.45)$$

where

$$E_{\alpha\beta} = C_{\alpha\mu\beta\nu}u^\mu u^\nu\,, \quad (2.46)$$

$$H_{\alpha\beta} = \frac{1}{2}\varepsilon_\alpha{}^{\mu\nu}C_{\mu\nu\beta\delta}u^\delta\,, \quad (2.47)$$

$$\bar{H}_{\alpha\beta} = \frac{1}{2}\varepsilon_\alpha{}^{\mu\nu}C_{\beta\delta\mu\nu}u^\delta\,, \quad (2.48)$$

are the "electric" part and "magnetic" parts of the Weyl tensor, respectively. Equation (2.45) generalizes the results in Ref. [41] to the case of non-null torsion[5]. Note, however, that, differently from the torsionless case, there are two different tensor quantities associated to the magnetic part of the Weyl tensor: $H_{\alpha\beta}$ and $\bar{H}_{\alpha\beta}$.

From the results in Eq. (2.23), we see that in the presence of torsion the tensors $E_{\alpha\beta}$, $H_{\alpha\beta}$ and $\bar{H}_{\alpha\beta}$ have the following properties:

$$\begin{aligned} H_{\alpha\beta} &= h_\alpha{}^\mu h_\beta{}^\nu H_{\mu\nu}\,, \qquad H_{\alpha\beta} = H_{(\alpha\beta)}\,, \\ \bar{H}_{\alpha\beta} &= h_\alpha{}^\mu h_\beta{}^\nu \bar{H}_{\mu\nu}\,, \qquad \bar{H}_{\alpha\beta} = \bar{H}_{(\alpha\beta)}\,, \\ E_{\alpha\beta} &= h_\alpha{}^\mu h_\beta{}^\nu E_{\mu\nu}\,, \qquad E^\alpha{}_\alpha = 0\,. \end{aligned} \quad (2.49)$$

Therefore, $E_{\alpha\beta}$, may not be a symmetric tensor and $H_{\alpha\beta}$ and $\bar{H}_{\alpha\beta}$ do not have to be trace-free. On the other hand, due to the properties of the Levi-Civita volume form and the fact that the Weyl tensor is, by definition, trace-free, even in the presence of torsion, the magnetic parts, $H_{\alpha\beta}$ and $\bar{H}_{\alpha\beta}$, are symmetric under the exchange of indices.

Continuing, if we are to apply the 1+3 formalism to study the dynamical evolution of matter fields permeating the space-time manifold, we will need to find the covariantly defined quantities associated with the fluid. These will be provided by the 1+3 decomposition of the stress-energy tensor $\mathcal{T}$. Without imposing any symmetries on $\mathcal{T}$ and using Eq. (2.38) we have

$$\mathcal{T}_{\alpha\beta} = \mu\, u_\alpha u_\beta + p\, h_{\alpha\beta} + q_{1\alpha}u_\beta + u_\alpha q_{2\beta} + \pi_{\alpha\beta} + \varepsilon_{\alpha\beta}{}^\gamma m_\gamma\,, \quad (2.50)$$

[5] Bear in mind that in Ref. [41] there is an incorrect sign, as noted and corrected in Ref. [45].

with

$$\begin{aligned} \mu &= u^\mu u^\nu \mathcal{T}_{\mu\nu} , & q_{1\alpha} &= -h_\alpha{}^\mu u^\nu \mathcal{T}_{\mu\nu} , \\ p &= \frac{1}{3} h^{\mu\nu} \mathcal{T}_{\mu\nu} , & q_{2\alpha} &= -u^\mu h^\nu_\alpha \mathcal{T}_{\mu\nu} , \\ \pi_{\alpha\beta} &= \mathcal{T}_{\langle\alpha\beta\rangle} , & m_\alpha &= \frac{1}{2} \varepsilon_\alpha{}^{\mu\nu} \mathcal{T}_{\mu\nu} , \end{aligned} \tag{2.51}$$

where we have used the angular brackets to represent the projected symmetric part without trace of a 2-tensor on $\mathcal{V}$, that is, for a tensor $\psi_{\alpha\beta}$,

$$\psi_{\langle\alpha\beta\rangle} \equiv \left[h^\mu{}_{(\alpha} h_{\beta)}{}^\nu - \frac{h_{\alpha\beta}}{3} h^{\mu\nu} \right] \psi_{\mu\nu} . \tag{2.52}$$

2.6 Structure equations

The kinematical quantities of a congruence of curves (2.36), the acceleration vector field (2.32) and the tensors found from the decomposition of the torsion, Weyl and stress-energy tensors, Eqs. (2.44), (2.46) - (2.48) and (2.51), completely describe the geometry of the manifold $\mathcal{M}$ and the properties of the matter fields that permeate it. We have then to find a complete set of differential equations that describe the evolution of these quantities along u and on $\mathcal{V}$.

In order to keep the equations as compact as possible, we will introduce the following notation: given a tensor field $\psi_{\alpha\ldots\beta}{}^{\gamma\ldots\sigma}$ we define

$$D_\mu \psi_{\alpha\ldots\beta}{}^{\gamma\ldots\sigma} \equiv h_\mu{}^\nu h_\alpha{}^\rho \ldots h_\beta{}^\delta h_\lambda{}^\gamma \ldots h_\varphi{}^\sigma \nabla_\nu \psi_{\rho\ldots\delta}{}^{\lambda\ldots\varphi} , \tag{2.53}$$

as the fully orthogonally projected covariant derivative on $\mathcal{V}$; on the other hand, a “dot” represents the covariant derivative along the integral curve of u, that is

$$\dot{\psi}_{\alpha\ldots\beta}{}^{\gamma\ldots\sigma} = u^\mu \nabla_\mu \psi_{\alpha\ldots\beta}{}^{\gamma\ldots\sigma} , \tag{2.54}$$

and we will use the expression in (2.37) for the orthogonal part of the evolution tensor B, $B_\perp$, in terms of the kinematical quantities, with $h_\gamma{}^\gamma = 3$.

Now, from Eqs. (2.30), (2.37) and (2.38) we find that the covariant derivative of the tangent vector field u is given by

$$\nabla_\alpha u_\beta = B_{\perp\alpha\beta} - W_{\alpha\beta} - u_\alpha a_\beta = \frac{1}{3} h_{\alpha\beta}\theta + \sigma_{\alpha\beta} + \omega_{\alpha\beta} - W_{\alpha\beta} - u_\alpha a_\beta . \tag{2.55}$$

Then, from the Ricci identity, Eq. (2.10), we find the propagation equations for the kinematical quantities[6],

$$\begin{aligned}\dot{\theta} - \dot{W}_{\alpha}{}^{\alpha} =& - R_{\alpha\beta}u^{\alpha}u^{\beta} - \left(\frac{1}{3}\theta^2 + \sigma_{\alpha\beta}\sigma^{\alpha\beta} + \omega_{\alpha\beta}\omega^{\beta\alpha}\right) + \nabla_{\alpha}a^{\alpha} + \\ &+ W^{\beta\alpha}\left[\frac{1}{3}h_{\alpha\beta}\theta + \sigma_{\alpha\beta} + \omega_{\alpha\beta}\right] + X_{\alpha}a^{\alpha} \,,\end{aligned} \tag{2.56}$$

$$\begin{aligned}h_{\mu}{}^{\alpha}h_{\nu}{}^{\beta}\left(\dot{\omega}_{\alpha\beta} - \dot{W}_{[\alpha\beta]}\right) =& - E_{[\mu\nu]} + \frac{1}{2}h_{\mu}{}^{\alpha}h_{\nu}{}^{\beta}R_{[\alpha\beta]} - \frac{2}{3}\theta\omega_{\mu\nu} - 2\sigma_{[\mu}{}^{\alpha}\omega_{\alpha|\nu]} + D_{[\mu}a_{\nu]} \\ &+ X_{[\mu}a_{\nu]} + \frac{1}{3}\theta W_{[\mu\nu]} + \left(\sigma_{[\mu|\delta} + \omega_{[\mu|\delta}\right)W^{\delta}{}_{|\nu]} \,,\end{aligned} \tag{2.57}$$

$$\begin{aligned}h_{\mu}{}^{\alpha}h_{\nu}{}^{\beta}\left(\dot{\sigma}_{\alpha\beta} - \dot{W}_{\langle\alpha\beta\rangle}\right) =& - E_{(\mu\nu)} + \frac{1}{2}R_{\langle\mu\nu\rangle} + D_{\langle\mu}a_{\nu\rangle} + \\ &+ a_{\langle\mu}a_{\nu\rangle} - \frac{2}{3}\sigma_{\mu\nu}\theta - \sigma_{\langle\mu|}{}^{\delta}\sigma_{\delta|\nu\rangle} - \omega_{\langle\mu|}{}^{\delta}\omega_{\delta|\nu\rangle} + \\ &+ X_{\langle\mu}a_{\nu\rangle} + \sigma_{\langle\mu|}{}^{\delta}W_{\delta|\nu\rangle} + \omega_{\langle\mu|}{}^{\delta}W_{\delta|\nu\rangle} + \frac{1}{3}W_{\langle\mu\nu\rangle}\theta \,,\end{aligned} \tag{2.58}$$

and the constraint equations,

$$\begin{aligned}\varepsilon^{\alpha\beta\gamma}D_{\alpha}\left(\omega_{\beta\gamma} - W_{\beta\gamma}\right) + \varepsilon^{\mu\nu\rho}a_{\rho}\omega_{\nu\mu} =& H_{\rho}{}^{\rho} + \varepsilon^{\mu\nu\rho}a_{\rho}\left(S_{\nu\mu} + W_{\nu\mu}\right) - \\ &- 2\bar{S}^{\mu\nu}\left(\frac{1}{3}h_{\nu\mu}\theta + \sigma_{\nu\mu} + \omega_{\nu\mu} - W_{\nu\mu}\right) \,,\end{aligned} \tag{2.59}$$

$$\begin{aligned}\varepsilon^{\alpha\beta\langle\mu|}D_{\alpha}\left(\sigma_{\beta}{}^{|\nu\rangle} + \omega_{\beta}{}^{|\nu\rangle} - W_{\beta}{}^{|\nu\rangle}\right) + \varepsilon^{\alpha\beta\langle\mu}a^{\nu\rangle}\omega_{\beta\alpha} =& H^{\langle\mu\nu\rangle} + 2\bar{S}^{\langle\mu|\delta}W_{\delta}{}^{|\nu\rangle} - \\ &- \varepsilon^{\alpha\beta\langle\mu}a^{\nu\rangle}\left(S_{\alpha\beta} + W_{\alpha\beta}\right) \\ &- 2\bar{S}^{\langle\mu|\delta}\left(\frac{1}{3}h_{\delta}{}^{|\nu\rangle}\theta + \sigma_{\delta}{}^{|\nu\rangle} + \omega_{\delta}{}^{|\nu\rangle}\right) \,,\end{aligned} \tag{2.60}$$

$$\begin{aligned}&\frac{2}{3}D_{\alpha}\theta - D_{\mu}\left(\sigma_{\alpha}{}^{\mu} + \omega_{\alpha}{}^{\mu} - W_{\alpha}{}^{\mu}\right) - \\ &-D_{\alpha}W_{\mu}{}^{\mu} - 2a^{\mu}\omega_{\alpha\mu} = -h^{\mu}{}_{\alpha}R_{\mu\beta}u^{\beta} - 2\varepsilon_{\lambda\rho\alpha}\bar{S}^{\lambda\mu}W_{\mu}{}^{\rho} + \\ &\qquad + 2a^{\mu}\left(S_{\mu\alpha} + W_{[\mu\alpha]}\right) + \\ &\qquad + 2\varepsilon_{\lambda\rho\alpha}\bar{S}^{\lambda\mu}\left(\frac{1}{3}h_{\mu}{}^{\rho}\theta + \sigma_{\mu}{}^{\rho} + \omega_{\mu}{}^{\rho}\right) \,.\end{aligned} \tag{2.61}$$

Equations (2.59) and (2.60) clearly exemplify how the presence of torsion modifies the geometry of the manifold and, consequently, the change in the evolution of a congruence of time-like curves. When comparing to the case of null torsion [38, 40], we see that, in the presence of a general torsion tensor field, the magnetic

[6]Equations (2.56), (2.57) and (2.58) follow from computing the projection $h_{\mu}{}^{\alpha}u^{\beta}h_{\nu}{}^{\gamma}R_{\alpha\beta\gamma\delta}u^{\delta}$ and evaluate, respectively, its trace, anti-symmetric part and symmetric part without trace. Equations (2.59), (2.60) and (2.61) follow from computing the projection $\varepsilon^{\alpha\beta\lambda}h_{\rho}{}^{\gamma}R_{\alpha\beta\gamma\delta}u^{\delta}$ and evaluate, respectively, its trace, symmetric part without trace and anti-symmetric part.

part of the Weyl tensor, H, is characterized by both these equations, in particular, contrary to the case of null-torsion, it also depends on the divergence of the vorticity tensor. Hence, from Eq. (2.59), we conclude that the presence of torsion contributes to a non-null divergence of the vorticity tensor, that is, the presence of torsion acts as a source of rotation of the congruence.

Provided the field equations of a gravity theory to relate the projection of the Ricci tensor with the stress-energy tensor, Eq. (2.56) represents the generalization of the Raychaudhuri equation for manifolds with non-null torsion [59], describing the evolution of the expansion of a congruence of curves. This result differs from previous versions of the torsion-full Raychaudhuri equation available in literature. More concretely, Refs. [33, 60–63] did not take properly into account the relation between the tensor B, (2.30), and the evolution of the separation vector between infinitesimally close curves of the congruence, assuming that the expression for $B_{\alpha\beta}$ is *a priori* the same as in the case of null torsion. On the other hand, few quite specific models [25–28, 64] where the component W of the torsion tensor is set to zero, accidentally result in the correct expression.

The propagation and constraint equations for the components of the Weyl tensor are found from the second Bianchi identity (2.12) or, equivalently, from Eq. (2.24). For the electric part we find[7]

$$
\begin{aligned}
&E_{\alpha}{}^{\mu}\left(\sigma_{\mu\beta}+\omega_{\mu\beta}\right)-h_{\alpha\mu}h_{\beta\nu}\dot{E}^{\mu\nu}-E_{\alpha\beta}\theta+\\
&\varepsilon_{\mu\beta}{}^{\nu}\left(D_{\nu}\bar{H}^{\mu}{}_{\alpha}+a_{\nu}\bar{H}^{\mu}{}_{\alpha}\right)+\varepsilon^{\mu}{}_{\alpha\delta}a^{\delta}H_{\mu\beta}+\\
&\quad+\varepsilon_{\alpha}{}^{\delta\mu}\varepsilon_{\beta}{}^{\lambda\nu}E_{\nu\mu}\left(\sigma_{\lambda\delta}+\omega_{\lambda\delta}-W_{\lambda\delta}\right)=-2\bar{S}_{\beta}{}^{\mu}\bar{H}_{\alpha\mu}+u_{\lambda}R^{\lambda\sigma}\bar{S}_{\beta}{}^{\mu}\varepsilon_{\sigma\mu\alpha}-\frac{1}{6}W_{\alpha\beta}R-\\
&\qquad-\varepsilon^{\mu\nu}{}_{\beta}X_{\mu}\bar{H}_{\alpha\nu}+\frac{1}{2}W^{\mu\nu}R^{\lambda\sigma}\varepsilon_{\sigma\nu\alpha}\varepsilon_{\mu\lambda\beta}+\\
&\qquad+\frac{1}{2}R_{\mu\nu}\left(W_{\alpha}{}^{\mu}-X_{\alpha}v^{\mu}\right)h^{\nu}{}_{\beta}+\\
&\qquad+\frac{1}{4}h_{\alpha\beta}R_{\nu\mu}\left(W_{\gamma}{}^{\gamma}h^{\mu\nu}-W^{\mu\nu}-2\varepsilon^{\mu}{}_{\delta\gamma}\bar{S}^{\gamma\delta}u^{\nu}\right)-\\
&\qquad-\frac{1}{2}h_{\alpha\beta}\left(-2\bar{S}_{\mu\nu}\bar{H}^{\mu\nu}-W^{\mu\nu}E_{\nu\mu}\right)+\\
&\qquad+h_{\alpha\mu}h_{\beta\nu}\left(u_{\delta}\nabla^{[\delta}R^{\mu]\nu}-\frac{1}{12}g^{\mu\nu}\dot{R}\right),
\end{aligned}
\tag{2.62}
$$

[7] Equation (2.62) is found from the projection $h_{\mu\gamma}h_{\nu\beta}u_{\delta}\nabla_{\lambda}C^{\gamma\delta\beta\lambda}$; Eq. (2.63) is follows from $h_{\delta\alpha}u_{\gamma}u_{\beta}\nabla_{\lambda}C^{\gamma\delta\beta\lambda}$.

and the constraint,

$$\begin{aligned}
D_\mu E_\alpha{}^\mu + \varepsilon^{\mu\gamma\delta}\bar{H}_{\mu\alpha}\left(\omega_{\delta\gamma} - W_{\delta\gamma} - \frac{1}{2}S_{\delta\gamma}\right) + \\
+\left(\sigma_{\delta\nu} + \omega_{\delta\nu} - \frac{1}{2}W_{\delta\nu}\right)\varepsilon^{\nu\beta\mu}h_{\alpha\beta}H_\mu{}^\delta = &- R_{\mu\gamma}\bar{S}^{\mu\beta}\varepsilon^\gamma{}_{\beta\alpha} + \frac{1}{2}R_{\beta\gamma}\bar{S}^{\mu\beta}\varepsilon^\gamma{}_{\mu\alpha} - \frac{1}{6}RX_\alpha + \\
&+ \frac{2}{3}R\varepsilon_{\mu\beta\alpha}\bar{S}^{\mu\beta} + 2\bar{S}^{\nu\beta}\varepsilon_{\nu\beta\mu}E_\alpha{}^\mu + \frac{1}{2}S_{\alpha\gamma}u^\beta R_\beta{}^\gamma + \\
&+ \frac{1}{2}h_{\delta\alpha}R^{\delta\mu}\left(\varepsilon_{\mu\nu\beta}\bar{S}^{\beta\nu} - \frac{1}{2}W_\gamma{}^\gamma u_\mu + \frac{1}{2}X_\mu\right) - \\
&- \frac{1}{4}R^{\gamma\beta}u_\beta\left(W_{\alpha\gamma} - X_\alpha u_\gamma\right) - \varepsilon_{\alpha\beta\nu}\bar{S}_\mu{}^\beta E^{\mu\nu} - \\
&- \frac{1}{2}E_{\alpha\nu}X^\nu + \frac{1}{2}D_\alpha\left(R_{\nu\beta}u^\nu u^\beta\right) + R_{\nu\beta}W_\alpha{}^{(\nu}u^{\beta)} - \\
&- \frac{1}{3}R_{\nu\beta}\theta u^{(\nu}h^{\beta)}{}_\alpha - R_{\nu\beta}u^{(\nu|}\left(\sigma_\alpha{}^{|\beta)} + \omega_\alpha{}^{|\beta)}\right) - \\
&- \frac{1}{2}h_\alpha{}^\delta u^\gamma\nabla_\gamma\left(R_{\nu\beta}h^\nu{}_\delta u^\beta\right) + \frac{1}{2}R_{\nu\beta}u^\nu u^\beta a_\alpha + \\
&+ \frac{1}{2}h_{\nu\alpha}R^{\nu\beta}a_\beta + \frac{1}{12}D_\alpha R\,;
\end{aligned} \tag{2.63}$$

for the magnetic parts[8],

$$\begin{aligned}
\left[4a_\mu E_\nu{}^{(\alpha|} + 2D_\mu E_\nu{}^{(\alpha|}\right]\varepsilon^{|\beta)\nu\mu} - 2h^{\mu(\alpha}h^{\beta)\nu}\dot{H}_{\mu\nu} + \\
+2H^{\mu(\alpha|}\left(\sigma_\mu{}^{|\beta)} + \omega_\mu{}^{|\beta)}\right) - \frac{1}{3}\left(4H^{\alpha\beta} + 2\bar{H}^{\alpha\beta}\right)\theta - \\
-2h^{\alpha\beta}\bar{H}^{\mu\nu}\left(\sigma_{\mu\nu} - W_{\mu\nu}\right) - 4h_\nu{}^{(\alpha}h^{\beta)}{}_\gamma\bar{H}_\mu{}^\nu W^{(\gamma\mu)} + \\
+2\bar{H}_\mu{}^\mu\left(W^{(\alpha\beta)} - h^{\alpha\beta}W_\gamma{}^\gamma - \sigma^{\alpha\beta} + \frac{1}{3}h^{\alpha\beta}\theta\right) + \\
+4\bar{H}_\mu{}^{(\alpha}\sigma^{\beta)\mu} + 2W_\mu{}^\mu\bar{H}^{\alpha\beta} = &2E_\lambda{}^{(\alpha|}\left[2\bar{S}^{|\beta)\lambda} + \varepsilon^{|\beta)\nu\lambda}X_\nu\right] + \\
&+ 2\bar{S}^{(\alpha\beta)}\left(R_{\mu\nu}u^\mu u^\nu + \frac{1}{3}R\right) - \\
&- 2h^{\mu(\alpha}\bar{S}^{\beta)\nu}R_{\nu\mu} + W_\mu{}^{(\alpha}\varepsilon^{\beta)\mu\nu}u^\lambda R_{\nu\lambda} - \\
&- \varepsilon^{\nu\lambda(\alpha}h^{\beta)\mu}X_\nu R_{\lambda\mu} + D_\delta\left(\varepsilon_\gamma{}^{\delta(\alpha}h^{\beta)}{}_\mu R^{\gamma\mu}\right),
\end{aligned} \tag{2.64}$$

describing the propagation of the H component of the Weyl tensor along the congruence, and the constraint

[8] Equation (2.64) is found from $h^\rho{}_\beta\varepsilon_{\mu\gamma\delta}\nabla_\lambda C^{\gamma\delta\beta\lambda}$; Eq. (2.65) follows from computing the projection $\varepsilon_{\alpha\gamma\delta}u_\beta\nabla_\lambda C^{\gamma\delta\beta\lambda}$.

equation

$$
\begin{aligned}
-2D_\mu H^\mu{}_\alpha + \frac{2}{3}\varepsilon_{\alpha\beta\delta}E^{\beta\delta}\theta + 2\varepsilon_{\alpha\beta}{}^\mu E^{\beta\delta}\sigma_{\delta\mu} + & \\
+4\varepsilon_{(\alpha|\beta}{}^\mu E^{\beta|\delta)}\left(\omega_{\delta\mu} - W_{\delta\mu}\right) = & \varepsilon_{\alpha\gamma\delta}\nabla^\delta\left(R^{\gamma\beta}u_\beta\right) + \varepsilon_{\alpha\gamma\delta}R^\gamma{}_\beta W^{\delta\beta} - \frac{1}{3}\varepsilon_{\alpha\gamma\delta}R^{\gamma\delta}\theta - \\
& - \varepsilon_{\alpha\gamma\delta}R^{\gamma\beta}\left(\sigma^\delta{}_\beta + \omega^\delta{}_\beta\right) + 2\bar{S}^\mu{}_\alpha R_{\mu\delta}u^\delta - \\
& - 2\bar{S}_\alpha{}^\beta R_{\beta\delta}u^\delta - 2\varepsilon^{\mu\nu\gamma}S_{\mu\nu}E_{\gamma\alpha} + \\
& + \varepsilon^{\mu\nu\gamma}S_{\mu\nu}R_{\gamma\beta}h^\beta{}_\alpha - \varepsilon^{\mu\nu}{}_\alpha S_{\mu\nu}u^\gamma u^\beta R_{\gamma\beta} - \\
& - \frac{1}{3}\varepsilon^{\mu\nu}{}_\alpha S_{\mu\nu}R - 4\bar{S}^{\gamma\beta}\varepsilon_{\gamma\beta\mu}H^\mu{}_\alpha \,,
\end{aligned}
\tag{2.65}
$$

for the divergence of H on $\mathcal{V}$.

Using the first Bianchi identity, Eq. (2.11), we find the following relation between the tensors H and $\bar{H}$,

$$
\begin{aligned}
H_\alpha{}^\beta - \bar{H}_\alpha{}^\beta + \varepsilon_\alpha{}^{\mu\beta}u^\nu R_{[\nu\mu]} = & -\frac{1}{2}\varepsilon_\alpha{}^{\mu\nu}D^\beta\left(W_{\nu\mu} + S_{\nu\mu}\right) + \varepsilon_{\alpha\mu\nu}X^\nu B_\perp^{\beta\mu} - \\
& - \varepsilon_{\alpha\mu\nu}X^\nu\left(W^{\beta\mu} + S^{\beta\mu}\right) - \frac{1}{2}\varepsilon_\alpha{}^{\mu\nu}a^\beta\left(W_{\nu\mu} + S_{\nu\mu}\right) + \\
& + 2h_\alpha{}^\gamma h^{\mu\beta}u^\nu\nabla_\nu\bar{S}_{\gamma\mu} + 2\bar{S}_\alpha{}^\beta\theta - 2h_\alpha{}^\beta\bar{S}^{\mu\nu}\left(S_{\mu\nu} + W_{[\mu\nu]}\right) + \\
& + \varepsilon_{\alpha\mu\nu}a^\nu\left(W^{(\mu\beta)} + S^{\mu\beta}\right) - h_\alpha{}^\beta u^\nu\nabla_\nu\bar{S}_\mu{}^\mu - h_\alpha{}^\beta\bar{S}_\mu{}^\mu\theta + \\
& + \varepsilon_{\alpha\mu}{}^\nu D_\nu\left(W^{(\beta\mu)} + S^{\beta\mu}\right) + \varepsilon_\alpha{}^{\mu\nu}X^\beta\omega_{\mu\nu} - 2\bar{S}^{\mu\beta}B_{\perp\mu\alpha} - \\
& - \frac{1}{2}\varepsilon_\alpha{}^{\mu\nu}X^\beta\left(W_{\mu\nu} + S_{\mu\nu}\right) + 2h_\alpha{}^\beta\bar{S}^{\mu\nu}B_{\perp\mu\nu} - 2\bar{S}_\alpha{}^\mu W_\mu{}^\beta \,,
\end{aligned}
\tag{2.66}
$$

showing how the presence of torsion is responsible for the degeneracy removal of the magnetic parts of the Weyl tensor[9]. It is interesting to notice that the difference between the magnetic parts of the Weyl tensor depends on the derivatives of the components of the torsion tensor – on $\mathcal{V}$ and along u, making it clear that both the value and the rate of change of the torsion field affect the difference between the tensors H and $\bar{H}$. Moreover, since in Eq. (2.66) we have an algebraic relation for the difference of the components of the tensors H and $\bar{H}$, Eqs. (2.64) - (2.66) characterize both H and $\bar{H}$, that is, we do not need to find propagation and constraint equations for $\bar{H}$ since, those will not be independent of Eqs. (2.64) - (2.66).

[9]Notice that in Eq. (2.66) the term with the Ricci tensor on the LHS could be removed by taking the symmetric part, however, this will add more terms and dense the notation on the RHS so, we opted to write the result as is.

Moving on, the remaining equations that characterize the torsion tensor components are[10]

$$
\begin{aligned}
h^{\alpha\nu}u^{\mu}R_{[\nu\mu]} = &-\left(\varepsilon_{\gamma\mu\nu}\bar{S}^{\nu\mu}+\frac{1}{2}X_{\gamma}\right)B_{\perp}^{\alpha\gamma}+\frac{1}{2}D^{\alpha}W_{\mu}{}^{\mu}-\varepsilon^{\alpha}{}_{\mu\nu}u^{\gamma}\nabla_{\gamma}\bar{S}^{\nu\mu}+\\
&+\frac{1}{2}W_{\mu}{}^{\mu}a^{\alpha}-\frac{1}{2}D_{\beta}W^{\alpha\beta}-\frac{1}{2}W^{\alpha\gamma}a_{\gamma}+\frac{1}{2}X^{\alpha}\theta+\\
&+\varepsilon_{\gamma}{}^{\alpha\mu}\bar{S}_{\mu\beta}\left(B_{\perp}^{\beta\gamma}-W^{\beta\gamma}\right)+S_{\gamma}{}^{\alpha}a^{\gamma}\,,
\end{aligned}
\tag{2.67}
$$

and

$$
\begin{aligned}
\varepsilon^{\alpha\mu\nu}R_{\mu\nu} = &2D_{\beta}\bar{S}^{\beta\alpha}-\varepsilon^{\alpha}{}_{\sigma\rho}W_{\mu}{}^{\mu}\left(\omega^{\sigma\rho}-W^{\sigma\rho}\right)-\varepsilon^{\alpha}{}_{\sigma\rho}W^{\rho}{}_{\beta}\left(B_{\perp}^{\beta\sigma}-W^{\beta\sigma}\right)-\\
&-\varepsilon^{\alpha}{}_{\sigma\rho}\left(X^{\sigma}a^{\rho}+S^{\rho\sigma}\theta\right)-\varepsilon^{\alpha}{}_{\sigma\rho}u^{\beta}\nabla_{\beta}S^{\rho\sigma}+2\bar{S}^{\alpha\mu}\left(X_{\mu}+a_{\mu}\right)+\\
&+\varepsilon^{\alpha}{}_{\sigma\rho}D^{\sigma}X^{\rho}-4\varepsilon^{\alpha}{}_{\sigma\rho}\left(\bar{S}_{\mu}{}^{\rho}\bar{S}^{\mu\sigma}-\bar{S}_{\mu}{}^{\rho}\bar{S}^{\sigma\mu}+\bar{S}_{\mu}{}^{\mu}\bar{S}^{\sigma\rho}\right)\,.
\end{aligned}
\tag{2.68}
$$

Equations (2.56) - (2.68) characterize the geometry of the manifold, containing exactly the same information as the Ricci and Bianchi identities. To find the set of equations describing the dynamical evolution of the matter fields in the manifold, we will consider the conservation law for the stress-energy tensor[11], given in general by

$$
\nabla_{\beta}\mathcal{T}^{\alpha\beta}=\Psi^{\alpha}\,,
\tag{2.69}
$$

where Ψ^{α} is some tensor to be determined by the field equations. From Eqs. (2.50) and (2.69), the projections along u and on $\mathcal{V}$ are

$$
\begin{aligned}
&\dot{\mu}+\left(\theta-W_{\alpha}{}^{\alpha}\right)(\mu+p)+\pi^{\alpha\beta}\left(\sigma_{\alpha\beta}-W_{\alpha\beta}\right)-\varepsilon^{\alpha\beta\gamma}m_{\gamma}\left(\omega_{\alpha\beta}-W_{\alpha\beta}\right)+\\
&\qquad\qquad+q_{1}^{\alpha}a_{\alpha}+\nabla_{\alpha}q_{2}^{\alpha}=-u_{\alpha}\Psi^{\alpha}\,,
\end{aligned}
\tag{2.70}
$$

$$
\begin{aligned}
&(\mu+p)\,a_{\alpha}+h_{\alpha}{}^{\beta}\left(D_{\beta}p+\dot{q}_{1\beta}+\nabla_{\mu}\pi_{\beta}{}^{\mu}+\varepsilon_{\beta}{}^{\mu\nu}D_{\mu}m_{\nu}-\varepsilon_{\beta}{}^{\mu\nu}m_{\mu}a_{\nu}\right)+\\
&\qquad+\left(q_{1\alpha}+\frac{1}{3}q_{2\alpha}\right)\theta-q_{1\alpha}W_{\beta}{}^{\beta}+q_{2}^{\beta}\left(\sigma_{\beta\alpha}+\omega_{\beta\alpha}-W_{\beta\alpha}\right)=h_{\alpha\beta}\Psi^{\beta}\,.
\end{aligned}
\tag{2.71}
$$

2.7 Conclusions and further discussion

Equations (2.56) – (2.71) in conjunction with the field equations of a theory of gravitation — relating the Ricci and the torsion tensors with the matter variables — and an equation of state for the matter model, form the

[10] Equations (2.67) and (2.68) are derived from the first Bianchi identity and, respectively, computing the projections $h^{\sigma\alpha}u^{\gamma}R_{[\alpha\beta\gamma]}{}^{\beta}$ and $h^{\sigma\alpha}h^{\rho\gamma}R_{[\alpha\beta\gamma]}{}^{\beta}$.

[11] Although at this point the imposition that the evolution equations for the matter variables are determined by (2.69) is given as *ad hoc*; provided the field equations of a gravity theory relating the Ricci and the stress-energy tensors, the conservation equations will follow from the second Bianchi identity. Hence, they are a pivotal component to guarantee the consistency of the whole system of equations.

most general set of structure equations for the 1+3 formalism for manifolds endowed with a metric compatible, affine connection. These generalize the results in Refs. [38, 40] for the theory of General Relativity, and in Refs. [47, 48] for $f(R)$ modified theories of gravity in Lorentzian manifolds with null torsion.

The full system of structure equations completely characterize the geometry and the matter fields that permeate the manifold. The fact that the covariantly defined quantities of the 1+3 formalism represent geometrical and physical quantities, lead to a clearer picture on how the properties of the fluids that permeate the manifold and its geometry are related and influence one another. As exemplified in the body of the chapter, this makes it possible to draw new insights, even without solving the system of equations.

A conclusion that can be taken directly from the structure equations is how greatly the inclusion of torsion changes the possible solutions. When comparing the full set of equations to the null-torsion case, it is obvious that the presence of torsion markedly affects the geometry of the manifold and the properties of the matter fluids. This can be seen explicitly in the form of the structure equations but also implicitly: it was found that the presence of torsion lifts a degeneracy in the magnetic parts of the Weyl tensor and removes some of the constraints on this tensor; moreover, as shown in the derivation of the evolution of the separation vector between infinitesimally close curves of a congruence, the W component of the torsion tensor directly appears in the definition of the kinematical quantities, creeping in all the structure equations, in particular, in the conservation equations of the stress-energy tensor, Eqs. (2.70) and (2.71). Anticipating the results of the subsequent chapters, all these new properties will significantly influence the possible solutions.

Chapter 3

Geometry of hypersurfaces

3.1 Introduction

Generically, covariant manifold decomposition methods relying on the threading of the manifold by one or more congruences of curves do not restrict that the orthogonal surfaces — to the congruences — have to be integral manifolds, hypersurfaces. This property makes these methods suitable to study very generic setups, showing clearly how the properties of the congruence and fluids permeating the space-time are intertwined, facilitating new insights and, in the cases where the manifold contains some preferred directions[1], greatly simplifying the equations.

There are, however, many cases where this is indeed verified: the manifold allows the existence of hypersurface orthogonal congruences. In this cases, the various operators: projectors and derivatives, are defined and smoothly spanned locally in some open neighborhood of each point — instead of being defined only at a tangent hyperplane. Then, the higher dimensional manifold induces a geometrical structure on the hypersurfaces, so that, the intrinsic properties of the sub-manifold and the way this is embedded in the higher dimensional space are related with the geometry of the latter. These supplementary results are often used alongside the structure equations found from the covariant manifold decomposition formalisms, providing constraints for the setup, without breaking the consistency of the whole system (cf., e.g., Refs. [40, 43, 45]).

Although in this thesis the motivation to study the geometry of sub-manifolds embedded in higher dimensional Lorentzian manifolds comes in line with the previous chapter, these results are overarching to many other areas of study both in mathematics and physics. For this reason, we chose to study the geometry of sub-manifolds in a separate chapter, hoping to emphasize the generality of these results, which can be used in a wide set of applications, not just those considered in this thesis.

In this chapter we will treat two distinct yet related topics. The first topic concerns the generalization of the Gauss-Codazzi-Ricci equations to manifolds with a metric compatible, affine connection. This set of equations

[1] By preferred directions we mean directions along which the description of the setup is simplified and some of the covariantly defined quantities are trivially zero. As we will see in Chapter 6, in general, these do not have to correspond to Killing vector fields.

provide a relation between the intrinsic geometry of sub-manifolds and the geometry of the ambient higher dimensional manifold. For the purposes of this thesis, these results will be used in Chapter 6 to relate the covariantly defined quantities of the 1+3 formalism with the components of the metric tensor of the space-time. It is, nonetheless, worth highlighting that these equations have extensive applications other than those mentioned here. Notably, the Gauss-Codazzi equations are one of the fundamental building blocks of the so called 3+1 formalism and numerical relativity, providing a set of constraint equations for the initial data, pivotal for the well-posedness of the initial-value problem [54, 58, 65–67].

Another topic we will consider in this chapter is the matching of Lorentzian manifolds at a common non-null boundary, so that the resulting space-time has a well defined geometry at the matching surface. In the study of gravitation it is often the case that the space-time is modeled by two distinct solutions which are then glued together at a common boundary. For instance, in the study of compact objects, the interior and exterior are permeated by distinct fluids — or vacuum — so that, each sub-space is modeled by different solutions of the field equations of the theory. It is then necessary to provide boundary conditions so that the whole space-time is a solution of the field equations and the geometric structure of the manifold is well defined at the boundary. In the absence of torsion, these conditions are provided by the Israel-Darmois matching formalism [68–70]. This, however, was developed for space-times with a Levi-Civita connection hence, due to the geometric nature of the torsion tensor field, the Israel-Darmois formalism needs to be extended for space-times endowed with a general metric compatible, affine connection. In the literature, a few works [71, 72] have considered this problem in different contexts, assuming specific classes of theories of gravitation. However, as we will see bellow, those results did not take fully into account the effects of the torsion field on the whole geometry of the manifold. Therefore, anticipating what follows, to ensure a smooth junction between space-times with a general metric compatible, affine connection, further constrains have to be imposed. Moreover, the treatment here is completely general, in particular, it does not rely on a specific theory of gravitation.

3.2 Hypersurface orthogonal congruences

We start by establishing under what conditions can hypersurface orthogonal congruences of curves be locally defined in a Lorentzian manifold with torsion.

Let $(\mathcal{M}, g, S)$ be a Lorentzian manifold of dimension $N \geq 2$ with metric g and torsion S. Given a point $p \in \mathcal{M}$, consider a subset $\mathcal{N}_p^* \subset T_p^*\mathcal{M}$ of dimension $N - m > 0$ and $\mathsf{W}_p \subset T_p\mathcal{M}$, with dimension $m \geq 1$, whose elements $w \in \mathsf{W}_p$ are such that $n^\flat(w) = 0$, for all $n^\flat \in \mathcal{N}_p^*$.

Given an open neighborhood $\mathcal{O}$ of $p \in \mathcal{M}$, let $\mathsf{W} \subset T\mathcal{M}$ be the union of all W_q, $\forall\, q \in \mathcal{O}$, and $\mathcal{N}^*$ represent the union of $\mathcal{N}_q^*$. We require $\mathcal{N}^*$ to be smooth, that is, $\mathcal{N}^*$ is spanned by $N - m$, C^∞ 1-forms. Then, it is well know (Frobenius' theorem) that W admits integral sub-manifolds if and only if, for all $w, z \in \mathsf{W}$ and for all $n^\flat \in \mathcal{N}^*$,

$$n^\flat(\mathcal{L}_w z) = 0\,, \tag{3.1}$$

where $\mathcal{L}_w z$ represents the Lie derivative of the vector field z along the integral curve of the vector field w.

Lemma 3.1. *Condition (3.1) is equivalent to*

$$\nabla_{[\alpha} n_{\beta]} + S_{\alpha\beta}{}^{\gamma} n_{\gamma} = \sum_{i=1}^{N-m} (n_i)_{[\alpha} (\chi_i)_{\beta]} \ , \tag{3.2}$$

where $n^\flat$, $n_i^\flat \in \mathcal{N}^*$ and χ_i are smooth arbitrary 1-forms.

Proof. Using the definitions of Lie derivative for vector fields and covariant derivative for a general affine, metric compatible connection we have that (2.8)

$$(\mathcal{L}_w z)^\gamma := [w, z]^\gamma = w^\beta \nabla_\beta z^\gamma - z^\beta \nabla_\beta w^\gamma - 2 S_{\alpha\beta}{}^{\gamma} w^\alpha z^\beta \,. \tag{3.3}$$

Then, from Eq. (3.1), we find for all $w, z \in$ W and for all $n^\flat \in \mathcal{N}^*$

$$w^\alpha z^\beta \left(\nabla_{[\beta} n_{\alpha]} - S_{\alpha\beta}{}^{\gamma} n_\gamma \right) = 0 \,. \tag{3.4}$$

Since the identity must be valid for all w and z, we get the equality (3.2). □

Lemma 3.1 states that, locally, W admits integral sub-manifolds if and only if all $n^\flat \in \mathcal{N}^*$ verify Eq. (3.2). Two particular cases of interest are when $\mathcal{N}^*$ is of dimension 1 or 2. In those cases, we have the useful result:

Lemma 3.2. *Consider a 1-form $n^\flat \in \mathcal{N}^*$ and the associated vector field n, with components in a local coordinate system $n^\alpha = g^{\alpha\beta} n_\beta$. If either*

(i) *$\mathcal{N}^*$ is of dimension 1 and $n^\flat(n) \neq 0$, or*

(ii) *$\mathcal{N}^*$ is of dimension 2 and $n^\flat(n) = 0$,*

then, a congruence of curves, defined in some open set $\mathcal{O} \subset \mathcal{M}$, with tangent vector field n is hypersurface orthogonal if and only if

$$3 n_{[\gamma} \nabla_\beta n_{\alpha]} + n_\sigma \left(S_{\beta\alpha}{}^{\sigma} n_\gamma + S_{\alpha\gamma}{}^{\sigma} n_\beta + S_{\gamma\beta}{}^{\sigma} n_\alpha \right) = 0 \,. \tag{3.5}$$

The proof of Lemma 3.2 follows directly from Eq. (3.2) in the cases when $N - m = 1$ and $N - m = 2$, for a particular choice of the 1-forms χ_i. Eq. (3.5) has the advantage of being independent of the arbitrary 1-forms in Eq. (3.2).

3.3 First and second fundamental forms

Having established the conditions for a manifold to admit the existence of hypersurface orthogonal congruences, an important particular case is when integral sub-manifolds have dimension $N-1$. Then, given a $(N-1)-$sub-manifold $\mathcal{V}$ of $\mathcal{M}$, we are interested in studying the relation between the geometry of $\mathcal{M}$ and $\mathcal{V}$.

Consider a point $p \in \mathcal{M}$, with coordinates $\{x^\alpha\}$ and $q \in \mathcal{V}$ with coordinates $\{y^a\}$, in some local coordinate systems. We will use Greek indices to refer to the components of tensor fields defined in $\mathcal{M}$ and Latin indices[2] for components of tensor fields in $\mathcal{V}$. Provided an embedding[3], $f : \mathcal{M} \to \mathcal{V}$, the Jacobian matrix associated with f with entries

$$e_a^\alpha = \frac{\partial x^\alpha}{\partial y^a}\,, \tag{3.6}$$

allows us to pull-back and push-forward tensorial quantities between $\mathcal{M}$ and $\mathcal{V}$, so that the geometrical structure of $\mathcal{M}$ induces, through f, a geometrical structure on $\mathcal{V}$. Notice that the set $\{e_a\}$ behave as vectors on $T\mathcal{M}$. Then, the metric g induces on $\mathcal{V}$ the metric h with

$$h_{ab} := e_a^\alpha e_b^\beta g_{\alpha\beta}\,. \tag{3.7}$$

The induced metric h is usually referred as first fundamental form of $\mathcal{V}$.

Consider now a congruence of non-null curves in $\mathcal{M}$ with tangent vector field n, orthogonal at each point to $\mathcal{V}$. We can define the projector, in $\mathcal{M}$, onto the orthogonal space to the congruence as

$$h_{\alpha\beta} = g_{\alpha\beta} - \epsilon n_\alpha n_\beta\,, \tag{3.8}$$

with $\epsilon = n_\alpha n^\alpha$. The projector $h_{\alpha\beta}$ can then be related with the induced metric of $\mathcal{V}$ by

$$h_{ab} = e_a^\alpha e_b^\beta h_{\alpha\beta}\,. \tag{3.9}$$

With the projector operator, we can readily find the fully orthogonal components of tensorial quantities on $\mathcal{M}$, that is, given a tensor $\psi^{\alpha\ldots\beta}$, the tensor

$$(\psi_\perp)^{\alpha\ldots\beta} \equiv h_\mu{}^\alpha \ldots h_\nu{}^\beta \psi^{\mu\ldots\nu}\,, \tag{3.10}$$

is completely orthogonal to the congruence. Then, using the Jacobian matrix, we can find the induced tensor on $\mathcal{V}$ associated with $(\psi_\perp)^{\alpha\ldots\beta}$,

$$\psi^{a\ldots b} = e_\alpha^a \ldots e_\beta^b \, (\psi_\perp)^{\alpha\ldots\beta}\ . \tag{3.11}$$

[2]We gently remind that Greek indices run from $\{0, N-1\}$, where N denotes the dimension of $\mathcal{M}$ and Latin indices run from $\{1, N-1\}$.

[3]Given two smooth manifolds $\mathcal{M}$ and $\mathcal{N}$, a smooth map $f : \mathcal{M} \to \mathcal{N}$ is said to be an immersion if $df : T\mathcal{M} \to T\mathcal{N}$ is injective. If f is an immersion and homeomorphic onto its image then f is said to be an embedding.

It is important to remark that, due to the properties of the partial derivatives: $e^a_\alpha n^\alpha = 0$; then, as expected, the induced quantities on $\mathcal{V}$ are associated with tensorial quantities orthogonal to the congruence, that is, the embedding map and its inverse associate tensors on $\mathcal{V}$ to tensors on $\mathcal{M}$, orthogonal to the congruence.

In addition to the induced metric, the map f also induces a connection on $\mathcal{V}$. Given a tensor field $\psi_{\alpha...\beta}$ on $\mathcal{M}$, completely orthogonal to n, we define the intrinsic covariant derivative of the tensor field $\psi_{a...b} \equiv e^\alpha_a ... e^\beta_b \psi_{\alpha...\beta}$ on $\mathcal{V}$ as

$$D_a \psi_{b...c} := e^\alpha_a e^\beta_b ... e^\gamma_c \nabla_\alpha \psi_{\beta...\gamma} \,. \tag{3.12}$$

Using this definition and Eq. (3.8) it is straightforward to show that the compatibility with the metric g of ∇ induces D to be compatible with the metric h:

$$D_a h_{bc} = e^\alpha_a e^\beta_b e^\gamma_c \nabla_\alpha h_{\beta\gamma} = e^\alpha_a e^\beta_b e^\gamma_c \nabla_\alpha \left(n_\beta n_\gamma \right) = 0 \,. \tag{3.13}$$

Now, from Eq. (3.11) we can write

$$D_a \psi_b = \partial_a \psi_b - C^c_{ab} \psi_c \,, \tag{3.14}$$

where the induced connection is given by

$$C^c_{ab} = e^\alpha_a \left(\nabla_\alpha e^\beta_b \right) e^c_\beta \,. \tag{3.15}$$

On the other hand, imposing that $D_a \psi_b$ is a tensor[4] on $\mathcal{V}$ we find that

$$C^c_{ab} = \Gamma^c_{ab} + K_{ab}{}^c \,. \tag{3.16}$$

Requiring the compatibility with metric yields

$$\Gamma^c_{ab} = \frac{1}{2} h^{ci} \left(\partial_a h_{ib} + \partial_b h_{ai} - \partial_i h_{ab} \right) \,, \tag{3.17}$$

$$K_{ab}{}^c = S_{ab}{}^c + S^c{}_{ab} - S_b{}^c{}_a \,, \tag{3.18}$$

with

$$S_{ab}{}^c := e^\alpha_a e^\beta_b e^c_\gamma S_{\alpha\beta}{}^\gamma = C^c_{[ab]} \,, \tag{3.19}$$

defining the induced torsion tensor on $\mathcal{V}$ as the totally projected torsion tensor of $\mathcal{M}$.

We introduce the second fundamental form, with components K_{ab}, of the hypersurface $\mathcal{V}$ as

$$K_{ab} := e^\alpha_a e^\beta_b \nabla_\alpha n_\beta \,. \tag{3.20}$$

[4]Imposing that $D_a \psi_b$ is a tensor of $\mathcal{V}$, it must transform accordingly under a change of coordinates. Requiring that the object Γ^c_{ab} transforms in such a way that the sum $\partial_a \psi_b - \Gamma^c_{ab} \psi_c$ is a tensor, we can write without loss of generality that the connection is given by Eq. (3.16), where $K_{ab}{}^c$ is some tensor field.

In Eq. (3.20) the covariant derivative of the normal vector field is evaluated using the connection on $\mathcal{M}$, so the second fundamental form characterizes the hypersurface $\mathcal{V}$ as a sub-manifold embedded in the higher dimensional manifold, $\mathcal{M}$. From this, K_{ab} is also usually referred to as the extrinsic curvature.

Computing the covariant derivative of a vector field $\psi \in TM$, such that $\psi^\alpha n_\alpha = 0$, along a curve orthogonal to n we find

$$e_a^\beta \nabla_\beta \psi^\alpha = e_b^\alpha D_a \psi^b - \epsilon K_{ac} \psi^c n^\alpha \,, \tag{3.21}$$

showing that the second fundamental form measures the purely normal component of the covariant derivative of a vector field, taken along a direction orthogonal to n. Using Eq. (3.21) in the case when $\psi = e_b$, leads to the useful result

$$e_a^\beta \nabla_\beta e_b^\alpha = C_{ab}^c e_c^\alpha - \epsilon K_{ab} n^\alpha \,, \tag{3.22}$$

famously known as the Gauss-Weingarten equation.

Now, although the previous results have the same form of the expressions found for manifolds with symmetric affine connections (cf., e.g., Ref. [65, 66]), the presence of torsion fundamentally change the various quantities and their behavior. In the presence of torsion, we find

$$K_{[ab]} = -e_a^\alpha e_b^\beta S_{\alpha\beta}{}^\gamma n_\gamma \,, \tag{3.23}$$

that is, the extrinsic curvature is no longer required to be a symmetric tensor. It is also worth mentioning that, in connection with what was done in Chapter 2, assuming the curves of the congruence to be affinely parameterized, from Eqs. (2.30), (2.33) and (3.8) we find that the trace

$$K_a{}^a = \nabla_\alpha n^\alpha = \theta - 2S_{\alpha\mu}{}^\mu n^\alpha \,, \tag{3.24}$$

that is, in the presence of a general torsion, the trace of the extrinsic curvature no longer measures the expansion of a congruence of curves (cf., e.g., Ref. [58]).

3.4 Gauss-Codazzi-Ricci embedding equations

Given the previous results, we are now in position to study the relation between the Riemann curvature tensor of $\mathcal{M}$ and that of the sub-manifold $\mathcal{V}$, defined by

$$R_{abc}{}^d \psi_d = D_a \left(D_b \psi_c \right) - D_b \left(D_a \psi_c \right) + 2S_{ab}{}^d D_d \psi_c \,, \tag{3.25}$$

where $\psi \in T^*\mathcal{V}$. From Eqs. (2.10), (3.8) and (3.22), we find

$$e_a^\alpha e_b^\beta e_c^\gamma e_d^\delta R_{\alpha\beta\gamma\delta} = R_{abcd} - \epsilon \left(K_{ac}K_{bd} - K_{bc}K_{ad} \right) \, , \tag{3.26}$$

$$e_a^\alpha e_b^\beta e_c^\gamma n^\delta R_{\alpha\beta\gamma\delta} = D_a K_{bc} - D_b K_{ac} + 2S_{ab}{}^d K_{dc} \, . \tag{3.27}$$

These relations generalize, respectively, the Gauss and Codazzi[5] equations for manifolds with a non-null torsion tensor field, relating the Riemann tensor of the manifold $\mathcal{M}$ with the Riemann tensor and the extrinsic curvature of $\mathcal{V}$. We see that the Gauss equation (3.26) for Lorentzian manifolds with torsion retains the same expression as in the torsionless case, so that the difference between the fully projected Riemann tensor on $\mathcal{M}$ and the intrinsic Riemann tensor on $\mathcal{V}$, is completely described by the second fundamental form. On the other hand, the presence of a general torsion field explicitly modifies the Codazzi equation (3.27), contributing to the tangent components of the Riemann tensor of $\mathcal{M}$. These results explicitly show that the intrinsic geometry of the sub-manifold $\mathcal{V}$ and how it is embedded in $\mathcal{M}$ are described not only by the first and second fundamental forms but also by the induced torsion tensor on $\mathcal{V}$.

Assuming that in some open set, the manifold $\mathcal{M}$ is foliated by a family of hypersurfaces $\{\mathcal{V}\}$, orthogonal at each point to a congruence of affinely parameterized curves with tangent vector field n, the remaining components of the Riemann tensor on $\mathcal{M}$ are given by

$$\begin{aligned} e_a^\alpha n^\beta e_c^\gamma n^\delta R_{\alpha\beta\gamma\delta} = {} & D_a a_c - K_{ad}K^d{}_c - \dot{K}_{ac} - \epsilon \, a_a a_c + K_{ad}\varphi^d{}_c + K_{dc}\varphi^d{}_a + \\ & + 2n^\mu S_{a\mu}{}^d K_{dc} + 2\epsilon \, n^\mu n^\nu S_{a\mu\nu} a_c \, , \end{aligned} \tag{3.28}$$

and, since the first Bianchi identity is modified by the presence of torsion, the extra equation

$$\begin{aligned} \frac{1}{2} e_a^\alpha e_b^\beta e_c^\gamma n^\delta \left(R_{\gamma\delta\alpha\beta} - R_{\alpha\beta\gamma\delta} \right) = {} & D_a \left(n^\mu S_{\mu(bc)} \right) - D_b \left(n^\mu S_{\mu(ac)} \right) + D_c \left(n^\mu S_{\mu[ba]} \right) - \frac{3}{2} \partial_{[a} K_{bc]} - \\ & + \left(S_{\mu[ba]} + \frac{1}{2} S_{ab\mu} \right) \left(e_i^\mu \varphi^i{}_c - \epsilon n^\mu a_c \right) - 2S_{b\mu}{}^\nu S_{\nu(ac)} n^\mu - \\ & - \frac{1}{2} \dot{S}_{abc} + \dot{S}_{c[ab]} + S_{\mu(bc)} \left(e_i^\mu \varphi^i{}_a - \epsilon n^\mu a_a - 2n^\nu S_{\nu a}{}^\mu \right) - \\ & - S_{\mu(ac)} \left(e_i^\mu \varphi^i{}_b - \epsilon n^\mu a_b \right) + S_{[a|c\delta} \left(e_i^\delta \varphi^i{}_{|b]} - \epsilon n^\delta a_{|b]} \right) - \\ & - S_{i(bc)} K_a{}^i - \epsilon S_{\mu a\nu} n^\mu n^\nu K_{(bc)} - \epsilon S_{[b|\mu\nu} n^\mu n^\nu K_{|c]a} + \\ & + S_{i(ac)} K_b{}^i - \epsilon S_{(b|\mu\nu} n^\mu n^\nu K_{a|c)} + 2S_{i\mu(b} S_{c)a}{}^i n^\mu - \\ & - 2S_{c\mu}{}^\nu S_{\nu[ab]} n^\mu - S_{i\mu a} n^\mu S_{bc}{}^i + S_{i[ab]} K_c{}^i \, , \end{aligned} \tag{3.29}$$

[5] Also referred as Codazzi-Mainardi equation.

where $\dot{K}_{ab} \equiv n^\alpha \partial_\alpha K_{ab}$, $a^c \equiv e^c_\alpha n^\beta \nabla_\beta n^\alpha$, ${\varphi^a}_b \equiv e^a_\alpha n^\beta \nabla_\beta e^\alpha_b$ and we have used the notation $S_{\ldots a \ldots} \equiv e^\mu_a S_{\ldots \mu \ldots}$ to represent partially projected components of the torsion tensor.

Equation (3.28) generalizes the Ricci embedding equation for Lorentzian manifolds with torsion. On the other hand, in the absence of torsion the LHS of Eq. (3.29) is identically null, recovering the Codazzi equation. We remark, that although the generalized Gauss equation is already present in the literature (cf. Ref. [73]), surprisingly, the Codazzi and Ricci equations — and (3.29) — to our knowledge, are not.

Contrary to the Gauss and Codazzi equations, in Eqs. (3.28) and (3.29) the considered components of the Riemann tensor on $\mathcal{M}$ are not functions of just the second fundamental form and the induced torsion on the hypersurfaces. This in fact was expected since, the full geometry of the higher dimensional manifold $\mathcal{M}$ can not be solely described by the intrinsic and extrinsic geometry of the sub-manifolds $\{\mathcal{V}\}$.

A direct application of Eqs. (3.26) - (3.29) follows from trying to find the relation between the Ricci scalar of the hypersurface, ${}^{(N-1)}R$, and that of the embedding manifold, R.

From the definition $R_{\alpha\gamma} \equiv {R_{\alpha\mu\gamma}}^\mu$ and Eq. (3.8) we find

$$R_{\alpha\gamma} = h^{bd} e^\beta_b e^\delta_d R_{\alpha\beta\gamma\delta} + \epsilon R_{\alpha\beta\gamma\delta} n^\beta n^\delta \,. \tag{3.30}$$

Taking the trace we then have

$$R = h^{ac} h^{bd} e^\alpha_a e^\beta_b e^\gamma_c e^\delta_d R_{\alpha\beta\gamma\delta} + 2\epsilon h^{ac} e^\alpha_a n^\beta e^\gamma_c n^\delta R_{\alpha\beta\gamma\delta} \,. \tag{3.31}$$

The Gauss and Ricci equations, (3.26) and (3.28), yield

$$\begin{aligned} R = {}& {}^{(N-1)}R - 2a_c a^c + 4n^\mu n^\nu S_{c\mu\nu} a^c + \\ & + 2\epsilon \left[D_c a^c - h^{ac} \dot{K}_{ac} - \frac{1}{2} K_{ab} K^{ba} + 2K_{(ab)} \varphi^{ba} - \frac{1}{2} K^2 - 2n^\mu S_{\mu a d} K^{da} \right] \,, \end{aligned} \tag{3.32}$$

where $K \equiv {K_a}^a$. From the definition of the second fundamental form, Eq. (3.20), we can also write

$$R = {}^{(N-1)}R - \epsilon \left(K^2 + K_{ab} K^{ba} \right) + 2\epsilon \left(\nabla_\alpha a^\alpha - \dot{K} - 2n^\mu {S_\mu}^{\beta\alpha} \nabla_\alpha n_\beta \right) \,, \tag{3.33}$$

which, assuming that n is the tangent vector field of a congruence, has the advantage of being easily related with the kinematical quantities defined in Chapter 2.

3.5 Junction conditions

With the introduction of the first and second fundamental forms we can find the conditions so that two Lorentzian manifolds can be glued together at a common boundary and the resulting manifold has a well defined geometrical structure.

Let $\mathcal{M}^+$ with metric ${}^{(+)}g$ and $\mathcal{M}^-$ with metric ${}^{(-)}g$ be two Lorentzian manifolds matched together at a common boundary, forming a new manifold $\mathcal{M}$. The manifold $\mathcal{M}$ is then partitioned by an hypersurface $\mathcal{S}$ in two regions: $\mathcal{M}^+$ and $\mathcal{M}^-$. Here we will restrict our attention to the cases when $\mathcal{S}$ is either time-like or space-like. Now, although points on the manifolds $\mathcal{M}^-$ and $\mathcal{M}^+$ are in general described by distinct coordinate systems, we will assume the existence of a common continuous coordinate system, $\{x^\alpha\}$, describing an open region that contains the hypersurface $\mathcal{S}$.

Consider that $\mathcal{S}$ is crossed by a congruence of curves parameterized by an affine parameter λ, with tangent vector field

$$n = n^\alpha \partial_\alpha = \frac{d}{d\lambda}\,, \tag{3.34}$$

pointing from $\mathcal{M}^-$ to $\mathcal{M}^+$. Without loss of generality we will consider $\lambda = 0$ at $\mathcal{S}$, negative at $\mathcal{M}^-$ and positive at $\mathcal{M}^+$. The vector field n is by definition orthogonal to $\mathcal{S}$ at every point then, introducing a coordinate system $\{y^a\}$ on $\mathcal{S}$, the tangent vectors to the hypersurface, $e_a \equiv \partial/\partial y^a$, verify

$$e^\alpha_a n_\alpha = 0\,, \tag{3.35}$$

where n_α are the components in the coordinate system $\{x^\alpha\}$ of the 1-form associated with the vector field n: $n^\flat = n_\alpha dx^\alpha = \epsilon\, d\lambda$ with $\epsilon = n_\alpha n^\alpha$. From the continuity of λ, $\{x^\alpha\}$ and $\{y^a\}$ we have the following identities

$$\begin{aligned} [n^\alpha]_\pm &= 0\,, \\ [e^\alpha_a]_\pm &= 0\,, \end{aligned} \tag{3.36}$$

where we introduced the notation $[\psi]_\pm$ to represent the difference of a field as seen from each sub-manifold at $\mathcal{S}$, i.e., $[\psi]_\pm \equiv \psi\left(\mathcal{M}^+\right)|_{\mathcal{S}} - \psi\left(\mathcal{M}^-\right)|_{\mathcal{S}}$. In what follows it is also useful to label a field ψ defined on the subspace $\mathcal{M}^+$ or $\mathcal{M}^-$ as ${}^{(+)}\psi \equiv \psi\left(\mathcal{M}^+\right)$ or ${}^{(-)}\psi \equiv \psi\left(\mathcal{M}^-\right)$, respectively.

Now, we will define that the components of the metric tensor of $\mathcal{M}$ and the entries of the connection in the coordinate system $\{x^\alpha\}$ are given by

$$g_{\alpha\beta} = \Theta\left(\lambda\right) {}^{(+)}g_{\alpha\beta} + \Theta\left(-\lambda\right) {}^{(-)}g_{\alpha\beta}\,, \tag{3.37}$$
$$C^\gamma_{\alpha\beta} = \Theta\left(\lambda\right) {}^{(+)}C^\gamma_{\alpha\beta} + \Theta\left(-\lambda\right) {}^{(-)}C^\gamma_{\alpha\beta}\,, \tag{3.38}$$

where ${}^{(+)}g_{\alpha\beta}$, ${}^{(+)}C^\gamma_{\alpha\beta}$, ${}^{(-)}g_{\alpha\beta}$ and ${}^{(-)}C^\gamma_{\alpha\beta}$ are the components of the metrics and entries of the connections in $\mathcal{M}^+$ and $\mathcal{M}^-$ and $\Theta\left(\lambda\right)$ is the Heaviside step function. Moreover, the Christoffel symbols and the torsion tensors are given formally by

$$\Gamma^\gamma_{\alpha\beta} = \Theta\left(\lambda\right) {}^{(+)}\Gamma^\gamma_{\alpha\beta} + \Theta\left(-\lambda\right) {}^{(-)}\Gamma^\gamma_{\alpha\beta}\,, \tag{3.39}$$
$$S_{\alpha\beta}{}^\gamma = \Theta\left(\lambda\right) {}^{(+)}S_{\alpha\beta}{}^\gamma + \Theta\left(-\lambda\right) {}^{(-)}S_{\alpha\beta}{}^\gamma\,. \tag{3.40}$$

Generically, Eq. (3.39) does not follow from Eq. (3.37), however it is easy to prove (cf., e.g., Refs. [58, 70]) that the Riemann tensor of the total space-time is well defined if and only if

$$[h_{ab}]_{\pm} = 0\,, \tag{3.41}$$

that is, the jump of the first fundamental form of $\mathcal{S}$, as described by each sub-manifold, is null; this constrain and Eq. 3.37 then imply Eq. (3.39). Equation (3.41) represents the first junction condition for two Lorentzian manifolds to be matched together at a common boundary.

Substituting Eq. (3.38) in Eq. (2.9) we find the following expression for the Riemann tensor[6]

$$R_{\alpha\beta\gamma}{}^{\rho} = \theta\left(\lambda\right){}^{(+)}R_{\alpha\beta\gamma}{}^{\rho} + \theta\left(-\lambda\right){}^{(-)}R_{\alpha\beta\gamma}{}^{\rho} + \delta\left(\lambda\right)\left(B_{\alpha\beta\gamma}{}^{\rho} + D_{\alpha\beta\gamma}{}^{\rho}\right)\,, \tag{3.42}$$

where $\delta\left(\lambda\right)$ represents the Dirac distribution and

$$B_{\alpha\beta\gamma}{}^{\rho} = \epsilon\left(n_{\beta}\left[\Gamma^{\rho}_{\alpha\gamma}\right]_{\pm} - n_{\alpha}\left[\Gamma^{\rho}_{\beta\gamma}\right]_{\pm}\right)\,, \tag{3.43}$$

$$D_{\alpha\beta\gamma}{}^{\rho} = \epsilon\left(n_{\beta}\left[K_{\alpha\gamma}{}^{\rho}\right]_{\pm} - n_{\alpha}\left[K_{\beta\gamma}{}^{\rho}\right]_{\pm}\right)\,, \tag{3.44}$$

are the singular parts of the Riemann tensor, with $K_{\alpha\beta}{}^{\gamma}$ being the contorsion tensor, Eq. (2.5).

Now, we will require that a smooth junction between two Lorentzian manifolds must guarantee that the discontinuities of all curvature tensors across the matching hypersurface have to be at most finite otherwise, the manifold $\mathcal{M}$ will be singular at $\mathcal{S}$. The remaining junction conditions will then follow from requiring that the singular parts of the Riemann tensor, Eqs. (3.43) and (3.44), are identically null.

Proposition 3.1. *The tensor D in Eq. (3.44) is identically null if and only if the jump of the contorsion tensor at $\mathcal{S}$ verifies*

$$\left[K_{\alpha\beta}{}^{\rho}\right]_{\pm} = \epsilon n_{\alpha}\left[n^{\mu}K_{\mu\beta}{}^{\rho}\right]_{\pm}\,. \tag{3.45}$$

Proof. The proof follows by direct inspection. Substituting Eq. (3.45) in Eq. (3.44) it follows that D is identically null. On the other hand, assuming $D_{\alpha\beta\gamma}{}^{\rho} = 0$, taking the contraction with n^{β} yields Eq. (3.45). □

Proposition 3.2. *Given Eqs. (3.41) and (3.45), the tensor B in Eq. (3.43) is identically null if and only if*

$$[K_{ab}]_{\pm} = 0\,. \tag{3.46}$$

Proof. Let us start by computing the jump of the partial derivatives of the metric components in the coordinate system $\{x^{\alpha}\}$, at $\mathcal{S}$. Using Eqs. (3.36)

$$\left[\partial_{\gamma}g_{\alpha\beta}\right]_{\pm} = e^{j}_{\gamma}\left[h_{j}{}^{i}e^{\mu}_{i}\partial_{\mu}g_{\alpha\beta}\right]_{\pm} + \chi_{\alpha\beta}n_{\gamma}\,, \tag{3.47}$$

[6] Here we mix the distribution associated to a tensor and the tensor itself which is, strictly speaking, an abuse of language. Our conclusions, however, are not influenced by this issue. See e.g. Ref. [70] and references therein for more details.

where χ is a symmetric two tensor completely orthogonal to n. Assuming the first junction condition, Eq. (3.41), we find

$$[\partial_\gamma g_{\alpha\beta}]_\pm = e^j_\gamma h_j{}^i e^\mu_i \partial_\mu [g_{\alpha\beta}]_\pm + \chi_{\alpha\beta} n_\gamma = \chi_{\alpha\beta} n_\gamma \,, \tag{3.48}$$

where the last equality is valid on the coordinate system $\{x^\alpha\}$. Writing $\chi_{\alpha\beta} = \epsilon n^\gamma [\partial_\gamma g_{\alpha\beta}]_\pm$, where, without loss of generality, n was assumed to have unit length, we have

$$\left[\Gamma^\gamma_{\alpha\beta}\right]_\pm = \frac{1}{2}\left(\chi^\gamma{}_\beta n_\alpha + \chi_\alpha{}^\gamma n_\beta - \chi_{\alpha\beta} n^\gamma\right) . \tag{3.49}$$

Substituting Eq. (3.49) in Eq. (3.43) yields

$$B_{\alpha\beta\gamma}{}^\rho = \frac{\epsilon}{2}\left[\chi_\alpha{}^\rho n_\beta n_\gamma + \chi_{\beta\gamma} n_\alpha n^\rho - \chi_{\alpha\gamma} n_\beta n^\rho - \chi_\beta{}^\rho n_\alpha n_\gamma\right] . \tag{3.50}$$

Using Eq. (3.36), by direct inspection it is straightforward to verify that B is zero if and only if $\chi_{\alpha\beta} = 0$.

All is left is to relate the tensor χ with the second fundamental form. From the definition (3.20) and Eqs. (3.36), (3.45) and (3.49) we have

$$\begin{aligned} [K_{ab}]_\pm &= e^\alpha_a e^\beta_b [\nabla_\alpha n_\beta]_\pm = -e^\alpha_a e^\beta_b \left[\Gamma^\gamma_{\alpha\beta} n_\gamma\right]_\pm - e^\alpha_a e^\beta_b [K_{\alpha\beta}{}^\gamma n_\gamma]_\pm \\ &= -e^\alpha_a e^\beta_b \left[\Gamma^\gamma_{\alpha\beta} n_\gamma\right]_\pm = \frac{\epsilon}{2} e^\alpha_a e^\beta_b \chi_{\alpha\beta} \,. \end{aligned} \tag{3.51}$$

We finish the proof with a small remark. Looking at Eq. (3.51) there seems to be an inconsistency since, the second fundamental form is not required to be symmetric in the presence of torsion and $\chi_{\alpha\beta}$ is by definition symmetric. Nonetheless, imposing the constraint (3.45) guarantees that the anti-symmetric part of K_{ab} is set to zero. This can be seen clearly if we introduce the metric part of the second fundamental form as, say $\tilde{K}_{ab}$, which is manifestly symmetric. Given that $[K_{ab}]_\pm = \left[\tilde{K}_{ab}\right]_\pm - e^\alpha_a e^\beta_b [K_{\alpha\beta}{}^\gamma]_\pm n_\gamma$, Eq. (3.54) then sets the second term on the RHS to zero, proving the consistency of Eq. (3.51). $\square$

Gathering the previous results,

Theorem 3.1. *Let $\mathcal{M}^-$ and $\mathcal{M}^+$ be two Lorentzian manifolds with boundary, endowed with a metric compatible, affine connection. $\mathcal{M}^-$ and $\mathcal{M}^+$ can be smoothly matched at a common, non-null hypersurface $\mathcal{S}$ when the following three conditions are verified:*

(i) *the induced metric at $\mathcal{S}$ is such that*

$$[h_{ab}]_\pm = 0\,; \tag{3.52}$$

(ii) *the jump of the extrinsic curvature of $\mathcal{S}$ is null, that is*

$$[K_{ab}]_\pm = 0\,; \tag{3.53}$$

(iii) *the jump of the contorsion tensor at $\mathcal{S}$ verifies*

$$[K_{\alpha\beta}{}^{\rho}]_\pm = \epsilon n_\alpha \left[n^\mu K_{\mu\beta}{}^{\rho}\right]_\pm\,, \tag{3.54}$$

where the vector field n represents the unit normal to $\mathcal{S}$ and $\epsilon \equiv n_\alpha n^\alpha$.

Notice that the third condition, Eq. (3.54), cannot be completely written in terms of quantities defined on $\mathcal{S}$ since, the contorsion tensor in general does not have to be orthogonal to the normal n. In fact, Eq. (3.54) can be written more concisely as $[h_\alpha{}^\mu K_{\mu\beta}{}^\rho]_\pm = 0$, where the projector onto $\mathcal{S}$ is defined in Eq. (3.8), highlighting that only the jump of a partial projection of the contorsion tensor must be null. This then implies that the common coordinate system $\{x^\alpha\}$ must exist. Nonetheless, this is not a loss of generality since, the whole construction is based on this assumption, in the same way that the remaining constraints, Eqs. (3.52) and (3.53), assume the existence of the coordinate system $\{y^a\}$ on $\mathcal{S}$, common to $\mathcal{M}^-$ and $\mathcal{M}^+$.

The result in Theorem 3.1, generalizes the Israel-Darmois junction conditions [68, 69] for the smooth junction of Lorentzian manifolds in the presence of torsion. This results, as we will see in Chapter 6, will greatly influence the allowed solutions for compact objects matched with a vacuum exterior.

3.6 Conclusions and further discussion

From the Frobenius' theorem we found necessary and sufficient conditions for a Lorentzian manifold endowed with a metric compatible, affine connection to admit the existence of a congruence of curves orthogonal to hypersurfaces. Then, we studied how the geometric structure of the higher dimensional manifold induces a structure on a sub-manifold, namely, the first and second fundamental forms and the induced torsion tensor field, completely characterizing the sub-manifold's intrinsic structure and how it is embedded in the ambient higher dimensional space. From these, it was found that the presence of the torsion field markedly influences the geometry of the sub-manifold: the second fundamental form is no longer required to be represented by a symmetric tensor. Moreover, it was found that the presence of torsion breaks the equality between the trace of the extrinsic curvature and the expansion of the congruence to which the sub-manifold is orthogonal to. This result highlights that both the second fundamental form and the torsion tensor field are needed to characterize how the sub-manifold is embedded in the ambient manifold.

Studying the relation between the intrinsic geometry of the sub-manifold and the way it is embedded in the higher dimensional manifold, we found the fundamental relations describing how the intrinsic Riemann tensor, extrinsic curvature and induced torsion of a sub-manifold are related with the Riemann curvature of the embedding space, forming the generalized Gauss-Codazzi-Ricci embedding equations for Lorentzian manifolds with non-null torsion. It was also found that the presence of the torsion field leads to an extra equation for the remaining components of the Riemann tensor.

Here we would like to bring attention to the fact that some authors (cf. Ref. [73]) claim that the requirement that the higher dimensional manifold must admit the existence of hypersurface orthogonal congruences is superfluous, so that, for manifolds where this is not verified it is still possible to study the induced geometrical structure on the orthogonal spaces and define the Gauss-Codazzi equations, pointwise. We will actively refute such possibility: to induce a geometrical structure on a sub-space, this space must be, in particular, a differential manifold otherwise, we can not define the notion of induced covariant derivatives in a neighborhood of a point, in particular there is no guarantee that a linear connection exists.

We studied how two Lorentzian manifolds endowed with a metric compatible, affine connection can be smoothly matched at a common boundary, finding a set of conditions for the smooth junction of two manifolds, generalizing the Israel-Darmois junction conditions [68–70] in the presence of torsion. On this note, the requirement that a smooth junction of two manifolds must imply that the discontinuities of the curvature tensor across the matching hypersurface have to be at most finite needs to be further explained. In the literature [71, 72] the matching of two manifolds with non-null torsion had already been studied, either in the context of the ECSK theory or other $f\left(R\right)$ theories of gravitation. These earlier results investigated the matching conditions by requiring that the field equations of the considered theories remained valid at the matching surface, so that the resulting manifold, was a solution of the theory, leading to a set of constrains. These conditions, though, contrary to the case of null torsion [70], are not enough to guarantee that the singular parts of the Riemann tensor, Eqs. (3.43) and (3.44), vanish. One might, nonetheless, argue that since the field equations are well defined, the remaining singular parts of the Riemann tensor would not influence the solution; this, however, is not the case. As is clearly shown in the 1+3 structure equations derived in Chapter 2, the presence of the torsion field unavoidably changes the geometry of the manifold, in particular, the components of the Weyl tensor. These, on the other hand, are non-linearly related with all the other variables that characterize the geometry and the matter fields that permeate the manifold. If we do not restrict the singular part of the Riemann tensor to be null for a smooth junction, some components of the Weyl tensor will diverge at the matching surface, affecting the whole set of equations and the smoothness of the solution at the matching. As we will see in Chapter 6, although the Weyl tensor does not appear directly in the field equations, it will appear in the continuity equations for the stress-energy tensor, hence its influence on the matter fields.

Part II

Singularities

Chapter 4

Singularity Theorems

4.1 Introduction

The existence and genericity of space-time singularities has been an issue of great debate since the early years of the theory of General Relativity, when these were thought to be a problem of some idealized and highly symmetric solutions of the Einstein field equations, and more realistic models were supposed to overcome such problems [74]. This debate propelled Raychaudhuri [11] and, independently, Komar [12], to study the nature of singularities and their relation to the space-time symmetries. In his seminal work [11], studying the kinematics of a fluid in the context of GR, Raychaudhuri derived an equation for the evolution of the expansion of time-like geodesic congruences. Applying this equation to a model with a particular physical source, he provided the first evidence that the occurrence of singularities was not a direct consequence of the imposed symmetries (see e.g. Ref. [53] for a review).

It was not until a decade later that the results derived by Penrose [13], and later extended by Hawking [14–16], established that under more general physical conditions, the formation of space-time singularities was inevitable. Relying on the Raychaudhuri equation, and imposing some rather general constraints on the space-time geometry, it was possible to establish sufficient conditions for a congruence of geodesics to focus on a point, usually referred as a focal point of the congruence (also called caustic[1]). Using the property that geodesics are extremal curves of the space-time interval, Penrose then related the formation of a focal point to the occurrence of a space-time singularity, in the sense that the geodesics of the congruence were inextensible beyond that point.

After the initial works of Hawking and Penrose, many extensions and reformulations to their singularity theorems have been made (see e.g. Refs. [53, 74] for a review in the context of space-times without torsion). In particular, there has been a growing interest in studying whether the inclusion of the torsion tensor field could prevent the formation of singularities [25–28]. This led to the formulation of singularity theorems for

[1]We will make no distinction between focal or conjugate points, see e.g. Ref. [53] for the precise definitions.

space-times with torsion [21, 75]. Indeed, using the incompleteness of time-like metric geodesics to probe the regularity of space-times in the ECSK theory of gravitation, Hehl *et al.* [29] found that the original Hawking-Penrose singularity theorems were equally valid in that theory, with the correction that the matter constraints had to be imposed on a modified stress-energy tensor. More recently, another approach was proposed by Cembranos *et al.* [76] where the occurrence of singularities was related with the existence of horizons, allowing the formulation of singularity theorems of Hawking-Penrose type for non-geodesic curves for particular Teleparallel and ECSK theories of gravitation.

In this chapter we will consider the more general setting of affine theories of gravitation with non-null torsion field. Using the results of the previous chapters, we will study what are the necessary conditions for focal points of a congruence of curves, not necessarily geodesics, to occur; in particular, what are the effects of the presence of torsion in the focusing of the congruence. Although a necessary step towards the proof of singularity theorems, those focusing results are not sufficient to prove singularity formation on a Lorentzian manifold, in general. In order to do so, we shall analyze the energy conservation equation for a general affine theory of gravitation. This is especially relevant in the cases of non-metric geodesic curves since, in such cases, one can not rely on the length extremality property of the curves. So, we will consider separately the cases of metric geodesics and accelerated curves. For the latter, we will first consider two cases of physically motivated stress-energy tensors: perfect fluids with particular equations of state and scalar fields. In each case, we derive a generalized Raychaudhuri-Komar singularity theorem, applicable to a wide class of affine theories of gravitation, without imposing any further symmetries on the manifold. Finally, we will consider the particular case of the ECSK theory, allowing us to extend our results to perfect fluids with a cosmological constant and without restrictions on the equation of state.

4.2 Curve focusing in Lorentzian manifolds with torsion

In this section, we will study necessary conditions for the formation of a focal point of an hypersurface orthogonal congruence, developing in the process new focus theorems for Lorentzian manifolds endowed with a metric compatible, affine connection. In order to do so, we will make use of the results in Section 3.2, where the necessary and sufficient conditions for a congruence of curves to admit the existence of orthogonal hypersurfaces were found. Those results are rather general, being valid for any type of curve, namely space-like, time-like or null, even for curves non-affinely parameterized, however, to use those results to study the formation of focal points, we have to specify the type of curve that is being considered. Then, we will treat separately the cases of time-like and null congruences.

4.2.1 Focusing of time-like congruences

Consider a congruence of time-like curves with tangent vector field u with acceleration a with components

$$a^\beta = u^\alpha \nabla_\alpha u^\beta \,. \tag{4.1}$$

Without loss of generality, we will assume u to be affinely parameterized, i.e.,

$$\begin{aligned} u^\alpha u_\alpha &= -1 \,, \\ u_\alpha a^\alpha &= 0 \,. \end{aligned} \tag{4.2}$$

Given the definitions of the expansion, shear and vorticity tensors of a congruence, Eq. (2.36), for the case of a time-like tangent vector field u, we have

$$\theta = \nabla_\nu u^\nu + 2S_{\sigma\nu}{}^\nu u^\sigma \,, \tag{4.3}$$

$$\sigma_{\alpha\beta} = h^\mu{}_{(\alpha} h_{\beta)}{}^\nu \left(\nabla_\mu u_\nu + 2S_{\sigma\mu\nu} u^\sigma\right) \,, \tag{4.4}$$

$$\omega_{\alpha\beta} = h^\mu{}_{[\alpha} h_{\beta]}{}^\nu \left(\nabla_\mu u_\nu + 2S_{\sigma\mu\nu} u^\sigma\right) \,, \tag{4.5}$$

where $h_{\alpha\beta}$ is the projector operator onto the orthogonal space, at each point, to u, Eq. (2.38). Then, we can show:

Lemma 4.1. *Given an affinely parametrized congruence of time-like curves in a Lorentzian manifold with torsion $(\mathcal{M}, g, S)$, a necessary and sufficient condition for the congruence to be hypersurface orthogonal is that the vorticity tensor is given by*

$$\omega_{\alpha\beta} = W_{[\alpha\beta]} + S_{\alpha\beta} \,, \tag{4.6}$$

where $W_{\alpha\beta}$ and $S_{\alpha\beta}$ are defined in (2.44)*.*

Proof. First, by using Eqs. (2.43) and (2.44), we find that Eq. (4.6) is equivalent to

$$\omega_{\alpha\beta} - 2u^\mu h^\nu{}_{[\alpha} h_{\beta]}{}^\sigma S_{\mu\nu\sigma} + h^\mu{}_\alpha h^\nu{}_\beta u^\sigma S_{\mu\nu\sigma} = 0 \,. \tag{4.7}$$

To show that Eq. (4.7) is a necessary and sufficient condition for the congruence to admit orthogonal hypersurfaces, it suffices to prove that this equation is equivalent to Eq. (3.2) for dimensions $N \geq 2$ and $N - m = 1$. In fact, contracting Eq. (3.2) with $h^\alpha{}_\mu h^\beta{}_\nu$ and using Eq. (4.5), we recover Eq. (4.7). On the other hand, substituting Eq. (2.38) in Eq. (4.5) and using Eq. (4.7) we find Eq. (3.2). □

Gathering the previous results, we are in position to prove a generalized focus theorem for time-like curves in manifolds with torsion:

Theorem 4.1. *Let (M, g, S) be a Lorentzian manifold with dimension $N \geq 2$ endowed with a torsion tensor field S such that $W_{(\alpha\beta)} = 0$. Given a congruence of hypersurface orthogonal time-like curves in (M, g, S), let c represent an affinely parametrized fiducial curve of the congruence, with tangent vector field u. A focal point of the congruence will form for a finite value of the affine parameter if the following three conditions are satisfied:*

(i) *At a point along c, the expansion of the congruence is negative;*

(ii) $R_{\alpha\beta}u^\alpha u^\beta \geq \left(W_{[\alpha\beta]} + S_{\alpha\beta}\right) S^{\alpha\beta}$*;*

(iii) $\nabla_\alpha a^\alpha + X_\alpha a^\alpha \leq 0$*,*

where $R_{\alpha\beta}$ is the Ricci tensor.

Proof. Using Eq. (4.6), for an hypersurface orthogonal, time-like congruence, the Raychaudhuri equation (2.56) can be written as[2]

$$\begin{aligned} \frac{d\theta}{d\tau} := \dot{\theta} = -R_{\alpha\beta}u^\alpha u^\beta - \left(\frac{1}{N-1}\theta^2 + \sigma_{\alpha\beta}\sigma^{\alpha\beta}\right) + \left(W_{[\alpha\beta]} + S_{\alpha\beta}\right) S^{\alpha\beta} + \\ + W^{(\alpha\beta)}\left(\frac{h_{\alpha\beta}}{N-1}\theta + \sigma_{\alpha\beta}\right) + \nabla_\alpha a^\alpha + \dot{W}_\alpha{}^\alpha + X_\alpha a^\alpha \,, \end{aligned} \tag{4.8}$$

where τ is an affine parameter. Assuming $W_{(\alpha\beta)} = 0$, Eq. (4.8) then reduces to

$$\dot{\theta} = -R_{\alpha\beta}u^\alpha u^\beta - \left(\frac{1}{N-1}\theta^2 + \sigma_{\alpha\beta}\sigma^{\alpha\beta}\right) + \left(W_{[\alpha\beta]} + S_{\alpha\beta}\right) S^{\alpha\beta} + \nabla_\alpha a^\alpha + X_\alpha a^\alpha \,. \tag{4.9}$$

Since $\sigma_{\alpha\beta}$ is orthogonal to u, we have $\sigma_{\alpha\beta}\sigma^{\alpha\beta} \geq 0$, therefore, under the conditions of the theorem,

$$\frac{d\theta}{d\tau} \leq -\frac{\theta^2}{N-1} \,, \tag{4.10}$$

which upon integration implies

$$\frac{1}{\theta(\tau)} \geq \frac{\tau}{N-1} + \frac{1}{\theta_0} \,, \tag{4.11}$$

where θ_0 is an initial value for θ. Hence, assuming the existence of a point along c where $\theta_0 < 0$, for a value $\tau \leq (N-1)/|\theta_0|$, then $\theta \to -\infty$, i.e., a focal point of the congruence will form. $\square$

Two important particular cases for space-times with torsion that are widely used in the physics literature and verify the conditions in Theorem 4.1 are models where: the torsion tensor is of the form $S_{\alpha\beta}{}^\sigma = S_{\alpha\beta}u^\sigma$

[2] For reasons explained in Section 2.4, in the derivation of Eq. (4.8) we assumed the space-time to be of dimension 4, nonetheless, that result is easily extendable to manifolds with dimension $N \geq 2$. For further details see Refs. [59, 77].

(see e.g. Refs. [25–29, 78–80]); the torsion tensor is completely anti-symmetric $S_{\alpha\beta\sigma} = S_{[\alpha\beta\sigma]}$, such that, $X^\alpha = 0$ and $W_{\alpha\beta} = W_{[\alpha\beta]} = -2S_{\alpha\beta}$ (see e.g. Refs. [33, 81, 82]).

To close this section, we make a few comments about the constraints on the torsion tensor in the above theorem. In the presence of a general torsion field, the Raychaudhuri equation has a very different functional form when compared to the case of a manifold without torsion. This added complexity makes it very difficult to infer the sign of the term with $W^{(\alpha\beta)}$ in Eq. (4.8) without knowing, *ab initio*, the quantities θ and $\sigma_{\alpha\beta}$. Therefore, in the case when $W^{(\alpha\beta)} \neq 0$ it does not seem possible to make a general statement regarding the formation of a focal point taking into account only the Raychaudhuri equation.

4.2.2 Focusing of null congruences

We now consider the case of a congruence of null curves. In this subsection, we will consider that the manifold is of dimension $N \geq 3$, moreover, to avoid confusion with the previous case, we will represent the tangent vector field to the curves of the congruence by k. Assuming the curves of the congruence to be to be affinely parameterized, we have

$$\begin{aligned} k^\alpha k_\alpha &= 0\,, \\ \nabla_k k^\alpha &= w^\alpha\,, \\ k_\alpha w^\alpha &= 0\,. \end{aligned} \tag{4.12}$$

We will follow a similar procedure to the previous section, however, instead of using the $1+3$ formalism, we shall use a $2+2$ type decomposition adapted to N–dimensional manifolds. To do so, we will have to find the proper projector to the subspace orthogonal to the tangent vector k. Since k is null, it is orthogonal to itself, therefore, the object in Eq. (2.38) does not meet the requirements for such projector. Let us then consider an auxiliary vector field ξ, with the following properties

$$\begin{aligned} \xi^\alpha \xi_\alpha &= 0\,, \\ \xi^\alpha k_\alpha &= -1. \end{aligned} \tag{4.13}$$

Then, we can define the following tensor

$$\tilde{h}_{\alpha\beta} := g_{\alpha\beta} + k_\alpha \xi_\beta + \xi_\alpha k_\beta\,, \tag{4.14}$$

such that

$$\begin{aligned} \tilde{h}_{\alpha\beta} k^\alpha &= 0\,, \\ \tilde{h}_{\alpha\beta} \xi^\alpha &= 0\,, \\ \tilde{h}_{\alpha\beta} \tilde{h}^{\alpha\gamma} &= \tilde{h}_\beta^\gamma\,, \\ \tilde{h}_{\alpha\beta} \tilde{h}^{\alpha\beta} &= N-2\,, \end{aligned} \tag{4.15}$$

where $\tilde{h}_{\alpha\beta}$ represents the projector onto the $N-2$ subspace orthogonal to both k and ξ.

Substituting Eq. (4.14) in Eq. (2.36) we find the following expressions for the kinematical quantities of the congruence associated with k, defined on the $N-2$ subspace, orthogonal to both k and ξ at each point:

$$\theta = \nabla_\alpha k^\alpha - 2S_{\alpha\gamma}{}^\alpha k^\gamma - 2\xi^\sigma S_{\sigma\gamma\rho} k^\gamma k^\rho + \xi^\gamma w_\gamma \,, \tag{4.16}$$

$$\sigma_{\alpha\beta} = \tilde{h}^\mu{}_{(\alpha} \tilde{h}_{\beta)}{}^\nu \left(\nabla_\mu k_\nu + 2S_{\sigma\mu\nu} k^\sigma \right) \,, \tag{4.17}$$

$$\omega_{\alpha\beta} = \tilde{h}^\mu{}_{[\alpha} \tilde{h}_{\beta]}{}^\nu \left(\nabla_\mu k_\nu + 2S_{\sigma\mu\nu} k^\sigma \right) \,. \tag{4.18}$$

Using the results in Section 3.2, we can now obtain necessary and sufficient conditions for the congruence associated with k to be hypersurface orthogonal:

Lemma 4.2. *Given an affinely parametrized congruence of null curves in a Lorentzian manifold $(\mathcal{M}, g, S)$, of dimension $N \geq 3$, a necessary and sufficient condition for the congruence to be hypersurface orthogonal is that the vorticity tensor is characterized by*

$$\omega_{\alpha\beta} = 2\tilde{h}^\mu{}_{[\alpha} \tilde{h}_{\beta]}{}^\nu S_{\sigma\mu\nu} k^\sigma - \tilde{h}^\mu{}_\alpha \tilde{h}^\nu{}_\beta S_{\mu\nu\sigma} k^\sigma \,. \tag{4.19}$$

Proof. As before, we give a proof sketch without including all the algebraic manipulations. In the considered case, Eq. (3.2) reads

$$\nabla_{[\alpha} k_{\beta]} + S_{\alpha\beta}{}^\gamma k_\gamma = k_{[\alpha} \left(\chi_1\right)_{\beta]} + \xi_{[\alpha} \left(\chi_2\right)_{\beta]} \,, \tag{4.20}$$

where χ_1 and χ_2 are arbitrary 1-forms. Contracting Eq. (4.20) with $\tilde{h}_\mu{}^\alpha \tilde{h}_\nu{}^\beta$ and using the properties (4.15) and Eq. (4.18), it is straightforward to find Eq. (4.19). On the other hand, starting from Eq. (4.19), using Eqs. (4.14) and (4.18) we recover (4.20). This then shows that Eq. (4.19) is equivalent to Eq. (4.20). □

Let us now write the Raychaudhuri equation for the congruence associated with k, valid on the $N-2$ subspace, orthogonal to both k and ξ. Assuming, without loss of generality, that the congruence of curves associated with k are affinely parameterized by a parameter λ, we have [59, 83–85]

$$\begin{aligned}
\frac{d\theta}{d\lambda} &= -R_{\alpha\beta} k^\alpha k^\beta - \left(\frac{1}{N-2}\theta^2 + \sigma_{\alpha\beta}\sigma^{\alpha\beta} + \omega_{\alpha\beta}\omega^{\beta\alpha} \right) + \\
&\quad + 2S_{\rho\alpha}{}^\beta k^\rho \left(\frac{\tilde{h}_\beta{}^\alpha}{N-2}\theta + \sigma_\beta{}^\alpha + \omega_\beta{}^\alpha \right) + 2k^\gamma \nabla_\gamma \left(S_\rho k^\rho \right) + \nabla_\alpha w^\alpha + \\
&\quad + k^\gamma \nabla_\gamma \left(\xi^\alpha w_\alpha \right) + \xi^\alpha \xi^\beta w_\alpha w_\beta + 2\xi^\beta w^\alpha \nabla_\alpha k_\beta + 2S_{\beta\gamma\rho} k^\beta \xi^\rho w^\gamma \\
&\quad + 2k^\gamma \nabla_\gamma \left(S_{\rho\alpha\sigma} k^\rho k^\sigma \xi^\alpha \right) + 2S_{\alpha\sigma\beta} k^\alpha k^\beta \xi^\gamma \nabla_\gamma k^\sigma \,.
\end{aligned} \tag{4.21}$$

For a completely general torsion tensor field and k vector field, there seems to be no relevant way to constraint Eq. (4.21) in order to infer the behavior of the expansion of the congruence associated with k. On the other hand, one could argue, on physical grounds, that Eq. (4.21) is only useful if k describes the paths of physical particles, hence, k must be tangent to metric geodesics (cf. Refs. [21, 86]), i.e., $k^\alpha \nabla^{(g)}_\alpha k^\beta = 0$, where $\nabla^{(g)}_\alpha$ represents the Levi-Civita connection. From the definition of w^α, Eq. (4.12), such choice requires $w^\alpha = 2S^\alpha{}_{\gamma\sigma} k^\gamma k^\sigma$. In the literature (see e.g. Refs. [82, 84]), there has also been interest in the case where $k^\alpha \nabla_\alpha k^\beta = 0$, which implies $w^\alpha = 0$. Unfortunately, in both cases, for our purposes, Eq. (4.21) still contains terms that can only be evaluated if we have full knowledge of the evolution of the metric and torsion tensors[3]. Therefore, for a completely general torsion tensor field, there seems to be no mathematically interesting way to construct a useful focus theorem based only on the Raychaudhuri equation. Notice that, although the auxiliary vector ξ is not completely determined by Eq. (4.13), we do not have enough freedom to remove all the problematic terms (whose signs we are not able to constraint *ab initio*). Moreover, requiring hypersurface orthogonality of the congruence associated with k (which we haven't done yet in this subsection) does not solve this problem.

Based on the discussion in previous paragraph, in the remaining of this section, we will consider the case when the torsion tensor field is completely anti-symmetric and the "acceleration" w is zero:

$$S_{\alpha\beta\sigma} = S_{[\alpha\beta\sigma]} \quad \text{and} \quad w = 0. \tag{4.22}$$

These assumptions allow us to describe both the cases $k^\alpha \nabla_\alpha k^\beta = 0$ and $k^\alpha \nabla^{(g)}_\alpha k^\beta = 0$ at once. We have then the following result:

Theorem 4.2. *Let (M, g, S) be a Lorentzian manifold of dimension $N \geq 3$ endowed with a totally anti-symmetric torsion tensor field S. Given a congruence of hypersurface orthogonal null curves in (M, g, S), let c represent the fiducial curve of the congruence, with tangent vector field k such that $\nabla_k k = 0$. A focal point of the congruence will form, for a finite value of the parameter of c, if the following two conditions are satisfied:*

(i) *At a point along c, the expansion of the congruence becomes negative;*

(ii) $R_{\alpha\beta} k^\alpha k^\beta + 2 S_{\alpha\beta\mu} S^{\alpha\beta\nu} k^\mu k_\nu \geq 0$.

Proof. Assuming, without loss of generality, that the fiducial curve is affinely parameterized by a parameter λ, given the premises of the theorem, the generalized Raychaudhuri equation (4.21) can be written as

$$\frac{d\theta}{d\lambda} = -R_{\alpha\beta} k^\alpha k^\beta - 2 S_{\alpha\beta\mu} S^{\alpha\beta\nu} k^\mu k_\nu - \left(\frac{1}{N-2} \theta^2 + \sigma_{\alpha\beta} \sigma^{\alpha\beta} \right) . \tag{4.23}$$

Repeating the reasoning of the proof of Theorem 4.1 we find

$$\frac{1}{\theta(\lambda)} \geq \frac{\lambda}{N-2} + \frac{1}{\theta_0} , \tag{4.24}$$

[3]Consider, for instance, the last terms in Eq. (4.21) such as $\xi^\gamma \nabla_\gamma k^\sigma$ and $k^\gamma \nabla_\gamma \xi^\sigma$.

where θ_0 is the initial value of θ. Hence, assuming the existence of a point along c where $\theta_0 < 0$, for a value $\lambda \leq (N-2)/|\theta_0|$, then $\theta \to -\infty$, i.e., a focal point of the congruence will form for a finite value of λ. $\square$

4.3 Singularity theorems in affine theories of gravity

The previous results concerned the formation of focal points of congruences of curves on Lorentzian manifolds. Nonetheless, the existence of such points does not necessarily imply the formation of singularities. In order to prove the existence of a singularity on the manifold, the formation of a focal point needs to be related to some kind of incompleteness of the manifold structure. To approach this problem there has been, at least, two lines of works (cf., e.g., Ref. [53]), in the sense that the occurrence of a focal point has been related either to the incompleteness of causal curves or, given a physical theory of gravitation, to the lack of regularity of some physical scalar invariant such as the energy density. In the former approach, metric geodesics are natural curves to consider from the geometric point of view, as they are curves of extremal length according to the metric tensor g. On the other hand, if one considers non-geodesic curves, it can be more feasible to follow the latter approach.

4.3.1 Geodesic curves

In Section 4.2, for the case of null congruences we considered that the torsion tensor field was totally anti-symmetric and the tangent vector field verified $\nabla_k k = 0$. In such case, the integral curves of the null vector k are metric geodesics, therefore, the results of the singularity theorems of Hawking and Penrose can be applied, assuming instead the conditions of Theorem 4.2. In fact, the argument also holds for time-like geodesics[4]. We summarize these observations as:

Lemma 4.3. *Given a congruence of time-like (resp. null) metric geodesics in (M, g, S), if the conditions of Theorem 4.1 (resp. Theorem 4.2) are verified, then the geodesics of the congruence are incomplete.*

The results of Lemma 4.3 imply, in particular, that the presence of torsion modifies the so-called *strong energy condition* of General Relativity, by including the explicit contribution of torsion in conditions 2 of Theorems 4.1 and 4.2. This observation was already behind the work of Trautman [25] (see also Ref. [26–29]) where it was found, in a particular model of the ECSK theory, that the presence of torsion could prevent the formation of singularities due to the violation of the modified strong energy condition.

[4] This was first noticed in Ref. [29], in the context of the ECSK theory.

4.3.2 Non-geodesic curves

An affine theory of gravity is characterized by field equations that relate the geometry of the manifold with the matter fields that permeate it. In general, at least one set of these field equations can be written in the form

$$f\left(g, S, \mathcal{T}, \text{other fields}\right) = 0\,, \tag{4.25}$$

where f is some functional, g represents the metric tensor, S the torsion tensor and $\mathcal{T}$ is a rank-2 tensor constructed from the matter fields which we call the *stress-energy tensor* and whose divergence verifies

$$\nabla_\beta \mathcal{T}^{\alpha\beta} = \Psi^\alpha\,, \tag{4.26}$$

for some vector field Ψ. Let us address the *other fields* part in Eq. (4.25). Here, we are thinking of theories like, for instance, scalar-tensor theories with a null torsion tensor field, where an extra scalar field non-minimally coupled to the geometry is considered and appears directly in the field equations. In those theories, it is still possible to write a metric stress-energy tensor for the matter fields so that, in the so-called Jordan frame, Eq. (4.26) is verified for an identically null Ψ (cf., e.g., Ref. [87]).

Considering an affine theory of gravitation, we use the following definition:[5]

Definition 4.1. A space-time (M, g, S) is said to be physically singular if there exists at least one frame (i.e., an observer) for which the stress-energy tensor $\mathcal{T}$ is not well defined in (M, g, S), for any coordinate system.

We remark that, classically, a stress-energy tensor consists of a rank-2 tensor with real entries. So, if independently of the coordinate system, some components of $\mathcal{T}$ either diverge or take complex values, this would correspond to a non-admissible physical object, in the sense that an observer is no longer able to make predictions using that physical theory.

Now, large classes of physical particles follow accelerated time-like curves and the effects of the acceleration might not be negligible in the extreme gravitational field regime. From a physical point of view, a congruence of curves represents the paths of matter particles, as such, a focal point of a congruence represents the focusing of a matter field. It is then natural to connect the formation of a focal point with the stress-energy tensor that characterizes the matter fluid that permeates the space-time. A prototype of this idea was first used by Raychaudhuri [11] and, independently, by Komar [12], and later formalized by Senovilla [53] for the case of congruences of time-like geodesics in torsionless space-times, in the context of the theory of General Relativity. Here, we extend the results of Ref. [53] to time-like congruences in space-times endowed with a metric compatible, affine connection, in the context of affine theories of gravitation.

Using the results in Eq. (2.50), for a congruence of time-like curves with tangent vector u, we can decom-

[5] In his review [53], Senovilla calls *matter singularity* to a singularity in the energy density term of the stress-energy tensor.

pose $\mathcal{T}$ as

$$\mathcal{T}_{\alpha\beta} = \mu\, u_\alpha u_\beta + p\, h_{\alpha\beta} + q_{1\alpha} u_\beta + u_\alpha q_{2\beta} + \pi_{\alpha\beta} + \varepsilon_{\alpha\beta}{}^{\gamma} m_\gamma\,, \tag{4.27}$$

where the components are given in Eq. (2.51). Before proceeding, notice that the components in Eq. (4.27) are defined covariantly such that, if it is found that some component is not defined at some point of the manifold, this indeed means that an observer co-moving with the fiducial curve of the congruence is unable to define a stress-energy tensor (in contrast with just an ill choice of coordinates).

Now, given Definition 4.1, we obtain the following result:

Theorem 4.3. *Consider a space-time (M, g, S) in the context of any affine theory of gravitation satisfying the conditions of Theorem 4.1, the conservation of energy $u^\alpha \nabla_\beta \mathcal{T}_\alpha{}^\beta = 0$ and containing either*

1. *A fluid with $\mathcal{T}_{\alpha\beta} = \mu u_\alpha u_\beta + p\, h_{\alpha\beta}$ and $p = k\mu^\gamma$, where $k \in \mathbb{R}_0^+$, $\gamma \in \mathbb{R}\setminus\{0\}$ and, at some instant, $\tau = \tau_0$, $\theta(\tau_0) < 0$ and $\mu(\tau_0) > 0$;*

2. *A scalar field with $\mathcal{T}_{\alpha\beta} = (\nabla_\alpha\psi)(\nabla_\beta\psi) - \frac{1}{2} g_{\alpha\beta}(\nabla_\mu\psi)(\nabla^\mu\psi) - g_{\alpha\beta} V(\psi)$, admitting level surfaces, such that its gradient $\nabla_\alpha\psi$ is time-like,*

then, the space-time will become physically singular at, or before, the formation of a focal point.

Corollary 4.1. *In case 1, if the fluid has $\gamma > 1$, then the space-time becomes physically singular before the formation of a focal point.*

Corollary 4.2. *Theorem 4.3 applies, in particular, to any minimally coupled affine theory of gravity without torsion, to scalar-tensor theories written in the Jordan frame in manifolds without torsion and to the ECSK theory for a torsion given by $S_{\alpha\beta\gamma} = S_{\alpha\beta} u_\gamma + u_{[\alpha} X_{\beta]} u_\gamma$.*

Proof. From Eq. (2.70) and setting $W_{\alpha\beta} = 0$, $u^\alpha \nabla_\beta \mathcal{T}_\alpha{}^\beta = 0$ can be written as

$$\dot{\mu} + \theta\,(\mu + p) + \pi^{\alpha\beta}\sigma_{\alpha\beta} - \varepsilon^{\alpha\beta\gamma} m_\gamma \omega_{\alpha\beta} + q_1^\alpha a_\alpha + \nabla_\alpha q_2^\alpha = 0\,, \tag{4.28}$$

where θ, $\sigma_{\alpha\beta}$ and $\omega_{\alpha\beta}$ are given by Eqs. (4.3) – (4.5) and $\dot{\mu} := u^\alpha \nabla_\alpha \mu$.

Case 1:

Considering the particular case of a fluid characterized by a stress-energy tensor whose non-zero components in the decomposition (4.27) are μ and p, then Eq. (4.28) reduces to

$$\dot{\mu} + \theta\,(\mu + p) = 0\,. \tag{4.29}$$

Assuming $p = k\mu^\gamma$, with $k \in \mathbb{R}_0^+$ and $\gamma \in \mathbb{R} \setminus \{0\}$, we may integrate Eq. (4.29) along the fiducial curve of the congruence, c, to find

$$\begin{cases} |\mu| = |\mu_0| \, e^{-(1+k)\int_{\tau_0}^{\tau} \theta(t)dt} \,, & \text{for } \gamma = 1, \\ |\mu^{1-\gamma} + k| = \left(\mu_0^{1-\gamma} + k\right) \, e^{-(1-\gamma)\int_{\tau_0}^{\tau} \theta(t)dt} \,, & \text{for } \gamma \neq 1, \end{cases} \tag{4.30}$$

where τ is an affine parameterization of c, $\mu \equiv \mu\left(c\left(\tau\right)\right)$ and $\mu_0 \equiv \mu\left(c\left(\tau_0\right)\right)$,

Assuming that the conditions of the focus Theorem 4.1 are verified, depending on the values of γ, we get different types of behavior. If $\gamma \leq 1$, taking the limit of Eq. (4.30), we find

$$\lim_{\tau \to \tau_f^-} |\mu| = +\infty \text{ , for } \gamma \leq 1 \,, \tag{4.31}$$

i.e., the space-time becomes physically singular at the formation of the focal point. On the other hand, if $\gamma > 1$ and assuming $\mu_0 > 0$, from Eq. (4.30) we find that $\mu^{1-\gamma}$ is a strictly decreasing function and

$$\lim_{\tau \to \tau_a^-} \frac{1}{\mu^{\gamma-1}} = 0 \,, \tag{4.32}$$

for some $\tau_a < \tau_f$. So, before the formation of a focal point we have that μ will diverge, hence, the space-time becomes physically singular. This proves part 1 of the theorem.

Case 2:

Consider now a real scalar field ψ characterized by a stress-energy tensor of the form

$$\mathcal{T}_{\alpha\beta} = \left(\nabla_\alpha \psi\right)\left(\nabla_\beta \psi\right) - \frac{1}{2} g_{\alpha\beta}\left(\nabla_\mu \psi\right)\left(\nabla^\mu \psi\right) - g_{\alpha\beta} V\left(\psi\right) \,, \tag{4.33}$$

where $V(\psi)$ represents a potential functional. Using the definitions in (2.51) and defining $\dot{\psi} := u^\alpha \nabla_\alpha \psi$, we find

$$\mu = \dot{\psi}^2 + \frac{1}{2}\left(\nabla_\mu \psi\right)\left(\nabla^\mu \psi\right) + V\left(\psi\right) \,, \tag{4.34}$$

$$p = \frac{1}{N-1}\mu + \frac{N-2}{2\left(N-1\right)}\left(\nabla_\mu \psi\right)\left(\nabla^\mu \psi\right) - \frac{N}{N-1} V\left(\psi\right) \,, \tag{4.35}$$

$$q_\alpha = q_{1\alpha} = q_{2\alpha} = -h^\mu{}_\alpha \left(\nabla_\mu \psi\right) \dot{\psi} \,, \tag{4.36}$$

$$\pi_{\alpha\beta} = \left[h^\mu{}_{(\alpha} h_{\beta)}{}^\nu - \frac{h_{\alpha\beta}}{N-1} h^{\mu\nu}\right]\left(\nabla_\mu \psi\right)\left(\nabla_\nu \psi\right) \,, \tag{4.37}$$

$$m_\alpha = 0 \,. \tag{4.38}$$

Let us now impose that the scalar field admits level surfaces such that its gradient, $\nabla_\alpha \psi$, is time-like. Since the choice of u is arbitrary, we can impose without loss of generality that u^α is given by $\nabla^\alpha \psi$, so that,

$h^{\beta}{}_{\alpha}\nabla_{\beta}\psi = 0$. This, then, implies that the quantities q_{α} and $\pi_{\alpha\beta}$ are identically null. Moreover,

$$\mu = \frac{1}{2}\dot{\psi}^2 + V(\psi) = \frac{1}{2} + V(\psi) \, , \tag{4.39}$$

$$p = \frac{1}{2}\dot{\psi}^2 - V(\psi) = \frac{1}{2} - V(\psi) \, , \tag{4.40}$$

where the last equalities follow from setting $u^{\alpha} := \nabla^{\alpha}\psi$ and $u^{\alpha}u_{\alpha} = -1.$

From Eq. (4.29), and integrating along the fiducial curve of the congruence, c, we find

$$\mu(\tau) = \mu(\tau_0) + \left| \int_{\tau_0}^{\tau} \theta dt \right| \, , \tag{4.41}$$

where τ is an affine parameterization of c. Assuming that the premises of the focus Theorem 4.1 are verified, the function θ will diverge to negative infinite when τ goes to some finite value τ_f. Then

$$\lim_{\tau \to \tau_f^-} \mu = +\infty \, , \tag{4.42}$$

that is, the space-time becomes physically singular at $\tau = \tau_f$. □

4.3.3 Singularities in the Einstein-Cartan-Sciama-Kibble theory

Theorem 4.3, relating a focal point to the formation of a singularity is rather general in the sense that it is applicable to a wide class of theories of gravity. However, we had to specify the particular equation of state, either for a perfect fluid or a scalar field, in order to draw those results. One reason for this is that we have just used a general conservation law for $\mathcal{T}$ and did not use the field equations of a particular theory of gravitation. Indeed, the field equations could enable us to directly relate the geometric conditions in the focus Theorem 4.1 with constraints on the matter fields.

In this section, we will show that provided a geometric theory of gravitation, it is possible to extend the results of Theorem 4.3 for perfect fluids without the need to impose an equation of state. In particular, we will consider the ECSK theory [22–24] defined in a Lorentzian manifold of dimension $N = 4$, characterized by the following field equations

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R + g_{\alpha\beta}\Lambda = 8\pi\mathcal{T}_{\alpha\beta} \, , \tag{4.43}$$

$$S^{\alpha\beta\gamma} + 2g^{\gamma[\alpha}S^{\beta]}{}_{\mu}{}^{\mu} = -8\pi\Delta^{\alpha\beta\gamma} \, , \tag{4.44}$$

where $\mathcal{T}_{\alpha\beta}$ represents the canonical stress-energy tensor, $\Delta^{\alpha\beta\mu}$ the intrinsic hypermomentum and Λ the cosmological constant. Moreover, the matter fields are constrained by the conservation laws [88]

$$\nabla_\beta \mathcal{T}_\alpha{}^\beta = 2S_{\alpha\mu\nu}\mathcal{T}^{\nu\mu} - \frac{1}{4\pi}S_{\alpha\mu}{}^\mu\Lambda + \frac{1}{8\pi}\left(S_{\alpha\mu}{}^\mu R - S^{\mu\nu\sigma}R_{\alpha\sigma\mu\nu}\right) , \tag{4.45}$$

found from the field equations and the second Bianchi identity, Eq. (2.13). Using Eq. (4.44) we can introduce the intrinsic hypermomentum tensor in the above conservation laws and recover the result in Ref. [21] (see also Ref. [86, 89] for similar results derived in the context of Poincaré gauge gravity theories [90–93]).

Computing the component $u^\alpha \nabla_\beta \mathcal{T}_\alpha{}^\beta$, we find

$$\begin{aligned} \dot{\mu} + \left(\theta - W^\alpha_\alpha\right)(\mu + p) + q^\alpha_1 a_\alpha + \nabla_\alpha q^\alpha_2 + \pi^{\alpha\beta}\left(\sigma_{\alpha\beta} - W_{(\alpha\beta)}\right) + m^{\alpha\beta}\left(\omega_{\beta\alpha} - W_{[\beta\alpha]}\right) \\ = -u^\alpha\left\{2S_{\alpha\mu\nu}\mathcal{T}^{\nu\mu} - \frac{1}{4\pi}S_{\alpha\mu}{}^\mu\Lambda + \frac{1}{8\pi}\left(S_{\alpha\mu}{}^\mu R - S^{\mu\nu\sigma}R_{\alpha\sigma\mu\nu}\right)\right\} . \end{aligned} \tag{4.46}$$

Now, from Eq. (4.46), it is easy to verify that for a generic torsion tensor field we will have on the RHS terms that depend on the Weyl tensor, making it very difficult to infer, in general, the behavior of the relevant quantities without actually solving the field equations[6]. Nevertheless, we are able to prove the following result:

Theorem 4.4. *Consider a space-time $(\mathcal{M}, g, S)$ in the context of the Einstein-Cartan-Sciama-Kibble theory of gravitation, satisfying the conditions of Theorem 4.1. The space-time will become physically singular at, or before, the formation of a focal point, if the following three conditions are verified:*

(i) *the torsion tensor field can be written as $S_{\alpha\beta}{}^\gamma = S_{\alpha\beta}u^\gamma + u_{[\alpha}X_{\beta]}u^\gamma$;*

(ii) *the matter fluid is characterized by a canonical stress-energy tensor of the form $\mathcal{T}_{\alpha\beta} = \mu u_\alpha u_\beta + p\, h_{\alpha\beta}$, and at some instant, $\tau = \tau_0$, $\theta(\tau_0) < 0 \wedge \mu(\tau_0) \geq 0$;*

(iii) *the cosmological constant, Λ, verifies $\Lambda + S_{\alpha\beta}S^{\alpha\beta} \geq 0$.*

Proof. We start by noting that, using condition (i), where the components $S_{\alpha\beta}$ and X_α were defined in Eq. (2.44), Eq. (4.46) reads

$$\dot{\mu} + \theta(\mu + p) + q^\alpha_1 a_\alpha + q^\alpha_2 X_\alpha + \nabla_\alpha q^\alpha_2 + \pi^{\alpha\beta}\sigma_{\alpha\beta} + \varepsilon^{\alpha\beta\gamma}m_\gamma\omega_{\beta\alpha} = 0 , \tag{4.47}$$

and using condition (ii), this equation reduces simply to Eq. (4.29). Now, consider h as the induced metric on each 3-hypersurface, orthogonal to u, with components h_{ab} in a local coordinate system. It is easy to show that the definition (4.3) for the expansion scalar is equivalent to

$$\theta = h^{ab}\mathcal{L}_u h_{ab} = \frac{1}{\sqrt{\det(h)}}\partial_u\sqrt{\det(h)} . \tag{4.48}$$

[6] This is also the case, even if we consider a torsion tensor field where $W_{(\alpha\beta)} = 0$, as in Theorem 4.1.

Defining $L := \left(\sqrt{\det\left(h\right)}\right)^{\frac{1}{3}}$ along the integral curve of u, we can rewrite Eq. (4.29) as

$$\frac{d}{dL}\left(\mu L^2\right) + L\left(\mu + 3p\right) = 0\,. \tag{4.49}$$

The second condition in Theorem 4.1, $R_{\alpha\beta}u^\alpha u^\beta \geq \left(W_{[\alpha\beta]} + S_{\alpha\beta}\right)S^{\alpha\beta}$, in the considered ECSK setup, implies

$$4\pi\left(3p + \mu\right) - \Lambda \geq S_{\alpha\beta}S^{\alpha\beta}\,. \tag{4.50}$$

Integrating Eq. (4.49) along the integral curve of u yields

$$\mu = \frac{\left(\mu L^2\right)|_{\tau_0}}{L^2} + \frac{1}{L^2}\int_L^{L_0} l(\mu + 3p)\,dl\,, \tag{4.51}$$

where τ_0 represents the value of the affine parameter of the integral curve of u, defined so that $L\left(\tau_0\right) = L_0 > 0$. Taking the limit when $L \to 0$, i.e., at the formation of a focal point, and assuming $\Lambda + S_{\alpha\beta}S^{\alpha\beta} \geq 0$ then, from Eq. (4.50), we find that the second term in the previous equation is positive therefore, $\mu \to +\infty$. □

4.4 Conclusions and further discussion

The results in this chapter show how the presence of torsion affects the formation of focal points of a congruence of curves on a Lorentzian manifold. In particular, we found how the torsion field explicitly appears in the conditions of the focus theorems which, in turn, can be used to study singularity formation. It is important to highlight that in the derivation of the focus theorems we used the definitions of the kinematical quantities as defined in Eqs. (2.30) and (2.36). As we showed, these quantities correctly describe the geometrical — hence, physical — kinematical behavior of a congruence of curves [59, 83–85]. Therefore, if it is found that the expansion scalar θ diverges to minus infinity at some point of the manifold, this indeed represents the formation of a caustic of the congruence. Some authors [63, 94] choose to use different definitions for the kinematical quantities, assuming the same expression for the deviation tensor B, Eq. (2.29), as for torsionless space-times. These quantities, though, do not have a geometrical meaning, therefore, for a general torsion tensor field, they can not be used by themselves to assert the dynamical behavior of the congruence.

Using the newly derived focus theorems, we extended the Raychaudhuri-Komar singularity theorem for a wide class of affine theories of gravitation in space-times endowed with a metric compatible, affine connection. Moreover, we discussed both the cases of metric geodesics and accelerated curves. This is an important point from the physical point of view. Take, for instance, the somewhat simpler case of the theory of General Relativity. In that case, single-pole massive particles will follow geodesics of the space-time [95, 96]. However, this is only verified if no gravitational radiation due to the particle's motion is taken into consideration [96, 97]. Moreover, not only gravitational radiation will spoil the geodesic motion of single-pole particles but also charged particles in curved space-times will be accelerated, as described by the deWitt-Brehme-Hobbs equation [98, 99]. Regarding the motion of massive particles in a space-time with non-vanishing torsion, the situation is even more complex. As shown in Refs. [86, 89], for an affine theory of gravitation minimally coupled to the matter fields, only single-pole massive particles with null intrinsic hypermomentum will follow geodesics of the space-time. However, this result was found by assuming that no radiation is emitted. It is then expected that, analogously to the case of space-times with null torsion, self-force effects will give rise to non-zero acceleration. The results in this chapter address these cases by showing how the presence of acceleration and torsion affects the formation of focal points of a congruence and space-time singularities.

Chapter 5

Influence of intrinsic spin in the formation of singularities for inhomogeneous effective dust space-times

5.1 Introduction

In Chapter 4 we studied how the presence of the torsion tensor field and acceleration affects the formation of singularities by extending the Raychaudhuri-Komar singularity theorem to a wide class of affine theories of gravitation. Albeit the generality of Theorems 4.3 and 4.4, providing sufficient conditions for the formation of singularities, if the premises of the theorems are not verified we can not predict, in general, the evolution of a physical system undergoing gravitational collapse without actually solving the field equations of a given theory. Trying to find necessary conditions for the formation of singularities due to gravitational collapse is, however, a very different problem: not only different geometric theories of gravitation are characterized by distinct relations between the matter sources and the geometry of the space-time, leading, ultimately, to distinct descriptions of gravitational collapse [100–111]; but also, the matter models that describe the matter source may play a pivotal role in the evolution of the system [112–116].

In this chapter we will focus on the study of the gravitational collapse of a perfect fluid composed of fermionic particles with non-null intrinsic spin in the ECSK theory [23, 24]. As explained in Chapter 1, the formalism provided by the Einstein-Cartan theory makes it possible to introduce intrinsic spin degrees of freedom in a geometric theory of gravitation by relating the torsion tensor field with the intrinsic spin of matter. This model is specially interesting because it encapsulates quantum corrections in a semi-classical limit [21, 117–119], providing a way to infer if quantum effects may avoid the formation of singularities. Indeed, in the framework of loop quantum gravity, it was found that for a universe permeated by a scalar field, quantum corrections modify the classical Friedmann equation and a cosmological singularity can be avoided [120, 121]. Moreover, some interesting but somewhat forgotten results from the 1970s [25–28] suggest that the introduction of spin in spatially homogeneous anisotropic models may prevent the formation of singularities, depending on the degree of isotropy of the collapsing space-time. More recently, the collapse process of a spatially homogeneous and isotropic spin fluid has been studied in [78], where it was shown that there exists a competition between fluid and spin source parameters, so that the collapse end state is determined via the dynamics of these source terms and both singular and non-singular solutions could be obtained. Moreover, the effects of spin have been shown to come into play in the cosmological context, where the torsion of the space-time generated by the intrinsic spin of fermionic particles induces gravitational repulsion in the early universe, where extremely high densities are present, possibly avoiding the formation of space-time singularities [122].

Nonetheless, spatially homogeneous and isotropic models represent extremely idealized scenarios. It would then be interesting to explore this debate further by considering inhomogeneous and anisotropic space-times and study how the symmetries of the space-time affect the evolution of gravitational collapse. Indeed, with the inclusion of intrinsic spin degrees of freedom, the situation can be more subtle: the effects of spin contribute, in a way, as a repulsive potential, therefore, collapsing solutions might not even exist for inhomogeneous space-times. Furthermore, even if some form of the energy conditions — arising from Theorem 4.4 — is verified, singularities may not be the end result of gravitational collapse. In this and the following chapters we will consider that the space-time manifold is of dimension 4.

5.2 Inclusion of spin effects in General Relativity

5.2.1 The Einstein-Cartan-Sciama-Kibble theory in brief

To include the spin effects in the theory of General Relativity we shall follow the semi-classical approach provided by ECSK theory and relate the spin of the matter fields with the space-time torsion tensor field. So, let us briefly review the theory and see how spin effects can be included in GR.

We start by recalling some of the expressions of Chapter 2. In the case of a manifold endowed with a metric compatible, affine connection, the covariant derivative of a vector field u is given by

$$\nabla_\alpha u^\beta = \partial_\alpha u^\beta + C^\beta_{\alpha\sigma} u^\sigma \,, \tag{5.1}$$

where the connection $C^\gamma_{\alpha\beta}$ can be written as a combination of the torsion tensor $S_{\alpha\beta}{}^\gamma$, Eq. (2.2), plus the usual metric Christoffel symbols $\Gamma^\gamma_{\alpha\beta}$, Eq. (2.4), that is

$$C^\gamma_{\alpha\beta} = \Gamma^\gamma_{\alpha\beta} + K_{\alpha\beta}{}^\gamma \,, \tag{5.2}$$

where the tensor field,

$$K_{\alpha\beta}{}^\gamma \equiv S_{\alpha\beta}{}^\gamma + S^\gamma{}_{\alpha\beta} - S_\beta{}^\gamma{}_\alpha \,, \tag{5.3}$$

is called *contorsion tensor*, Eq. (2.5). From the definition (2.9),

$$R_{\alpha\beta\gamma}{}^{\rho} = \partial_\beta C^\rho_{\alpha\gamma} - \partial_\alpha C^\rho_{\beta\gamma} + C^\rho_{\beta\sigma} C^\sigma_{\alpha\gamma} - C^\rho_{\alpha\sigma} C^\sigma_{\beta\gamma} \,, \tag{5.4}$$

and Eq. (5.2), we can write the Riemann tensor in terms of the metric Riemann tensor, with components $\widetilde{R}_{\alpha\beta\gamma}{}^{\rho}$ and defined just in terms of the Christoffel symbols, and the contorsion tensor as

$$R_{\alpha\beta\gamma}{}^{\rho} = \widetilde{R}_{\alpha\beta\gamma}{}^{\rho} + 2\widetilde{\nabla}_{[\beta} K_{\alpha]\gamma}{}^{\rho} + 2K_{[\beta|\sigma}{}^{\rho} K_{|\alpha]\gamma}{}^{\sigma} \,, \tag{5.5}$$

where the square brackets denote anti-symmetrization in the indicated indexes and $\widetilde{\nabla}_\alpha$ represents the covariant derivative of a tensor quantity using only the metric connection $\Gamma^\gamma_{\alpha\beta}$. Contracting Eq. (5.5) gives the following expression for the Ricci tensor

$$R_{\alpha\gamma} = \widetilde{R}_{\alpha\gamma} + 2\widetilde{\nabla}_{[\beta} K_{\alpha]\gamma}{}^{\beta} + 2K_{[\beta|\sigma}{}^{\beta} K_{|\alpha]\gamma}{}^{\sigma} \,. \tag{5.6}$$

Now, the field equations of the ECSK theory follow from the Einstein-Hilbert action[1]

$$\begin{aligned} A &= \frac{1}{16\pi} \int \sqrt{-g}\, g^{\alpha\gamma} R_{\alpha\gamma}\, d^4x + \int \sqrt{-g}\, \mathcal{L}_{\text{Matter}}\, d^4x = \\ &= \frac{1}{16\pi} \int \sqrt{-g}\, g^{\alpha\gamma} \left\{ \widetilde{R}_{\alpha\gamma} + K_{\beta\sigma}{}^{\beta} K_{\alpha\gamma}{}^{\sigma} - K_{\alpha\sigma}{}^{\beta} K_{\beta\gamma}{}^{\sigma} \right\} d^4x + \int \sqrt{-g}\, \mathcal{L}_{\text{Matter}}\, d^4x \,, \end{aligned} \tag{5.7}$$

where $\mathcal{L}_{\text{Matter}}$ represents the matter Lagrangian density and in the second equality we have omitted the total derivative terms. Following the Palatini's approach [123–126], where the action is varied independently with respect to the space-time metric and the connection, we find two sets of field equations. Making the variation of the action in Eq. (5.7) with respect to the contorsion tensor we find

$$S^{\alpha\beta\mu} + 2g^{\mu[\alpha} S^{\beta]} = -8\pi \Delta^{\alpha\beta\mu} \,, \tag{5.8}$$

where $S_\alpha \equiv S_{\mu\alpha}{}^{\alpha}$ and the quantity

$$\Delta^{\alpha\beta\mu} = \frac{1}{\sqrt{-g}} \frac{\delta\left(\sqrt{-g}\mathcal{L}_{\text{Matter}}\right)}{\delta K_{\mu\beta\alpha}} \,, \tag{5.9}$$

is usually called the *intrinsic hypermomentum*, as it encapsulates all the information of the microscopic structure of the matter that composes the fluid [127–129]. Notice that, in Eq. (5.8), the torsion tensor field is not a dynamical field, in the sense that the LHS contains no derivatives of the torsion tensor and indeed appears as a purely algebraic equation. As such, in vacuum, where the matter Lagrangian density $\mathcal{L}_{\text{Matter}}$ is null, from

[1] In this chapter we assume a null cosmological constant.

Eq. (5.8), we see that the torsion tensor is identically null so, all solutions of the ECSK theory in vacuum are also solutions of the theory of General Relativity.

Varying the action in Eq. (5.7) with respect to the metric tensor yields the modified Einstein field equations [29]

$$\begin{aligned}\widetilde{G}_{\mu\nu} - 4S_\sigma K_{\mu\nu}{}^\sigma - 2g^{\beta\delta}K_{\mu\sigma\delta}K_{\beta\nu}{}^\sigma + 2g_{\mu\nu}S_\sigma S^\sigma + \\ + \frac{1}{2}g_{\mu\nu}K_{\rho\sigma\delta}K^{\delta\rho\sigma} - 4S_\mu S_\nu - g^{\sigma\rho}K_{\beta\rho\nu}K_{\sigma\mu}{}^\beta = 8\pi\, T_{\mu\nu}\,,\end{aligned} \tag{5.10}$$

where $\widetilde{G}_{\mu\nu} \equiv \widetilde{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\widetilde{R}$ represents the metric Einstein tensor, defined just using the metric connection, and

$$T_{\mu\nu} = g_{\mu\nu}\mathcal{L}_{\text{Matter}} - 2\frac{\delta\mathcal{L}_{\text{Matter}}}{\delta g^{\mu\nu}}\,, \tag{5.11}$$

is the metric stress-energy tensor of the matter fluid[2].

5.2.2 The Weyssenhoff fluid

To relate the torsion tensor field with the intrinsic spin of matter particles, we shall consider a perfect fluid with non-null intrinsic spin degrees of freedom that permeates a region of the space-time. A description of such fluid is given by the Weyssenhoff model [130]. Although the Weyssenhoff fluid was first proposed as a phenomenological theory, a variational theory of an ideal spinning fluid has been developed in Refs. [131, 132] providing a Lagrangian for the Weyssenhoff semi-classical spin fluid which can, in turn, be used to compute the corresponding hypermomentum and stress-energy tensors through Eqs. (5.9) and (5.11). It is then found that the Weyssenhoff fluid is characterized by the following relation between the intrinsic hypermomentum tensor and spin [131, 132]

$$\Delta^{\mu\nu\alpha} = -\frac{1}{8\pi}\Delta^{\mu\nu}u^\alpha\,, \tag{5.12}$$

where u^α is the fluid's 4-velocity and $\Delta^{\mu\nu}$ is the anti-symmetric spin density tensor which, in turn, verifies the following constraint

$$\Delta^{\alpha\beta}u_\beta = 0\,, \tag{5.13}$$

known as the *Frenkel condition* [133, 134]. Substituting Eqs. (5.12) and (5.13) in Eq. (5.8), we find that the torsion tensor is related to the intrinsic spin of matter as

$$S^{\alpha\beta\mu} = \Delta^{\alpha\beta}u^\mu\,, \tag{5.14}$$

[2] In this chapter we will use a different notation for the stress-energy tensor (cf. Eq. (4.43)). The reason is to avoid confusing the metric stress-energy tensor in Eq. (5.11), with the canonical stress-energy tensor, Eq. (4.43). Nevertheless, the results in Section 4.3 are equally applicable, in particular Theorem 4.4.

making it clear that, in this model, spin constitutes the effective source of the torsion field. Equations (5.13) and (5.14) allow us then to rewrite Eq. (5.10) as

$$
\begin{aligned}
\widetilde{G}_{\mu\nu} + \Delta_{\alpha\beta}\Delta^{\alpha\beta} u_\mu u_\nu + \frac{1}{2} g_{\mu\nu} \Delta_{\alpha\beta}\Delta^{\alpha\beta} \\
- 2\Delta_{\mu\sigma}\Delta_{\nu}{}^{\sigma} = 8\pi\, T_{\mu\nu}\,.
\end{aligned}
\tag{5.15}
$$

Now, the spin tensor $\Delta^{\mu\nu}$ and the stress-energy tensor $T_{\mu\nu}$ that appear in Eqs. (5.12) - (5.15) refer to the spin, energy and momentum of microscopic particles. However, we are interested in studying the macroscopic behavior of an ideal spinning fluid. Therefore, to find the field equations that describe a macroscopic gravitational field due to a Weyssenhoff fluid, we will compute a space-time average of the tensors $\Delta^{\mu\nu}$ and $T_{\mu\nu}$ over an element of volume of the fluid, respectively $\langle \Delta^{\mu\nu} \rangle$ and $\langle T_{\mu\nu} \rangle$.

Assuming that the spinning fluid is composed of microscopic particles with randomly oriented spin, we get the averaged quantities [135]

$$
\langle \Delta^{\mu\nu} \rangle = 0\,, \tag{5.16}
$$

$$
\langle \Delta^{\alpha\beta}\Delta_{\alpha\beta} \rangle = 32\pi^2 s^2\,, \tag{5.17}
$$

$$
\langle \Delta_{\mu}{}^{\alpha}\Delta_{\nu\alpha} \rangle = \frac{32}{3}\pi^2 \left(g_{\mu\nu} + u_\mu u_\nu \right) s^2\,, \tag{5.18}
$$

where s^2 represents the average of the square of the spin density of the fluid. Moreover, from the Lagrangian density given in Ref. [132] for a locally irrotational fluid, we find

$$
\langle T_{\mu\nu} \rangle = -\frac{8\pi}{3} u_\mu u_\nu s^2 - \frac{8\pi}{3} g_{\mu\nu} s^2 + (\mu + p)\, u_\mu u_\nu + p\, g_{\mu\nu}\,, \tag{5.19}
$$

where μ and p represent the energy density and pressure of the fluid, respectively. Substituting Eqs. (5.16) - (5.19) in the average version of Eq. (5.15) yields

$$
\widetilde{G}_{\mu\nu} = 8\pi\, \left[\left(\mu + p - 4\pi s^2 \right) u_\mu u_\nu + \left(p - 2\pi s^2 \right)\, g_{\mu\nu} \right]\,. \tag{5.20}
$$

Notice that the field equations in Eq. (5.20), for a zero-vorticity Weyssenhoff fluid in the ECSK theory, are equivalent to those in GR for a perfect fluid with additional contributions from the spin to the energy density and pressure. Therefore, at the level of the metric, a perfect fluid composed of particles with non-null intrinsic spin in the ECSK theory can be, classically, described by the field equations of the theory of GR assuming that the fluid is modeled by the effective stress-energy tensor

$$
T_{\mu\nu} = \left(\mu_{\text{eff}} + p_{\text{eff}} \right) u_\mu u_\nu + p_{\text{eff}}\, g_{\mu\nu}\,, \tag{5.21}
$$

with

$$\mu_{\text{eff}} = \mu - 2\pi s^2 \qquad \text{and} \qquad p_{\text{eff}} = p - 2\pi s^2, \tag{5.22}$$

where we have omitted the angular brackets. Nevertheless, one should bear in mind that all matter related quantities are to be considered as averages. Moreover, for the remaining of this chapter we will also omit the tilde to indicate tensor quantities computed only using the symmetric part of the connection since, as was shown, we can effectively treat the problem at hand using the field equations of the theory of General Relativity for an effective stress-energy tensor.

5.2.3 The stress-energy tensor for effective uncharged spinning dust

We shall now restrict our attention to the case where the fluid's pressure varies in such a way that it cancels the contribution coming from spin effects, so that the whole fluid effectively behaves as dust, i.e.,

$$p_{\text{eff}} = p - 2\pi s^2 = 0\,, \tag{5.23}$$

so that,

$$T^{\mu\nu}_{\text{eff}} = \mu_{\text{eff}} u^\mu u^\nu\,, \tag{5.24}$$

where μ_{eff} represents the effective energy density of the matter fluid.

Now, let us consider that the matter is composed of N species of uncharged fermionic particles with mass m_i and spin $s_i = \pm\hbar/2$ and assume that the interactions between the microscopic constituents of the fluid are negligible. Clearly, the distribution of spin and mass are related to each other. The particle number density for each species is given by [29]

$$n_i = \frac{\mu_i}{m_i} = \left(\frac{4s_i^2}{\hbar^2}\right)^{\frac{1}{2}}\,, \tag{5.25}$$

where μ_i and s_i^2 represent, respectively, the averaged energy density and the average of the squared spin density of a single species, such that

$$\mu_{i\text{eff}} = \mu_i - 2\pi s_i^2\,, \qquad \text{and} \qquad \mu_{\text{eff}} = \sum_{i=1}^{N} \mu_{i\text{eff}}\,. \tag{5.26}$$

Using Eq. (5.25) in Eqs. (5.22) and (5.26) , we find that for each species

$$\mu_{i\text{eff}} = \mu_i \left(1 - \frac{\mu_i}{\bar{\mu}_i}\right)\,, \qquad p_{i\text{eff}} = p_i - \frac{\mu_i^2}{\bar{\mu}_i}\,, \tag{5.27}$$

where the total pressure of the perfect fluid is formally given by

$$p_{\text{eff}} = \sum_{i=1}^{N} p_{i\text{eff}} \,, \tag{5.28}$$

and

$$\bar{\mu}_i = \frac{2m_i^2}{\pi\hbar^2} \tag{5.29}$$

represents a critical mass-density which sets a scale at which the spin effects have to be taken into account [29].

In the perfect fluid approximation, the various constituents of the matter are non-interacting. Therefore, the total pressure of the fluid will be null if the various terms in Eq. (5.28) are null. Substituting Eq. (5.27) in Eq. (5.23), we find that the constituents of the fluid are characterized by an equation of state of the form

$$p_i = \frac{1}{\bar{\mu}_i}\mu_i^2 \,, \tag{5.30}$$

that is, a polytrope of order 2.

5.3 Spin effects in the gravitational evolution

To study the influence of spin in singularity formation, we will analyze the evolution of an uncharged effective dust fluid with non-null spin degrees of freedom composed only of fermionic particles.

When considering the case of the gravitational collapse of a compact object in astrophysics, we consider a model of Oppenheimer-Snyder type [10] having a collapsing interior given by the Szekeres metric [136, 137], which is inhomogeneous, matched to a vacuum exterior. Examples of such models were shown to exist in [101, 138–142]. Whereas, when considering a cosmological model, we shall assume that the coordinates in the Szekeres metric are globally defined and that the universe can be either initially expanding or contracting. In this case, part of the problem will be to figure out if, during evolution, there can be a bounce in the collapsing phase or a turning point, followed by re-collapse, in the expanding phase.

The Szekeres space-time represents the solutions of the Einstein field equations (EFE) for a space-time permeated by irrotational dust whose line element can be written in the form

$$ds^2 = -d\tau^2 + e^{\alpha(\tau,r,p,q)}dr^2 + e^{\beta(\tau,r,p,q)}\left(dp^2 + dq^2\right) \,, \tag{5.31}$$

where α and β are C^2-functions of the coordinates $\{\tau, r, p, q\}$. This metric has no Killing vectors, in general [143]. Historically, the Szekeres models are split in two classes: one that generalizes the Lemaître-Tolman-Bondi (LTB) metric [6, 144, 145] and another that generalizes the Kantowski-Sachs (KS) metric [146]. We shall treat them separately.

5.3.1 Szekeres space-times: LTB-like

The LTB-like Szekeres models are characterized by a line element of the form [136, 137]

$$ds^2 = -d\tau^2 + \frac{\left(R' - \frac{RE'}{E}\right)^2}{\epsilon + f\left(r\right)} dr^2 + \frac{R^2}{E^2}\left(dp^2 + dq^2\right) \, , \tag{5.32}$$

where the prime indicates the derivative with respect to r, $\epsilon = \{-1, 0, 1\}$, $E = E\left(r, p, q\right)$ and $f\left(r\right)$ are arbitrary C^2-functions such that $f\left(r\right) + \epsilon > 0$, while the function $R = R\left(\tau, r\right)$ satisfies the Friedmann like equation

$$\dot{R}^2\left(\tau, r\right) = f\left(r\right) + \frac{2M\left(r\right)}{R\left(\tau, r\right)} \, , \tag{5.33}$$

with the overdot denoting the derivative with respect to the time coordinate τ and $M = M\left(r\right)$ is another arbitrary function.

Given the line element in Eq. (5.32) and Eq. (5.24) we find, from the EFE, the following relation between the energy density of the effective dust source and the functions that describe the geometry of the space-time

$$8\pi \, \mu_{\text{eff}} = 2\frac{EM' - 3ME'}{R^2\left(ER' - RE'\right)} \, . \tag{5.34}$$

This expression can be re-written as

$$\mu_{\text{eff}}\left(\tau, r, p, q\right) = \frac{r^2 \, F\left(r, p, q\right)}{R^2\left(\tau, r\right) W\left(\tau, r, p, q\right)} \, , \tag{5.35}$$

where

$$r^2 F\left(r, p, q\right) = \frac{1}{4\pi}\left[E\left(r, p, q\right) M'\left(r\right) - 3M\left(r\right) E'\left(r, p, q\right)\right] \, , \tag{5.36}$$

and

$$W\left(\tau, r, p, q\right) = E\left(r, p, q\right) R'\left(\tau, r\right) - R\left(\tau, r\right) E'\left(r, p, q\right) \, , \tag{5.37}$$

so, $f\left(r\right)$, $M\left(r\right)$ and $E\left(r, p, q\right)$ will be the free initial functions that characterize the space-time.

5.3.1.1 Regularity conditions and the influence of spin in singularity formation

Since we are interested in studying the influence of spin in the formation of curvature singularities, in addition to the previous assumptions we shall consider some further requirements on the initial regularity of the space-time:

Assumption 5.1. *At the initial time $\tau = \tau_0$:*

(i) $R(\tau_0, r)$ *is an increasing function of the coordinate* r.

(ii) *the space-time is non-singular.*

(iii) *for every triplet* (r, p, q)*, we assume* $W(\tau_0, r, p, q)$ *and* $\mu_{\text{eff}}(\tau_0, r, p, q)$ *to be non-null.*

Under these assumptions, we can set by convention

$$R(\tau_0, r) = r. \tag{5.38}$$

When it exists, it is also useful to introduce $\tau_c(r) > \tau_0$, defined as the value of the time coordinate at which the function R of the shell with coordinate r goes to zero.

Under our assumptions, from Eqs. (5.35), and in the coordinate system defined by Eq. (5.38), it is straightforward to see that the function $F(r, p, q)$ must be finite and non-null. This leads us to conclude that a necessary condition for the divergence of $\mu_{\text{eff}}(\tau, r, p, q)$ for a given (r, p, q) is that either $R(\tau, r)$ or $W(\tau, r, p, q)$ go to zero at some instant of time $\tau_S > \tau_0$. The former is associated with a shell-focusing singularity, whereas the latter represents the formation of a shell-crossing singularity [147].

Moreover, since the functions $R(\tau, r)$ and $W(\tau, r, p, q)$ are continuous in the time coordinate τ before the formation of a singularity, for each triad (r, p, q), the effective energy density function $\mu_{\text{eff}}(\tau, r, p, q)$ is also a continuous function in the coordinate τ. We are then able to prove

Lemma 5.1. *Under our assumptions, if in a continuous gravitational collapse there exists a curve $\tau = \tau_s(r) > \tau_0$ such that for each (r, p, q) within the matter cloud either* $\lim_{\tau\to\tau_s(r)^-} R(\tau, r) = 0$ *or* $\lim_{\tau\to\tau_s(r)^-} W(\tau, r, p, q) = 0$ *then,* $\lim_{\tau\to\tau_s(r)^-} \mu_{\text{eff}}(\tau, r, p, q) = -\infty$.

Proof. We split the proof in two parts:

1. First, we prove that for each triad (r, p, q), if either $\lim_{\tau\to\tau_s(r)^-} R(\tau, r) \to 0$ or $\lim_{\tau\to\tau_s(r)^-} W(\tau, r, p, q) \to 0$ for some instant $\tau_s > \tau_0$, then we have that $\lim_{\tau\to\tau_s(r)^-} \mu_{\text{eff}}(\tau, r, p, q) \to \pm\infty$. As was discussed previously, a necessary condition for divergent μ_{eff} is that either $W(\tau, r, p, q)$ or $R(\tau, r)$ go to zero at some instant $\tau = \tau_s$. In Appendix B it is shown that if R is a real function, these conditions are also sufficient for the divergence of μ_{eff}, in particular, even in the case where $\lim_{\tau\to\tau_s(r)^-} W \to \infty$, the limit $\lim_{\tau\to\tau_s(r)^-} R^2 R' \to 0$ is always verified, hence from (5.37) $\lim_{\tau\to\tau_s(r)^-} R^2 W \to 0$ and from (5.35) $\lim_{\tau\to\tau_s(r)^-} \mu_{\text{eff}}(\tau, r, p, q) \to \pm\infty$, as $F(r, p, q)$ is finite. Therefore, since $F(r, p, q)$ takes finite values, from Eq. (5.35), fixing (r, p, q), if either $\lim_{\tau\to\tau_s(r)^-} R(\tau, r) \to 0$ or $\lim_{\tau\to\tau_s(r)^-} W(\tau, r, p, q) \to 0$ for some instant $\tau_s > \tau_0$, then we have $\lim_{\tau\to\tau_s(r)^-} \mu_{\text{eff}}(\tau, r, p, q) \to \pm\infty$.

2. We now show that we can only have $\mu_{\text{eff}}(\tau, r, p, q) \to -\infty$. For a dust spacetime composed only of fermions, the effective mass-energy density is given by the sum in Eq. (5.26) and it takes infinite values if, at least, one term of the sum, $\mu_{i\text{eff}}$, is infinite. Although we shall restrain ourselves from imposing any of the usual

energy conditions, we shall consider that if any of the parameters that characterize the fluid take complex values at any point of the space-time, the solution is unphysical. Therefore, for $\mu_i \in \mathbb{R}\setminus\{0\}$, from Eq. (5.27), we find that each $\mu_{i\text{eff}}$ may only tend to $-\infty$, hence, for each (r,p,q), $\lim_{\tau\to\tau_s(r)^-} \mu_{\text{eff}}(\tau,r,p,q) \to -\infty$. □

Now, from Lemma 5.1, we get the following result:

Theorem 5.1. *Given a Szekeres space-time with line element (5.32) permeated by an uncharged perfect fluid composed only of fermionic particles, characterized by an equation of state such that, the fluid effectively behaves as dust, if Assumption 5.1 is verified and sign $(F(r,p,q)) =$ sign $(W(\tau_0,r,p,q))$ then, a curvature singularity will not form.*

Proof. The proof follows from Lemma 5.1 and the continuity of the functions R, W and μ_{eff}.

Consider that sign $(F(r,p,q)) =$ sign $(W(\tau_0,r,p,q))$. Let us start by showing that the function $W(\tau,r,p,q)$ will not be zero for any $\tau \in\,]\tau_0,\tau_c(r)]$. We argue by contradiction.

Assume that there exists, for each r, an instant $\tau_1(r) \in\,]\tau_0,\tau_c(r)]$ such that $\lim_{\tau\to\tau_1^-} W(\tau_1,r,p,q) = 0$. Then, from Lemma 5.1, $\lim_{\tau\to\tau_1^-} \mu_{\text{eff}}(\tau,r,p,q) \to -\infty$. From our assumptions, $W(\tau,r,p,q)$ and $R(\tau,r)$ are continuous functions in the τ coordinate for all $\tau \in\,]\tau_0,\tau_1[$ therefore, for that domain, the effective mass-energy density, μ_{eff}, is a continuous function in the coordinate τ, concluding that there exists a $\tau_2 \in\,]\tau_0,\tau_1[$ such that $\mu_{\text{eff}}(\tau_2,r,p,q) < 0$. On the other hand, continuity of $W(\tau,r,p,q)$ implies that the sign of $W(\tau_2,r,p,q)$ must be equal to the sign of $W(\tau_0,r,p,q)$. Therefore, since $R(\tau_2,r) \in \mathbb{R}$ and sign $(F(r,p,q)) =$ sign $(W(\tau_2,r,p,q))$, we conclude, from Eq. (5.35), $\mu_{\text{eff}}(\tau_2,r,p,q) > 0$, contradicting what was shown before. Hence, $W(\tau,r,p,q)$ will not be zero for any $\tau \in\,]\tau_0,\tau_c(r)]$.

The case when the function $R(\tau,r)$ goes to zero before the function $W(\tau,r,p,q)$ also can not occur since, from Eq. (5.35), Lemma 5.1 would be violated. □

Remark 5.1. The condition sign $(F(r,p,q)) =$ sign $(W(\tau_0,r,p,q))$ of Theorem 5.1 is equivalent to consider that the effective stress-energy tensor (5.24) verifies the weak energy condition at the initial time τ_0.

Remark 5.2. To clarify the statement of the Theorem 5.1, let us consider the simpler case of an effective dust fluid composed of only one type of fermions. In this case, from Eqs. (5.27) and (5.35) we can solved for μ, such that

$$\mu(r,p,q) = \frac{\bar{\mu}}{2}\left(1 \pm \sqrt{1 - \frac{4r^2F}{\bar{\mu}R^2W}}\right), \tag{5.39}$$

hence, given sign $(F(r,p,q)) =$ sign $(W(\tau_0,r,p,q))$, from the results of Appendix B, if, for each r, either $W(\tau,r,p,q)$ or $R(\tau,r)$ go to zero for some instant $\tau \in\,]\tau_0,\tau_c(r)]$, the energy density of the fluid will take complex values hence, it is considered unphysical.

5.3.2 Szekeres space-times: KS-like

The KS-like Szekeres models are characterized by the general line element [136]

$$ds^2 = -d\tau^2 + \left[X(\tau, r) - R(\tau)\frac{E_1(r, p, q)}{E_0(p, q)}\right]^2 dr^2 + \frac{R(\tau)^2}{E_0(p, q)^2}\left(dp^2 + dq^2\right) , \tag{5.40}$$

where the function $R(\tau)$ verifies

$$\dot{R}(\tau)^2 = \frac{2M}{R(\tau)} + k_1 , \tag{5.41}$$

with M being an arbitrary constant to be given as initial data and the constant k_1 is determined by

$$(\partial_p E_0)^2 + (\partial_q E_0)^2 - E_0\left(\partial_p^2 E_0 + \partial_q^2 E_0\right) = k_1 . \tag{5.42}$$

Moreover, the function $X(\tau, r)$ is a solution of following equation:

$$X\,\ddot{R} + \dot{R}\,\dot{X} + R\,\ddot{X} = \frac{k_2(r)}{2} , \tag{5.43}$$

with

$$2\partial_q E_0 \partial_q E_1 + 2\partial_p E_0 \partial_p E_1 - \left(\partial_q^2 E_0 + \partial_p^2 E_0\right) E_1 - E_0\left(\partial_p^2 E_1 + \partial_q^2 E_1\right) = k_2(r) , \tag{5.44}$$

where we have omitted some functional dependencies to avoid saturating the notation.

From the EFE, the effective energy density of the spinning cloud verifies

$$\mu_{\text{eff}} = \frac{G(r, p, q)}{R^2(\tau)\,H(\tau, r, p, q)} , \tag{5.45}$$

where

$$H(\tau, r, p, q) = X(\tau, r)\ E_0(p, q) - R(\tau)\ E_1(r, p, q) . \tag{5.46}$$

Such space-times are completely determined by the initial functions $E_0(p, q)$and $E_1(r, p, q)$ and the constant M.

5.3.2.1 Regularity conditions and the influence of spin in singularity formation

As in the previous section, we shall consider some further constraints on the initial regularity of the space-time:

Assumption 5.2. *At the initial time, $\tau = \tau_0$:*

(i) *the space-time is non-singular.*

(ii) *for every triplet (r, p, q), we assume $H(\tau_0, r, p, q)$ and $\mu_{\text{eff}}(\tau_0, r, p, q)$ to be non-null.*

Now, given the above initial regularity constraints and looking at Eqs. (5.45) and (5.46), as well as to the solutions in Appendix B, it can be seen that, following the same reasoning applied in the proofs of subsection 5.3.1.1, we have the following result:

Theorem 5.2. *Given a Szekeres space-time with line element (5.40) filled with an uncharged perfect fluid composed only of fermionic particles, characterized by an equation of state such that, the fluid effectively behaves as dust, if Assumption 5.2 are verified and sign* $(G(r,p,q)) =$ *sign* $(H(\tau_0,r,p,q))$*, then a curvature singularity will not form.*

5.4 Special Cases

In this section, we shall discuss the effects of spin in the formation of singularities for some particular cases of the Szekeres space-times, where the analysis of the previous section can be extended.

5.4.1 Lemaître-Tolman-Bondi space-time

The Lemaître-Tolman-Bondi space-time [6, 144, 145] is a solution of the EFE for a spherically symmetric neutral dust source characterized, in co-moving coordinates, by the line element

$$ds^2 = -d\tau^2 + \frac{R'(\tau,r)^2}{1+f(r)}dr^2 + R(\tau,r)^2\, d\Omega^2\,, \tag{5.47}$$

where $d\Omega^2 \equiv d\theta^2 + \sin^2\theta\, d\varphi^2$ represents the line element of the unit 2-sphere. The function $R(\tau,r)$ verifies Eq. (5.33) and represents the circumference radius at an instant τ and radial coordinate r and $f(r) > -1$ is an arbitrary C^2-function of the radial coordinate. The LTB metric can be found from the Szekeres solution, Eq. (5.32), by setting $E^2(r,p,q) = (1+p^2+q^2)/4$ and $\epsilon = 1$.

From the EFE, we find that Eq. (5.35) is re-written for the LTB space-time as

$$\mu_{\text{eff}}(\tau,r) = \frac{r^2\, F(r)}{R^2 R'}\,, \tag{5.48}$$

where $F(r)$ is another arbitary function of r and it is related to the function $M(r)$, Eq. (5.33), by $M'(r) = 4\pi r^2\, F(r)$, or

$$M(r) = 4\pi \int_0^r F(\mathrm{r})\ \mathrm{r}^2\, d\mathrm{r}. \tag{5.49}$$

It is worth remarking that, in this case, the function $M(r)$ represents the mass contained within a shell with coordinate r (cf., e.g., Ref. [148]).

In the case of an LTB space-time, the function $W(\tau,r) \equiv R'(\tau,r)$ and in the coordinate system defined by Eq. (5.38) we have $W(\tau_0,r) = +1$. Therefore, for the LTB space-time imposing $F(r) > 0$ is equivalent to impose $M'(r) > 0$.

The result in Theorem 5.1, albeit important, does not provide an answer in the cases where $F(r) < 0$. One might think that such cases are unphysical since the effective null energy condition[3] would be violated. There are, however, known physical phenomena that seem to violate the energy conditions (see e.g. Ref. [149]). Moreover, cases of collapsing space-times with negative mass have been studied several times in the past (see e.g. Ref. [150, 151] and references therein).

Let us then consider that $M'(r)$ takes negative values for all r. In this case we have, from Eq. (5.49), that the mass function $M(r)$ is negative for all r. Assuming that $f(r) \geq -2M/r$, otherwise Eq. (5.33) has no real solutions, it is clear, from Eq. (5.33), that one of two scenarios will occur: either $\dot{R}(\tau_0, r) > 0$, in which case the effective dust matter will continue to expand indefinitely; or $\dot{R}(\tau_0, r) < 0$ where at some instant of time, say τ_b, $\dot{R}(\tau_b, r)$ will go to zero. From Eq. (5.33), it is easy to show that $\ddot{R}(\tau, r)$ is always positive, therefore, at τ_b, the function R will have a minimum. Hence, the system will bounce back and start to expand indefinitely.

We summarize these conclusions in the next proposition, which softens the conditions of Theorem 5.1 for the case of LTB:

Proposition 5.1. *Given an LTB space-time permeated by an uncharged, collapsing perfect fluid, composed only of fermionic particles, characterized by an equation of state such that, the fluid effectively behaves as dust, if Assumptions 5.1 are verified and sign* $(M'(r))$ *is the same for all* r *within the space-time, the circumferential radius function* $R(\tau, r)$ *will not go to zero.*

Aside the simpler case where $M'(r) < 0$ for all r, there may be configurations where there are regions of the space-time with positive effective energy density and other regions with negative effective energy density, however, it is easy to show that such configurations are solutions of the Einstein field equations only if surface layers, separating the different regions, are present. Such cases shall not be considered here.

To conclude this subsection, we note that if $M'(r)$ takes negative values, shell-crossing singularities may occur.

5.4.1.1 The evolution of the collapse

In the previous subsection it was shown that, under our assumptions, and if the sign of the function $M'(r)$ is the same for all r, an uncharged effective dust cloud composed only of fermionic particles, in an LTB space-time will not form a shell-focusing singularity. Then, for $\dot{R}(\tau_0, r) < 0$, and assuming that no shell-crossing singularities occur for sign $(M'(r)) < 0$, there are only three possibilities for the behavior of the uncharged effective dust cloud:

(I) there are no global in time solutions of the EFE for a collapsing uncharged effective dust matter;

[3] See Section 7.5 for further details.

(II) the gravitational collapse will lead to a bounce of the matter cloud;

(III) the gravitational collapse of the matter cloud will settle in a stable configuration.

As we shall see below, the third scenario will never occur, and the first and second cases may occur depending on the initial data:

(I) Consider the initial data $M(r) > 0$ and $f(r) \geq 0$ or $M(r) = 0$ and $f(r) > 0$. In both cases, from Eq. (5.33), the function $\dot{R}$ can not go to zero, that is, for a given r, $R(\tau, r)$ is either a strictly decreasing or a strictly increasing function of τ. However, if R is a decreasing function in the τ coordinate, it will eventually go to zero, violating the result of Proposition 5.1. Hence, in these cases, the only possible physical solution is when the matter expands indefinitely.

(II) Another possibility is the case where $M(r) < 0$ and $f(r) > 0$, with $2M(r)/R(\tau_0, r) + f(r) > 0$ for a given r within the matter cloud. In this case, from Eq. (5.33), there are two possible behaviors depending on the initial value of $\dot{R}(\tau)$: either $\dot{R}(\tau_0, r) > 0$, in which case the effective dust cloud will continue to expand indefinitely; or $\dot{R}(\tau_0, r) < 0$, where at some instant of time, τ_b, $\dot{R}(\tau_b, r)$ will go to zero. From Eq. (5.33), it is easy to show that $\ddot{R}(\tau, r)$ is always positive. Therefore, at τ_b, the function R will attain a minimum, different from zero, concluding that the system will bounce and start to expand from then on. Note that this solution is characterized by $M(r) < 0$, therefore the weak (hence, the strong) energy condition is not verified and there is no inconsistency with the result in Theorem 4.4.

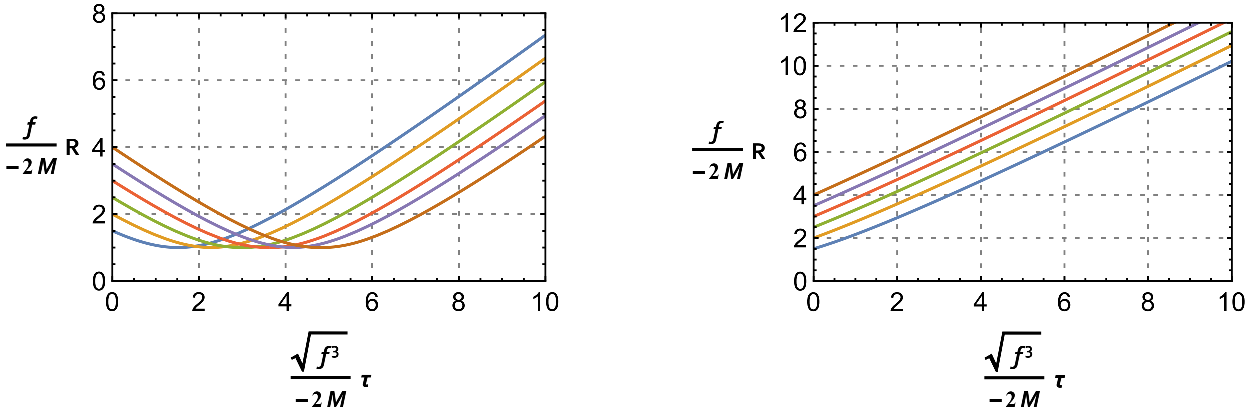
(III) Yet another possibility is the case where $M(r) > 0$ and $f(r) < 0$, with $2M(r)/R(\tau_0, r) + f(r) > 0$: if initially $\dot{R}(\tau_0, r) < 0$, then a singularity will eventually form; on the other hand, if $\dot{R}(\tau_0, r) > 0$, $\dot{R}$ will go to zero at some $\tau \in]\tau_0, +\infty[$, however, from Eq. (5.33) we see that $\ddot{R}(\tau, r)$ is always negative in this case, hence, the system will cease to expand further and start to collapse, eventually forming a singularity. In both scenarios the result in Theorem 5.1 is violated, hence, such initial data does not correspond to a physical solution. This is because $M(r) > 0$ implies that at some region $F(r) > 0$, therefore, if a singularity occurs the energy density of the constituents of the fluid would have to take complex values.

As examples, in Figures 1 and 2, we show the behavior of R as a function of τ for various fixed values of the coordinate r, depending on the initial sign of $\dot{R}(\tau_0, r)$, found by numerically solving Eq. (5.33) in particular cases when $M(r) < 0$ and $f(r) > 0$ and $M(r) > 0$ and $f(r) < 0$, respectively.

5.4.2 Friedmann-Lemaître-Robertson-Walker space-time

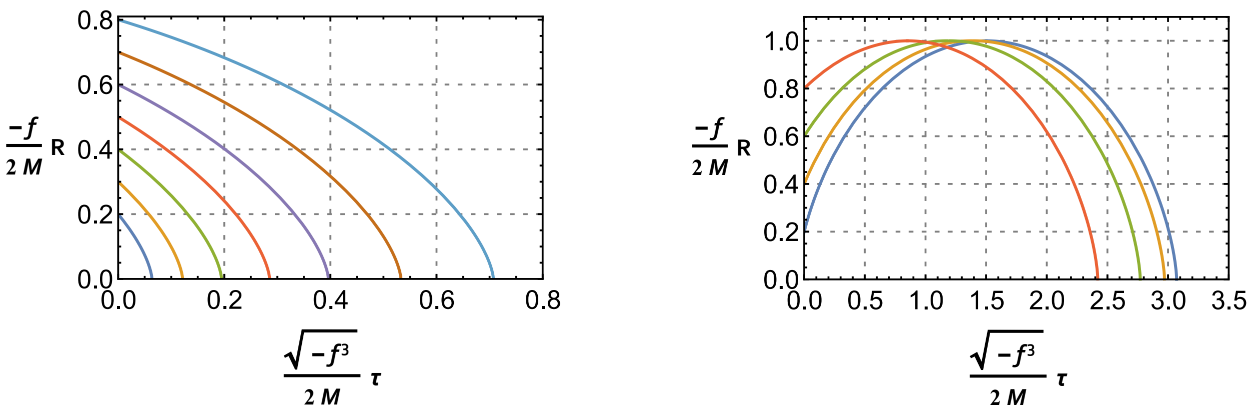
Another example of notable interest is the Friedmann-Lemaître-Robertson-Walker model [3–9], where the space-time, sourced by spatially homogeneous and isotropic dust, is characterized by the line element

$$ds^2 = -d\tau^2 + a(\tau)^2 \left(\frac{dr^2}{1 + k\, r^2} + r^2 d\Omega^2 \right) , \tag{5.50}$$



(a) (b)

Figure 1: Behavior of the function $R(\tau, r)$ for various fixed values of the coordinate r, depending on the initial value of $\dot{R}$, in the case of $M(r) < 0$ and $f(r) > 0$. Panel (a) At the initial time $\dot{R}(\tau_0, r) < 0$. Panel (b) At the initial time $\dot{R}(\tau_0, r) > 0$.



(a) (b)

Figure 2: Behavior of the function $R(\tau, r)$ for various values of the coordinate r depending of the initial value of $\dot{R}$ in the case of $M(r) > 0$ and $f(r) < 0$. Panel (a) At the initial time $\dot{R}(\tau_0, r) < 0$. Panel (b) At the initial time $\dot{R}(\tau_0, r) > 0$.

where $a(\tau)$ is the scale factor and $k = \{-1, 0, 1\}$. This solution is a particular case of the Szekeres line element, Eq. (5.32), by setting $E^2(r, p, q) = (1 + p^2 + q^2)/4$, $\epsilon = 1$, $f(r) \equiv k\,r^2$ and $R(\tau, r) \equiv a(\tau)\,r$. We then have for an effective dust cloud

$$\mu_{\text{eff}}(\tau) = \frac{F_0}{a^3(\tau)}\,, \tag{5.51}$$

with F_0 being a non-zero constant, and $a(\tau)$ satisfying the well-known Friedmann equation [3]

$$\dot{a}^2(\tau) = \frac{2M(r)}{r^3\,a(\tau)} + k\,, \tag{5.52}$$

where

$$M(r) = M_0 r^3, \qquad \text{with} \qquad M_0 = \frac{4\pi}{3} F_0 = \frac{4\pi}{3} a(\tau_0)^3\,\mu_{\text{eff}}(\tau_0)\ . \tag{5.53}$$

Moreover, this space-time is itself a particular case of the LTB space-time studied in the previous subsection. Therefore, all the previous results are valid for this particular case.

From Proposition 5.1 we know that independently of the sign of M_0, or equivalently F_0, a singularity does not occur. As discussed in subsection 5.4.1.1, the actual evolution of the space-time depends of the values of the constants k and M_0. More concretely,

(i) for $M_0 > 0 \wedge k \geq 0$, the space-time will expand indefinitely;

(ii) for $M_0 < 0 \wedge k > 0$, depending on whether $\dot{a}(\tau_0) > 0$ or $\dot{a}(\tau_0) < 0$, the space-time will either expand indefinitely or start by a collapsing phase which is followed by a nonsingular bounce and then entering an expanding phase, respectively;

in all other cases, from Eq. (5.52), we find that the functions $a(\tau)$ or $\mu(\tau)$ will take complex values, hence, the solutions to the Friedmann equations are considered unphysical (see also Section 6.2 of Ref. [152]).

By comparison with our setup, we recall that in Ref. [25], the effects of spin in singularity formation was studied by considering a dust cloud composed of fermionic particles in a FLRW space-time whose spins were assumed to be aligned in a given preferred spatial direction. In that case, the field equations can be explicitly solved and a closed form expression for the scale factor $a(\tau)$ was found, showing explicitly that there is a minimum positive value for $a(\tau)$ thus, concluding that a singularity does not form. Moreover, cosmological as well as astrophysical consequences of introducing spin and torsion in gravitation has been studied in Refs. [78, 109, 153]. In Ref. [153], a closed homogeneous and isotropic universe filled with fermionic matter has been considered and it was shown that the effects of spin-torsion coupling produces a gravitational repulsion in the early universe, preventing the formation of cosmological singularity. In Refs [78, 109], the effects of spin on the dynamics of collapse for a closed [78] and flat [109] FLRW backgrounds has been studied. It was shown that, under certain conditions, the formation of space-time singularities can be avoided through a non-singular bounce, which can either be hidden behind a horizon or visible to external observers, depending on the initial radius or mass of the collapsing body.

All the previous mentioned cases differ from our setup since we assume that spins are randomly oriented and the fluid effectively behaves as dust. However, as was shown above, our conclusions are similar.

5.4.3 Senovilla-Vera space-time

As a special inhomogeneous case of KS-like Szekeres space-times, we study the Senovilla-Vera space-time [154] characterized by a line element of the form

$$ds^2 = -d\tau^2 + dr^2 + r^2 d\varphi^2 + \left(\beta + \frac{\tau^2 + r^2}{\gamma}\right)^2 dz^2 \,, \tag{5.54}$$

where $\gamma \in \mathbb{R}\setminus\{0\}$, $\beta \in \mathbb{R}_{\geq 0}$, $r > 0$, $z \in \,]-\infty, +\infty[$ and $\varphi \in [0, 2\pi[$. A space-time exterior to (5.54) was found in Ref. [142]. Assuming that the space-time is permeated by effective dust, we find that the effective energy density is given by

$$8\pi\mu_{\text{eff}}\left(\tau, r\right) = -\frac{4}{\beta\gamma + r^2 + \tau^2} \,. \tag{5.55}$$

In the cases where $\gamma > 0 \wedge \beta \geq 0$ or $\gamma < 0 \wedge \gamma\beta + r^2 > 0$, it is clear that there are no collapsing solutions. Let us then consider the cases where $\gamma < 0 \wedge \gamma\beta + r^2 < 0$. Setting $R = 1$, $E_0 = 1$, $X = \tau^2/\gamma$ and $E_1 = -\left(r^2 + \beta\gamma\right)/\gamma$ in Eq. (5.40), we recover the line element (5.54). The functions G and H, in Eqs. (5.45) and (5.46), are in this case

$$H\left(\tau, r\right) = \beta + \frac{\tau^2 + r^2}{\gamma} \,, \qquad G = -\frac{1}{2\pi\gamma} \,. \tag{5.56}$$

Now, for $\gamma < 0$ we see that the function $H\left(\tau, r\right)$ is a strictly decreasing function of the coordinate τ. It is then clear that, for $\gamma\beta + r^2 < 0$, such model is not a solution of the EFE for effective dust, since this would imply the formation of a curvature singularity when $\tau = \sqrt{\gamma\beta + r^2}$, violating Proposition 5.2.

We can see this more clearly by considering that the fluid is composed of only one type of fermionic particles. In that case, from Eqs. (5.27) and (5.55) we have that the energy density of the perfect fluid is given by

$$\mu\left(\tau, r\right) = \frac{\bar{\mu}}{2}\left[1 \pm \sqrt{1 + \frac{2}{\bar{\mu}\pi\left(\gamma\beta + \tau^2 + r^2\right)}}\right] \,, \tag{5.57}$$

where $\bar{\mu}_i$ is given by Eq. (5.29) and the discriminant is assumed to be non-negative. From this relation, we see that in the case of $\gamma < 0 \wedge \gamma\beta + r^2 < 0$, when $\tau > \sqrt{\beta|\gamma| - r^2 - \frac{2}{\bar{\mu}\pi}}$, the mass-energy density of the fluid will take complex values, that is, such solution is unphysical.

5.4.4 LRS Bianchi type I space-time

As a final particular case, we will study a locally rotationally symmetric (LRS) Bianchi type I space-time with line element

$$ds^2 = -d\tau^2 + A(\tau)^2 \, dx^2 + B(\tau)^2 \left(dy^2 + dz^2\right), \tag{5.58}$$

which is a spatially homogeneous particular case of the KS-like Szekeres metric. Interestingly, this space-time was used to study the influence of spin in singularity formation by assuming that the space-time is filled with a dust fluid (not effective dust) with non-null spin, such that, the spins of the dust particles are aligned along a preferred direction [26, 27].

In line with our previous examples, we assume an effective dust fluid and, in that case, from the EFE we find

$$A(\tau) = \frac{c}{(b-\tau)^{\frac{1}{3}}} + (b-\tau)^{\frac{2}{3}} \ , \tag{5.59}$$

$$B(\tau) = (b-\tau)^{\frac{2}{3}} \ , \tag{5.60}$$

with b and c being integration constants, and

$$8\pi \, \mu_{\text{eff}}(\tau) = \frac{4}{3\Gamma(\tau)^3} \, , \tag{5.61}$$

where

$$\Gamma(\tau)^3 = A(t)\, B(t)^2 = (c+b-t)(b-t) \ . \tag{5.62}$$

We note that the case $c = 0$ corresponds to the flat FLRW metric.

Now, Eqs. (5.58), (5.59) and (5.60) can be found from Eq. (5.40) by setting $E_0 = E_1 = 1$, $R(\tau) = B(\tau)$ and $X(\tau) = A(\tau) + B(\tau)$, hence

$$H(\tau) = A(\tau) = \frac{c}{(b-\tau)^{\frac{1}{3}}} + (b-\tau)^{\frac{2}{3}} \ , \tag{5.63}$$

and the function $G = 1/6\pi$. To apply the results found in the previous section, at the initial time, for simplicity $\tau = 0$, $H(0)$ must be positive (to match the sign of G) and finite, that is

$$\begin{cases} c > -b & \text{, for } b > 0 \\ c < -b & \text{, for } b < 0 \end{cases} . \tag{5.64}$$

Therefore, if the constraints in Eq. (5.64) are verified, Proposition 5.2 tells us that a shell-focusing or shell-crossing singularity will not be formed. As in the previous subsection, this can be readily verified: In the case $b < 0$ this conclusion is trivial, since $A(\tau)$ and $B(\tau)$ will never be zero; On the other hand, for $b > 0$, assuming that the fluid is composed of only one type of fermionic particles, the energy density is given by

$$\mu\left(\tau\right)=\frac{\bar{\mu}}{2}\left[1\pm\sqrt{1-\frac{2}{3\pi\bar{\mu}\,\Gamma\left(\tau\right)^{3}}}\,\right]\ , \tag{5.65}$$

so that, as $\tau\to b$ or $\tau\to b+c$ (whichever occurs first), the energy density will take complex values, and the resulting solution is unphysical. Let us also remark that if Eq. (5.64) is not verified but $\Gamma\left(0\right)\neq0$ then,

$$\begin{cases}\lim_{\tau\to b}\Gamma\left(\tau\right)\to-\infty & \text{, for } b>0\ \wedge\ c<-b\\ \lim_{\tau\to c+b}\Gamma\left(\tau\right)\to-\infty & \text{, for } b<0\ \wedge\ c>-b\end{cases}\ , \tag{5.66}$$

and all solutions of the EFE are real and a singularity will form.

5.5 Concluding remarks

We have considered models of gravitational collapse of inhomogeneous and anisotropic (effective) dust fluid on a space-time described by a Szekeres metric. We have found that, under certain conditions on the initial data, the formation of a singularity may be avoided due to the presence of spin. Comparing our results with those in the literature for spatially homogeneous space-times, it was shown that not only the geometry of the space-time, but also the equation of state of the fluid, play a pivotal role in the evolution of the space-time and singularity formation. Moreover, it was shown that even if the effective stress-energy tensor of the spinning fluid verifies the strong energy condition, a singularity can be avoided. Notice that this is inline with the results of Chapter 4 since, the cases considered in this chapter where the strong energy condition is verified, the presence of intrinsic spin prevents the formation of trapped surfaces — regions where the expansion scalar is negative.

Some particular cases of the Szekeres model were considered in order to either extend the previous results or show explicitly the evolution of the effective dust space-time. The results found for the various cases are summarized in Table 1.

The model of an effective dust fluid represents a critical case, providing but a first step towards a deeper study of this important question: In what conditions may spin effects prevent the formation of singularities? We expect that deviations from the critical case will give rise to a broader variety of dynamics and outcomes of gravitational collapse, including the formation of black holes or naked singularities [151, 155, 156].

Particular solution	**Parameters**	**Possible behavior**		**Other**
LTB	$\{M(r)\,,\,f(r)\}$	$M(r) > 0 \wedge\, f(r) \geq 0$	The matter cloud will expand indefinitely.	If $M'(r) < 0$, a shell-crossing singularity may form.
		$M(r) = 0 \wedge\, f(r) > 0$		
		$M(r) < 0\, \wedge\, f(r) > 0$	If $\dot{R}(\tau_0, r) \geq 0$, the matter cloud will expand indefinitely. If $\dot{R}(\tau_0, r) < 0$, the matter cloud will collapse, bounce with $R(\tau, r) \neq 0$ and expand indefinitely.	
Senovilla-Vera	$\{\beta,\,\gamma\}$	$\gamma > 0 \wedge \beta \geq 0$	The matter cloud will expand indefinitely.	
		$\gamma < 0 \wedge \gamma\beta + r^2 > 0$		
LRS Bianchi type I	$\{b,\,c\}$	$b < 0\, \wedge\, c + b < 0$	The matter cloud will expand indefinitely.	
		$b > 0\, \wedge\, c + b < 0$	A singularity will form in finite time.	
		$b < 0\, \wedge\, c + b > 0$		

Table 1: Evolution of an effective dust matter on various space-times, particular solutions of the Szekeres metric. The space-times are assumed to verify the premises of Theorem 5.1 or Theorem 5.2. All unmentioned possible initial data either lead to unphysical solutions or correspond to the trivial static case.

Part III

Compact objects

Chapter 6

Static compact objects in Einstein-Cartan theory

6.1 Introduction

Compact objects, in particular neutron stars, represent one of richest environments to probe fundamental physics due their extreme gravitational fields, densities and the state of the matter that composes them, especially, at the core. The recent detection of gravitational waves due to the coalescing of two orbiting neutron stars [157] opened a new window to study their tidal deformations, allowing the study of the properties of the matter fields that compose this kind of objects. Nonetheless, the usage of neutron stars as a physics laboratory is only possible if we have a deep knowledge of their properties. In particular, it is important to understand how the intrinsic spin[1] of the fermionic matter particles affects the behavior of such bodies.

In an astrophysical context, the effects of intrinsic spin were first considered when Chandrasekhar [158] established the maximum mass an ideal white dwarf could hold due to the electron degeneracy pressure, before it underwent continuous gravitational collapse (see also Ref. [159] for the rotating case). In the subsequent years, similar limits relying on the Pauli exclusion principle were proposed for other types of compact objects, namely neutron stars, showing that the intrinsic spin of matter particles markedly influences astrophysical objects (cf., e.g., Ref. [160]). Nevertheless, the way in which the presence of intrinsic spin affects the properties of astrophysical bodies, remains largely unknown.

In an affine theory of gravity, the gravitational field is represented by the geometry of the space-time which is, in turn, determined by the energy and momentum of the matter fields. Mathematically, all classical matter properties are described by a stress-energy tensor that acts as a source in the field equations. Since spin can be considered as an intrinsic angular momentum of the matter particles, one would expect that this property could also be encoded in an stress-energy tensor. However, in GR it appears immediately clear that there is

[1]We should remark that here, and in the following, the word "spin" will be used exclusively to represent the quantum spin of the particles that source the gravitational field equations. In no case the word spin will be associated to any form of rotation of the compact objects we will analyze.

no obvious way to introduce the intrinsic spin in a way that is consistent with the conservation laws for the total angular momentum. A way around this problem is to endow the space-time with additional geometrical structure, providing extra degrees of freedom to model intrinsic spin and its relation with the gravitational field. This is the fundamental idea behind the so-called Einstein-Cartan-Sciama-Kible theory of gravitation. In this theory the connection is not imposed be symmetric so that, the anti-symmetric part of the connection defines an extra tensor field: torsion. In this way, it is possible to impose a local Poincaré gauge symmetry on the tangent space of each point of the manifold such that the matter intrinsic spin can be related with the torsion tensor field. Indeed, theories of gravity with a non-symmetric connection (generically called Einstein-Cartan theories) were proposed even before the discovery of intrinsic spin. Sciama and Kibble [23, 24] introduced the idea of connecting the torsion tensor with the matter intrinsic angular momentum, paving the way to a geometrized treatment of spin.

Early works on the ECSK theory focused on the effects of intrinsic spin in the evolution of gravitational collapse and the possibility of avoidance of singularities [25–29, 92]. Only by the end of the 1970's, solutions for spherically symmetric space-times were found [34–36]. The solutions in Refs. [34–36] were obtained by directly solving the field equations for the ECSK theory. Such approach, though, leads to great difficulties in searching for exact solutions. In this thesis we will adopt a different method and consider the formalism provided by the 1+1+2 space-time decomposition [39, 42–44]. As stated in Chapter 2, covariant space-time decomposition approaches were initially devised to explore the properties of cosmological models and their perturbations (see e.g. Ref. [37–48]) and only recently they have been employed to deal with space-times of astrophysical interest [49–52]. In Ref. [51, 52], this approach was used to construct - in the context of GR - a covariant version of the Tolman-Oppenheimer-Volkoff (TOV) equations. The new equations allowed to pinpoint the mathematical nature of the problem of determining interior solutions for compact objects and, for instance, the treatment of stars with anisotropic pressure.

In this chapter, we will derive exact solutions of the ECSK theory to model static compact objects permeated by a perfect fluid with non-null spin degrees of freedom. These results will allow us to study how the presence of intrinsic spin of matter and its coupling with the geometry of the space-time affects the possible solutions. Moreover, by generalizing the TOV equations, we will be able to provide a set of algorithms to generate new exact solutions, linking seemingly unrelated solutions.

6.2 The 1+1+2 formalism

As was studied in Chapter 2, given a Lorentzian manifold of dimension 4 and a congruence of time-like curves with tangent vector u we can foliate the manifold in 3-hypersurfaces, $\mathcal{V}$, orthogonal at each point to the curves of the congruence, such that all quantities are defined by their behavior along the direction of u and in $\mathcal{V}$, through the introduction of the projector operator

$$h_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta \,, \tag{6.1}$$

where $g_{\alpha\beta}$ represents the components of the space-time metric in a coordinate system and $u_\alpha u^\alpha = -1$, with properties defined in Eq. (2.39). This procedure defines the 1+3 space-time decomposition. The 1+1+2 formalism [39, 42–44] builds from the 1+3 decomposition by defining a congruence of spatial curves with tangent vector field e such that any quantity in the sub-space $\mathcal{V}$ is defined by its behavior along e and in the 2-surfaces $\mathcal{W}$. We shall refer to $\mathcal{W}$ as the "sheet". We can then introduce a projector onto $\mathcal{W}$ by

$$N_{\alpha\beta} = h_{\alpha\beta} - e_\alpha e_\beta \,, \tag{6.2}$$

where $e_\alpha e^\alpha = 1$, and such that

$$\begin{aligned} N_{\alpha\beta} &= N_{\beta\alpha} \,, & N_{\alpha\beta}N^{\beta\gamma} &= N_\alpha^\gamma \,, \\ N_{\alpha\beta}u^\alpha &= N_{\alpha\beta}e^\alpha = 0 \,, & N_\alpha^\alpha &= 2 \,. \end{aligned} \tag{6.3}$$

Given the Levi-Civita volume form $\varepsilon_{\alpha\beta\gamma\sigma}$, Eq. (2.15), we can define the projected volume forms onto $\mathcal{V}$ and $\mathcal{W}$, respectively as[2]

$$\begin{aligned} \varepsilon_{\alpha\beta\gamma} &= \varepsilon_{\alpha\beta\gamma\sigma}u^\sigma \,, \\ \varepsilon_{\alpha\beta} &= \varepsilon_{\alpha\beta\gamma}e^\gamma \,, \end{aligned} \tag{6.4}$$

with the following properties

$$\begin{aligned} \varepsilon_{\alpha\beta\gamma} &= \varepsilon_{[\alpha\beta\gamma]} \,, & \varepsilon_{\alpha\beta} &= \varepsilon_{[\alpha\beta]} \,, \\ \varepsilon_{\alpha\beta\gamma}u^\gamma &= 0 \,, & \varepsilon_{\alpha\beta}u^\alpha &= \varepsilon_{\alpha\beta}e^\alpha = 0 \,, \\ \varepsilon_{\alpha\beta\gamma}\varepsilon^{\mu\nu\gamma} &= h_\alpha{}^\mu h_\beta{}^\nu - h_\beta{}^\mu h_\alpha{}^\nu \,, & \varepsilon_\alpha{}^\gamma \varepsilon_{\beta\gamma} &= N_{\alpha\beta} \,, \\ \varepsilon_{\mu\nu\alpha}\varepsilon^{\mu\nu\beta} &= 2h_\alpha^\beta \,, & \varepsilon_{\alpha\beta\gamma} &= e_\alpha \varepsilon_{\beta\gamma} - e_\beta \varepsilon_{\alpha\gamma} + e_\gamma \varepsilon_{\alpha\beta} \,. \end{aligned} \tag{6.5}$$

Then, using the results in Appendix C, the covariant derivatives of the tangent vectors u and e can be written as

$$\begin{aligned} \delta_\alpha u_\beta &= N_\alpha^\sigma N_\beta^\gamma \nabla_\sigma u_\gamma = N_{\alpha\beta}\left(\frac{1}{3}\theta - \frac{1}{2}\Sigma\right) + \Sigma_{\alpha\beta} + \varepsilon_{\alpha\beta}\Omega \,, \\ D_\alpha u_\beta &= h_\alpha^\sigma h_\beta^\gamma \nabla_\sigma u_\gamma = \delta_\alpha u_\beta + \left(\frac{1}{3}\theta + \Sigma\right) e_\alpha e_\beta \\ &\quad + 2\Sigma_{(\alpha}e_{\beta)} - \varepsilon_{\alpha\lambda}\Omega^\lambda e_\beta + e_\alpha \varepsilon_{\beta\lambda}\Omega^\lambda \,, \\ \nabla_\alpha u_\beta &= D_\alpha u_\beta - u_\alpha\left(\mathcal{A}e_\beta + \mathcal{A}_\beta\right) \,, \end{aligned} \tag{6.6}$$

[2]For readability, we repeat the definition and some of the properties of the 3-volume form $\varepsilon_{\alpha\beta\gamma}$, introduced back in Chapter 2, Eq. (2.40).

and

$$
\begin{aligned}
\delta_\alpha e_\beta =& \frac{1}{2} N_{\alpha\beta}\phi + \zeta_{\alpha\beta} + \varepsilon_{\alpha\beta}\xi \,, \\
D_\alpha e_\beta =& \delta_\alpha e_\beta + e_\alpha \bar{a}_\beta \,, \\
\nabla_\alpha e_\beta =& D_\alpha e_\beta - u_\alpha \alpha_\beta - \mathcal{A} u_\alpha u_\beta + \left(\frac{1}{3}\theta + \Sigma\right) e_\alpha u_\beta + \\
&+ \left(\Sigma_\alpha - \varepsilon_{\alpha\sigma}\Omega^\sigma\right) u_\beta \,,
\end{aligned}
\tag{6.7}
$$

where θ is the expansion scalar, Eq. (2.36), associated with the congruence to which u is tangent to and $W_{\alpha\beta}$ is one of the components of the 1+3 decomposition of the torsion tensor, Eq. (2.44), orthogonal to u.

From Eqs. (6.6) and (6.7) we can write the kinematical quantities, Eq. (2.36), associated with the congruences and their acceleration in terms of the 1+1+2 covariantly defined quantities. Namely,

$$
\begin{aligned}
\sigma_{\alpha\beta} &= \Sigma_{\alpha\beta} + 2\Sigma_{(\alpha} e_{\beta)} + \Sigma\left(e_\alpha e_\beta - \frac{1}{2} N_{\alpha\beta}\right) + W_{\langle\alpha\beta\rangle} \,, \\
\omega_{\alpha\beta} &= \varepsilon_{\alpha\beta\gamma}\left(\Omega e^\gamma + \Omega^\gamma\right) + W_{[\alpha\beta]} \,, \\
a_\alpha &= \nabla_u u_\alpha = \mathcal{A} e_\alpha + \mathcal{A}_\alpha \,,
\end{aligned}
\tag{6.8}
$$

represent, respectively, the shear, vorticity and acceleration tensors associated with the congruence to which u is tangent to, and

$$
\begin{aligned}
\phi_g &= \phi + 2 S_{\gamma\mu\nu} e^\gamma N^{\mu\nu} \,, \\
\zeta_{g\,\alpha\beta} &= \zeta_{\alpha\beta} + 2 S_{\gamma\mu\nu} e^\gamma N^\mu_{\langle\alpha|} N^\nu_{|\beta\rangle} \,, \\
\xi_{g\,\alpha\beta} &= \varepsilon_{\alpha\beta}\left(\xi + S_{\gamma\mu\nu} e^\gamma \varepsilon^{\mu\nu}\right) \,,
\end{aligned}
\tag{6.9}
$$

represent, respectively, the expansion, shear and vorticity scalar, associated with the congruence to which e is tangent to.

We shall also need to find the various contributions along u, e and on $\mathcal{W}$ of the stress-energy tensor $\mathcal{T}_{\alpha\beta}$. At this point we shall not assume $\mathcal{T}_{\alpha\beta}$ to have any symmetry. Hence,

$$
\begin{aligned}
\mathcal{T}_{\alpha\beta} &= \mu\, u_\alpha u_\beta + p\, h_{\alpha\beta} + q_{1\alpha} u_\beta + u_\alpha q_{2\beta} + \pi_{\alpha\beta} + \varepsilon_{\alpha\beta}{}^\gamma m_\gamma \\
&= \mu\, u_\alpha u_\beta + Q_{1\alpha} u_\beta + u_\alpha Q_{2\beta} + Q_1\, e_\alpha u_\beta + Q_2\, u_\alpha e_\beta + \Pi_{1\alpha} e_\beta + e_\alpha \Pi_{2\beta} + \\
&\quad + p_r\, e_\alpha e_\beta + p_\perp\, N_{\alpha\beta} + \Pi_{\alpha\beta} + \varepsilon_{\alpha\beta}\mathbb{M} \,,
\end{aligned}
\tag{6.10}
$$

with

$$
\begin{aligned}
q_{1\alpha} &= -h^{\sigma}_{\alpha} u^{\gamma} \mathcal{T}_{\sigma\gamma} \,, & \mu &= u^{\sigma} u^{\gamma} \mathcal{T}_{\sigma\gamma} \,, \\
q_{2\alpha} &= -u^{\sigma} h^{\gamma}_{\alpha} \mathcal{T}_{\sigma\gamma} \,, & p &= \frac{1}{3} h^{\alpha\beta} \mathcal{T}_{\alpha\beta} \,, \\
Q_{1\alpha} &= -N^{\sigma}_{\alpha} u^{\gamma} \mathcal{T}_{\sigma\gamma} \,, & p_r &= p + \Pi = e^{\sigma} e^{\gamma} \mathcal{T}_{\sigma\gamma} \,, \\
Q_{2\alpha} &= -u^{\sigma} N^{\gamma}_{\alpha} \mathcal{T}_{\sigma\gamma} \,, & p_{\perp} &= p - \frac{1}{2}\Pi = \frac{1}{2} N^{\sigma\gamma} \mathcal{T}_{\sigma\gamma} \,, \\
\Pi_{1\alpha} &= N^{\sigma}_{\alpha} e^{\gamma} \mathcal{T}_{\sigma\gamma} \,, & Q_1 &= -e^{\sigma} u^{\gamma} \mathcal{T}_{\sigma\gamma} \,, \\
\Pi_{2\alpha} &= e^{\sigma} N^{\gamma}_{\alpha} \mathcal{T}_{\sigma\gamma} \,, & Q_2 &= -u^{\sigma} e^{\gamma} \mathcal{T}_{\sigma\gamma} \,, \\
\pi_{\alpha\beta} &= h^{\sigma}_{\langle\alpha} h_{\beta\rangle}{}^{\gamma} \mathcal{T}_{\sigma\gamma} \,, & \Pi &= \frac{1}{3} \mathcal{T}_{\alpha\beta} \left(2e^{\alpha} e^{\beta} - N^{\alpha\beta}\right) \,, \\
m_{\alpha\beta} &= h^{\sigma}_{[\alpha} h_{\beta]}{}^{\gamma} \mathcal{T}_{\sigma\gamma} \,, & \mathbb{M} &= \frac{1}{2} \varepsilon^{\mu\nu} \mathcal{T}_{\mu\nu} \,, \\
\Pi_{\alpha\beta} &= \mathcal{T}_{\{\alpha\beta\}} \,,
\end{aligned}
\tag{6.11}
$$

where the angular and curly parentheses notation is defined in Eqs. (2.52) and (C.5). Moreover, the following relations are useful

$$
\begin{aligned}
q_{1,2\alpha} &= Q_{1,2\alpha} + Q_{1,2}\, e_{\alpha} \,, \\
\pi_{\alpha\beta} &= \Pi_{\alpha\beta} + \Pi \left(e_{\alpha} e_{\beta} - \frac{1}{2} N_{\alpha\beta} \right) + \Pi_{1(\alpha}\, e_{\beta)} + \Pi_{2(\alpha}\, e_{\beta)} \,.
\end{aligned}
\tag{6.12}
$$

Lastly, we will need to find the 1+1+2 decomposition of the Weyl tensor. Starting from the Eq. (2.45) and using Eq. (2.49), the full 1+1+2 decomposition of the electric and magnetic components: $E_{\alpha\beta}$, $H_{\alpha\beta}$ and $\bar{H}_{\alpha\beta}$, is given by

$$
E_{\alpha\beta} = \mathcal{E} \left(e_{\alpha} e_{\beta} - \frac{1}{2} N_{\alpha\beta} \right) + \mathcal{E}_{\alpha} e_{\beta} + e_{\alpha} \bar{\mathcal{E}}_{\beta} + \mathcal{E}_{\alpha\beta} + \varepsilon_{\alpha\beta} \mathbb{E} \,, \tag{6.13}
$$

$$
H_{\alpha\beta} = \frac{1}{2} N_{\alpha\beta} \mathsf{H} + e_{\alpha} e_{\beta} \mathcal{H} + \mathcal{H}_{\alpha} e_{\beta} + e_{\alpha} \mathcal{H}_{\beta} + \mathcal{H}_{\alpha\beta} \,, \tag{6.14}
$$

$$
\bar{H}_{\alpha\beta} = \frac{1}{2} N_{\alpha\beta} \bar{\mathsf{H}} + e_{\alpha} e_{\beta} \bar{\mathcal{H}} + \bar{\mathcal{H}}_{\alpha} e_{\beta} + e_{\alpha} \bar{\mathcal{H}}_{\beta} + \bar{\mathcal{H}}_{\alpha\beta} \,, \tag{6.15}
$$

with

$$
\begin{aligned}
\mathcal{E} &= E_{\mu\nu}e^{\mu}e^{\nu} = -N^{\mu\nu}E_{\mu\nu}\,, & \mathcal{E}_{\alpha} &= N_{\alpha}{}^{\mu}e^{\nu}E_{\mu\nu}\,,\\
\mathbb{E} &= \frac{1}{2}\varepsilon^{\mu\nu}E_{\mu\nu}\,, & \bar{\mathcal{E}}_{\alpha} &= e^{\mu}N_{\alpha}{}^{\nu}E_{\mu\nu}\,,\\
\mathsf{H} &= N^{\mu\nu}H_{\mu\nu}\,, & \mathcal{E}_{\alpha\beta} &= E_{\{\alpha\beta\}}\,,\\
\mathcal{H} &= e^{\mu}e^{\nu}H_{\mu\nu}\,, & \mathcal{H}_{\alpha} &= N_{\alpha}{}^{\mu}e^{\nu}H_{\mu\nu}\,,\\
\bar{\mathsf{H}} &= N^{\mu\nu}\bar{H}_{\mu\nu}\,, & \bar{\mathcal{H}}_{\alpha} &= N_{\alpha}{}^{\mu}e^{\nu}\bar{H}_{\mu\nu}\,,\\
\bar{\mathcal{H}} &= e^{\mu}e^{\nu}\bar{H}_{\mu\nu}\,, & \mathcal{H}_{\alpha\beta} &= H_{\{\alpha\beta\}}\,,\\
& & \bar{\mathcal{H}}_{\alpha\beta} &= \bar{H}_{\{\alpha\beta\}}\,.
\end{aligned}
\tag{6.16}
$$

6.3 Decomposition of the field equations

We are now in position to apply the 1+1+2 framework to study solutions of the ECSK theory, characterized by the field equations (4.43) and (4.44):

$$
R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R + \Lambda g_{\alpha\beta} = 8\pi\mathcal{T}_{\alpha\beta}\,, \tag{6.17}
$$

$$
S^{\alpha\beta\gamma} + 2g^{\gamma[\alpha}S^{\beta]}{}_{\mu}{}^{\mu} = -8\pi\Delta^{\alpha\beta\gamma}\,, \tag{6.18}
$$

where $\mathcal{T}_{\alpha\beta}$ represents the canonical stress-energy tensor, $\Delta^{\alpha\beta\mu}$ the intrinsic hypermomentum and we will assume a null cosmological constant, Λ; and the conservation laws (4.45)

$$
\nabla_{\beta}\mathcal{T}_{\alpha}{}^{\beta} = 2S_{\alpha\mu\nu}\mathcal{T}^{\nu\mu} + \frac{1}{8\pi}\left(S_{\alpha\mu}{}^{\mu}R - S^{\mu\nu\sigma}R_{\alpha\sigma\mu\nu}\right)\,. \tag{6.19}
$$

We will consider that the source for the above field equations is an uncharged Weyssenhoff fluid [130]. The Weyssenhoff fluid provides a semi-classical description of a perfect fluid composed of fermions, such that the fluid is characterized by its energy density, pressure and spin density. Its canonical stress-energy tensor is given by

$$
\mathcal{T}_{\alpha\beta} = \mu u_{\alpha}u_{\beta} + p\,h_{\alpha\beta} - \left(\mathcal{A}e^{\mu} + \mathcal{A}^{\mu}\right)S_{\mu\alpha}u_{\beta}\,, \tag{6.20}
$$

where μ and p represent the energy density and pressure of the fluid, respectively.

Following Refs. [131, 132], the hypermomentum tensor for the Weyssenhoff spin fluid can be written, in our conventions, as (5.12)

$$
\Delta^{\alpha\beta\gamma} = -\frac{1}{8\pi}\Delta^{\alpha\beta}u^{\gamma}\,, \tag{6.21}
$$

where u represents the proper 4-velocity of an element of volume of the fluid and the anti-symmetric spin density tensor, $\Delta^{\alpha\beta}$, verifies $\Delta^{\alpha\beta}u_\beta = 0$. From Eq. (6.18) we find (5.14),

$$S^{\alpha\beta\gamma} = \Delta^{\alpha\beta}u^\gamma\,, \tag{6.22}$$

hence, comparing Eq. (6.22) with Eq. (2.43) we see that the Weyssenhoff fluid model implies that the tensors $\bar{S}^{\alpha\beta}$, $W^{\alpha\beta}$ and X^α, Eq. (2.44), are null and $S^{\alpha\beta} = \Delta^{\alpha\beta}$. In this way, the decomposition of the torsion tensor will coincide with the decomposition of $S^{\alpha\beta}$. Taking into account that $S_{\alpha\beta} \equiv S_{[\alpha\beta]}$ we find

$$S_{\alpha\beta} = \varepsilon_{\alpha\beta}\tau + 2s_{[\alpha}e_{\beta]}\,, \tag{6.23}$$

with

$$\begin{aligned}\tau &= \frac{1}{2}\varepsilon^{\mu\nu}S_{\mu\nu}\,,\\ s_\alpha &= N^\sigma_{\ \alpha}e^\gamma S_{\sigma\gamma}\,.\end{aligned} \tag{6.24}$$

6.3.1 The symmetries of the problem

We are interested in solutions of the ECSK theory that are static and locally rotationally symmetric (LRS). Following Ref. [37], a space-time is said to be locally rotationally symmetric in a neighborhood $B(q)$ of a point q, if there exists a non-discrete sub-group G of the Lorentz group in the tangent space of each $q' \in B(q)$ which leaves u, the curvature tensor and their derivatives (up to third order) invariant. Assuming G to be one-dimensional, we can set at each point the vector field e to have the same direction as an axis of symmetry. Then, LRS implies that all covariantly defined space-like vectors must have the same direction of e — otherwise they would not be invariant under G. Thus, the vector quantities $\{\bar{a}_\beta,\, \alpha_\beta,\, \Sigma_\beta,\, \Omega_\beta\,,\, A_\beta\}$ are null in such space-times. Also the shear tensors of the congruences of curves associated with u and e projected onto the sheet: $\Sigma_{\alpha\beta}$ and $\zeta_{\alpha\beta}$, must be null since, there can not be any preferred direction at the sheet[3].

From the definition of LRS space-times, the Riemann curvature tensor must also be invariant under G therefore, the vector components of the Weyl tensor $\{\mathcal{E}_\alpha,\, \bar{\mathcal{E}}_\alpha,\, \mathcal{H}_\alpha,\, \bar{\mathcal{H}}_\alpha\}$ must also be identically null. Since the Riemann tensor also depends on the torsion tensor, the latter must also be invariant under the action of G. Therefore, from Eqs. (6.22) and (6.23), the tensor field s^α, Eq. (6.24), must vanish. In light of this results and taking into account Eq. (6.22) we also conclude that for an LRS space-time the intrinsic hypermomentum tensor is simply given by

$$\Delta_{\alpha\beta} = \varepsilon_{\alpha\beta}\delta\,, \tag{6.25}$$

where $\delta = \frac{1}{2}\varepsilon^{\alpha\beta}\Delta_{\alpha\beta}$, with the constraint $\tau = \delta$.

Now, an LRS space-time is said to be of class I (LRS I) if the congruence of the curves associated with the

[3]It should be remarked here that, as shown in Eqs. (6.8) and (6.9), the presence of a generic torsion tensor field affects the definition of the kinematical quantities [59, 83–85]. As such, in the presence of a general torsion, LRS implies that the geometric shear vector fields on the sheet: $\Sigma_{g\,\alpha\beta}$ and $\zeta_{g\,\alpha\beta}$, must be null and not the quantities $\Sigma_{\alpha\beta}$ and $\zeta_{\alpha\beta}$. However, as mentioned in Appendix C, for a Weyssenhoff fluid those are equal hence, from here on we shall refer to $\Sigma_{\alpha\beta}$ and $\zeta_{\alpha\beta}$ as the shear tensors onto the sheet of the congruences associated with u and e, being implicit that we assume the Weyssenhoff model.

vector field e — defined to have the same direction as the axis of symmetry — is hypersurface orthogonal. If the congruence of curves associated with the vector field u is also hypersurface orthogonal, the space-time is said to be LRS of class II (LRS II).[4] From the results in Chapter 4, Eq. (4.6), for a torsion tensor given by Eq. (6.22) we have that e will be hypersurface orthogonal if and only if

$$\xi = 0 \,, \tag{6.26}$$

and that u will be hypersurface orthogonal if and only if

$$\begin{aligned} \Omega &= \tau \,, \\ s_\alpha &= 0 \,, \end{aligned} \tag{6.27}$$

where we opted to highlight that s_α will also be null from the imposition that the congruence of u is hypersurface orthogonal.

Before proceeding we should point out the fact that $s_\alpha = 0$ has an interesting effect on the nature of the Weyssenhoff fluid. Comparing Eq. (6.20) with Eq. (6.10) we conclude that for a Weyssenhoff fluid the only non-null covariantly defined quantities in Eq. (6.11) are μ, p and $q_{1\alpha} = -\left(\mathcal{A}e^\mu + \mathcal{A}^\mu\right) S_{\mu\alpha}$. Now, since in an LRS space-time both $\mathcal{A}_\alpha$ and s_α are null, it implies that $q_{1\alpha} = -\left(\mathcal{A}e^\mu + \mathcal{A}^\mu\right) S_{\mu\alpha} = 0$, that is, the contributions of intrinsic spin in the Weyssenhoff fluid model for an LRS space-time, will not appear in the canonical stress-energy tensor. From this result, one might (wrongly) conclude that torsion has no role in the dynamics of the setup. In reality, torsion will still markedly influence the behavior of the matter fields. Indeed, for instance, when comparing to space-times with null torsion, where LRS II space-times are necessarily irrotational (cf., e.g., Ref. [37]), the presence of a non-null torsion of the form of Eq. (6.22) will induce a non-null vorticity of the congruence of curves associated with u, Eq. (6.27). Thus, although in the considered setup spin does not appear in the canonical stress-energy tensor, it will still markedly change the geometry of the space-time.

An additional assumption we will consider is that the space-time is static. Now, a space-time is said to be stationary if it admits the existence of a time-like Killing vector field ψ. If the congruence of time-like curves associated with ψ are also hypersurface orthogonal the space-time is said to be static. Given that the choice of the vector field u is arbitrary, we can write, at each point, $\psi = C\, u$, where $C = C\left(x^\alpha\right)$ is a generic non-null smooth function of the coordinates. The Killing equation $\mathcal{L}_\psi g_{\alpha\beta} = 0$ in presence of torsion can be written as

$$\nabla_{(\alpha}\psi_{\beta)} + 2S_{\sigma(\alpha\beta)}\psi^\sigma = 0 \,, \tag{6.28}$$

[4] Following Ref. [37], a space-time is said to be LRS II when it has locally rotational symmetry and the vector fields u and e are hypersurface orthogonal. Just so happens, in space-times with null-torsion, an hypersurface orthogonal, time-like congruence has null vorticity (cf. Eq. (4.6)). As such, in the literature, LRS II space-times are characterized and usually referred as space-times with locally rotational symmetry and vorticity free u and e vector fields. As was shown in Chapter 4 this is not the case for space-times with non-null torsion where an hypersurface orthogonal congruence does not have null vorticity. In this thesis, we will follow the naming convention of Ref. [37]. This has at least one advantage: when comparing results with the null torsion case, we simply have to compare with the same named class; for instance, static spherically symmetric space-times, with or without torsion, always fall in the category of static LRS II space-times.

for any metric compatible affine connection. Assuming Eq. (6.22), contracting Eq. (6.28) with $h_\mu^\alpha h_\nu^\beta$ and $h^{\alpha\beta}$ we have

$$\{\theta,\, \Sigma,\, \Sigma_\alpha,\, \Sigma_{\alpha\beta}\} = 0\,, \tag{6.29}$$

and $u^a \partial_a C\left(x^\alpha\right) = 0$. All is left now is to impose the condition that ψ is hypersurface orthogonal. However, if u is hypersurface orthogonal, so is any $\psi = C\, u$. Hence, for the space-time to be static, Eqs. (6.27) must hold.

Lastly, computing the quantities: $N_\mu^\alpha N_\nu^\gamma u^\beta R_{\alpha\beta\gamma\delta} u^\delta$, ${\varepsilon_\mu}^{\alpha\beta} R_{\alpha\beta\gamma\delta} u^\delta$ and $\varepsilon^{\mu\gamma} u^\beta R_{\alpha\beta\gamma\delta} e^\delta$, we also find that in the considered setup

$$\left\{\mathbb{E},\, \mathcal{E}_{\alpha\beta},\, \mathcal{H}_{\alpha\beta}, \bar{\mathcal{H}}_{\alpha\beta}\right\} = 0\,. \tag{6.30}$$

Therefore, gathering the previous results we find that stationary LRS I or LRS II space-times permeated by an uncharged Weyssenhoff fluid are characterized by the following set of quantities {μ, p, ϕ, Ω, $\mathcal{A}$, τ, $\mathcal{E}$, H, H̄, $\mathcal{H}$, $\bar{\mathcal{H}}$}.

6.3.2 Structure equations

6.3.2.1 General 1+3 equations in the Einstein-Cartan-Sciama-Kibble theory for a torsion of the form ${S_{\alpha\beta}}^\gamma = S_{\alpha\beta} u^\gamma$

To find the structure equations for a Weyssenhoff fluid in the considered setup we have to find the various non-trivial projections of the Ricci and Bianchi identities onto u, e and $\mathcal{W}$. We cannot, however, stress enough how much it simplifies the computation if we to start from the 1+3 structure equations derived in Chapter 2, applied to the ECSK theory, instead of starting directly from the Ricci and Bianchi identities — Eqs. (2.10), (2.12) and (2.13). Let us then find the 1+3 structure equations in the case of a space-time with a torsion tensor field of the form ${S_{\alpha\beta}}^\gamma = S_{\alpha\beta} u^\gamma$, in the context of the ECSK theory. From Eqs. (2.56) – (2.71) and (6.17) – (6.19) we find:

The evolution equations for the kinematical quantities associated with u,

$$\dot{\theta} = -\,4\pi\left(\mu + 3p\right) + \Lambda - \left(\frac{1}{3}\theta^2 + \sigma_{\alpha\beta}\sigma^{\alpha\beta} + \omega_{\alpha\beta}\omega^{\beta\alpha}\right) + \nabla_\alpha a^\alpha\,, \tag{6.31}$$

$${h_\mu}^\alpha {h_\nu}^\beta \dot{\omega}_{\alpha\beta} = -\,E_{[\mu\nu]} + \frac{1}{2}\varepsilon_{\mu\nu\gamma} m^\gamma - \frac{2}{3}\theta\omega_{\mu\nu} - 2{\sigma_{[\mu}}^\alpha \omega_{\alpha|\nu]} + D_{[\mu}\, a_{\nu]}\,, \tag{6.32}$$

$$\begin{aligned} {h_\mu}^\alpha {h_\nu}^\beta \dot{\sigma}_{\alpha\beta} = &-\,E_{(\mu\nu)} + 4\pi\left(\pi_{\mu\nu}\right) + D_{\langle\mu}\, a_{\nu\rangle} + \\ &+ a_{\langle\mu}\, a_{\nu\rangle} - \frac{2}{3}\sigma_{\mu\nu}\theta - {\sigma_{\langle\mu|}}^\delta \sigma_{\delta|\nu\rangle} - {\omega_{\langle\mu|}}^\delta \omega_{\delta|\nu\rangle}\,, \end{aligned} \tag{6.33}$$

and the constraint equations,

$$\varepsilon^{\mu\nu\rho} D_\mu \omega_{\nu\rho} + \varepsilon^{\mu\nu\rho} a_\rho \omega_{\nu\mu} = H_\rho{}^\rho + \varepsilon^{\mu\nu\rho} a_\rho S_{\nu\mu}\,, \tag{6.34}$$

$$\varepsilon^{\alpha\beta\langle\mu|} D_\alpha \left(\sigma_\beta{}^{|\nu\rangle} + \omega_\beta{}^{|\nu\rangle}\right) + \varepsilon^{\alpha\beta\langle\mu} a^{\nu\rangle} \omega_{\beta\alpha} = H^{\langle\mu\nu\rangle} - \varepsilon^{\alpha\beta\langle\mu} a^{\nu\rangle} S_{\alpha\beta}\,, \tag{6.35}$$

$$\frac{2}{3} D_\alpha \theta - D_\mu \left(\sigma_\alpha{}^\mu + \omega_\alpha{}^\mu\right) - 2a^\mu \omega_{\alpha\mu} = 8\pi q_{1\alpha} + 2a^\mu S_{\mu\alpha}\,; \tag{6.36}$$

The evolution equations for the electric and magnetic parts of the Weyl tensor,

$$\begin{aligned}
&E_\alpha{}^\mu \left(\sigma_{\mu\beta} + \omega_{\mu\beta}\right) - h_{\alpha\mu} h_{\beta\nu} \dot{E}^{\mu\nu} - E_{\alpha\beta}\theta + \\
&+\varepsilon_{\mu\beta}{}^\nu \left(D_\nu \bar{H}^\mu{}_\alpha + a_\nu \bar{H}^\mu{}_\alpha\right) + \varepsilon^\mu{}_{\alpha\delta} a^\delta H_{\mu\beta} + \\
&\quad +\varepsilon_\alpha{}^{\delta\mu} \varepsilon_\beta{}^{\lambda\nu} E_{\nu\mu} \left(\sigma_{\lambda\delta} + \omega_{\lambda\delta}\right) = \frac{4\pi}{3} h_{\alpha\beta}\dot{\mu} + 4\pi h_{\alpha\mu} h_{\beta\nu} \dot{\pi}^{\mu\nu} + 4\pi \varepsilon_{\alpha\beta}{}^\gamma \dot{m}_\gamma + \\
&\qquad + 4\pi \left(q_{1\alpha} a_\beta + a_\alpha q_{2\beta}\right) + 4\pi D_\alpha q_{2\beta} + \\
&\qquad + 4\pi \left(\frac{1}{3} h_{\alpha\delta}\theta + \sigma_{\alpha\delta} + \omega_{\alpha\delta}\right) \times \\
&\qquad \times \left[h^\delta{}_\beta \left(\mu + p\right) + \pi^\delta{}_\beta + \varepsilon^\delta{}_{\beta\gamma} m^\gamma\right]\,,
\end{aligned} \tag{6.37}$$

$$\begin{aligned}
&\bar{H}_\mu{}^\mu \left(\frac{1}{3} h^{\alpha\beta}\theta - \sigma^{\alpha\beta}\right) + 2\bar{H}_\mu{}^{(\alpha}\sigma^{\beta)\mu} - h^{\alpha\beta}\bar{H}^{\mu\nu}\sigma_{\mu\nu} + \\
&\quad + \left[2a_\mu E_\nu{}^{(\alpha|} + D_\mu E_\nu{}^{(\alpha|}\right] \varepsilon^{|\beta)\nu\mu} - h^{\mu(\alpha} h^{\beta)\nu} \dot{H}_{\mu\nu} + \\
&\quad +H^{\mu(\alpha|} \left(\sigma_\mu{}^{|\beta)} + \omega_\mu{}^{|\beta)}\right) - \frac{1}{3} \left(2H^{\alpha\beta} + \bar{H}^{\alpha\beta}\right)\theta = 4\pi \varepsilon_\gamma{}^{\delta(\alpha|} D_\delta \pi^{|\beta)\gamma} + \\
&\qquad + 4\pi D^{(\alpha} m^{\beta)} - 4\pi h^{\alpha\beta} D_\delta m^\delta\,,
\end{aligned} \tag{6.38}$$

and the constraint equations,

$$\begin{aligned}
&D_\beta E_\alpha{}^\beta + \varepsilon^{\beta\gamma\delta}\bar{H}_{\beta\alpha} \left(\omega_{\delta\gamma} - \frac{1}{2} S_{\delta\gamma}\right) + \\
&\quad + \left(\sigma_{\delta\nu} + \omega_{\delta\nu}\right) \varepsilon^{\nu\beta\gamma} h_{\alpha\beta} H_\gamma{}^\delta = 4\pi D_\alpha p + \frac{8\pi}{3} D_\alpha \mu + 4\pi \left[\pi_\alpha{}^\beta - \varepsilon^\beta{}_{\alpha\gamma} m^\gamma\right] a_\beta + \\
&\qquad + 4\pi \left(q_{2\lambda} + q_{1\lambda}\right) \left(\sigma_\alpha{}^\lambda + \omega_\alpha{}^\lambda + \frac{1}{3} h_\alpha{}^\lambda \theta\right) + \\
&\qquad + 4\pi h_\alpha{}^\gamma \dot{q}_{1\gamma} + 4\pi \left(\mu + p\right) a_\alpha - 4\pi S_{\alpha\gamma} q_2^\gamma\,,
\end{aligned} \tag{6.39}$$

$$
\begin{aligned}
&4\varepsilon_{(\alpha|\beta}{}^{\gamma}E^{\beta|\delta)}\omega_{\delta\gamma}+2\varepsilon_{\alpha\beta}{}^{\gamma}E^{\beta\delta}\sigma_{\delta\gamma}-\\
&\quad -2D_{\gamma}H^{\gamma}{}_{\alpha}+\frac{2}{3}\varepsilon_{\alpha\beta\delta}E^{\beta\delta}\theta=-8\pi\varepsilon_{\alpha\gamma\delta}\left[D^{\delta}q_1^{\gamma}+\omega^{\delta\gamma}\left(\mu+p\right)\right]-\\
&\qquad -8\pi\varepsilon_{\alpha\gamma\delta}\left(\pi^{\gamma\beta}+\varepsilon^{\gamma\beta}{}_{\nu}m^{\nu}\right)\left(\sigma^{\delta}{}_{\beta}+\omega^{\delta}{}_{\beta}\right)-\\
&\qquad -\frac{16\pi}{3}\theta m_{\alpha}-\frac{8\pi}{3}\varepsilon^{\mu\nu}{}_{\alpha}S_{\mu\nu}\left(\mu+3p-\frac{\Lambda}{4\pi}\right)+\\
&\qquad +8\pi\varepsilon^{\mu\nu\gamma}S_{\mu\nu}\left(\pi_{\gamma\alpha}+\varepsilon_{\gamma\alpha\nu}m^{\nu}\right)-2\varepsilon^{\sigma\nu\gamma}S_{\sigma\nu}E_{\gamma\alpha}\,,
\end{aligned}
\tag{6.40}
$$

$$
\begin{aligned}
H_{\alpha}{}^{\beta}-\bar{H}_{\alpha}{}^{\beta}+4\pi\varepsilon_{\alpha}{}^{\mu\beta}\left(q_{1\mu}-q_{2\mu}\right)=&-\frac{1}{2}\varepsilon_{\alpha}{}^{\mu\nu}D^{\beta}S_{\nu\mu}-\frac{1}{2}\varepsilon_{\alpha}{}^{\mu\nu}a^{\beta}S_{\nu\mu}+\\
&+\varepsilon_{\alpha\mu\nu}a^{\nu}S^{\mu\beta}+\varepsilon_{\alpha\mu}{}^{\nu}D_{\nu}S^{\beta\mu}\,;
\end{aligned}
\tag{6.41}
$$

The equations that characterize the torsion tensor,

$$
4\pi\left(q_2^{\alpha}-q_1^{\alpha}\right)=S_{\gamma}{}^{\alpha}a^{\gamma}\,,
\tag{6.42}
$$

$$
16\pi m^{\alpha}=-\varepsilon^{\alpha}{}_{\sigma\rho}\left(S^{\rho\sigma}\theta+\dot{S}^{\rho\sigma}\right)\,;
\tag{6.43}
$$

and the conservation of energy and momentum,

$$
\dot{\mu}+\theta\left(\mu+p\right)+2q_1^{\alpha}a_{\alpha}+D_{\alpha}q_2^{\alpha}+\pi^{\alpha\beta}\sigma_{\alpha\beta}+\varepsilon^{\alpha\beta\gamma}m_{\gamma}\omega_{\beta\alpha}=0,
\tag{6.44}
$$

$$
\begin{aligned}
&h_{\alpha}{}^{\beta}\left[\nabla_{\beta}p+\dot{q}_{1\beta}+\nabla_{\sigma}\pi_{\beta}{}^{\sigma}+\varepsilon_{\beta}{}^{\sigma\gamma}\left(D_{\sigma}m_{\gamma}-m_{\sigma}a_{\gamma}\right)\right]+\\
&\quad +\left(\mu+p\right)a_{\alpha}+\theta\left(q_{1\alpha}+\frac{q_{2\alpha}}{3}\right)+q_2^{\beta}\left(\sigma_{\beta\alpha}+\omega_{\beta\alpha}\right)=-\frac{1}{8\pi}\left(\bar{H}_{\alpha}{}^{\rho}S^{\gamma\delta}\varepsilon_{\rho\gamma\delta}\right)-S_{\alpha}{}^{\beta}q_{2\beta}\,.
\end{aligned}
\tag{6.45}
$$

Let us briefly discuss some of the results that can be inferred directly from this set of structure equations. From Eq. (6.42) we see that although the heat flow components q_1 and q_2 might be different, in the considered setup we have $q_1^{\beta}a_{\beta}=q_2^{\beta}a_{\beta}$. This result is consistent with the stress-energy tensor for the Weyssenhoff fluid model, Eq. (6.20), where q_2 is null and, indeed, the contraction of $q_1^{\beta}a_{\beta}$ also vanishes identically. On the other hand, relating the torsion component $S_{\alpha\beta}$ with the intrinsic spin of the matter field, Eq. (6.43) represents the conservation law for the spin density [86, 89, 132].

Incidentally, this form of the structure equations allows us to easily compare them with the results in the literature and test their validity. Setting the torsion terms in Eqs. (6.31) – (6.45) to zero and imposing the stress-energy tensor to be symmetric, such that $m_{\alpha}=0$ and $q_{1\alpha}=q_{2\alpha}$, we recover the expressions for the structure equations for GR [38, 40]. On the other hand, we see that our results differ from the ones in Ref. [161]. In this reference the authors failed to realize that in the presence of torsion, the Weyl tensor is characterized by three tensors, namely, the "magnetic" part of the Weyl tensor is described by two distinct tensors; moreover, it is quite surprising that the authors did not verify that the "electric" and "magnetic" parts of the Weyl tensor did not carry all the usual symmetries found in space-times with null torsion.

6.3.2.2 Propagation and constraint equations

We are now in position to find the structure equations for a stationary, locally rotationally symmetric space-time filled by a Weyssenhoff fluid in the case where the congruence of space-like curves associated with e are hypersurface orthogonal, that is, in the case when $\xi = 0$. The non-trivial, independent propagation equations are[5]

$$\hat{p} + \mathcal{A}\left(\mu + p\right) = -\frac{1}{4\pi}\tau\bar{\mathcal{H}}\,, \tag{6.46}$$

$$\hat{\mathcal{A}} + \mathcal{A}\left(\mathcal{A} + \phi\right) + 2\Omega^2 = 4\pi\left(\mu + 3p\right)\,, \tag{6.47}$$

$$\hat{\phi} + \frac{1}{2}\phi^2 + \mathcal{E} = -\frac{16\pi}{3}\mu\,, \tag{6.48}$$

$$\hat{\mathcal{E}} + \frac{3}{2}\mathcal{E}\phi + \Omega\mathsf{H} + 2\left(\tau - \Omega\right)\bar{\mathcal{H}} = \frac{8\pi}{3}\hat{\mu}\,, \tag{6.49}$$

$$\hat{\mathcal{H}} - \frac{1}{2}\phi\left(\mathsf{H} - 2\mathcal{H}\right) + \mathcal{E}\left(3\Omega - 2\tau\right) = -8\pi\Omega\left(\mu + p\right) + \frac{8\pi}{3}\tau\left(\mu + 3p\right)\,, \tag{6.50}$$

$$2\hat{\Omega} + \Omega\phi = \mathsf{H}\,, \tag{6.51}$$

and the constraint equations

$$\mathcal{E} + \mathcal{A}\phi + 2\Omega^2 = \frac{8\pi}{3}\left(\mu + 3p\right)\,, \tag{6.52}$$

$$2\mathcal{A}\left(\Omega - \tau\right) - \Omega\phi + \mathcal{H} = 0\,, \tag{6.53}$$

$$\Omega\left(\phi - 2\mathcal{A}\right) + \bar{\mathsf{H}} = 0\,, \tag{6.54}$$

$$\mathsf{H} + \bar{\mathsf{H}} + 2\bar{\mathcal{H}} = 0\,, \tag{6.55}$$

where we have used the notation $\hat{F} \equiv e^{\alpha}\nabla_{\alpha}F$, for a scalar field F.

Let us discuss the cases when the congruence associated with u is either hypersurface orthogonal or not, separately. Consider the cases when $\Omega \neq \tau$. In such cases we find from Eqs. (6.46) – (6.55) the following relation between Ω and τ

$$\left(\phi - \mathcal{A}\right)\left(\tau - \Omega\right) + \hat{\tau} - \hat{\Omega} = 0\,, \tag{6.56}$$

leading us to conclude that the difference between τ and Ω can be uniquely described by the behavior of the variables ϕ and $\mathcal{A}$. Notice that, if at an initial instant Ω and τ are different then, unless the term $\phi - A$ diverges, there will be no point in which they are equal. Conversely, if $\Omega = \tau$ at a point these two quantities

[5] Equations (6.46) – (6.55) were derived as follows: equation (6.46) from the momentum conservation (6.45); equation (6.47) from the Raychaudhuri equation (6.31); equation (6.48) from the contraction $e^{\alpha}N^{\beta\gamma}R_{\alpha\beta\gamma}{}^{\rho}e_{\rho}$; equation (6.49) from (6.39); equation (6.50) from (6.40); equations (6.51) and (6.53) from (6.34); equation (6.52) from (6.33); equations (6.54) and (6.55) follow from (6.41).

will be equal at any point.

The relations $\Omega = \tau$ or Eq. (6.56), for stationary LRS II or LRS I space-times, respectively, have the advantage of not depending directly on the magnetic components of the Weyl tensor and they can replace one of Eqs. (6.46) – (6.55). As we shall see, it is useful to remove Eq. (6.50).

Finally, to close the system we will need an equation of state that relates the pressure of the fluid with its energy density: $p = p(\mu)$; and an equation that relates the energy density of the fluid with the intrinsic hypermomentum: $\delta = \delta(p(\mu), \mu)$.

6.4 Generalized TOV equation for stationary LRS I and LRS II space-times

With the full set of structure equations we are finally in position to make the derivation of the generalized TOV equations. Let us start by introducing the scalar function

$$K = \frac{8\pi}{3}\mu - \mathcal{E} + \frac{1}{4}\phi^2 - 3\Omega^2 + 2\Omega\tau\,, \tag{6.57}$$

with the following property

$$\hat{K} = -\phi K\,, \tag{6.58}$$

found from the structure equations. Equation (6.57) generalizes the expressions in Refs. [43, 162].[6] Moreover, following from the fact that the Gauss equation is unchanged by the presence of torsion, Eq. (3.26), we show in Appendix (C.2) that, in the cases where the vector fields u and e are hypersurface orthogonal, the quantity K represents the Gaussian curvature of the 2-sheet orthogonal to both u and e.

Now, following the treatment in Refs. [51, 52], without loss of generality, let us re-parameterize the integral curves of e using, in general, a non-affine parameter ρ, such that for an arbitrary scalar field F

$$\hat{F} = \phi F_{,\rho}\,. \tag{6.59}$$

In particular we have

$$K_{,\rho} = -K\,. \tag{6.60}$$

[6] Notice that in Ref. [162] there is a small typographic error.

Introducing the following set of variables

$$
\begin{aligned}
\mathbb{X} &= \frac{\phi_{,\rho}}{\phi}\,, & \mathbb{B}_1 &= \frac{\mathsf{H}}{\phi^2}\,, & \mathcal{M} &= 8\pi\frac{\mu}{\phi^2}\,,\\
\mathbb{Y} &= \frac{\mathcal{A}}{\phi}\,, & \mathbb{B}_2 &= \frac{\bar{\mathsf{H}}}{\phi^2}\,, & \mathcal{P} &= 8\pi\frac{p}{\phi^2}\,,\\
\mathbb{E} &= \frac{\mathcal{E}}{\phi^2}\,, & \mathbb{D}_1 &= \frac{\mathcal{H}}{\phi^2}\,, & \Delta &= \frac{\delta}{\phi}\,,\\
\mathbb{T} &= \frac{\tau}{\phi}\,, & \mathbb{D}_2 &= \frac{\bar{\mathcal{H}}}{\phi^2}\,, & &\\
\mathbb{W} &= \frac{\Omega}{\phi}\,, & \mathcal{K} &= \frac{K}{\phi^2}\,, & &
\end{aligned}
\tag{6.61}
$$

we can re-write Eqs. (6.46) – (6.55) as

$$2\mathbb{Y}_{,\rho} + 2\mathbb{Y}\left(\mathbb{X}+\mathbb{Y}+1\right) = \mathcal{M} + 3\mathcal{P} - 4\mathbb{W}^2\,, \tag{6.62}$$

$$\mathcal{K}_{,\rho} + \mathcal{K}\left(2\mathbb{X}+1\right) = 0\,, \tag{6.63}$$

$$\mathcal{P}_{,\rho} + \mathcal{P}\left(2\mathbb{X}+\mathbb{Y}\right) + \mathbb{Y}\mathcal{M} = 2\mathbb{T}\mathbb{W}\left(\mathbb{X}+\mathbb{Y}\right) + 2\mathbb{T}\mathbb{W}_{,\rho}\,, \tag{6.64}$$

$$2\mathbb{W}_{,\rho} + \mathbb{W}\left(2\mathbb{X}+1\right) = \mathbb{B}_1\,, \tag{6.65}$$

with the constraints

$$\mathcal{M} + 3\mathcal{P} - 3\mathbb{Y} - 3\mathbb{E} - 6\mathbb{W}^2 = 0\,, \tag{6.66}$$

$$2\mathcal{M} + 2\mathbb{X} + 2\mathcal{P} - 2\mathbb{Y} - 4\mathbb{W}^2 + 1 = 0\,, \tag{6.67}$$

$$4\mathbb{Y} + 4\mathbb{W}\left(2\mathbb{T} - \mathbb{W}\right) - 4\mathcal{P} - 4\mathcal{K} + 1 = 0\,, \tag{6.68}$$

$$\mathbb{D}_1 + \mathbb{W}\left(2\mathbb{Y}-1\right) - 2\mathbb{Y}\mathbb{T} = 0\,, \tag{6.69}$$

$$\mathbb{B}_2 + \mathbb{W}\left(1 - 2\mathbb{Y}\right) = 0\,, \tag{6.70}$$

$$\mathbb{B}_1 + \mathbb{B}_2 + 2\mathbb{D}_2 = 0\,, \tag{6.71}$$

$$\mathbb{T} = \Delta\,, \tag{6.72}$$

and, depending on whether we are considering stationary LRS I or LRS II space-times, we have the extra equation

$$\begin{cases} \mathbb{W}_{,\rho} - \mathbb{T}_{,\rho} = \left(1 - \mathbb{Y} + \mathbb{X}\right)\left(\mathbb{T} - \mathbb{W}\right) & \text{, if LRS I,}\\ \mathbb{W} = \mathbb{T} & \text{, if LRS II.} \end{cases} \tag{6.73}$$

The system is closed provided and equation of state such that $\mathcal{P} = \mathcal{P}\left(\mathcal{M}\right)$ and a relation such that $\Delta = \Delta\left(\mathcal{P}(\mathcal{M}), \mathcal{M}\right)$.

Now, using Eqs. (6.67) and (6.68) to eliminate $\mathbb{X}$ and $\mathbb{Y}$ in Eqs. (6.64) and (6.73) we find

$$\begin{aligned}
\mathcal{P}_{,\rho} &= -\mathcal{P}^2 + \mathcal{P}\left[\frac{7}{4} - 3\mathcal{K} + \mathbb{W}\left(8\Delta - 7\mathbb{W}\right)\right] + \\
&\quad + 2\Delta\left[\left(2\mathcal{K} - 1\right)\mathbb{W} - 4\mathbb{W}^2\left(\Delta - \mathbb{W}\right) + \mathbb{W}_{,\rho}\right] + \\
&\quad + \mathcal{M}\left(\frac{1}{4} - \mathcal{K} + \mathcal{P} - \mathbb{W}^2\right), \\
\mathcal{K}_{,\rho} &= -2\mathcal{K}\left(3\mathbb{W}^2 - 2\Delta\mathbb{W} + \mathcal{K} - \mathcal{M} - \frac{1}{4}\right), \\
\mathcal{P} &= \mathcal{P}\left(\mathcal{M}\right), \\
\Delta &= \Delta\left(\mathcal{P}(\mathcal{M}), \mathcal{M}\right),
\end{aligned} \tag{6.74}$$

and

$$\begin{cases} \mathbb{W}_{,\rho} - \Delta_{,\rho} = \frac{1}{2}\left(\Delta - \mathbb{W}\right)\left(1 - 2\mathcal{M} - 2\mathcal{P} + 4\mathbb{W}^2\right) & \text{, if LRS I} \\ \mathbb{W} = \Delta & \text{, if LRS II} \end{cases} \tag{6.75}$$

which represent the covariant TOV equations. The system is completed by the extra relations:

$$\mathcal{K} - \frac{1}{4} + \mathcal{P} + \mathbb{W}\left(\mathbb{W} - 2\Delta\right) = \mathbb{Y}, \tag{6.76}$$

$$\mathcal{K} - \frac{3}{4} - \mathcal{M} + \mathbb{W}\left(3\mathbb{W} - 2\Delta\right) = \mathbb{X}, \tag{6.77}$$

$$\mathcal{M} + \mathbb{W}\left(6\Delta - 9\mathbb{W}\right) - 3\left(\mathcal{K} - \frac{1}{4}\right) = 3\mathbb{E}, \tag{6.78}$$

$$2\mathbb{W}_{,\rho} + \mathbb{W}\left(2\mathcal{K} - 2\mathcal{M} - 4\Delta\mathbb{W} + 6\mathbb{W}^2 - \frac{1}{2}\right) = \mathbb{B}_1, \tag{6.79}$$

$$\mathbb{W}\left(2\mathcal{K} + 2\mathcal{P} - 4\Delta\mathbb{W} + 2\mathbb{W}^2 - \frac{3}{2}\right) = \mathbb{B}_2, \tag{6.80}$$

$$\mathbb{B}_1 + \mathbb{B}_2 + 2\mathbb{D}_2 = 0, \tag{6.81}$$

$$\mathbb{W}\left(6\Delta\mathbb{W} - 2\mathbb{W}^2 - 4\Delta^2 + 1\right) + 2\left(\Delta - \mathbb{W}\right)\left(\mathcal{K} + \mathcal{P} - \frac{1}{4}\right) = \mathbb{D}_1. \tag{6.82}$$

6.4.1 The static case

The full set of Eqs. (6.74) – (6.82) completely describe the geometry of a stationary LRS I or LRS II space-time filled by an Weyssenhoff fluid. Let us now consider the particular cases when the space-time is static, that is the case when $\mathbb{W} = \mathbb{T} = \Delta$.

Introducing the following quantities

$$\begin{aligned} \mathscr{M} &= \mathcal{M} - \Delta^2\,, \\ \mathscr{P} &= \mathcal{P} - \Delta^2\,, \\ \mathscr{E} &= \mathbb{E} + \frac{2}{3}\Delta^2\,, \end{aligned} \tag{6.83}$$

Eqs. (6.74) are given by

$$\mathscr{P}_{,\rho} = -\mathscr{P}^2 + \mathscr{P}\left[\mathscr{M} + 1 - 3\left(\mathcal{K} - \frac{1}{4}\right)\right] - \mathscr{M}\left(\mathcal{K} - \frac{1}{4}\right)\,, \tag{6.84}$$

$$\mathcal{K}_{,\rho} = -2\mathcal{K}\left(\mathcal{K} - \frac{1}{4} - \mathscr{M}\right)\,. \tag{6.85}$$

$$\mathbb{Y} = \mathcal{K} - \frac{1}{4} + \mathscr{P}\,, \tag{6.86}$$

$$\mathbb{X} = \mathcal{K} - \frac{3}{4} - \mathscr{M}\,, \tag{6.87}$$

$$3\mathscr{E} = \mathscr{M} - 3\left(\mathcal{K} - \frac{1}{4}\right)\,, \tag{6.88}$$

which match exactly the expressions found in the theory of General Relativity (cf. Ref. [51]) for an effective energy density and pressure and the corrected electric part of the Weyl tensor: $\mathscr{M}$,$\mathscr{P}$ and $\mathscr{E}$. Note that the extra constraints for the magnetic components of the Weyl tensor

$$2\Delta_{,\rho} + 2\Delta\left(\mathcal{K} - \mathscr{M} - \frac{1}{4}\right) = \mathbb{B}_1\,, \tag{6.89}$$

$$\mathbb{B}_2 + \Delta\left[1 - 2\mathscr{P} - 2\left(\mathcal{K} - \frac{1}{4}\right)\right] = 0\,, \tag{6.90}$$

$$\mathbb{B}_1 + \mathbb{B}_2 + 2\mathbb{D}_2 = 0\,, \tag{6.91}$$

$$\mathbb{D}_1 = \Delta\,, \tag{6.92}$$

imply that the geometry of the space-time is fundamentally different from the corresponding one in GR. Nonetheless, the fact that Eqs. (6.84) – (6.88) have the same form for the corrected quantities in Eq. (6.83) lead us to the notable result:

Proposition 6.1. *At the level of the metric, all static, locally rotationally symmetric of class II solutions of the theory of General Relativity for a perfect fluid with stress-energy tensor* $\left(\mathcal{T}^{GR}\right)_{\alpha\beta} = \mu\, u_\alpha u_\beta + p h_{\alpha\beta}$, *are also solutions of Einstein-Cartan-Sciama-Kibble theory sourced by a Weyssenhoff fluid with stress-energy tensor* $\left(\mathcal{T}^{EC}\right)_{\alpha\beta} = \left(\mu + \frac{\delta^2}{8\pi}\right)\, u_\alpha u_\beta + \left(p + \frac{\delta^2}{8\pi}\right) h_{\alpha\beta}$.

It is important to stress that, because of the nature of the corrections in Eq. (6.83), solutions which are unacceptable in GR due, for example, to negative energy densities or pressure, might still correspond to physically acceptable ones in the ECSK case.

In the remaining of this chapter we will consider the case of static spherically symmetric space-times, hence, we will study solutions of Eqs. (6.83) – (6.92).

6.5 1+1+2 junction of static LRS II space-times with torsion

To apply the generalized TOV equations of the previous section to find and study exact solutions suitable to model astrophysical objects, we will need to impose boundary conditions. These will be provided by the junction formalism studied in Chapter 3. Let us then express conditions (3.52) – (3.54) covariantly in the specific case of two static LRS II space-times endowed with a torsion tensor field of the form $S_{\alpha\beta}{}^{\gamma} = \varepsilon_{\alpha\beta}u^{\gamma}\tau$, where τ is a generic function of the space-time coordinates.

In what follows we will be interested in the case when the interior and exterior space-times are to be smoothly matched at a time-like hypersurface, $\mathcal{S}$, orthogonal to the vector field e. Then, condition (3.52) reads

$$[N_{\alpha\beta} - u_\alpha u_\beta]_\pm = 0\,, \tag{6.93}$$

where $N_{\alpha\beta}$ verifies Eq. (6.3). Using Eq. (6.7), in the considered setup, Eq. (3.53) is simply

$$\left[\frac{1}{2}\phi N_{\alpha\beta} - \mathcal{A}u_\alpha u_\beta\right]_\pm = 0\,, \tag{6.94}$$

which, contracting with the induced metric at $\mathcal{S}$ and using Eq. (6.93), gives

$$[\phi + \mathcal{A}]_\pm = 0\,. \tag{6.95}$$

From Eqs. (6.93) – (6.95) we find that at the matching surface the following constraints have to be met

$$[\phi]_\pm = 0\,, \tag{6.96}$$

$$[\mathcal{A}]_\pm = 0\,, \tag{6.97}$$

implying, for $\phi \neq 0$,

$$[\mathbb{Y}]_\pm = 0\,. \tag{6.98}$$

Given that e is continuous across $\mathcal{S}$, we can integrate Eq. (6.60), finding $K = k_0 e^{-\rho}$. Using Eq. (6.61) and (6.96) we have

$$[\mathcal{K}]_\pm = 0\,. \tag{6.99}$$

Using the previous results in Eq. (6.86) we arrive at

$$\left[8\pi p - \delta^2\right]_\pm = 0\,. \tag{6.100}$$

Finally, for the specific type of torsion that we consider in this chapter, condition (3.54) imposes

$$[\delta]_{\pm} = 0\,, \tag{6.101}$$

then, from Eq. (6.100),

$$[p]_{\pm} = 0\,. \tag{6.102}$$

We have then found that for a smooth matching between two static LRS II space-times endowed with a torsion tensor field of the form $S_{\alpha\beta}{}^{\gamma} = \varepsilon_{\alpha\beta} u^{\gamma} \tau$, both the pressure of the fluid and the intrinsic spin density, as seen from each space-time, must match at $\mathcal{S}$.

6.6 Exact solutions for static LRS II space-times

Given the set of structure equations (6.83) – (6.92) that describe the behavior of a static, LRS II space-time permeated with a Weyssenhoff fluid, let us now find and study some exact solutions.

As was stated before, the system of structure equations is not closed until an equation of state and an expression for the spin density are provided. Let us then consider some particular relations for the pressure, energy and spin densities of the fluid in order to gain some insight into the behavior of compact objects in a fully relativistic theory with non-null intrinsic spin.

For the remaining of the chapter we will consider only the particular case of spherically symmetric space-times. Moreover, in what follows we will refer to static, spherically symmetric compact objects as "stars". Although this is an abuse of language, it is also a trend in the literature since such systems are expected to be a good model for slowly varying astrophysical bodies.

6.6.1 Effective constant energy-density and the Buchdahl limit

We start by considering the case of a system where the effective energy density is assumed to be constant, that is

$$8\pi\mu - \delta^2 = \tilde{\mu}_0\,, \tag{6.103}$$

where $\tilde{\mu}_0 \in \mathbb{R}$. Notice that, contrary to the case of null torsion, the above assumption does not have to imply that the energy density, μ, is constant.

Using Eqs. (6.58) and (6.59) we have

$$K(\rho) = \frac{e^{-\rho}}{r_0^2}\,, \tag{6.104}$$

where r_0 is an integration constant. Eq. (6.104) then yields

$$\mathscr{M}(\rho) = \tilde{\mu}_0 r_0^2 e^{\rho} \mathcal{K}(\rho) \ . \tag{6.105}$$

Eq. (6.105) allows us to solve Eq. (6.85), finding

$$\mathcal{K}(\rho) = \frac{3}{12 - 4\tilde{\mu}_0 r_0^2\, e^{\rho} + 3\mathcal{K}_0 e^{-\frac{\rho}{2}}}\,, \tag{6.106}$$

where $\mathcal{K}_0$ is yet another integration constant. Setting $\mathcal{K}_0 = 0$ to avoid a conical singularity at $\rho \to -\infty$ [66], the structure equations yield

$$p(\rho) - \frac{\delta(\rho)^2}{8\pi} = -\frac{\tilde{\mu}_0 \left(P_0 + 3\sqrt{3 - \tilde{\mu}_0 r_0^2 e^{\rho}}\right)}{24\pi \left(P_0 + \sqrt{3 - \tilde{\mu}_0 r_0^2 e^{\rho}}\right)}\,, \tag{6.107}$$

$$\mathcal{A}(\rho) = -\frac{\tilde{\mu}_0 r_0 e^{\frac{\rho}{2}}}{\sqrt{3}\left(P_0 + \sqrt{3 - \tilde{\mu}_0 r_0^2 e^{\rho}}\right)}\,, \tag{6.108}$$

$$\phi(\rho) = \frac{2}{r_0\sqrt{3}} e^{-\frac{\rho}{2}} \sqrt{3 - \tilde{\mu}_0 r_0^2 e^{\rho}}\,, \tag{6.109}$$

$$\mathcal{E} = -\frac{2}{3}\delta^2\,, \tag{6.110}$$

where we have chosen the direction of e so that ϕ is positive, and the value of the integration constant P_0 is to be determined by the boundary conditions.

Let us now assume that relations (6.103) - (6.110) describe the interior of a compact object matched at a boundary $\mathcal{S}$ to an exterior space-time modeled by the Schwarzschild vacuum solution. From Eqs. (6.101) and (6.102) we find that the quantity in Eq. (6.107) must be zero at the boundary. Setting, without loss of generality, the matching hypersurface to be at $\rho = 0$, we find

$$P_0 = -3\sqrt{3 - \tilde{\mu}_0 r_0^2}\,. \tag{6.111}$$

The matching conditions, Eqs. (6.96) and (6.99), imply that interior and exterior observers agree on the value of circumferential radius of $\mathcal{S}$, say r_0, and the Schwarzschild parameter, M, is given by

$$M = \frac{\tilde{\mu}_0 r_0^3}{6}\,. \tag{6.112}$$

Moreover, from condition (6.101) we find that the spin density must go to zero at the matching surface, that is, $\delta(\rho = 0) = 0$.

Given the previous results, we are now in position to study some effects arising from the presence of intrinsic spin in compact objects. In the remaining of this subsection, for clarity, we shall write the results in terms of the circumferential radius r. Using the fact that, in the considered setup, the quantity K, Eq. (6.57), represents the Gaussian curvature of the 2-sheet, we have that the parameter ρ and r are related by

$$\rho = 2\ln\left(\frac{r}{r_0}\right) , \tag{6.113}$$

where we have set the value of the arbitrary scaling factor to be r_0.

Now, defining the central pressure $p_c := p\left(\rho \to -\infty\right)$, from Eqs. (6.107) and (6.111), we have

$$p_c = -\frac{\mu_c\left(1-\sqrt{1-\frac{2M}{r_0}}\right)}{1-3\sqrt{1-\frac{2M}{r_0}}} + \frac{\delta_c^2\left(1-2\sqrt{1-\frac{2M}{r_0}}\right)}{4\pi\left(1-3\sqrt{1-\frac{2M}{r_0}}\right)} , \tag{6.114}$$

where $\mu_c \equiv \mu\left(\rho \to -\infty\right)$ and $\delta_c \equiv \delta\left(\rho \to -\infty\right)$. If we compared directly the above expression to a similar system in GR, we would see that the second term on the RHS of Eq. (6.114) represents an explicit contribution due to the presence of intrinsic spin. However, there is a subtlety: Eq. (6.112) indicates that the presence of spin also modifies the matching radius r_0 and the value of the Schwarzschild parameter M, making it difficult to draw conclusions only on (6.114).

A clearer idea of the differences between our case and GR can be obtained by computing the maximum mass that can be held by a star with constant radius. Considering Eq. (6.103) and if neither the densities μ_c and δ_c diverge, the central pressure in Eq. (6.114) will go to infinity when $r_0 = \frac{9}{4}M$ or, using Eq. (6.112), when

$$M_{\text{max}} = \frac{4}{9\sqrt{3\pi}}\left(\mu - \frac{\delta^2}{8\pi}\right)^{-\frac{1}{2}} . \tag{6.115}$$

This results makes it clear that, when compared to a system with the same energy density μ in GR, the presence of intrinsic spin increases the maximum allowed mass.

In analogy with the calculation of the Buchdahl limit in GR we can generalize this discussion to non-constant corrected energy density. Consider the quantity $\tilde{\mu} := 8\pi\mu - \delta^2$ and assume it to be non-negative and $d\tilde{\mu}/dr \leq 0$, for $r \in [0, r_0]$. Following the same reasoning of Ref. [1] (see Appendix (C.3) for the derivation) we can find an upper limit for the amount of mass a star with constant radius can hold:

$$\frac{m\left(r_0\right)}{r_0} \leq \frac{4}{9} , \tag{6.116}$$

with

$$m\left(r_0\right) = \frac{1}{2}\int_0^{r_0} \tilde{\mu}\left(r\right)\, r^2 dr . \tag{6.117}$$

At first sight, the expression in Eq. (6.116) matches the one found by Buchdahl [1] for GR. However, there is a correction due to the presence of spin in the function $m\left(r_0\right)$, Eq. (6.117), leading us to conclude that for the same value of the circumferential radius, r_0, a star can hold more matter in the presence of intrinsic spin than in the null-spin case. It is also worth mentioning that the quantity $m(r_0)$ agrees with the value of the Schwarzschild parameter of the exterior space-time, therefore, the gravitational mass of such objects is determined not only by the energy density, μ, but also by the spin density, δ, which was expected because of the specific way in which intrinsic spin gravitates in the considered setup.

As a final comment, although *a priori* there is nothing that forces δ^2 to be smaller than $8\pi\mu$, it is expected that in stars, even neutron stars, $\delta^2 \ll \mu$ (see Refs. [29, 163]), hence, $\tilde{\mu} \geq 0$, as was assumed in the derivation of Eq. (6.116). On the other hand, the requirement that $d\tilde{\mu}/dr \leq 0$ might not be as physically reasonable as in the case of GR since, as we will see bellow, the presence of intrinsic spin allows for a richer possible behavior for the matter variables.

6.6.2 Spin held stars

In the previous subsection we have considered a classical model for a relativistic star which is similar to the simplest model for this type of objects in GR. However, the presence of intrinsic spin allows for solutions which are not contemplated in the Einstein's theory. The prototype of such objects is a star which is supported only by the gravitation of the spin of the Weyssenhoff fluid. In the remaining of the subsection, we will analyze this case and prove the following result

Proposition 6.2. *There are no static, spherically symmetric solutions of the Einstein-Cartan-Sciama-Kibble theory sourced by a Weyssenhoff fluid with null isotropic pressure that have all the following properties*

(i) $\delta(r)$ *is non-null for* $r \in [0, r_0[$ *and* $\delta(r_0) = 0$*, for some* $r_0 > 0$*;*

(ii) $\delta^2(r)$ *is a monotonically decreasing function for all* $r \in [0, r_0]$*;*

(iii) *the spin and energy density functions:* $\delta(r)$ *and* $\mu(r)$*, are at least of class* C^1 *and the function* $\mathcal{A}(r)$ *is differentiable for all* $r \in [0, r_0]$*;*

(iv) *the function* $M(r) := \frac{1}{2}\int_0^r [8\pi\,\mu(r) - \delta^2(r)]\, x^2 dx$ *is such that* $2M(r) < r$*, for all* $r \in]0, r_0]$*.*

To prove Proposition 6.2 we will consider first the behavior of the quantities of interest in a neighborhood of the center, $r = 0$, and then on the boundary of the star. In doing so, in order to make the reasoning more intuitive, we shall consider here that the integral curves of the vector field e are parameterized by the circumferential radius r.

Defining the quantities

$$\begin{aligned} \tilde{\mu}(r) &= 8\pi\,\mu(r) - \delta^2(r)\ , \\ \tilde{p}(r) &= 8\pi\,p(r) - \delta^2(r)\ , \end{aligned} \tag{6.118}$$

we then find from the structure equations

$$\frac{r}{2}\phi\left(r\right)\tilde{p}_{,r}=-\mathcal{A}\left(\tilde{\mu}+\tilde{p}\right)\,, \tag{6.119}$$

$$\frac{r}{2}\phi\left(r\right)\mathcal{A}_{,r}+\mathcal{A}^2+\mathcal{A}\phi=\frac{1}{2}\left(\tilde{\mu}+3\tilde{p}\right)\,, \tag{6.120}$$

$$\tilde{p}=\mathcal{A}\phi-K+\frac{1}{4}\phi^2\,, \tag{6.121}$$

with

$$K\left(r\right)=\frac{1}{r^2}\,, \tag{6.122}$$

$$\phi\left(r\right)=\frac{2}{r}\sqrt{1-\frac{2M\left(r\right)}{r}}\,, \tag{6.123}$$

and

$$M\left(r\right)=\frac{1}{2}\int_0^r\tilde{\mu}\left(x\right)x^2dx\,, \tag{6.124}$$

where, without loss of generality, we chose the direction of e so that $\phi\left(r\right)$ is non-negative. Moreover, from Eqs. (6.121) and (6.123), we find the useful relation

$$\mathcal{A}\phi=\frac{2M\left(r\right)}{r^3}+\tilde{p}\,. \tag{6.125}$$

We will consider now the case of a static, spherically symmetric compact object held entirely by intrinsic spin, that is, the case when $p\left(r\right)=0$ and $\tilde{p}\left(r\right)=-\delta^2\left(r\right)$, smoothly matched to an exterior space-time modeled by a vacuum solution of the ECSK field equations. Moreover, we will assume that for $r>0$, $2M\left(r\right)<r$, otherwise the scalar $\phi\left(r\right)$ would take complex values.

6.6.2.1 Behavior at the center

Assuming that the functions $\mu\left(r\right)$, $\delta^2\left(r\right)\in C^1$ we can write in a small enough neighborhood of $r=0$:

$$\begin{aligned}\mu\left(r\right)&=\mu\left(0\right)+\mu_{,r}\left(0\right)\,r\,,\\ \delta^2\left(r\right)&=\delta^2\left(0\right)+\left(\delta^2\right)_{,r}\left(0\right)\,r\,,\end{aligned} \tag{6.126}$$

where comma represents partial — or total — derivative with respect to the variable in front. From Eqs. (6.126), we find that in a small enough neighborhood of $r=0$, the mass function (6.124) is described by

$$2M\left(r\right)=\frac{\tilde{\mu}\left(0\right)}{3}r^3+\frac{1}{4}\left(\tilde{\mu}_{,r}\left(0\right)\right)r^4\,. \tag{6.127}$$

In particular, we find that $M\left(r\right)$ goes to zero at least as fast as r^3.

Now, Eqs. (6.119) and (6.125) yield

$$\frac{2}{r}\left(1-\frac{2M\left(r\right)}{r}\right)\frac{d\delta^2}{dr}=\left(\frac{2M\left(r\right)}{r^3}-\delta^2\right)\left(\tilde{\mu}-\delta^2\right)\,. \tag{6.128}$$

In a region where $r\in\left[0,\epsilon\right[$, with $\epsilon\ll 1$, the RHS of this equation takes values in $\mathbb{R}$, therefore

$$\left(\delta^2\right)_{,r}\left(0\right)=0\,. \tag{6.129}$$

Repeating the same reasoning in Eq. (6.125) we find that

$$\mathcal{A}\left(0\right)=0\,. \tag{6.130}$$

Let us now assume that there exists an $r_a>0$ where for $r\in\left]0,r_a\right[$, $\mathcal{A}\left(r\right)>0$. From Eq. (6.119) we will find that in this region $\tilde{\mu}+\tilde{p}\leq 0$ which implies that $\tilde{\mu}+3\tilde{p}<0$. Then, from Eq. (6.120) we find that $\mathcal{A}_{,r}\left(r\right)<0$, for all $r\in\left]0,r_a\right[$. This, however, violates the initial hypothesis since, $\mathcal{A}\left(0\right)=0$ and we assume that $\mathcal{A}\left(r\right)>0$, for $r\in\left]0,r_a\right[$, that is, $\mathcal{A}_{,r}\left(r\right)$ would have to be positive for some $r\in\left[0,r_a\right[$.

Another possibility is that for a region $r\in\left[0,r_a\right]$, $\mathcal{A}\left(r\right)=0$ and for $r\in\left]r_a,r_c\right]$ with $r_c>r_a$, $\mathcal{A}\left(r\right)>0$. If this were the case, since for $r\in\left]r_a,r_c\right]$, $\mathcal{A}\left(r\right)>0$, there would exist a value $r_b\in\left]r_a,r_c\right]$ such that $\mathcal{A}_{,r}\left(r_b\right)>0$. Using this in Eq. (6.120), at $r=r_b$ we find

$$\tilde{\mu}+3\tilde{p}>0\Rightarrow\tilde{\mu}+\tilde{p}>0\,, \tag{6.131}$$

but from Eq. (6.119) and imposing that $\left(\delta^2\right)_{,r}\leq 0$ we find that: $\left.\tilde{\mu}+\tilde{p}\right|_{r=r_b}\leq 0$, contradicting (6.131).

Another possibility is that $\mathcal{A}\left(r\right)=0$, for all $r\in\left[0,r_0\right]$. From Eqs. (6.119) and (6.120) this simply represents a vacuum solution as such it does not represent a solution for a compact object.

Gathering this results we conclude that there exists an $r_d>0$ such that in the region $\left[0,r_d\right[$, $\mathcal{A}\left(r\right)\leq 0$ and it must take negative values in some sub-region.

6.6.2.2 Behavior at the boundary

Let us now define the boundary of the compact object as the hypersurface at which the spin density goes to zero, that is, $\delta^2\left(r_0\right)=0$. In such hypersurface we have three possible behaviors for the function $\mathcal{A}$:

(i) $\mathcal{A}\left(r_0\right)<0$;

(ii) $\mathcal{A}\left(r_0\right)>0$;

(iii) $\mathcal{A}\left(r_0\right)=0$.

Let us consider each case separately.

(i) The case $\mathcal{A}(r_0) < 0$

From Eq. (6.125) we have that at $r = r_0$

$$\mathcal{A}\phi|_{r=r_0} = \frac{2M(r_0)}{r_0^3} < 0\,. \tag{6.132}$$

Therefore, from Eq. (6.124) there exists a region $]r_f, r_g[$ where

$$\tilde{\mu}(r) < 0\,, \tag{6.133}$$

then $\tilde{\mu}+\tilde{p} < 0$, in that region. From Eq. (6.119), to guarantee that the intrinsic spin density is a monotonically decreasing function of r, we find that $\mathcal{A}(r) \geq 0$, for $r \in\,]r_f, r_g[$. So, either $\mathcal{A}(r) = 0 \wedge \mathcal{A}_{,r}(r) = 0$ for all $r \in\,]r_f, r_g[$, that is, the function $\mathcal{A}(r)$ takes the value zero and stays zero for all $r \in\,]r_f, r_g[$; or $\mathcal{A}(r) > 0$ for some $r \in\,]r_f, r_g[$. The former case is not possible: from Eq. (6.120), $\tilde{\mu}(r) + 3\tilde{p}(r) = 0$, hence, $\tilde{\mu}(r) \geq 0$, for all $r \in\,]r_f, r_g[$, which contradicts the inequality (6.133). As for the latter — the case when $\mathcal{A}(r) > 0$, for some $r \in\,]r_f, r_g[$ — in the previous sub-section it was shown that for some sub-region of $[0, r_d[$, $\mathcal{A}(r) \leq 0$, therefore the region $]r_f, r_g[$ cannot be a sub-region of $[0, r_d[$. With this said, since $\mathcal{A}(r)$ is a differentiable function, there exists a region with, say, $r = r_e < r_0$, where $\mathcal{A}(r_e) > 0 \wedge \mathcal{A}_{,r}(r_e) > 0$. Then, from Eq. (6.120)

$$\tilde{\mu} + 3\tilde{p}|_{r=r_e} > 0 \Rightarrow \tilde{\mu} + \tilde{p}|_{r=r_e} > 0\,. \tag{6.134}$$

However, substituting this results in Eq. (6.119) we find: $(\delta^2)_{,r}(r_e) > 0$, which contradicts the assumption that the spin density is a monotonically decreasing function.

(ii) The case $\mathcal{A}(r_0) > 0$

For the case when $\mathcal{A}(r_0) > 0$, we can simply repeat the proof in the previous sub-subsection and conclude in the same way that the assumptions are violated in a region. We just remark that the point with radial coordinate $r = r_e$, in the proof, can always be chosen such that $r_e < r_0$ since, for whatever the value of $\mathcal{A}(r_0) > 0$, there is a point where $0 < \mathcal{A}(r < r_0) < \mathcal{A}(r_0)$.

(iii) The case $\mathcal{A}(r_0) = 0$

In this the case when $\mathcal{A}(r_0) = 0$ we have, from Eq. (6.125) that

$$M(r_0) = 0\,. \tag{6.135}$$

From this we have three possibilities:

(a) $\tilde{\mu}(r) = 0$, for $r \in [0, r_a]$;

(b) $\tilde{\mu}(r) < 0$, for $r \in\,]0, r_a]$;

(c) $\tilde{\mu}(r) > 0$, for $r \in]0, r_a]$;

for some $r_a > 0$.

Let us consider each case individually.

(a) In the case when $\tilde{\mu}(r) = 0$, for $r \in [0, r_a]$ we have from Eq. (6.125) that $\mathcal{A}(r) < 0$. However, using this result in (6.119) we see that it implies that the spin density is an increasing function of r, violating the hypothesis.

(b) In the case when the corrected energy density is such that $\tilde{\mu}(r) < 0$, for $r \in]0, r_a]$, from (6.124) we have that the mass function is negative, in this region. From Eq. (6.125) we than conclude that $\mathcal{A}(r) < 0$, $r \in]0, r_a]$. However, going back to Eq. (6.119) we find that the spin density is an increasing function of r, violating the hypothesis.

(c) Finally, consider the case when $\tilde{\mu}(r) > 0$, for $r \in]0, r_a]$. From (6.124), this implies that the mass function is positive in this region. Since Eq. (6.135) must be verified, there must be a region where $\tilde{\mu}(r) < 0$. We can then repeat the arguments of the case $\mathcal{A}(r_0) < 0$, which lead to the conclusion that the hypothesis would be violated in some region inside the star.

Gathering the previous results we have proven the result in Proposition 6.2.

We end this Section by remarking that if instead of imposing $\tilde{p} = -\delta^2$ we only imposed that $p < \delta^2$, that is, the thermodynamical pressure is always smaller than the correction due to the spin density, then all the previous results are valid if $\tilde{p}_{,r} \geq 0$. Notice, however, that in this scenario this condition, simply measures the gradient of the quantity $8\pi p - \delta^2$.

6.6.3 Reconstructing exact solutions

As in the case of the theory of General Relativity, when torsion is present it is possible to use the covariant TOV equations to generate exact solutions via reconstruction algorithms [51, 52]. The idea is to assign a given metric tensor and deduce the corresponding behavior of the energy density, pressure and spin density.

Analyzing Eqs. (6.62) – (6.63) and using Eqs. (6.66) and (6.68), shows that differently from the case of anisotropic compact objects in General Relativity [52], the structure equations cannot be solved for the spin density. This implies that the reconstruction algorithm can only be used if an additional relation is provided, either relating the spin density to the other matter variables or an equation of state for matter.

In the following we will show some applications of this algorithms which return some interesting solutions from a physical point of view.

6.6.3.1 Connecting the spin density to the energy density: "Buchdhal stars"

A natural additional relation is to have the intrinsic spin density to be proportional to the energy density of the Weyssenhoff fluid. In this case, however, the junction conditions that we have derived in Section 6.5 pose the problem to have both the energy density and the pressure to be zero at the boundary. A class of solutions which are devised to have exactly this property was given by Buchdhal [164]. We will now reconstruct this solutions in the case of Eqs. (6.84) – (6.88).

Consider a spherically symmetric space-time characterized by the line element

$$ds^2 = -A\left(w\right)dt^2 + B\left(w\right)dw^2 + C\left(w\right)\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right), \tag{6.136}$$

where

$$\begin{aligned} \eta\left(w\right) &= \frac{\left(a-1\right)\sin\left(Rw\right)}{Rw}, \qquad & A\left(w\right) &= \frac{a\left(1+a-\eta\right)}{1+a+\eta}, \\ B\left(w\right) &= \frac{\left(1+a+\eta\right)}{a(1+a-\eta)} & C\left(w\right) &= \frac{w^2(1+a+\eta)^2}{4a^2}, \end{aligned} \tag{6.137}$$

with $a \in \mathbb{R}\setminus\{0\}$ and the parameter w is connected to ρ by the relation

$$e^{\rho} = \frac{w^2}{4a^2}\left(1+a+\eta\right)^2 . \tag{6.138}$$

Notice that the circumferential radius, r, vanishes when $w = 0$.

From Eqs. (6.62), (6.63), (6.85) and (6.86), assuming $\mathbb{W} = \Delta$ and $\Delta^2 = \gamma\mathcal{M}$, we find

$$\mathcal{M} = \frac{2\mathcal{K}_{,\rho} + 4\mathcal{K}^2 - \mathcal{K}}{4(1-\gamma)\mathcal{K}}, \tag{6.139}$$

$$\mathcal{P} = Y - \mathcal{K} + \frac{1}{4} - \frac{2\mathcal{K}_{,\rho} + 4\mathcal{K}^2 - \mathcal{K}}{4(1-\gamma)\mathcal{K}}, \tag{6.140}$$

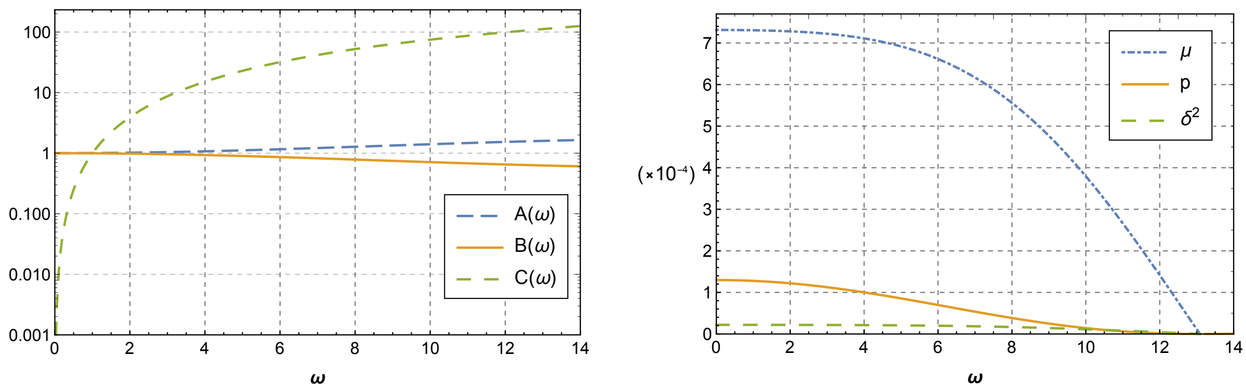
$$0 = (2Y+1)\mathcal{K}_{,\rho} - 4\mathcal{K}^2 - \mathcal{K}\left[4Y_{,\rho} + 4(Y-1)Y - 1\right] . \tag{6.141}$$

The form of Y and $\mathcal{K}$ that satisfies the constraint (6.141) can be found directly from their definition in a general coordinate system (see Refs. [51, 52])

$$\begin{aligned} Y &= \frac{1}{2}\frac{CA_{,w}}{AC_{,w}} = \frac{\left(1+a\right)w\eta_{,w}}{2\left(\eta-a-1\right)\left(1+a+\eta+w\eta_{,w}\right)}, \\ \mathcal{K} &= \frac{BC}{\left(C_{,w}\right)^2} = \frac{a\left(1+a+\eta\right)}{\left(1+a-\eta\right)\left(1+a+\eta+w\eta_{,w}\right)^2} . \end{aligned} \tag{6.142}$$

Using Eqs. (6.139), (6.140) and (6.142), the energy density and the pressure are then given by

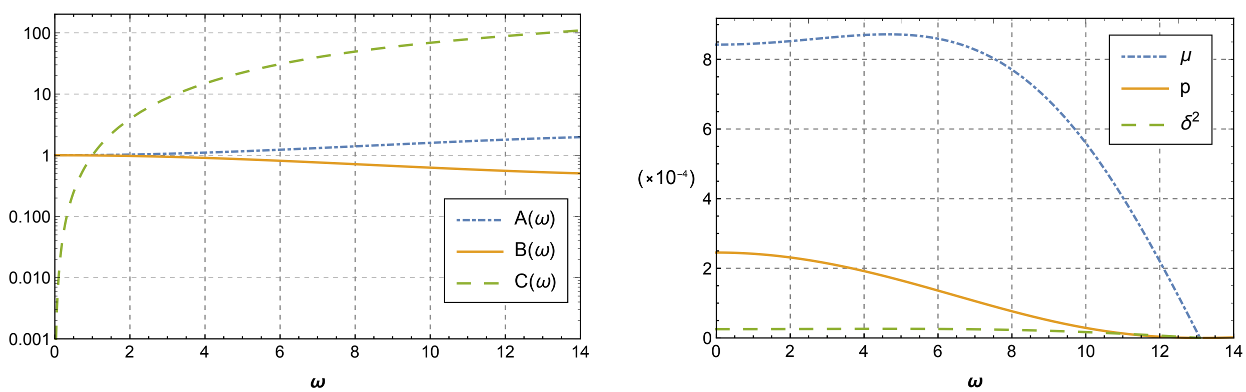
$$\begin{aligned} \mu &= \frac{aR^2\eta\left(3\eta-2-2a\right)}{8\pi\left(\gamma-1\right)\left(1+a+\eta\right)^2}, \\ p &= \frac{aR^2\eta\left[2\gamma\left(2\eta-a-1\right)-\eta\right]}{8\pi\left(\gamma-1\right)\left(1+a+\eta\right)^2} . \end{aligned} \tag{6.143}$$



(a) Coefficients of the metric in Eqs. (6.136) and (6.137).

(b) Thermodynamic quantities in Eqs. (6.143) and the intrinsic spin density.

Figure 3: Plots of the behavior of the metric components, (a) and matter variables (b) associated with the solution in Eqs. (6.136) and (6.137) for $a = 1.6$, $\gamma = 0.03/(8\pi)$ and $R = 0.24$.



(a) Coefficients of the metric in Eqs. (6.136) and (6.137).

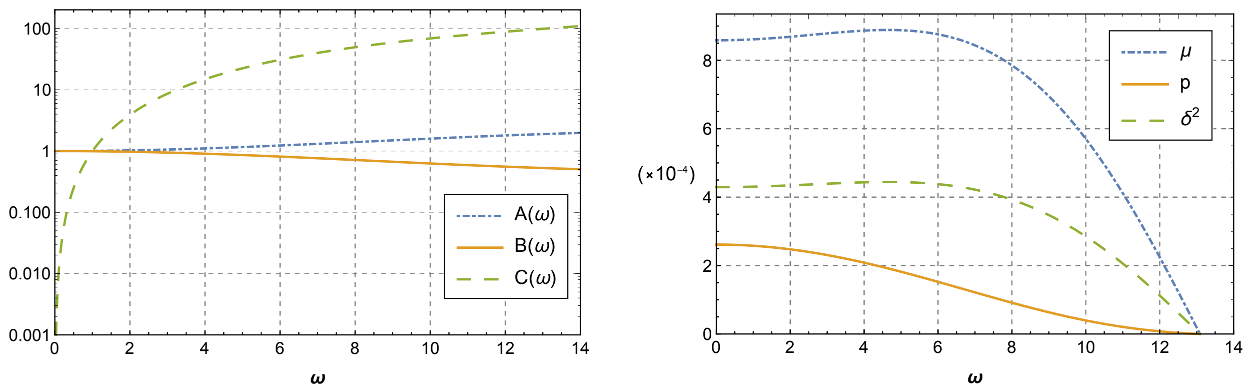
(b) Thermodynamic quantities in Eqs. (6.143) and the intrinsic spin density.

Figure 4: Plots of the behavior of the metric components, (a) and matter variables (b) associated with the solution in Eqs. (6.136) and (6.137) for $a = 1.9$, $\gamma = 0.03/(8\pi)$ and $R = 0.24$.

As said, this family of solutions have, by construction, the property that the pressure, energy and spin densities all vanish at a particular hypersurface. In Figures 3 – 5 we present the behavior of these quantities for a few combinations of the parameters, showing that the values of the parameters a and γ have a direct impact in the profile of the densities, whereas, the parameter R defines the value of w for which the matter variables go to zero. Moreover, from the plots it is clear that the presence of intrinsic spin markedly changes the type of behavior the matter may have. In particular, for certain values of the parameters a and γ the functions μ, p or δ might not be monotonically decreasing functions of the coordinate w.

6.6.3.2 Connecting the spin density to the pressure

Another option that reduces the number of conditions related to the junction, is to associate the spin density to the pressure. This choice, which at first might appear unnatural, corresponds to the case in which the intrinsic



(a) Coefficients of the metric in Eqs. (6.136) and (6.137).

(b) Thermodynamic quantities in Eqs. (6.143) and the intrinsic spin density.

Figure 5: Plots of the behavior of the metric components, (a) and matter variables (b) associated with the solution in Eqs. (6.136) and (6.137) for $a = 1.9$, $\gamma = 0.5/(8\pi)$ and $R = 0.24$.

spin depends on the equation of state. We can imagine that particles with intrinsic spin will create different structures not unlikely to the ones that characterize the crystalline phases of water ice (see, e.g., Ref. [165]). Our ansatz refers to this kind of effects.

The reconstruction equations in this case, setting $\Delta^2 = \gamma\mathcal{P}$, read

$$\mathcal{M} = \frac{4\mathcal{K}^2 + 2\mathcal{K}_{,\rho} - \mathcal{K}}{4(1-\gamma)\mathcal{K}} - \frac{\gamma\left[\mathcal{K}_{,\rho} + \mathcal{K}\left(4\mathcal{K} - 2Y - 1\right)\right]}{2(1-\gamma)\mathcal{K}}\,, \tag{6.144}$$

$$\mathcal{P} = \frac{1 - 4\mathcal{K} + 4Y}{4(1-\gamma)}\,, \tag{6.145}$$

$$0 = (2Y+1)\,\mathcal{K}_{,\rho} - 4\mathcal{K}^2 - \mathcal{K}\left[4Y_{,\rho} + 4(Y-1)Y - 1\right]\,. \tag{6.146}$$

Let us now consider a metric in which the $(0,0)$ coefficient, A, is given by

$$A = A_0\left(a + b r_0^2 e^{\rho}\right)^2\,, \tag{6.147}$$

where a, b and r_0 are arbitrary constants. From the definition of Y one obtains

$$Y = \frac{1}{2}\frac{A_{,\rho}}{A} = \frac{b r_0^2 e^{\rho}}{a + b r_0^2 e^{\rho}}\,, \tag{6.148}$$

and from Eq. (6.146) it follows that

$$\mathcal{K} = \frac{\left(a + 3 b r_0^2 e^{\rho}\right)^{\frac{2}{3}}}{\mathcal{K}_0 e^{\rho} + 4\left(a + 3 b r_0^2 e^{\rho}\right)^{\frac{2}{3}}}\,, \tag{6.149}$$

where $\mathcal{K}_0$ is an integration constant. In terms of the circumferential radius r, this result corresponds to the line element

$$ds^2 = -A(r)\,dt^2 + B(r)\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right), \tag{6.150}$$

with

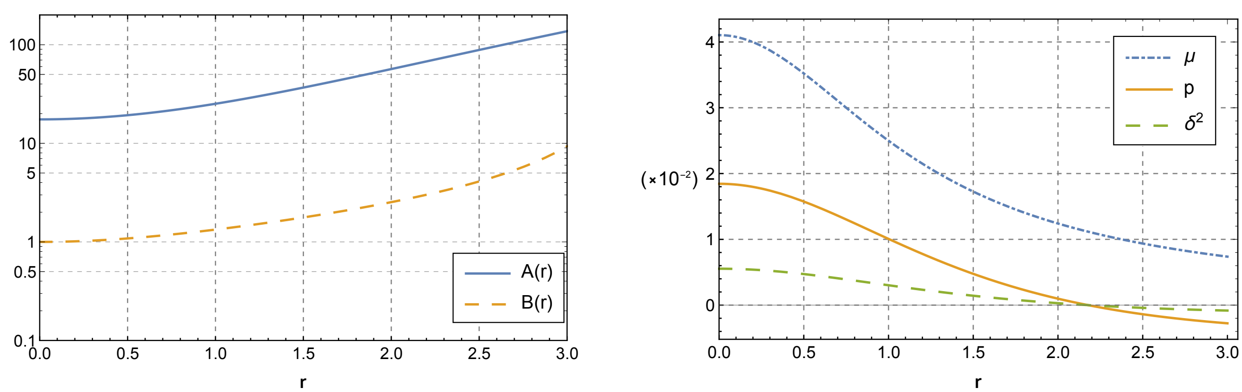
$$\begin{aligned} A(r) &= A_0\left(a + br^2\right)^2, \\ B(r) &= \left(1 + \frac{c\,r^2}{\left(a + 3br^2\right)^{\frac{2}{3}}}\right)^{-1} . \end{aligned} \tag{6.151}$$

The energy density and the pressure are given by

$$\begin{aligned} \mu =& \frac{b^2r^2\left[5c\,(1-4\gamma)\,r^2 - 12\gamma\left(a+3br^2\right)^{\frac{2}{3}}\right] - 4ab\left[\gamma\left(a+3br^2\right)^{\frac{2}{3}} + 2c\,(2\gamma-1)\,r^2\right]}{8\pi(\gamma-1)\left(a+br^2\right)\left(a+3br^2\right)^{\frac{5}{3}}} + \\ &+ \frac{a^2c\,(3-4\gamma)}{8\pi(\gamma-1)\left(a+br^2\right)\left(a+3br^2\right)^{\frac{5}{3}}}\,, \end{aligned} \tag{6.152}$$

$$p = \frac{4b\left(a+3br^2\right)^{\frac{2}{3}} + ac + 5bcr^2}{8\pi(1-\gamma)\left(a+br^2\right)\left(a+3br^2\right)^{\frac{2}{3}}} .$$

We give in Figure 6 the behavior of this solution for specific values of the parameters, showing the existence of an hypersurface where both p and δ^2 vanish, so that we can smoothly match such solution with a vacuum exterior space-time.



(a) Coefficients of the metric in Eqs. (6.150) and (6.151).

(b) The thermodynamic quantities in Eqs. (6.152).

Figure 6: Plots of the behavior of the metric components, (a) and matter variables (b) associated with the solution in Eqs. (6.150) – (6.152) in the case $a = 5$, $b = 1$, $c = -1$, $\gamma = 0.3/(8\pi)$ and $A_0 = 0.7$.

Another example, based on the same assumptions, can be given considering

$$A = A_0\left(a + \sqrt{c - be^{\rho}}\right)^2 , \tag{6.153}$$

which corresponds to

$$Y=-\frac{be^{\rho}}{2\sqrt{c-be^{\rho}}\left(a+\sqrt{c-be^{\rho}}\right)}\ , \tag{6.154}$$

Eq. (6.146) then gives

$$\mathcal{K}=\frac{c\psi\left(a\sqrt{c-be^{\rho}}-2be^{\rho}+c\right)}{\left(c-be^{\rho}\right)\left[4\psi\left(a\sqrt{c-be^{\rho}}-2be^{\rho}+c\right)-b\,d\,e^{\rho}\right]}\,, \tag{6.155}$$

with

$$\psi=\left(\frac{\sqrt{a^2+8c}+a+4\sqrt{c-be^{\rho}}}{\sqrt{a^2+8c}-a-4\sqrt{c-be^{\rho}}}\right)^{\frac{a}{\sqrt{a^2+8c}}}\ . \tag{6.156}$$

Using the area radius r, we find the following solution for the metric (6.150)

$$\begin{aligned}
A&=A_0\left[a+y\left(r\right)\right]^2\ ,\\
B&=\frac{4c\left[ay\left(r\right)+2y\left(r\right)^2-c\right]}{y\left(r\right)^2\left[4ay\left(r\right)+8y\left(r\right)^2+d\,\psi\left(r\right)\left(y\left(r\right)^2-c\right)-4c\right]}\,,
\end{aligned} \tag{6.157}$$

where

$$\begin{aligned}
y\left(r\right)&=\sqrt{c-\frac{br^2}{r_0^2}}\,,\\
\psi\left(r\right)&=\left(\frac{\sqrt{a^2+8c}+a+4y\left(r\right)}{\sqrt{a^2+8c}-a-4y\left(r\right)}\right)^{\frac{a}{\sqrt{a^2+8c}}}\ ,
\end{aligned} \tag{6.158}$$

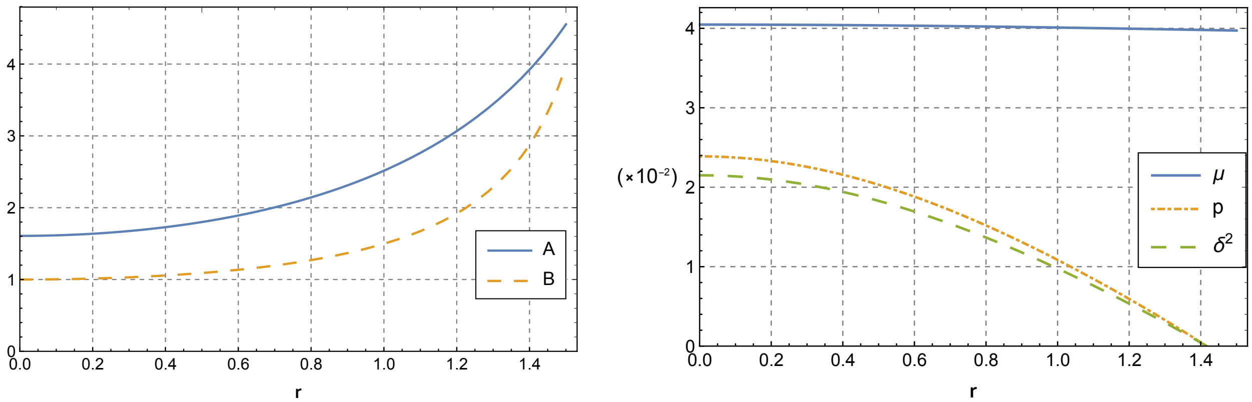
with the following expressions for the energy density and pressure of the fluid

$$\begin{aligned}
\mu=&\frac{b\,d\,\gamma\left[6y^5+7ay^4+2y^3\left(a^2-3c\right)+c^2\left(2y+a\right)-4acy^2\right]\psi}{16\pi c(\gamma-1)r_0^2(a+y)[c-y(a+2y)]^2}+\\
&+\frac{b[2\gamma(2a+3y)-3(a+y)]}{8\pi cr_0^2(\gamma-1)(a+y)}-\frac{bd\left[6y^4+3ay^3-5cy^2+2c^2\right]\psi}{32\pi c(\gamma-1)r_0^2[c-y(a+2y)]^2}\,,
\end{aligned} \tag{6.159}$$

$$p=\frac{b\,d\,y\left(ay-2c+3y^2\right)\psi}{32\pi c(\gamma-1)r_0^2(a+y)\left(ay-c+2y^2\right)}+\frac{b(a+3y)}{8\pi cr_0^2(\gamma-1)(a+y)}\ .$$

In Figure 7 we show the behavior of this solution for specific values of the parameters. Notice that also this solution admits the existence of a common hypersurface where both p and δ vanish.

Before finishing this section we remark that, as shown by Figures 3 – 7, in all considered cases it is possible to find values of the parameters for which all the thermodynamical quantities and spin density are positive, hence, all the classical energy conditions are valid.



(a) The coefficients of the metric in Eq. (6.150) and (6.157).

(b) The thermodynamic quantities in Eqs. (6.159).

Figure 7: Plots of the behavior of the metric components, (a) and matter variables (b) associated with the solution in Eqs. (6.150), (6.157) and (6.152) in the case $A_0 = 1$, $a = -3$, $b = 1$, $c = 3$, $d = 0.03$, $r_0 = 1$ and $\gamma = 0.9/(8\pi)$.

6.7 Generating theorems

As discussed in Ref. [51], the form of the structure equations (6.84) – (6.92) is especially useful to find algorithms for generating new solutions from previous known ones.

Consider a solution for the structure equations (6.84) – (6.92) characterized by the functions

$$\{\mathcal{P}_0, \mathcal{M}_0, \Delta_0, K_0, \mathcal{E}_0, \mathbb{X}_0, \mathbb{Y}_0, (\mathbb{B}_1)_0\,, (\mathbb{B}_2)_0\,, (\mathbb{D}_1)_0\,, (\mathbb{D}_2)_0\}\;. \tag{6.160}$$

Given the quantities

$$\begin{aligned}
\mathcal{M} &= \mathcal{M}_0 + \mathcal{M}_1\,,\\
\mathcal{P} &= \mathcal{P}_0 + \mathcal{P}_1\,,\\
\Delta^2 &= {\Delta_0}^2 + {\Delta_1}^2\,,\\
\mathcal{K} &= \mathcal{K}_0 + \mathcal{K}_1\,,
\end{aligned} \tag{6.161}$$

where $\{\mathcal{P}_1, \mathcal{M}_1, \Delta_1, \mathcal{K}_1\}$ are sufficiently smooth arbitrary functions, let us search conditions on the deforming functions so that, the set $\{\mathcal{P}, \mathcal{M}, \Delta, \mathcal{K}, \mathcal{E}, \mathbb{X}, \mathbb{Y}, \mathbb{B}_1, \mathbb{B}_2, \mathbb{D}_1, \mathbb{D}_2\}$ is a solution of the structure equations.

Substituting Eq. (6.161) in Eq. (6.85) we find

$$\partial_\rho \mathcal{K}_1 + 2{\mathcal{K}_1}^2 + 2\mathcal{K}_0\left({\Delta_1}^2 - \mathcal{M}_1\right) - \mathcal{K}_1\left(2\mathcal{M}_0 - 4\mathcal{K}_0 - 2{\Delta_0}^2 + 2{\Delta_1}^2 - 2\mathcal{M}_1 + \frac{1}{2}\right) = 0\,. \tag{6.162}$$

This equation has the form of a Riccati differential equation to which, in general, there are no known closed form solutions. We can, nonetheless, consider particular cases so that the previous equation reduces to a Bernoulli differential equation, where general closed form solutions exist.

6.7.1 Case 1

Let us first consider that

$$\mathcal{M}_1 = \Delta_1^2 \,. \tag{6.163}$$

In this case, Eq. (6.162) can be readily integrated for $\mathcal{K}_1$, such that

$$\mathcal{K}_1 = 0 \;\vee\; \mathcal{K}_1\left(\rho\right) = \frac{\text{Exp}\left[\int_{\rho_0}^{\rho} \Lambda \, dx\right]}{\mathcal{K}_\star + 2\int_{\rho_0}^{\rho} \text{Exp}\left[\int_{y_0}^{y} \Lambda \, dx\right] \, dy} \,, \tag{6.164}$$

where $\mathcal{K}_\star$ is an integration constant and

$$\Lambda = 2\mathcal{M}_0 - 2{\Delta_0}^2 - 4\mathcal{K}_0 + \frac{1}{2} \,. \tag{6.165}$$

Using Eq. (6.163) in Eq. (6.84) we find

$$\partial_\rho \mathcal{P}_1 + \mathcal{P}_1 \left(3\mathcal{K}_0 + 3\mathcal{K}_1 - \mathcal{M}_0 + 2\mathcal{P}_0 - {\Delta_0}^2 - \frac{7}{4}\right) + {\mathcal{P}_1}^2 + \frac{1}{4}\mathcal{F} = 0 \,, \tag{6.166}$$

with

$$\begin{aligned} \mathcal{F} = {} & 4\mathcal{M}_0\left({\Delta_1}^2 + \mathcal{K}_1\right) - 8\left(\mathcal{P}_0 + \mathcal{P}_1\right){\Delta_1}^2 + 12\mathcal{P}_0\mathcal{K}_1 + 4{\Delta_0}^2{\Delta_1}^2 - \\ & - 4\partial_\rho {\Delta_1}^2 + +4{\Delta_1}^4 + 7{\Delta_1}^2 - 12{\Delta_1}^2\mathcal{K}_0 - 16{\Delta_0}^2\mathcal{K}_1 - 12{\Delta_1}^2\mathcal{K}_1 \,. \end{aligned} \tag{6.167}$$

For Eq. (6.166) to reduce to a Bernoulli like differential equation we will require $\mathcal{F}\left(\rho\right) = 0$, that is

$$\begin{aligned} & \partial_\rho {\Delta_1}^2 - {\Delta_1}^4 - \mathcal{K}_1\left(\mathcal{M}_0 + 3\mathcal{P}_0 - 4{\Delta_0}^2\right) - \\ & \quad -{\Delta_1}^2\left(\frac{7}{4} + \mathcal{M}_0 - 3\mathcal{K}_0 - 3\mathcal{K}_1 - 2\mathcal{P}_0 - 2\mathcal{P}_1 + {\Delta_0}^2\right) = 0 \,, \end{aligned} \tag{6.168}$$

which, by setting $\mathcal{K}_1 = 0$ or $\mathcal{M}_0 + 3\mathcal{P}_0 - 4{\Delta_0}^2 = 0$, can be formally solved, such that

$$\Delta_1^2\left(\rho\right) = 0 \;\vee\; {\Delta_1}^2\left(\rho\right) = \frac{\text{Exp}\left[\int_{\rho_0}^{\rho} \Phi \, dx\right]}{\Delta_\star - \int_{\rho_0}^{\rho} \text{Exp}\left[\int_{y_0}^{y} \Phi \, dx\right] \, dy} \,, \tag{6.169}$$

where $\Delta_\star$ is an integration constant and

$$\Phi = \mathcal{M}_0 - 3\mathcal{K}_0 - 3\mathcal{K}_1 - 2\mathcal{P}_0 - 2\mathcal{P}_1 + {\Delta_0}^2 + \frac{7}{4}\,. \tag{6.170}$$

Consequently, from Eq. (6.166), we find

$$\mathcal{P}_1(\rho) = 0 \ \vee \ \mathcal{P}_1(\rho) = \frac{\text{Exp}\left[\int_{\rho_0}^{\rho} \Gamma\, dx\right]}{\mathcal{P}_\star + \int_{\rho_0}^{\rho} \text{Exp}\left[\int_{y_0}^{y} \Gamma\, dx\right]\, dy}\,, \tag{6.171}$$

with

$$\Gamma = \mathcal{M}_0 - 2\mathcal{P}_0 + {\Delta_0}^2 - 3\mathcal{K}_0 - 3\mathcal{K}_1 + \frac{7}{4}\,, \tag{6.172}$$

and $\mathcal{P}_\star$ is an integration constant.

Before we conclude this subsection, we should stress that Eqs. (6.164), (6.169) and (6.171) present two possible solutions for the considered functions and all combinations of those solutions verify the structure equations with $\mathcal{M}_1 = \Delta_1^2$, leading, *a priori*, to distinct solutions.

6.7.2 Case 2

Another possibility to solve Eq. (6.162) is the case when

$$(\mathcal{K}_0 + \mathcal{K}_1)\left(2\Delta_1^2 - 2\mathcal{M}_1\right) = G(\rho)\,\mathcal{K}_1 + Q(\rho)\,{\mathcal{K}_1}^2\,, \tag{6.173}$$

where $G(\rho)$ and $Q(\rho)$ are sufficiently smooth, arbitrary functions. Setting

$$\begin{aligned} 2\Delta_1^2 - 2\mathcal{M}_1 &= \mathcal{K}_1 Q(\rho)\ , \\ G(\rho) &= \mathcal{K}_0 Q(\rho)\ , \end{aligned} \tag{6.174}$$

and substituting Eqs. (6.173) and (6.174) in Eq. (6.162) we find

$$\partial_\rho \mathcal{K}_1 + \mathcal{K}_1\left[2{\Delta_0}^2 - 2\mathcal{M}_0 + 4\mathcal{K}_0 + \mathcal{K}_0 Q(\rho) - \frac{1}{2}\right] + \left[2 + Q(\rho)\right]{\mathcal{K}_1}^2 = 0\,, \tag{6.175}$$

which, provided an expression for $Q(\rho)$ can be solved for $\mathcal{K}_1$, or vice-versa.

Now, to solve the remaining equations for the functions Q, $\mathcal{P}_1$ and Δ_1, we will consider that the original solution is such that $\mathcal{M}_0 = \mathcal{P}_0 = \Delta_0 = 0$, that is, the original space-time is described by a vacuum solution of the field equations. From Eq. (6.84) we then find

$$\partial_\rho \mathcal{P}_1 + {\mathcal{P}_1}^2 + \mathcal{P}_1\left[3\mathcal{K}_0 + 3\mathcal{K}_1 - \frac{7}{4}\right] + \mathcal{J}(\rho) = 0\,, \tag{6.176}$$

where

$$\begin{aligned}\mathcal{J}(\rho) = &-\partial_\rho {\Delta_1}^2 + {\Delta_1}^4 + \frac{1}{2}Q(\rho)\,\mathcal{K}_1\left[\mathcal{P}_1 - \mathcal{K}_0 - \mathcal{K}_1 + \frac{1}{4}\right] \\ &+ {\Delta_1}^2\left(-2\mathcal{P}_1 - \frac{1}{2}Q(\rho)\mathcal{K}_1 - 3\mathcal{K}_0 - 3\mathcal{K}_1 + \frac{7}{4}\right).\end{aligned} \tag{6.177}$$

As before, to reduce Eq. (6.176) to a Bernoulli differential equation we will impose $\mathcal{J}(\rho) = 0$. Unfortunately, this equation itself is also not possible to solve in general since it has the form of a Riccati differential equation. Let us then further impose the last term in the first line of the previous equation to be zero. Solving for $\mathcal{K}_1$, we have

$$\mathcal{K}_1 = \mathcal{P}_1 - \mathcal{K}_0 + \frac{1}{4}\,, \tag{6.178}$$

where we have ignored the solutions where $Q = 0 \vee \mathcal{K}_1 = 0$ since they lead to a particular case of subsection 6.7.1.

Considering the constraint that originally we have a vacuum solution, substituting Eq. (6.178) in Eq. (6.175) we find,

$$Q(\rho) = -\frac{8\left(2\partial_\rho\mathcal{P}_1 + 4{\mathcal{P}_1}^2 + \mathcal{P}_1\right)}{\left(4\mathcal{P}_1 + 1\right)\left(4\mathcal{P}_1 - 4\mathcal{K}_0 + 1\right)}\,. \tag{6.179}$$

Gathering the previous results, we find the following expressions for the remaining deformations:

$$\begin{aligned}\mathcal{P}_1 &= \frac{e^\rho}{\mathcal{P}_\star + 4e^\rho}\,,\\ \Delta_1^2 = 0 \vee {\Delta_1}^2 &= \frac{\mathrm{Exp}\left[-\int_{\rho_0}^{\rho}\Phi\,dx\right]}{\Delta_\star - \int_{\rho_0}^{\rho}\mathrm{Exp}\left[-\int_{y_0}^{y}\Phi\,dx\right]\,dy}\,,\\ \mathcal{M}_1 &= \Delta_1^2 + \frac{2\partial_\rho\mathcal{P}_1 + 4{\mathcal{P}_1}^2 + \mathcal{P}_1}{4\mathcal{P}_1 + 1}\,,\end{aligned} \tag{6.180}$$

where $\mathcal{P}_\star$ and $\Delta_\star$ are integrating constants and

$$\Phi = 2\mathcal{P}_1 + \frac{1}{2}Q(\rho)\mathcal{K}_1 + 3\mathcal{K}_0 + 3\mathcal{K}_1 - \frac{7}{4}\,. \tag{6.181}$$

Notice that we did not consider the case when $\mathcal{P}_1 = 0$ since it would lead to the case when $Q(\rho) = 0$, which, as mentioned before, represents a particular case of subsection 6.7.1. Let us also remark that, for solutions generated using the above equations, the functional form of the pressure, $\mathcal{P} \equiv \mathcal{P}_1$, is independent of the original solution and completely determined up to a constant. Moreover, notice that the pressure — in such solutions — is only null when $\rho \to -\infty$.

6.7.3 Case 3

Let us now consider the deformations in Eq. (6.161) with the extra constraint

$$\mathbb{Y} = \mathbb{Y}_0\,, \tag{6.182}$$

that is, we will impose that the function $\mathbb{Y}$ is unchanged between the original and the perturbed space-time. This is a generalization of the deformations considered in Refs. [51, 166], for non-null spin density. Substituting Eqs. (6.161) and (6.182) in Eq. (6.86) we find that

$$\mathcal{P}_1 = \Delta_1^2 - \mathcal{K}_1\,. \tag{6.183}$$

Using Eqs. (6.161), (6.182) and (6.183) in Eqs. (6.85) and (6.84) we find the following relations for $\mathcal{M}_1$ and $\mathcal{K}_1$

$$\mathcal{M}_1 = \frac{\mathcal{K}_1\,(2\mathbb{Y}_0+3)}{2\mathbb{Y}_0+1} + \Delta_1^2\,, \tag{6.184}$$

$$\mathcal{K}_1 = \frac{\text{Exp}\left[-\int_{\rho_0}^{\rho}\Phi\,dx\right]}{\mathcal{K}_\star - \int_{\rho_0}^{\rho}\frac{4}{2\mathbb{Y}_0+1}\text{Exp}\left[-\int_{y_0}^{y}\Phi\,dx\right]\,dy} - \mathcal{K}_0\,, \tag{6.185}$$

where

$$\Phi = \frac{\mathcal{K}_0\,(6+4\mathbb{Y}_0)}{2\mathbb{Y}_0+1} + 2\Delta_0^2 - 2\mathcal{M}_0 - \frac{1}{2}\,, \tag{6.186}$$

and $\mathcal{K}_\star$ is an integration constant. Eqs. (6.183) – (6.186) generalize the results in Ref. [51] in the presence of a non-null spin density.[7]

Contrary to the previous cases, the Eqs. (6.183) – (6.186) do not completely determine the system since the function Δ_1^2 is unconstrained. Notice that $\mathcal{K} \equiv \mathcal{K}_1$ is determined uniquely by the unperturbed solution and Δ_1^2 will only affect $\mathcal{M}_1$ and $\mathcal{P}_1$. Therefore, provided an unperturbed solution, the metric of the perturbed space-time is completely determined by Eqs. (6.182) and (6.185). As already pointed out, Δ_1^2 will not only affect the energy density and the pressure of the fluid but also the Weyl tensor components. Thus, although the metric of the space-time is independent of Δ_1^2, the geometry is profoundly influenced by the presence of intrinsic spin.

[7] Notice that there is an error in the expression for $\mathcal{M}_1$ in Ref. [51]. The correct expression is found by setting $\Delta_0^2 = \Delta_1^2 = 0$ in Eqs. (6.184) – (6.186).

6.8 Conclusions

In this chapter we have introduced and used the 1+1+2 formalism to derive the structure equations for LRS I and LRS II, stationary space-times with a Weyssenhoff like torsion field in the context the ECSK theory of gravitation. The structure of the covariant equations show in detail how the intrinsic spin interacts with the space-time via the torsion tensor. In particular, the presence of a torsion tensor field separates the magnetic part of the Weyl tensor in two distinct tensors, which behave differently. Even in the case of static LRS II space-times, the magnetic parts of the Weyl tensor do not vanish and some of its components depend on both the value and spatial derivative of the spin density. This suggests, in particular, that the effects of intrinsic spin on the matter fluid, even in the regimes expected to be found in neutron stars, may not be negligible, as it was previously thought (see e.g. Ref. [36, 163]), even in the case in which the contributions due to the spin is very small.

The 1+1+2 equations were then used to derive the covariant TOV equations for ECSK gravity for LRS I and LRS II space-times. In the case of LRS II space-times, the equations are structurally very similar to the ones of GR. Indeed this similarity allows to recast them into the same form of the GR TOV equations via a redefinition of the matter variables and the electric part of the Weyl tensor. As a consequence we found that at the level of the metric it is possible to map static, locally rotationally symmetric solutions of class II from the ECSK theory to the ones of the theory of General Relativity. Moreover, due to this mapping and the re-scale in the matter variables, some GR solutions which are physically irrelevant become, in the context of ECSK gravity, interesting.

When we examine in detail the physical properties of physically relevant solutions, the differences between the ECSK theory and GR become once more evident. This is particularly true looking at the junction conditions. We found that the requirement that all of the components of the Riemann tensor have finite discontinuities across the separation surface $\mathcal{S}$ leads to additional constraints with respect to the torsionless case. This is especially evident when looking at the structure equations for stationary LRS I and LRS II space-times sourced by a Weyssenhoff fluid. In these equations the magnetic parts of the Weyl tensor depend explicitly on the derivatives of the torsion tensor and the classical Israel-Darmois junction conditions of GR do not guarantee these terms to be finite across $\mathcal{S}$. As consequence of the generalized junction conditions, in the considered setup, observers at the interior and exterior space-times must measure the same value for the spin density at $\mathcal{S}$, turning the task of finding physically relevant solutions even more daunting.

Using the full set of structure equations and boundary conditions provided by the junction formalism, we were able to study various properties of possible solutions. We started by analyzing how the presence of intrinsic spin changes the Buchdahl limit for the maximum compactness of a star. We concluded that the spin-geometry coupling allows stars with a given circumferential radius to hold more matter than the corresponding GR ones. Next we considered the case of static, spherically symmetric compact objects entirely held by the matter intrinsic spin, smoothly matched to a vacuum exterior. This scenario was expected to represent a good model for cold neutron stars, where the thermodynamical pressure is negligible when compared to the spin density. We found, surprisingly, that such objects cannot be simultaneously static, spherically symmetric and smoothly matched to a vacuum exterior. This is a strong result and it is necessary to discuss in detail the hypothesis that led to such conclusion. More specifically our conclusion may not be valid if:

(i) the spin density is not a monotonically decreasing function of the radial coordinate inside the star;

(ii) we consider a non-vacuum exterior space-time;

(iii) we replace the uncharged Weyssenhoff fluid model;

(iv) we allow the presence of a thin shell.

The first possibility might lead to a total energy density and a pressure density which is not monotonically decreasing. While this is not a strong enough reason to discard this case, we expect these oscillation to make the solution unstable under small perturbations. The second case suggests that if ECSK theory had a non trivial vacuum (vortical) solution, one could smoothly match the interior to it, bypassing the requirement of the spin density to vanish at an hypersurface. At present there is no evidence that such solution might/should exist. Indeed the theory is expected to reduce to GR in vacuum. For what concerns hypothesis (iii), the Weyssenhoff fluid can be advocated to be a good model for the matter fluids that might constitute cold neutron stars. However, in this work we made the simplifying assumption that the fluid is electrically neutral. If instead a charged Weyssenhoff fluid model is considered, we expect that other effects will appear — such as anisotropic pressure — which may drastically change the behavior of the fluid. As for the last possibility, although a smooth junction with a vacuum exterior might represent a more reasonable scenario, it might be argued that neutron stars may have a well defined surface, therefore it is not completely unreasonable to consider the presence of a thin shell of matter at the matching surface.

On top of the zero pressure solution considered above, we have also considered solutions in which pressure is non zero. Using reconstruction algorithms we have been able to obtain various classes of solutions for the interior of static, spherically symmetric compact objects that can be smoothly matched to a Schwarzschild exterior. One family of those solutions, which we dubbed Buchdahl stars, represent a very interesting scenario: they admit the existence of a common hypersurface where the pressure, spin density and energy density all vanish. This model, studied for the first time by Buchdahl for gaseous stars in GR [164], represents the scenario where the fluid that composes a star will smoothly dissipate away from a denser core and transition to vacuum. These solutions also provided a key example for the effects that intrinsic spin may have on the behavior of the fluid. Figures 3 – 5 clearly exemplify that even if the spin density is much smaller than the other matter variables, it allows for a much richer behavior for the fluid.

Finally, as in the case of GR, also in the ECSK theory it is possible to derive generating theorems. In this chapter we have presented several algorithms to generate new exact solutions from previously known ones. We should stress here that the results we have obtained followed from the idea of finding conditions so that the Riccati differential equations would reduce to Bernoulli equations. Although this scheme allowed us to find various generating algorithms, we make no claim that we have exhausted all possibilities for finding new ones. On this note, the integrability conditions for Riccati type equations in Refs. [167–169] were also considered; however, these did not lead to useful results, in the considered context.

Chapter 7

Electrically charged static shells: maximally extended Reissner-Nordström space-time outside a Minkowski core

7.1 Introduction

In Chapter 3 we studied the conditions for two Lorentzian manifolds with boundary, $\mathcal{M}^+$ and $\mathcal{M}^-$, endowed with a metric compatible affine connection, to be smoothly matched at a common boundary, $\mathcal{S}$, so that the resulting space-time $\mathcal{M}$ has a well defined geometry at the matching surface. A natural follow up question is if those conditions are necessary for the space-time $\mathcal{M}$ to be a solution of the field equations of a given theory of gravitation. The answer is, surprisingly, no. Although the constraints on the geometry of the space-time at the matching surface depend on the specific theory of gravitation (cf., e.g., Refs. [68–72, 170–174]), these are in general less restrictive than those found in Chapter 3, by relating the infinite discontinuity of the Riemann curvature tensor with the presence of a thin shell of matter at $\mathcal{S}$.

In the case of the theory of General Relativity, these conditions are usually referred as the Israel-Darmois junction formalism [68, 69], forming a set of constraints on the geometry of $\mathcal{S}$, so that the resulting space-time is a solution of the Einstein field equations in GR. The conditions imposed by the Israel-Darmois formalism allow for the study of solutions of the theory of General Relativity by matching two previously known ones. Indeed, in the literature the formalism has been extensively used as a tool to analyze a plethora of distinct setups, namely, it has allowed for the study of the thermodynamical properties of black holes [175–180], the Hawking radiation [181–183], gravitational collapse [184–187] and the properties of solutions of astrophysical interest [188–193].

One such possibility provided by the junction formalism is the scenario of a spherically symmetric thin shell of matter separating two vacuum solutions [184, 187, 194–199]. The discovery that the universe is best described by an inflationary model has led to speculation of what were the initial conditions that led to the inflationary phase that we experience in the present. One idea is that the universe underwent various inflationary phases in the course of some cosmological phase transitions [194]. These phase transitions could have given rise to bubbles that separate an old inflationary phase from a new phase [184]. These bubbles, leftovers of the early universe, could survive and be themselves sources of an exterior space-time. In this spirit, the case of sufficiently thin bubbles, thin shells, with a vacuum Minkowski interior has been previously considered and an intriguing possibility was found that static thin shells at the opposite side of the Einstein-Rosen bridge could be the source of an exterior Schwarzschild space-time for an outside observer, provided that such shells were hold by tension [195]. In this chapter we will extend the analysis in Ref. [195] and consider the case of charged static thin shells separating an interior Minkowski from an exterior Reissner-Nordström space-time [200]. As we will see, given the complexity of the Reissner-Nordström model, various scenarios are possible depending on the properties of the matter fluid that composes the shell.

7.2 Space-time junction formalism

Let us then start by briefly revising the Israel-Darmois formalism for the theory of GR.

Repeating some of the concepts introduced in Section 3.5, consider two space-time manifolds with boundary, $\mathcal{M}^+$ with metric ${}^{(+)}g$ and $\mathcal{M}^-$ with metric ${}^{(-)}g$. Let us then consider that the space-times $\mathcal{M}^+$ and $\mathcal{M}^-$ are glued together at a common boundary, forming a new space-time, $\mathcal{M}$, partitioned by an hypersurface $\mathcal{S}$ in two regions, $\mathcal{M}^+$ and $\mathcal{M}^-$. The hypersurface $\mathcal{S}$ can be either time-like or space-like. Although the case of null boundary hypersurface can also be considered [170], it will not be studied in this chapter.

Now, we will assume that it is possible to define a common coordinate system, $\{x^a\}$, on both sides of the hypersurface $\mathcal{S}$ and we choose n, the unit normal to $\mathcal{S}$, to point from $\mathcal{M}^-$ to $\mathcal{M}^+$. The normal vector field is such that it is space-like or time-like if the hypersurface is either time-like or space-like, respectively. Then, assuming $\{y^a\}$ to represent a local coordinate system around each point on $\mathcal{S}$, the normal vector field n must be orthogonal — at each point — to the tangent vectors to the hypersurface $\mathcal{S}$, $e_a \equiv \partial/\partial y^a$, such that

$$e^{\alpha}_{a} n_{\alpha} = 0\,. \tag{7.1}$$

Now, in order to join the two regions $\mathcal{M}^+$ and $\mathcal{M}^-$ at $\mathcal{S}$, so that the union of ${}^{(+)}g$ and ${}^{(-)}g$ forms a valid solution to the Einstein field equations, the Israel formalism states that two junctions conditions must be verified at the matching surface $\mathcal{S}$:

(i) The induced metric as seen from each region $\mathcal{M}^-$ and $\mathcal{M}^+$, ${}^{(\pm)}h_{ij} := {}^{(\pm)}g_{\alpha\beta}e_i^\alpha e_j^\beta$, must be the same, that is:[1]

$$[h_{ij}]_\pm = 0\,; \tag{7.2}$$

(ii) If the extrinsic curvature, $K_{ab} \equiv e_a^\alpha e_b^\beta \nabla_\alpha n_\beta$, is not the same on both sides of the boundary $\mathcal{S}$, then a thin shell with stress-energy tensor

$$S_{ab} = -\frac{\varepsilon}{8\pi}\left([K_{ab}]_\pm - h_{ab}\,[K]_\pm\right)\,, \tag{7.3}$$

where ε is defined by $n^\alpha n_\alpha = \varepsilon$ and $K \equiv h^{ab}K_{ab}$, is present at $\mathcal{S}$.

In this chapter, we will then study the properties of a static thin shell composed of a perfect fluid[2] that separates an interior Minkowski space-time and an exterior Reissner-Nordström space-time using the Israel junction formalism. The Reissner-Nordström solution describes the space-time outside an electrically charged non-rotating mass. Depending on the absolute value of the charge to be less, equal or greater than the mass — in geometrized units — three distinct space-times, respectively: non-extremal, extremal and overcharged, are described by this solution (cf. Figure 8). All three cases shall be analyzed.

7.3 Non-extremal thin shells

7.3.1 Electric thin shells outside the event horizon: normal and tension shells in a non-extremal Reissner-Nordström space-time

7.3.1.1 Induced metric and extrinsic curvature of $\mathcal{S}$ as seen from $\mathcal{M}^-$

Defined the setup of the problem, let us start by analyzing the interior Minkowski space-time, $\mathcal{M}^-$, whose line element in spherical coordinates is given by

$$ds_-^2 = -dt^2 + dr^2 + r^2 d\Omega^2\,, \tag{7.4}$$

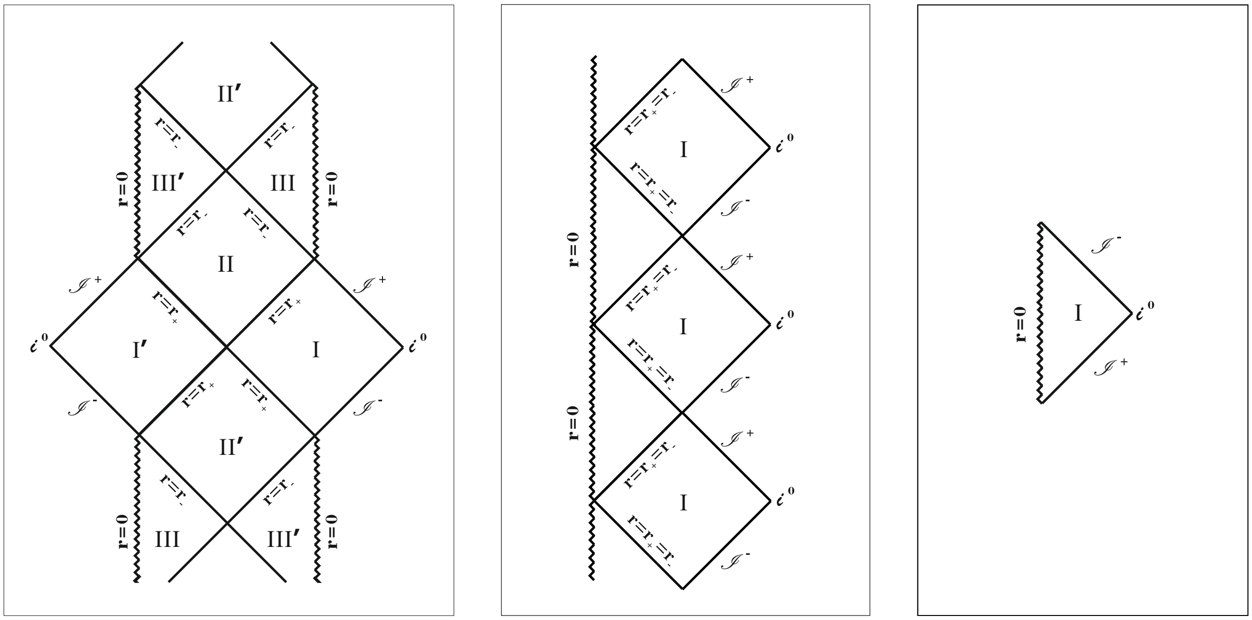
where $d\Omega^2 \equiv d\theta^2 + \sin^2(\theta)\,d\varphi^2$.

Now, we assume the hypersurface $\mathcal{S}$ to be static[3] but, in general, it can be either time-like or space-like. However, since we are considering the Minkowski space-time, it is not possible to have a static space-like surface hence, $\mathcal{S}$ must be time-like. It is, then, convenient to choose the coordinates on $\mathcal{S}$ to be $\{y^a\} =$

[1] We gently remind the notation introduced back in Chapter 3 and which will be used here. We will write $[\psi]_\pm$ to represent the difference of a field as seen from each sub-manifold at $\mathcal{S}$, i.e., $[\psi]_\pm \equiv \psi\left(\mathcal{M}^+\right)\big|_{\mathcal{S}} - \psi\left(\mathcal{M}^-\right)\big|_{\mathcal{S}}$; moreover, we will use the notation ${}^{(+)}\psi \equiv \psi\left(\mathcal{M}^+\right)$ or ${}^{(-)}\psi \equiv \psi\left(\mathcal{M}^-\right)$, to refer to a field ψ defined in $\mathcal{M}^+$ or $\mathcal{M}^-$, respectively.

[2] Here we will assume that the total stress-energy tensor defined as the sum, $S_{ab} = M_{ab} + E_{ab}$, where M_{ab} represents the matter stress-energy tensor and E_{ab} represent the electromagnetic stress-energy tensor are such that S_{ab} can be written as: $S_{ab} = \sigma u_a u_b + p\left(h_{ab} + u_a u_b\right)$.

[3] Static has seen from an observer free-falling in the interior Minkowski space-time.



(a) (b) (c)

Figure 8: Carter-Penrose diagrams of the distinct space-times described by the Reissner-Nordström solution. Panel (a) Non-extremal Reissner-Nordström space-time. Panel (b) Extremal Reissner-Nordström space-time. Panel (c) Overcharged Reissner-Nordström space-time.

(τ, θ, φ), where τ is the proper time measured by an observer co-moving with $\mathcal{S}$ and it follows that $e_\tau \equiv u$, where u is the 4-velocity of an observer co-moving with the shell.

The hypersurface $\mathcal{S}$, as seen from the interior $\mathcal{M}^-$ space-time, is parametrized by τ, such that the surface's radial coordinate is described by a function $R(\tau)$. The fact that $\mathcal{S}$ is assumed to be static implies,

$$\frac{dR}{d\tau} = 0\,, \tag{7.5}$$

from which we have that

$$u^\alpha_- = \left(\frac{dt}{d\tau}, 0, 0, 0\right)\,, \tag{7.6}$$

where u^α_- represent the components of the 4-velocity u as seen from the interior space-time $\mathcal{M}^-$. Since $\mathcal{S}$ is a time-like hypersurface, it must verify

$$^{(-)}g_{\alpha\beta} u^\alpha_- u^\beta_- = -1\,. \tag{7.7}$$

Using Eqs. (7.6) and (7.7) we find that $dt/d\tau = \pm 1$. Imposing that u points to the future leads to the choice of the plus sign, thus

$$u^\alpha_- = (1, 0, 0, 0)\,. \tag{7.8}$$

Eq. (7.8) can now be used to write the induced metric on $\mathcal{S}$, such that

$$ds^2\big|_{\mathcal{S}} = -d\tau^2 + R^2 d\Omega^2 \,. \tag{7.9}$$

Having found the expression for the 4-velocity of an observer co-moving with $\mathcal{S}$, we can now use Eqs. (7.1) and (7.8) and the condition that n is space-like[4],

$$n^\alpha n_\alpha = +1 \,, \tag{7.10}$$

to find the expression for the unit normal as seen from $\mathcal{M}^-$, n_-^α, hence

$$n_{-\alpha} = \lambda \left(0, 1, 0, 0\right) \,, \tag{7.11}$$

where λ is a normalization factor. Using Eqs. (7.10) and (7.11) yields $\lambda = \pm 1$. Since we are studying the case where the interior Minkowski space-time is spatially compact and enclosed by the hypersurface $\mathcal{S}$, we must choose the plus sign, such that, the expression for the outward pointing unit normal to $\mathcal{S}$ is given by

$$n_{-\alpha} = \left(0, 1, 0, 0\right) \,. \tag{7.12}$$

We are now in position to compute the components of the extrinsic curvature of $\mathcal{S}$ as seen from $\mathcal{M}^-$, ${}^{(-)}K_{ab}$. In the case where the matching surface $\mathcal{S}$ is time-like, static and spherically symmetric, the non-null components of the extrinsic curvature are given by

$$\begin{aligned} K_{\tau\tau} &= -a^\alpha n_\alpha \,, & (7.13)\\ K_{\theta\theta} &= \nabla_\theta n_\theta \,, & (7.14)\\ K_{\varphi\varphi} &= \nabla_\varphi n_\varphi \,, & (7.15) \end{aligned}$$

where ∇_α is the Levi-Civita connection and

$$a^\alpha \equiv u^\beta \nabla_\beta u^\alpha \,, \tag{7.16}$$

is the acceleration of an observer co-moving with $\mathcal{S}$.

Taking into account Eqs. (7.1), (7.4) and (7.9) we find that the non-trivial components of the exterior

[4] For simplicity and without loss of generality we will assume that the integral curves of n are affinely parameterized.

curvature as seen from the interior Minkowski space-time are given by

$$
\begin{aligned}
{}^{(-)}K^{\tau}{}_{\tau} &= 0\,, &(7.17)\\
{}^{(-)}K^{\theta}{}_{\theta} &= \frac{1}{R}\,, &(7.18)\\
{}^{(-)}K^{\varphi}{}_{\varphi} &= \frac{1}{R}\,, &(7.19)
\end{aligned}
$$

where the induced metric (7.9) was used to raise the indices.

7.3.1.2 Induced metric, and extrinsic curvature of $\mathcal{S}$ as seen from $\mathcal{M}^{+}$

To proceed we have now to find the expressions for the induced metric on $\mathcal{S}$ and the extrinsic curvature components as seen from the exterior space-time, $\mathcal{M}^{+}$. As was explained in Section 7.2, the exterior Reissner-Nordström space-time may be non-extremal, extremal or overcharged depending on the charge to mass ratio of the shell. Let us start by considering the non-extremal case.

In our study we shall allow the shell to be placed at any sub-region of the non-extremal Reissner-Nordström space-time (see Figure 8a). In this regard, following Ref. [196], it is useful to find a coordinate system that is well behaved — without coordinate singularities — in any of those sub-regions. Although it is possible to find a coordinate system that covers the entire Reissner-Nordström space-time without coordinate singularities (cf., e.g., Refs. [201–203]) the expressions become very complicated to work with and, in our analysis, no real advantages arise from using such coordinate system. In Appendix D.1, we define, instead, two coordinate patches, each well behaved in a neighborhood of the event horizon, at $r = r_{+}$, or in a neighborhood of the Cauchy horizon, at $r = r_{-}$. While the main motivation to find a new coordinate system is the fact that it is well behaved in a neighborhood of a coordinate singularity, hence allowing the shell to be as close to the pseudo-singularity as we want, another reason for introducing a new coordinate system is the fact that the radial Schwarzschild coordinate, r, is ambiguous when describing an event in one region or in its parallel — denoted with prime in Figure 8a — which can lead to difficulties. Moreover, as we will see, the introduction of such coordinate patch will allow a concise treatment of the case where the shell is placed in one region or in its parallel. Due to the formalism that we use, we will need two coordinates patches to describe the various sub-regions of the exterior Reissner-Nordström space-time hence, we shall separate the study of a shell placed in a region described by one coordinate patch and the other.

Let us then start by studying the properties of a static shell placed in a region described by the coordinate patch defined in Appendix D.1.1 — the coordinate patch without the pseudo-singularity at $r = r_{+}$. In this region, the metric of the Reissner-Nordström space-time in Kruskal-Szekeres coordinates was found to be given by

$$
ds^2 = \frac{16M^2}{r^2} R_{+}^2 e^{-\frac{r}{R_{+}}} \left(\frac{r - r_{-}}{2M}\right)^{1+\left(r_{-}/r_{+}\right)^2} \left(dX^2 - dT^2\right) + r^2\left(X, T\right) d\Omega^2\,, \tag{7.20}
$$

where

$$\begin{aligned} r_\pm &= M \pm \sqrt{M^2 - Q^2}\,, \\ R_\pm &= \frac{r_\pm^2}{r_+ - r_-}\,, \end{aligned} \tag{7.21}$$

and r is given implicitly by

$$X^2 - T^2 = e^{\frac{r}{R_+}} \left(\frac{r - r_+}{2M}\right) \left(\frac{r - r_-}{2M}\right)^{-\left(r_-/r_+\right)^2} \,. \tag{7.22}$$

The shell's radial coordinate when measured by an observer at $\mathcal{M}^+$ is described by a function $\rho\left(\tau\right)$, where τ is the proper time of an observer co-moving with the surface $\mathcal{S}$; which, since we assume it to be static, is such that

$$\frac{d\rho}{d\tau} = 0\,. \tag{7.23}$$

Considering Eq. (7.22), Eq. (7.23) implies that the X and T coordinates of a point on $\mathcal{S}$ must verify

$$X^2 - T^2 = \text{constant}\,. \tag{7.24}$$

Taking the derivative of Eq. (7.24) in order to the proper time we find the important relation

$$\frac{\partial X}{\partial \tau} = \frac{T}{X}\frac{\partial T}{\partial \tau}\,. \tag{7.25}$$

In our previous analysis of the $\mathcal{M}^-$ space-time, we found that the hypersurface $\mathcal{S}$ must be time-like, then, due to the first junction condition, $\mathcal{S}$ must also be time-like when seen from the exterior $\mathcal{M}^+$ space-time. Therefore, the components of the 4-velocity of an observer co-moving with it as seen from $\mathcal{M}^+$ are,

$$u_+^\alpha = \left(\frac{\partial T}{\partial \tau}, \frac{\partial X}{\partial \tau}, 0, 0\right)\,, \tag{7.26}$$

and it must verify

$${}^{(+)}g_{\alpha\beta}u_+^\alpha u_+^\beta = -1\,. \tag{7.27}$$

Using Eqs. (7.25) – (7.27) we can find the components of u,

$$\frac{\partial T}{\partial \tau} = \pm\sqrt{\frac{g^{XX}\,X^2}{X^2 - T^2}}\,, \tag{7.28}$$

$$\frac{\partial X}{\partial \tau} = \pm\sqrt{\frac{g^{XX}\,T^2}{X^2 - T^2}}\,, \tag{7.29}$$

such that

$$u_+^\alpha = \sqrt{\frac{g^{XX}}{X^2 - T^2}}\,(X, T, 0, 0)\ , \tag{7.30}$$

where the sign was chosen so that u points to the future and g^{XX} is the XX component of the inverse metric of Eq. (7.20). Notice that the expression found for the components of u, Eq. (7.30), only makes sense, physically, if $X^2 > T^2$. Looking at Eq. (7.22), that implies that either the shell is located in the region I or in the region I′ (cf. Figure 8a). This is a consequence of the shell being assumed static. If we were to consider a dynamic shell or a different interior space-time, shells in the black hole region or at the horizon could also be treated.

Before we proceed let us remark that in our analysis we have not specified if we consider the shell to be in the sub-region I or I′, however, in our choice of the plus sign in Eq. (7.30), such that u points to the future, we had to take into account each sub-region, that is, in this case the choice of the plus sign of u_+^α has the same meaning for both sub-regions but in general that does not have to be true, as different signs might have to be chosen depending on the studied sub-region. It is, however, important to mention that in our treatment the choice of the sign of u_+^α is irrelevant but, as we will see, the choice of the sign of the components for the normal, n_+^α, for each sub-region is not.

Eq. (7.30) can now be used to find the induced metric on the hypersurface $\mathcal{S}$, such that

$$ds^2\big|_{\mathcal{S}} = -d\tau^2 + \rho^2 d\Omega^2\,. \tag{7.31}$$

Matching Eq. (7.9) with Eq. (7.31) we find that ρ, the radial coordinate of $\mathcal{S}$ when measured by an observer at $\mathcal{M}^+$, and R, the radial coordinate of $\mathcal{S}$ when measured by an observer at $\mathcal{M}^-$, must be equal. We shall then use R to describe the radial coordinate of the shell for either the interior and exterior space-time.

Now, using the fact that the unit normal to $\mathcal{S}$, as seen from the exterior space-time $\mathcal{M}^+$, is space-like implies,

$$n_+^\alpha n_{+\alpha} = +1\,, \tag{7.32}$$

and taking into account Eqs. (7.1) and (7.30), we find

$$n_{+\alpha} = \pm\sqrt{\frac{g_{XX}}{X^2 - T^2}}\,(-T, X, 0, 0)\ . \tag{7.33}$$

To proceed we must choose the sign for the normal. The choice of the sign is related with the direction of the normal and we shall impose that it points in the direction of increasing X coordinate. This implies that the choice of the sign is different if we consider the shell to be in the sub-region I or I′ (cf. Figure 20). One of the simplifications that the use of the Kruskal-Szekeres coordinates introduces is that the choice of the sign can be written in a concise manner, such that

$$n_{+\alpha} = \text{sign}\,(X)\sqrt{\frac{g_{XX}}{X^2 - T^2}}\,(-T, X, 0, 0) \ , \tag{7.34}$$

where sign (X) is the signum function of the coordinate X of the shell. Notice however, that the usage of this notation is simply to treat in a concise way the two possible directions of the normal of the shell. Physically, there is nothing different between a shell placed in either region (with positive or negative values of X).

Found the normal to the hypersurface $\mathcal{S}$ as seen from the exterior non-extremal Reissner-Nordström space-time, we are now in position to find the non-null components of the extrinsic curvature. Following the results in Appendix D.2.1 we have

$$^{(+)}K^{\tau}{}_{\tau} = \frac{\text{sign}\,(X)}{2R^2\Upsilon}\left(r_+ + r_- - 2\frac{r_- r_+}{R}\right) \ , \tag{7.35}$$

$$^{(+)}K^{\theta}{}_{\theta} = {}^{(+)}K^{\varphi}{}_{\varphi} = \frac{\text{sign}\,(X)}{2R_+}\frac{\sqrt{g_{XX}\left(X^2 - T^2\right)}}{R} \ , \tag{7.36}$$

where the parameter Υ is given by

$$\Upsilon = \sqrt{\frac{\left(R - r_+\right)\left(R - r_-\right)}{R^2}} \ . \tag{7.37}$$

In Figure 9 we show the Carter-Penrose diagrams [31, 204] of the Minkowski — non-extremal Reissner-Nordström junction space-time either for a junction surface with normal pointing towards the event horizon or in the direction of spatial infinity.

7.3.1.3 Non-extremal thin shell's properties outside the event horizon

Found the necessary components of the extrinsic curvature of the hypersurface $\mathcal{S}$ as seen from each space-time, we are now in position to find the properties of a perfect fluid thin shell placed outside the event horizon of a non-extremal Reissner-Nordström space-time. The shell's stress-energy tensor is given by

$$S^{ab} = \sigma u^a u^b + p\left(h^{ab} + u^a u^b\right) \ , \tag{7.38}$$

where σ is the energy per unit area and p is the pressure of the fluid. From our choice of coordinates on $\mathcal{S}$, $\{y^a\} = (\tau, \theta, \varphi)$, we have

$$S^{\tau}_{\tau} = -\sigma \, , \tag{7.39}$$

$$S^{\theta}_{\theta} = p \, . \tag{7.40}$$

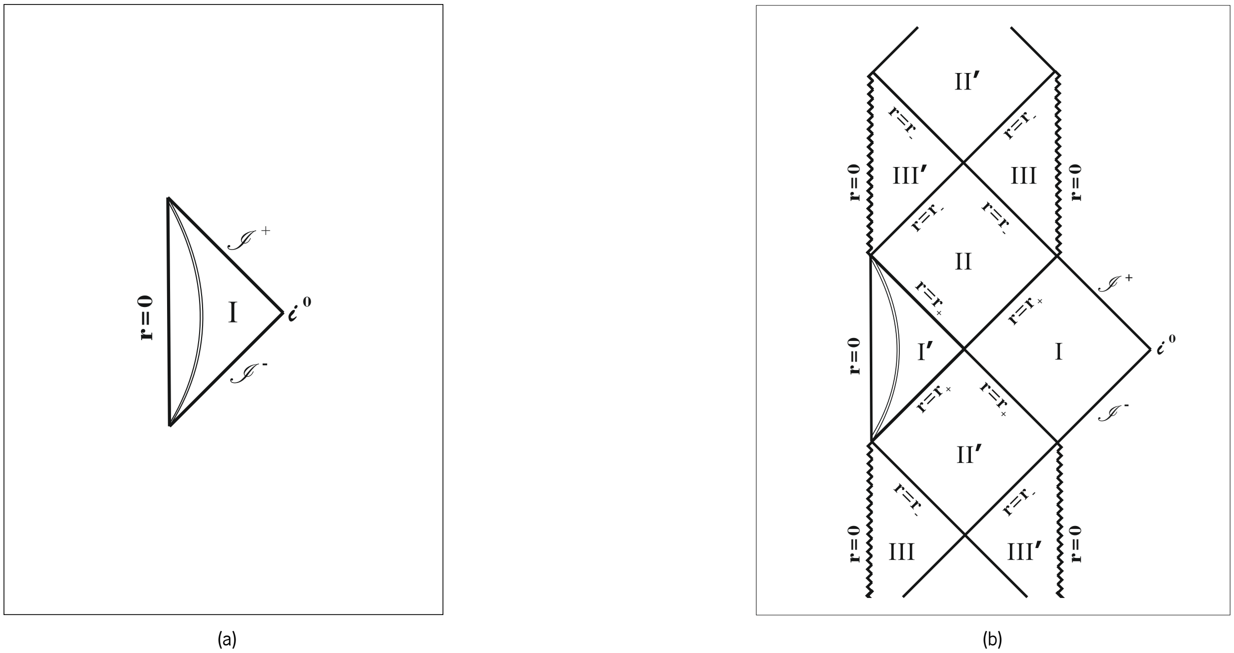


Figure 9: Carter-Penrose diagrams of a Minkowski — non-extremal Reissner-Nordström junction space-time through a time-like shell. Panel (a) The normal points in the direction of spatial infinity. Panel (b) The normal points towards the event horizon.

Now, comparing Eqs. (7.39) and (7.40) with the second junction condition, Eq. (7.3), taking into account the components of the induced metric, Eq. (7.9), and the fact that in our case $\left[K^{\theta}_{\theta}\right]_{\pm} = \left[K^{\varphi}_{\varphi}\right]_{\pm}$we find that

$$\sigma = -\frac{1}{4\pi}\left[K^{\theta}_{\theta}\right]_{\pm} , \tag{7.41}$$

$$p = \frac{1}{8\pi}\left[K^{\tau}_{\tau}\right]_{\pm} - \frac{\sigma}{2} . \tag{7.42}$$

Substituting in Eqs. (7.41) and (7.42) the values for the components of the extrinsic curvature found in the previous sections we find

$$8\pi\sigma = \frac{2}{R}\left(1 - \text{sign}\,(X)\,\Upsilon\right) , \tag{7.43}$$

$$8\pi p = \frac{\text{sign}\,(X)}{2R\Upsilon}\left[\left(1 - \text{sign}\,(X)\,\Upsilon\right)^2 - \frac{r_- r_+}{R^2}\right] , \tag{7.44}$$

where the parameter Υ is given by Eq. (7.37).

Before we discuss our results let us verify their consistency by checking what happens in the limit when the charge Q goes to zero. In this limit, from Eq. (7.21), we see that $r_- \rightarrow 0$ and $r_+ \rightarrow 2M$ hence, the outside space-time is described by the Schwarzschild solution. Then, Eqs. (7.43) and (7.44) yield

$$\lim_{Q\to 0} 8\pi\sigma = \frac{2}{R}\left(1 - \text{sign}\,(X)\sqrt{1-\frac{2M}{R}}\right)\,, \tag{7.45}$$

$$\lim_{Q\to 0} 8\pi p = \frac{\text{sign}\,(X)}{2R}\sqrt{\frac{R}{R-2M}}\left[\left(1 - \text{sign}\,(X)\sqrt{1-\frac{2M}{R}}\right)^2\right]\,, \tag{7.46}$$

matching the expressions found in Ref. [195] where a Minkowski — Schwarzschild space-times junction was considered.

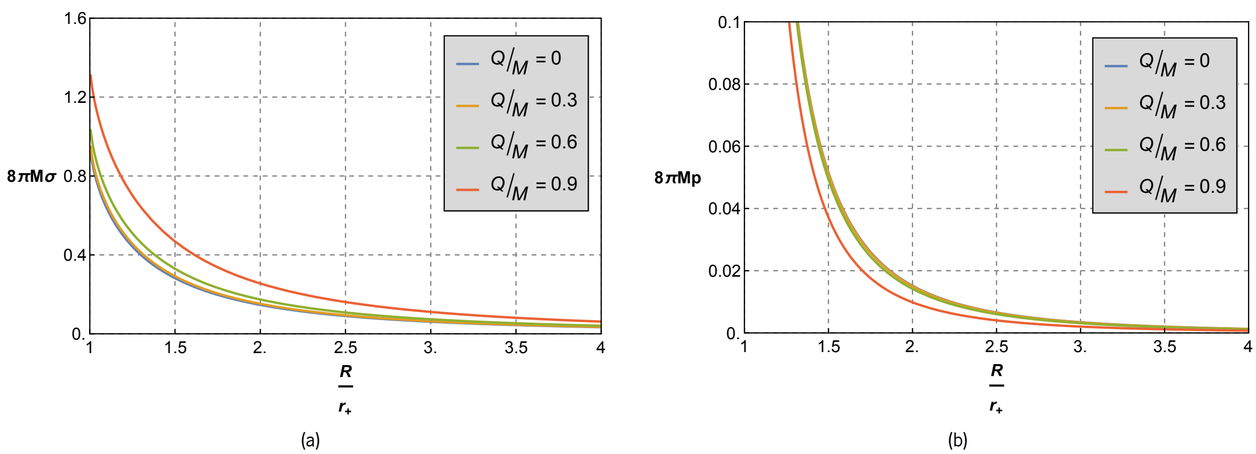


Figure 10: Properties of a perfect fluid thin shell placed with normal pointing in the direction of spatial infinity (see Figure 9a) of a non-extremal Reissner-Nordström space-time with an interior Minkowski space-time. Panel (a) Energy density. Panel (b) Pressure support.

In Figures 10 and 11 we show the behavior of the energy density, σ, and the pressure (tension), p (γ), that supports a thin shell for the possible resulting space-times for various values of the ratio Q/M. From Figure 11b we see that the thin shell is supported not by pressure but by tension. Physically this result was actually expected since an observer momentarily co-moving with the shell but detached from it will infall towards the black hole region of the exterior Reissner-Nordström space-time, hence, a perfect fluid thin shell placed at the junction hypersurface, in order to be static, must by supported by tension. This reasoning also explains the behavior exhibited in Figure 10b where the thin shell is supported by pressure. Notice, also, that both the pressure and the tension of such type of shells go to infinity as $R \to r_+$, but the energy density is constant, in both cases, at that limit.

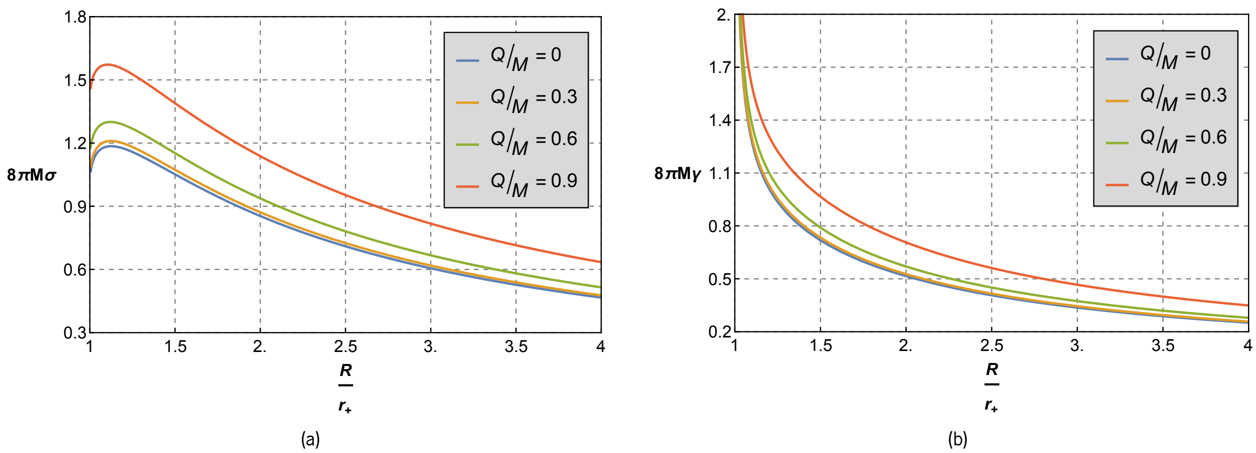


Figure 11: Properties of a perfect fluid thin shell with normal pointing towards the event horizon (see Figure 9b) of a non-extremal Reissner-Nordström space-time with an interior Minkowski space-time. Panel (a) Energy density. Panel (b) Tension support.

7.3.2 Electric thin shells inside the Cauchy horizon: normal and tension shells in an interior non-extremal Reissner-Nordström space-time

7.3.2.1 Induced metric, and extrinsic curvature of $\mathcal{S}$ as seen from $\mathcal{M}^+$

To study the case of a static time-like thin shell placed in a region covered by the coordinate patch without the pseudo-singularity at $r = r_-$ we have now to repeat the same recipe followed in the previous section. We will see, however, that many of the previous results are also valid for the second coordinate patch, greatly simplifying our work. From the discussion in Appendix D.1.3, the line element for the Reissner-Nordström space-time in Kruskal-Szekeres coordinates is:

$$ds^2 = \frac{16M^2}{r^2} R_-^2 e^{\frac{r}{R_-}} \left(\frac{r_+ - r}{2M}\right)^{1+\left(r_+/r_-\right)^2} \left(dX^2 - dT^2\right) + r^2\left(X,T\right) d\Omega^2 \,, \tag{7.47}$$

where $r_\pm$ and R_- are given by Eq. (7.21) and r is given implicitly by

$$X^2 - T^2 = e^{-\frac{r}{R_-}} \left(\frac{r_- - r}{2M}\right) \left(\frac{r_+ - r}{2M}\right)^{-\left(r_+/r_-\right)^2} . \tag{7.48}$$

Considering a static shell, from Eq. (7.48) we take that the X and T coordinates of the shell must verify

$$X^2 - T^2 = \text{constant} \,. \tag{7.49}$$

Now, as was argued in the previous section, a static shell in the Minkowski space-time must be time-like as seen from both interior and exterior space-times. This restriction and Eq. (7.49) imply that Eqs. (7.28) and

(7.29) are also valid in this coordinate patch, hence, the components of the 4-velocity of an observer co-moving with the shell as seen from the exterior space-time, are given by

$$u_+^\alpha = -\sqrt{\frac{g^{XX}}{X^2 - T^2}}\,(X, T, 0, 0)\ , \tag{7.50}$$

where, in this case, g^{XX} is the XX component of the inverse of the metric in Eq. (7.47). We see that Eq. (7.50) only makes sense, physically, if $X^2 - T^2 > 0$, which, taking into account Eq. (7.48), allow us to conclude that the shell must then be placed either at the sub-region III or III′ (see Figure 8a). This conclusion was actually expected since it is not possible to have a static time-like hypersurface in the black-hole region or at the Cauchy horizon, $r = r_-$. Let us remark that the minus sign arises from the convention that the 4-velocity points to the future for both sub-regions III and III′.

Making use of Eqs. (7.47) and (7.50) to find the induced metric on $\mathcal{S}$ as seen by an observer at $\mathcal{M}^+$ and imposing the first junction condition, Eq. (7.2), we find that the shell's radial coordinate R is the same as measured by an observer at $\mathcal{M}^-$ or $\mathcal{M}^+$ and the induced metric on $\mathcal{S}$ is given by Eq. (7.9).

Combining Eqs. (7.1), (7.10) and (7.50) we find the expression for the unit normal to the hypersurface $\mathcal{S}$, as seen from the exterior space-time $\mathcal{M}^+$, to be

$$n_{+\alpha} = \pm\sqrt{\frac{g_{XX}}{X^2 - T^2}}\,(-T, X, 0, 0)\ . \tag{7.51}$$

Now, to specify the sign of the normal to $\mathcal{S}$ for each sub-region we shall consider two cases: the case where the normal n points towards the Cauchy horizon at $r = r_-$ and the case where the normal points towards the singularity at $r = 0$. These two cases can be treated in a concise way by assuming, for example, a shell placed either in the sub-region III or III′ and the normal pointing in the direction of decreasing X coordinate, such that[5]

$$n_{+\alpha} = \text{sign}\,(X)\sqrt{\frac{g_{XX}}{X^2 - T^2}}\,(T, -X, 0, 0)\ . \tag{7.52}$$

Then, using the results from Appendix D.2.2, we find the non-null components of the extrinsic curvature of $\mathcal{S}$ as seen from the exterior space-time to be given by

$${}^{(+)}K^{\tau}{}_{\tau} = \frac{\text{sign}\,(X)}{2R^2\Upsilon}\left[r_+ + r_- - 2\frac{r_- r_+}{R}\right]\ , \tag{7.53}$$

$${}^{(+)}K^{\theta}{}_{\theta} = {}^{(+)}K^{\varphi}{}_{\varphi} = \frac{\text{sign}\,(X)}{2R}\frac{\sqrt{g_{XX}\,(X^2 - T^2)}}{R_-}\ , \tag{7.54}$$

where Υ is given by Eq. (7.37).

In Figure 12 we show the Carter-Penrose diagrams of the Minkowski — Reissner-Nordström junction space-

[5]We emphasize, as in the previous subsection, that physically the important characteristic is the sign of the normal and not where the shell itself is placed, as sub-regions differing by the sign of X have the same physical properties, as measured by an observer in each sub-region.

time either for a junction surface whose outside normal points towards the Cauchy horizon or towards the singularity at $r = 0$. Notice that in the former case, Figures 12a.1 and 12a.2, we have the equivalent possibility of a thin shell placed at only one sub-region or at both. Although the scenarios represented in these diagrams are very different: as there might or not be a singularity at $r = 0$, for an outside observer with coordinate $r > R$ the geometry of the space-time — hence, the gravitational field — will be the same.

7.3.2.2 Non-extremal thin shell's properties inside the Cauchy horizon

Determined the components of the extrinsic curvature of the hypersurface $\mathcal{S}$ as seen from each space-time, we are now in position to find the properties of a perfect fluid thin shell placed inside of the Cauchy horizon. As was found in subsection 7.3.1.2, the energy density and the pressure of a perfect fluid thin shell are related to the extrinsic curvature components by Eqs. (7.41) and (7.42). Then, from Eqs. (7.18) and (7.54) we find

$$8\pi\sigma = \frac{2}{R}\left(1 - \text{sign}\left(X\right)\Upsilon\right) \,, \tag{7.55}$$

and, substituting Eqs. (7.17), (7.53) and (7.55) in Eq. (7.42),

$$8\pi p = \frac{\text{sign}\left(X\right)}{2R\Upsilon}\left[\left(1 - \text{sign}\left(X\right)\Upsilon\right)^2 - \frac{r_- r_+}{R^2}\right] \,, \tag{7.56}$$

where the parameter Υ is given by Eq. (7.37).

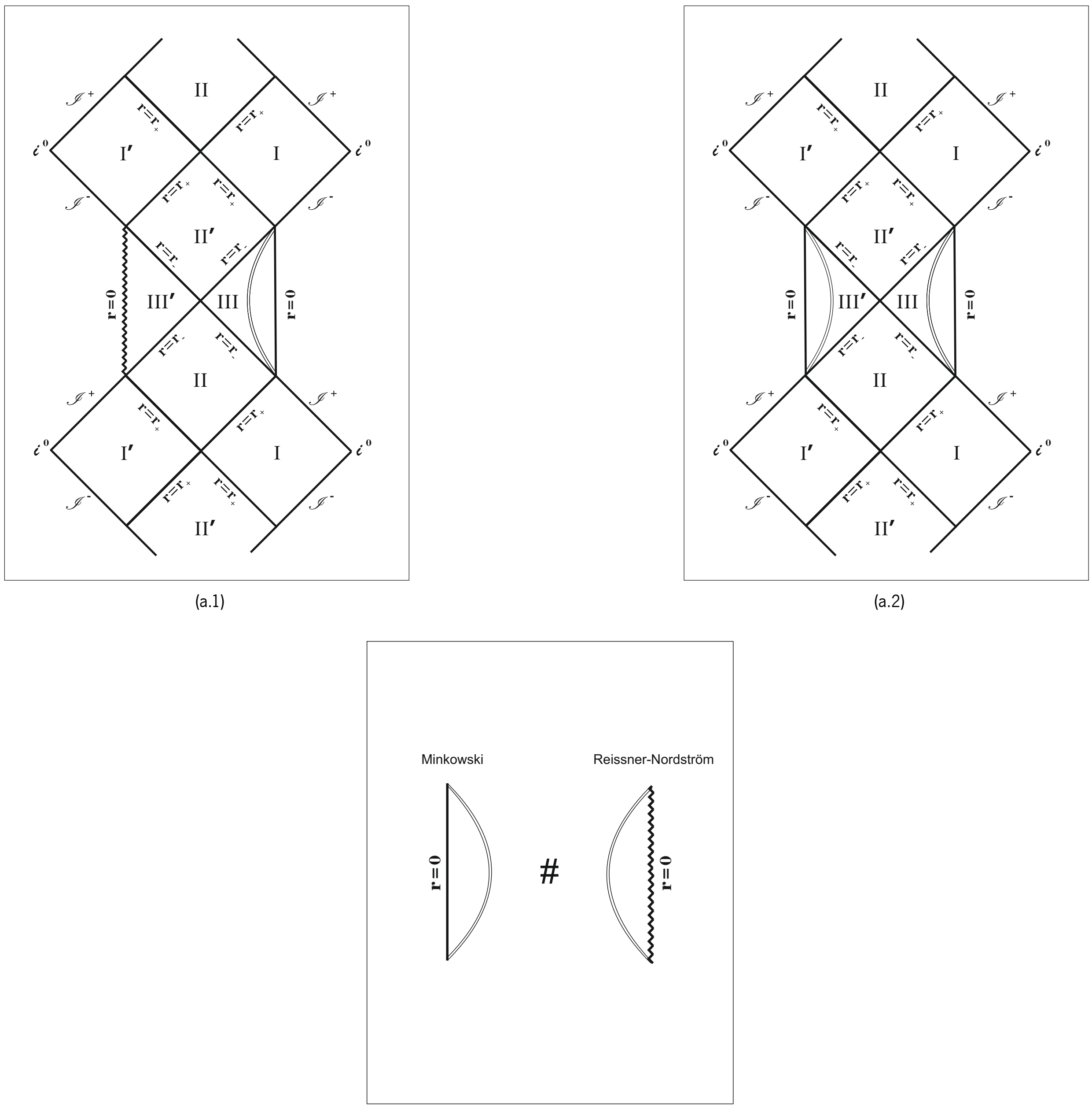
Now, the expressions found for the energy density for a shell placed inside the Cauchy horizon, Eqs. (7.55) and (7.56), are the same as Eqs. (7.43) and (7.44), however, the behavior of the properties of the shell will be different since, the radial coordinate of the shell, R, in this case ranges between zero and r_-. In Figures 13 and 14 we present the behavior of the energy density and pressure of a shell associated with the diagrams in Figure 12.

Let us analyze each case separately. In the case of a thin shell associated with the diagrams 12a.1 or 12a.2 we see from Figure 13 that, depending on the radial coordinate of the shell, the energy density might take negative values. Indeed, from Eq. (7.55) we find that for

$$R < \frac{Q^2}{2M} \,, \tag{7.57}$$

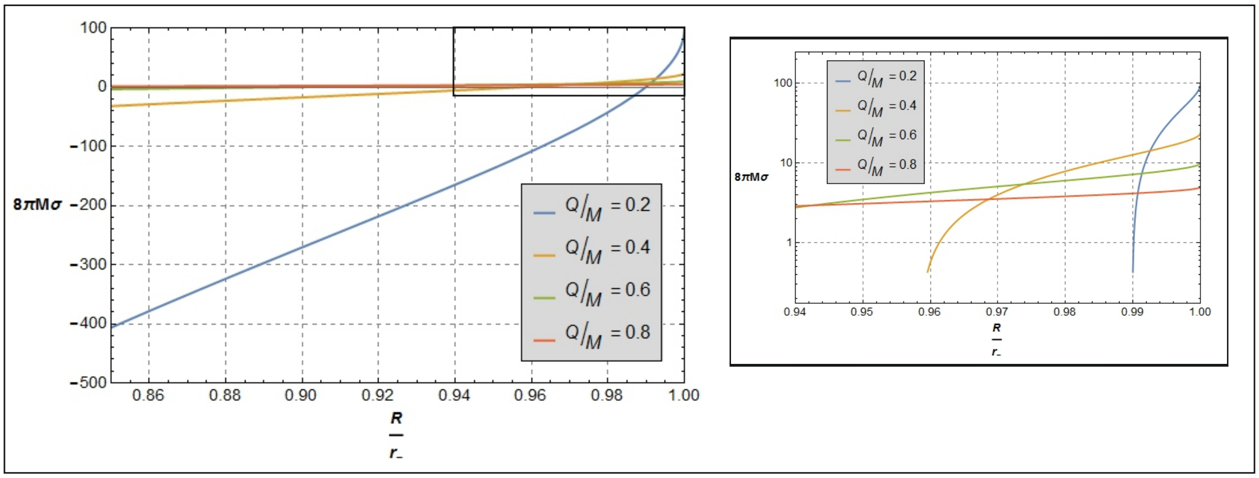
the energy density, σ, is negative. On the other hand, this kind of thin shell is always supported by tension, γ. This translates the well known fact that the Reissner-Nordström singularity at $r = 0$ is repulsive [201]. Moreover, we see that both the energy density and the pressure of the shell diverge to negative infinity as we get closer to the would be singularity, $R \to 0$. In the limit of $R \to r_-$ the pressure also diverges to negative infinity but the energy density, σ, tends to $(4\pi\, r_-)^{-1}$.

In the case of a thin shell associated with the diagram 12b, we see from Figure 14 that the energy density

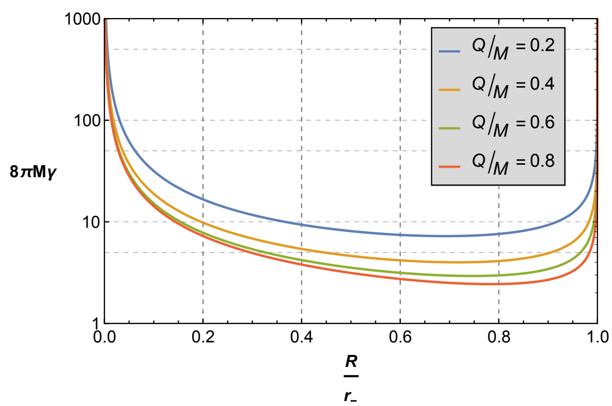


(b)

Figure 12: Carter-Penrose diagrams of the Minkowski — non-extremal Reissner-Nordström junction space-time through a time-like thin shell in sub-regions III and III′. Panels (a.1)/(a.2) The outside normal to the shell points towards the Cauchy horizon at $r = r_{-}$. Panel (b) The outside normal to the shell points towards the singularity at $r = 0$.



(a)



(b)

Figure 13: Properties of a perfect fluid thin shell with circumferential radius $R < r_-$ and outside normal pointing towards the Cauchy horizon at $r = r_-$, separating a non-extremal Reissner-Nordström space-time with an interior Minkowski space-time. Panel (a) Energy density. The image on the right is a zoom of the region inside the rectangle in the image on the left. Panel (b) Tension support.

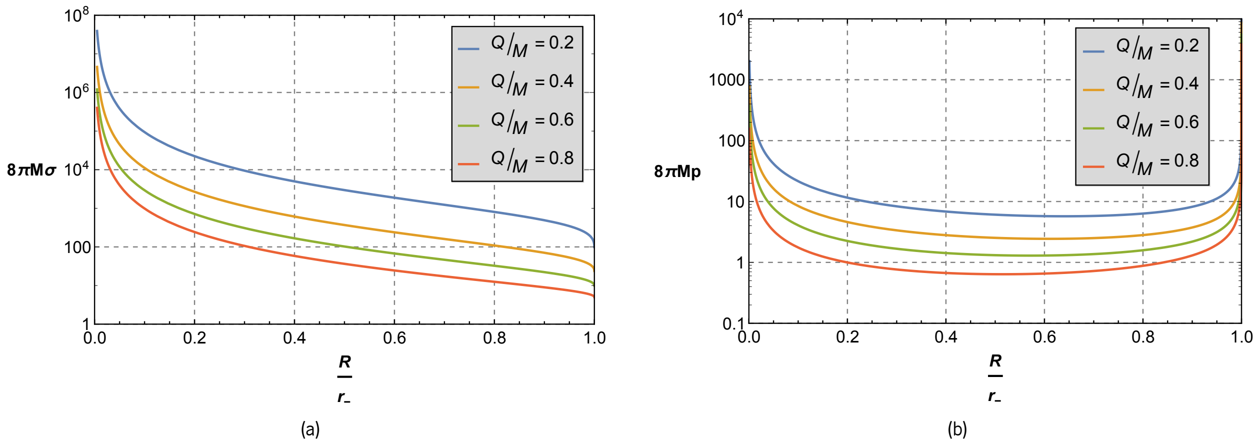


Figure 14: Properties of a perfect fluid thin shell with circumferential radius $R < r_-$ and outside normal pointing towards the singularity, separating a non-extremal Reissner-Nordström space-time with an interior Minkowski space-time. Panel (a) Energy density. Panel (b) Pressure support.

of the shell is always positive and the shell is supported by pressure. In this case, as the radial coordinate of the shell, R, goes to zero, both the energy density and pressure of the shell diverge to positive infinity. On the other hand, as $R \to r_-$ the pressure diverges to positive infinity but, as in the previous case, the energy density tends to $(4\pi\, r_-)^{-1}$.

7.4 Extremal and overcharged thin shells

In the previous sections we have found the expressions for the energy density and the pressure of a perfect fluid static thin shell with a Minkowski flat interior in a non-extremal $(M^2 > Q^2)$ exterior Reissner-Nordström space-time. Let us now see if the results can be extended for the extremal and overcharged Reissner-Nordström models.

In the construction of the Kruskal-Szekeres coordinates X and T, it was assumed that the Reissner-Nordström space-time was non-extremal (cf. Appendix D.1). Indeed if we try to extended our results to describe the extremal and overcharged cases, we realize that the expressions found for these coordinates do not make much sense since, they may take imaginary values. This translates the well known fact that the Kruskal-Szekeres coordinates that describe an extremal or overcharged Reissner-Nordström space-time can not be found by a limiting case of those that describe a non-extremal space-time [201, 202]. Albeit this fact, we see that the expressions for the energy density and pressure support of the shell, Eqs. (7.43) and (7.44) and Eqs. (7.55) and (7.56), only depend on the radial coordinate of the shell, R, and on the sign of the coordinate X. Now, the term sign (X) was introduced in the expression for the outside unit normal to the shell, Eqs. (7.34) and (7.52), to encapsulate in a concise and clear geometric way the information about the direction of the normal in the exterior space-time. That is, we related the sub-region where the shell was considered with the two possible directions that the normal could point to. This means, therefore, that we could substitute the term sign (X) in the expressions for the energy density and the pressure of the shell by a parameter $\xi = \{-1, 1\}$, defined as $\xi = +1$ if the outside unit normal to the shell points in the direction of increasing radial coordinate r, measured by an observer in the exterior $\mathcal{M}^+$ space-time, and $\xi = -1$ if the outside unit normal to the shell points in the direction of decreasing radial coordinate r, again, measured by an observer in the exterior $\mathcal{M}^+$ space-time. This definition is consistent with our previous conventions (cf. Eqs. (7.34) and (7.52)). As such, we can rewrite the expressions for the energy density and the pressure as

$$8\pi\sigma = \frac{2}{R}\left(1 - \xi\Upsilon\right) , \tag{7.58}$$

$$8\pi p = \frac{\xi}{2R\Upsilon}\left[\left(1 - \xi\Upsilon\right)^2 - \frac{r_- r_+}{R^2}\right] , \tag{7.59}$$

where the parameter Υ is given by Eq. (7.37) and the radial coordinate of the shell $R \in]0, r_-[\cup]r_+, +\infty[$, for the non-extremal and extremal cases, and $R \in]0, +\infty[$ for the overcharged case.

In Figure 15 we present the Carter-Penrose diagrams for all possible cases of an interior Minkowski — exterior extremal Reissner-Nordström junction space-time and, for the sake of clarity, we present separately in Figure 16 the diagrams in the case of an exterior overcharged space-time. In Figures. 17 and 18 we show the behavior of the energy density and pressure support described by Eqs. (7.58) and (7.59), for each case.

Let us now analyze the properties of the shells for each case. We see in Figure 17 that the energy density σ may either take positive or negative values, depending on its radial coordinate. From Eq. (7.58) we find that if the radial coordinate of the shell is in the range defined by Eq. (7.57), σ is negative. Another interesting feature of such extremal shells is the behavior of σ at $R = r_\pm = M$. Although, as was argued in the previous sections, a static time-like shell with a Minkowski core in an exterior Reissner-Nordström space-time can not be placed at the horizon, we see that the one-sided limits exist. From Figure 17b we conclude that overcharged shells are always hold by tension, whereas, extremal shells are supported by tension only if considered inside the horizon and — although not visible due to the use of logarithmic scale — a perfect fluid time-like static thin shell placed outside the horizon has zero pressure support. This result confirms the discussion in Refs. [202, 205] where the metric for the outer region to the horizon of the extremal Reissner-Nordström space-time, $r > r_+$, represents the exterior field of a spherical charged dust cloud in equilibrium between gravitational attraction and electrostatic repulsion.

As for shells whose properties' behavior is presented in Figure 18, we see that their energy density is always positive. For extremal shells, however, we, again, verify a discontinuity in the derivative of σ at the horizon but the one-sided limits exist. Overcharged shells are always hold by tension, γ, but an interesting feature is the change in the behavior of γ at $R > M$, where the tension support of such shells is smaller as the Q/M ratio increases. Another interesting behavior is that such extremal shells placed inside the horizon at $r = M$ have zero pressure support.

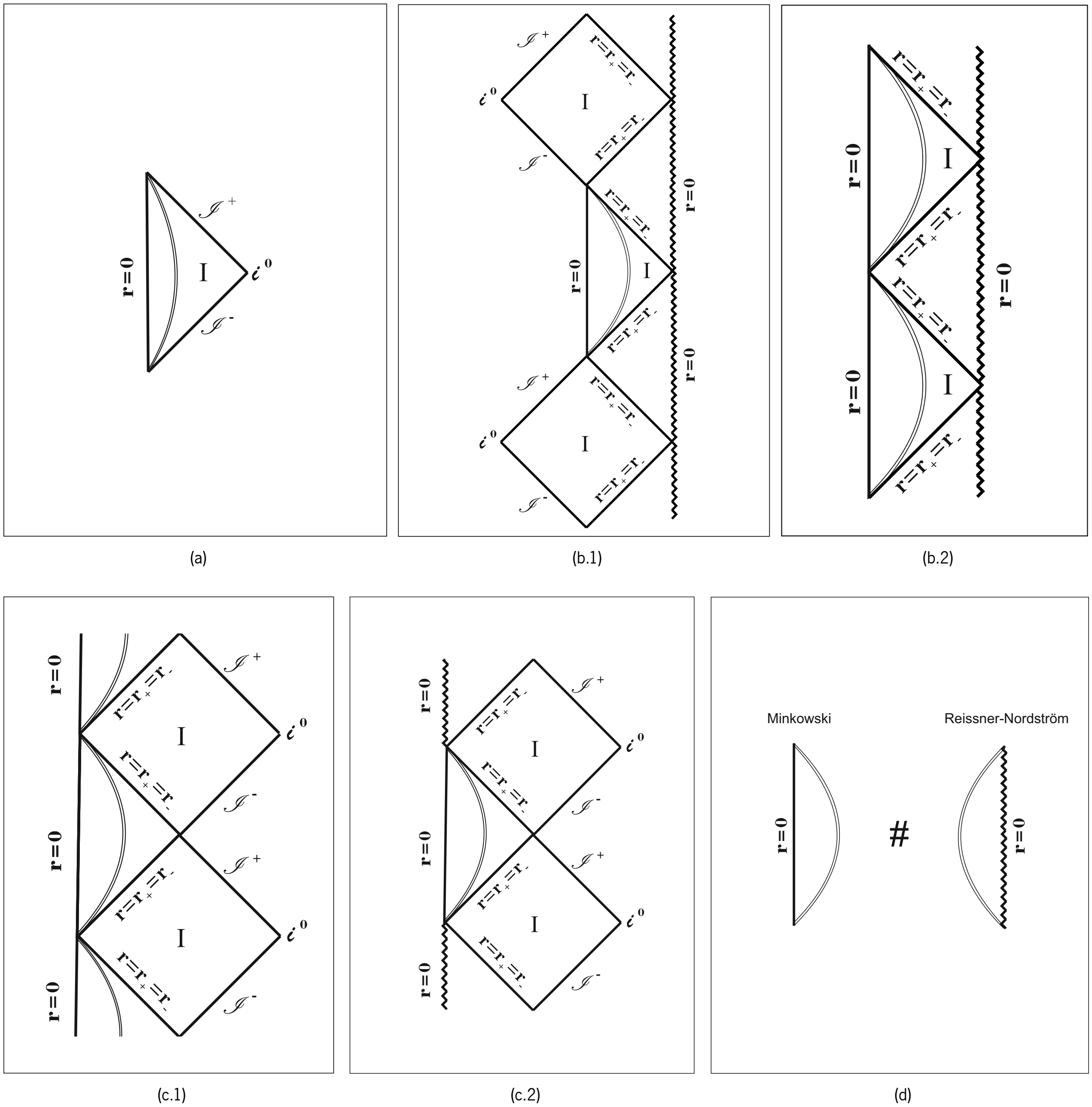
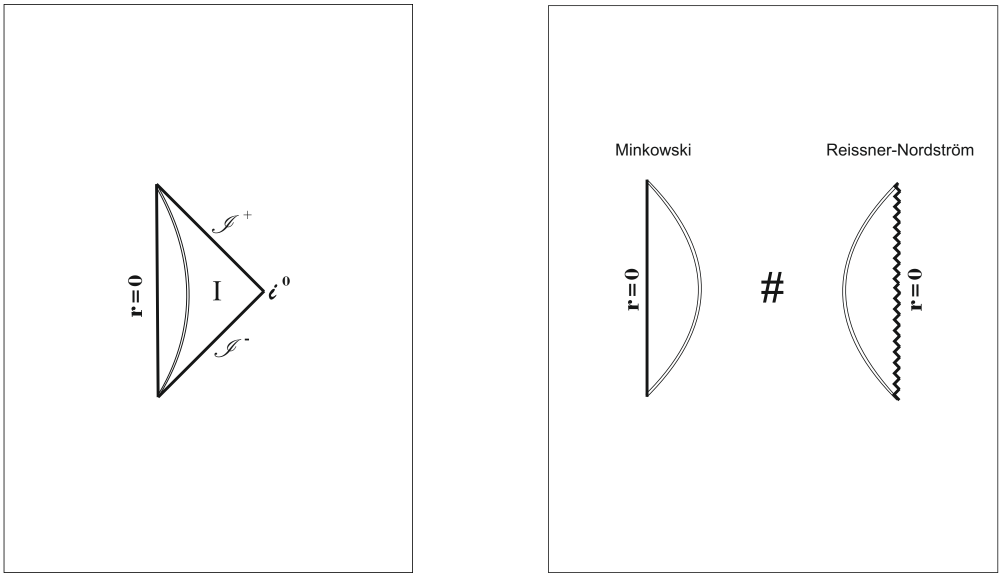


Figure 15: Carter-Penrose diagrams of the Minkowski — extremal Reissner-Nordström junction space-time through a time-like shell with constant radial coordinate R. Panels (a) Shell with $R > r_+$ and the outside normal points towards spatial infinity. Panel (b.1)/(b.2) Shell with $R > r_+$ and the outside normal points towards the event horizon. Panel (c.1)/(c.2) Shell with $R < r_+$ and the outside normal points towards the event horizon. Panel (d) Shell with $R < r_+$ and the outside normal points towards the singularity at $r = 0$.



(a) (b)

Figure 16: Carter-Penrose diagrams of the Minkowski — overcharged Reissner-Nordström junction space-time through a time-like shell with constant radial coordinate R. Panel (a) Outside normal to the shell points towards spatial infinity. Panel (b) Outside normal to the shell points towards the singularity at $r = 0$.

7.5 Energy conditions for electric thin shells

7.5.1 Energy conditions

The analysis of the properties of time-like, static, perfect fluid thin shells with a Minkowski interior and a Reissner-Nordström exterior showed that both the energy density and pressure support depend of the direction of the outside pointing normal. Moreover, we saw that depending on the value of the radial coordinate of the shell, the energy density and pressure may take negative values. In this section, we shall address the question of in what conditions are such boundary layers possible to exist in nature, at least from a classical point of view. To do so, we shall verify for what values of the shell's properties are the various energy conditions verified, if at all.

The energy conditions are a set of restrictions on the stress-energy tensor and, in the case of a perfect fluid, lead to specific constraints on the energy density and pressure. Although various restrictions can be assumed, we shall only study the null, weak, strong and dominant energy conditions. Moreover, each energy condition may be considered to hold at any point of the space-time or along a flowline, where the specified energy condition is only verified on average — allowing for pointwise violations (cf., e.g., Refs. [16, 58, 149, 206, 207]). Here we will consider the pointwise version of the energy conditions.

Let us briefly explain the physical motivation for each energy condition and their implications on the properties of a perfect fluid thin shell:

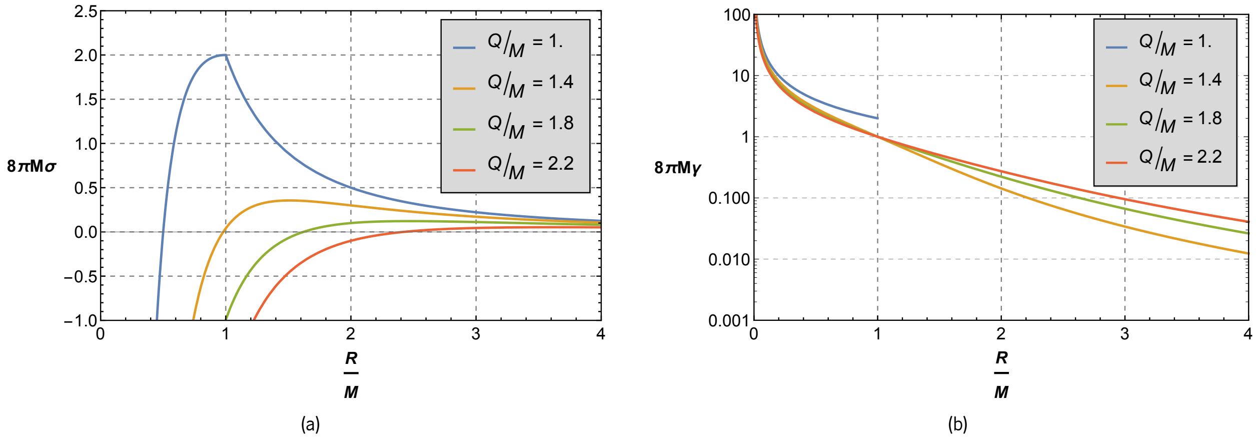


Figure 17: Properties of an extremal and overcharged perfect fluid thin shell with a Minkowski core described by the Penrose diagrams in Figures 15a, 15c.1, 15c.2 and 16a. Panel (a) Energy density. Panel (b) Tension support.

Null energy condition

The null energy condition (NEC) represents the restriction that the energy density of any matter distribution in space-time experienced by a ligh-ray is non-negative. This is represented by

$$\mathcal{T}_{\alpha\beta}k^{\alpha}k^{\beta} \geq 0\,, \tag{7.60}$$

for any future pointing null vector field k. For a perfect fluid thin shell with stress-energy tensor given by Eq. (7.38) this implies

$$\sigma + p \geq 0\,. \tag{7.61}$$

Weak energy condition

The weak energy condition (WEC) is a more restrictive version of the NEC where it is imposed that the energy density of any matter distribution in space-time measured by any time-like observer must be non-negative, then

$$\mathcal{T}_{\alpha\beta}v^{\alpha}v^{\beta} \geq 0\,, \tag{7.62}$$

for any future pointing, time-like vector field. In the case of a perfect fluid this leads to the following restrictions

$$\begin{aligned} \sigma &\geq 0\,,\\ \sigma + p &\geq 0\,. \end{aligned} \tag{7.63}$$

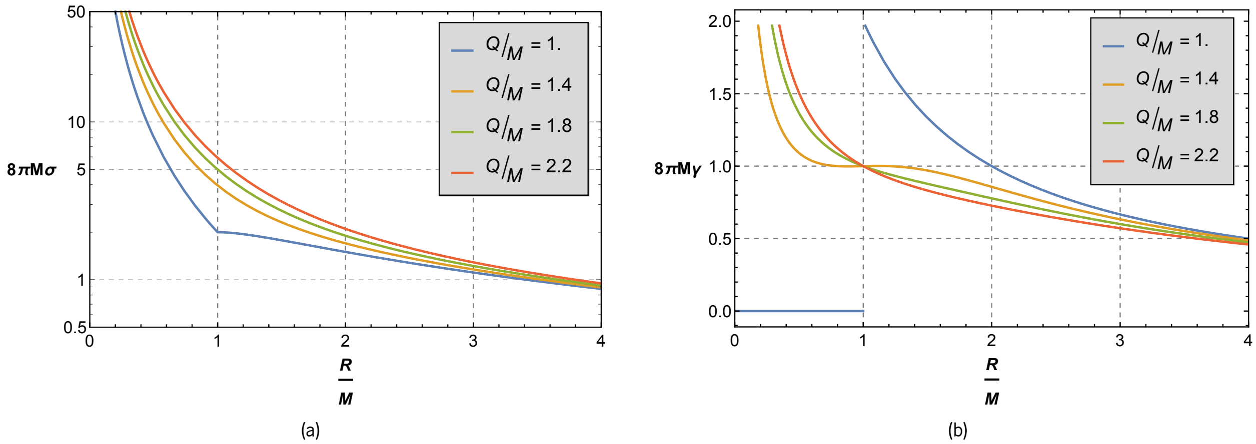


Figure 18: Properties of an extremal and overcharged perfect fluid thin shell with a Minkowski core described by the Penrose diagrams in Figures 15b.1, 15b.2, 15d and 16b. Panel (a) Energy density. Panel (b) Tension support.

Dominant energy condition

The dominant energy condition represents the statement that in addition to the WEC being verified, the flow of energy can never be observed to be faster than light, that is, in addition to Eq. (7.62) the vector field Y with components $Y^\alpha = -\mathcal{T}_\beta{}^\alpha v^\beta$, verifies $Y^\alpha Y_\alpha \leq 0$, for any time-like future pointing vector field v. In the case of a perfect fluid thin shell, this implies

$$\begin{aligned} \sigma &\geq 0\,, \\ \sigma &\geq |p|\,. \end{aligned} \tag{7.64}$$

Strong energy condition

The strong energy condition represents the restriction that nearby time-like geodesics are always focused towards each other — essentially guaranteeing that gravity is always perceived to be attractive by any time-like observer. In the case of GR, this is found by guaranteeing[6]

$$\left(\mathcal{T}_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \mathcal{T}_\mu{}^\mu\right) v^\alpha v^\beta \geq 0\,, \tag{7.65}$$

for any time-like vector field v. For a perfect fluid thin shell we find,

$$\begin{aligned} \sigma + p &\geq 0\,, \\ \sigma + 2p &\geq 0\,. \end{aligned} \tag{7.66}$$

[6]Notice that condition (7.65) follows from the premise 2 in the focus theorem 4.1 in the case of GR. Then, it is clear that the constraint on the stress-energy tensor imposed by the strong energy condition depends on the theory of gravitation that is considered.

7.5.2 Limiting radii

From Eqs. (7.61) – (7.66) we see that the energy conditions imply various restrictions on the energy density and pressure of a perfect fluid. In the considered setup, we found that the properties of the perfect fluid thin shells are functions of the radial coordinate of the shell only — Eqs. (7.58) and (7.59); hence, the constrains imposed by the energy conditions on the thin shell for the various possible junction space-times will lead to restrictions on the radial coordinate of the boundary layer. Anticipating what follows, we present the expressions for the limiting radii that arise from solving the inequalities (7.61) – (7.66) for the various junction space-times, namely, R_{I}, $R_{\mathsf{I}'}$ and R_{III}:

$$R_{\mathsf{I}} = \frac{M}{36}\left[25 + 3\left(\frac{Q}{M}\right)^2 + \frac{9\left(\frac{Q}{M}\right)^4 - 570\left(\frac{Q}{M}\right)^2 + 625}{\Delta_{\mathsf{I}}} + \Delta_{\mathsf{I}}\right], \tag{7.67}$$

$$R_{\mathsf{I}'} = \frac{M}{4}\left[3 + \left(\frac{Q}{M}\right)^2 + \frac{\left(\frac{Q}{M}\right)^4 - 10\left(\frac{Q}{M}\right)^2 + 9}{\Delta_{\mathsf{I}'}} + \Delta_{\mathsf{I}'}\right], \tag{7.68}$$

$$R_{\mathsf{III}} = \frac{M}{72}\left[50 + 6\left(\frac{Q}{M}\right)^2 - \frac{(1 - i\sqrt{3})\left[9\left(\frac{Q}{M}\right)^4 - 570\left(\frac{Q}{M}\right)^2 + 625\right]}{\Delta_{\mathsf{I}}} - \left(1 + i\sqrt{3}\right)\Delta_{\mathsf{I}}\right], \tag{7.69}$$

with

$$\Delta_{\mathsf{I}}^3 = 27\left(\frac{Q}{M}\right)^6 + 216\left(\frac{Q}{M}\right)^3\sqrt{9\left(\frac{Q}{M}\right)^4 + 366\left(\frac{Q}{M}\right)^2 - 375} + 5211\left(\frac{Q}{M}\right)^4 - 21375\left(\frac{Q}{M}\right)^2 + 15625\,, \tag{7.70}$$

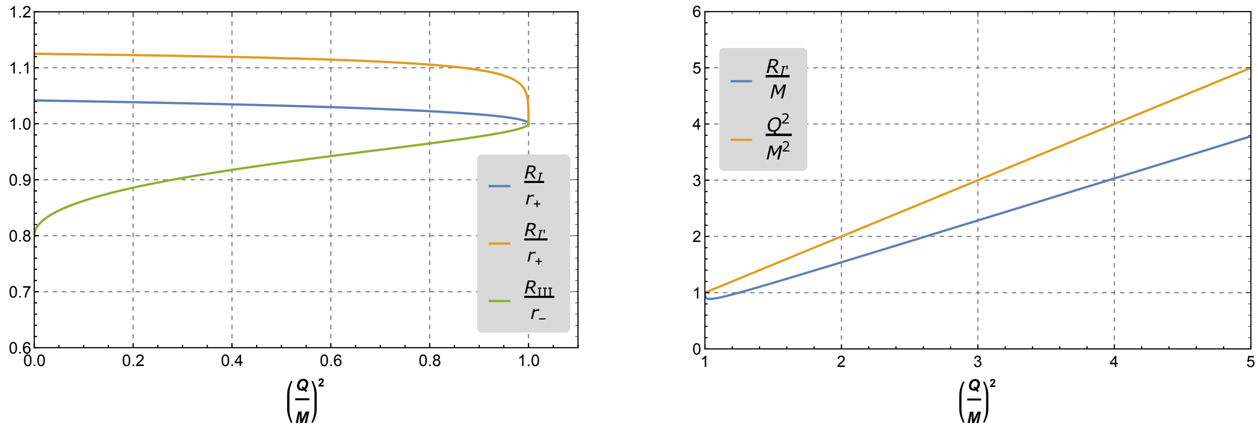
$$\Delta_{\mathsf{I}'}^3 = 8\left(\frac{Q}{M}\right)^3\sqrt{\left(\left(\frac{Q}{M}\right)^2 - 1\right)^2} + \left(\frac{Q}{M}\right)^6 + 17\left(\frac{Q}{M}\right)^4 - 45\left(\frac{Q}{M}\right)^2 + 27\,, \tag{7.71}$$

where we refer to the real root of the cubic equations. Let us remark that although the expression for R_{III} is written in terms of the imaginary unit i, for the range of values of the ratio Q/M of interest, this function takes purely real values. Moreover, although it is not clear from the expressions, the values of the radii R_{I}, $R_{\mathsf{I}'}$ and R_{III} are independent of the sign of Q, as expected.

For completeness, in Figure 19 we present the behavior of the various limiting radii defined in Eqs. (7.67) – (7.69) as functions of the ratio Q/M.

7.5.3 Analysis

Let us then use Eqs. (7.58) and (7.59) in the inequalities (7.61) – (7.66) for each kind of shell. In Table 2 we summarize the results.



(a) (b)

Figure 19: Behavior of the various radii, whose expressions are given by Eqs. (7.67) – (7.69), found by imposing the null, weak, dominant and strong energy conditions to the thin shells present at the matching surface of the various junction space-times.

7.5.3.1 Energy conditions of non-extremal electric thin shells

For the case of non-extremal electric thin shells with radial coordinate $R > r_+$, we conclude that shells with normal pointing in the direction of spatial infinity, Figure 9a, always verify the null, weak and strong energy conditions, and, except for shells in a region between $R > r_+$ and $R < R_{\mathrm{I}}$, the dominant energy condition is also verified; for shells whose normal points in the direction of the event horizon, Figure 9b, interestingly, only the strong energy condition is always violated, independently of the radius of the shell. Moreover, the limiting radius $R_{\mathrm{I}'}$, Eq. (7.68), also determines the value of the circumferential radius of the shell for which its energy density is maximum, explaining the “bumps” in Figure 11a.

For non-extremal electric thin shells with radial coordinate $R < r_-$, we have found that, if the outward normal points in the direction of the Cauchy horizon, Figures 12a.1 and 12a.2, none of the considered energy conditions are verified; on the other hand, the somehow exotic, case of shells whose outside normal points to the singularity at $r = 0$, Figure 12b, verify all the tested energy conditions in the domain, except for the dominant energy condition which is only verified by shells with $0 < R \leq R_{\mathrm{III}}$.

7.5.3.2 Energy conditions of extremal electric thin shells

As for the extremal shells, we find that, thin shells in the cases of Figures 15a, 15b.1, 15b.2 and 15d always verify the null, weak and dominant energy conditions. However, shells associated with the diagrams 15b.1 and 15b.2 will violate the strong energy condition. On the other hand, extremal thin shells represented by the diagrams 15c.1 and 15c.2 will always violate the tested energy conditions.

		Null energy condition	Weak energy condition	Dominant energy condition	Strong energy condition
Non-Extremal	9a	$R > r_+$	$R > r_+$	$R \geq R_{\mathrm{I}}$	$R > r_+$
	9b	$R \geq R_{\mathrm{I'}}$	$R \geq R_{\mathrm{I'}}$	$R \geq R_{\mathrm{I'}}$	Never verified
	12a.1/12a.2	Never verified	Never verified	Never verified	Never verified
	12b	$0 < R < r_-$	$0 < R < r_-$	$0 < R \leq R_{\mathrm{III}}$	$0 < R < r_-$
Extremal	15a	$R > r_+$	$R > r_+$	$R > r_+$	$R > r_+$
	15b.1/15b.2	$R > r_+$	$R > r_+$	$R > r_+$	Never verified
	15c.1/15c.2	Never verified	Never verified	Never verified	Never verified
	15d	$0 < R < r_+$	$0 < R < r_+$	$0 < R < r_+$	$0 < R < r_+$
Overcharged	16a	$R \geq R_{\mathrm{I'}}$	$R \geq R_{\mathrm{I'}}$	$R \geq R_{\mathrm{I'}}$	$R \geq \frac{Q^2}{M}$
	16b	$R > 0$	$R > 0$	$R > 0$	$R \leq \frac{Q^2}{M}$

Table 2: Range of values of the radial coordinate of a static, time-like, perfect fluid thin shell, in the various allowed sub-regions of the exterior Reissner-Nordström space-time for which, the null, weak, dominant and strong energy conditions are verified.

7.5.3.3 Energy conditions of overcharged electric thin shells

Lastly, in the overcharged case of Figure 16a, we conclude that if the radial coordinate of the shell is greater or equal to $R_{\mathrm{I'}}$, Eq. (7.68), it is possible to have overcharged thin shells with a Minkowski flat core that verify the null, weak and dominant energy conditions. If the radial coordinate of the shell is also greater or equal to Q^2/M then all the studied energy conditions are verified by this type of thin shells. As for shells in the case of Figure 16b we see that the null, weak and dominant energy conditions are always verified and the strong energy condition is verified for shells with $R \leq Q^2/M$.

These results for the strong energy condition indicate that in the overcharged Reissner-Nordström space-time the singularity is repulsive in a core region with $r < Q^2/M$. This result extends that of Refs. [201, 202, 205] where it was found that the non-extremal and extremal cases are characterized by a repulsive region delimited, respectively, by the Cauchy or event horizon. Here, although there are no horizons, we see that the same conclusion holds. To our knowledge, this was first realized in Ref. [208] when studying the radial motion of infalling particles in a Kerr-Newmann space-time, although the authors did not discuss the limiting radius of the repulsive region. We also highlight the discussion in Refs. [209, 210], where the former wrongly showed that there was no repulsion in the Reissner-Nordström space-time and the latter corrected the conclusion. Here we confirm that conclusion and find the limiting radius of the repulsive region.

7.6 Conclusions

In this last chapter, we studied the possibility of generating a Reissner-Nordström space-time by a thin matter shell with a Minkowski core. By considering the maximum analytical extension of the Reissner-Nordström space-time, this setup lead to a plethora of possible solutions of the Einstein field equations, depending on the properties of the matter fluid.

Indeed, either in the non-extremal, extremal or overcharged cases we found scenarios where for an observer exterior to the shell, locally, the geometry of the space-time is indistinguishable from that of the Reissner-Nordström model, generated by a central charged mass. In the non-extremal case, the formalism provided the possibility of regularizing the space-time, either removing the central singularity by placing the shell in a region outside the event horizon or inside the Cauchy horizon, although in the latter case, the matter fluid composing the shell will always violate the pointwise versions of the null, weak, strong and dominant energy conditions. The same conclusions hold in the case of extremal shells inside the event horizon.

It was also considered the possibility of having a shell separating a Minkowski and a Reissner-Nordström space-times such that the circumferential radius function would decrease in the direction of each sub-space. In these cases, we could have either the presence of an event horizon or a naked singularity. In the former, it was found that the strong energy condition must always be violated; whereas, surprisingly, in the latter scenario, it is always possible to find domains for the shell's properties where all the considered energy conditions are verified.

Appendix A

Divergence of the Weyl tensor

Given a 4-tensor field $\psi_{\alpha\beta\gamma\delta}$ in a 4–dimensional Lorentzian manifold, such that

$$\begin{aligned} \psi_{\alpha\beta\gamma\delta} &= \psi_{[\alpha\beta]\gamma\delta}\,, \\ \psi_{\alpha\beta\gamma\delta} &= \psi_{\alpha\beta[\gamma\delta]}\,, \end{aligned} \tag{A.1}$$

that is, anti-symmetric in the first and second pair of indices, we define the left and right Hodge dual, respectively, as

$$\begin{aligned} {}^*\psi_{\alpha\beta\gamma\delta} &= \frac{1}{2}\varepsilon_{\alpha\beta}{}^{\mu\nu}\psi_{\mu\nu\gamma\delta}\,, \\ \psi^*{}_{\alpha\beta\gamma\delta} &= \frac{1}{2}\psi_{\mu\nu\gamma\delta}\,\varepsilon^{\mu\nu}{}_{\gamma\delta}\,, \end{aligned} \tag{A.2}$$

where $\varepsilon_{\alpha\beta\gamma\delta}$ represents the Levi-Civita volume form, Eq. (2.15). Note that the left and right Hodge dual is well defined for any 4-tensor field, not only those that verify Eqs. (A.1), however, this requirement guarantees that the mapping is invertible, that is, no information is lost and applying the left (right) dual twice, returns the same tensor — up to a sign.

Now, using the fact that the Weyl tensor is trace-free and the properties of the Levi-Civita volume form, Eqs. (2.18) – (2.21), we have the following identity

$$ {}^*C^*{}_{\alpha\beta\gamma\delta} = \frac{1}{4}\varepsilon_{\alpha\beta}{}^{\mu\nu}C_{\mu\nu\rho\sigma}\varepsilon^{\rho\sigma}{}_{\gamma\delta} = -C_{\gamma\delta\alpha\beta}\,. \tag{A.3}$$

Defining the quantities

$$\begin{aligned} g_{\alpha\beta\gamma\delta} &\equiv g_{\alpha[\gamma}\,g_{\delta]\beta}\,, \\ E_{\alpha\beta\gamma\delta} &\equiv R_{\alpha[\gamma}\,g_{\delta]\beta} - R_{\beta[\gamma}\,g_{\delta]\alpha}, \end{aligned} \tag{A.4}$$

we have that

$$\begin{aligned} {}^*g^*{}_{\alpha\beta\gamma\delta} &= -g_{\alpha\beta\gamma\delta}\,, \\ {}^*E^*{}_{\alpha\beta\gamma\delta} &= E_{\gamma\delta\alpha\beta} - Rg_{\gamma\delta\alpha\beta}\,, \end{aligned} \tag{A.5}$$

where R represents the Ricci scalar.

Equations (A.3) – (A.5) can now be used to find an expression for the divergence of the Weyl tensor in terms of the Riemann and torsion tensor. Defining

$$Q_{\alpha\beta\gamma\delta\rho} = 2S_{[\alpha\beta|}{}^{\sigma}R_{|\gamma]\sigma\delta\rho}\,, \tag{A.6}$$

the second Bianchi identity, Eq. (2.13),

$$\nabla_{\mu}{}^{*}R^{*\,\mu\nu\gamma\delta} = \frac{1}{4}\varepsilon^{\alpha\beta\sigma\nu}Q_{\alpha\beta\sigma\lambda\rho}\varepsilon^{\lambda\rho\gamma\delta}\,. \tag{A.7}$$

Substituting Eqs. (2.22), (A.3) and (A.5) in Eq. (A.7) we find

$$\nabla_{\alpha}C^{\gamma\delta\beta\alpha} = \frac{1}{4}\varepsilon^{\mu\nu\lambda\beta}Q_{\mu\nu\lambda\sigma\rho}\varepsilon^{\sigma\rho\gamma\delta} - \nabla_{\alpha}\left(R^{\gamma[\alpha}g^{\beta]\delta} - R^{\delta[\alpha}g^{\beta]\gamma} - \frac{2}{3}R\,g^{\gamma[\alpha}g^{\beta]\delta}\right)\,. \tag{A.8}$$

Using the following relation

$$\nabla_{\alpha}R - 2\nabla_{\mu}R_{\alpha}{}^{\mu} = 3Q_{\alpha\mu\nu}{}^{\mu\nu}\,, \tag{A.9}$$

found from contracting the second Bianchi identity, Eq. (A.8) then yields

$$\begin{aligned}\nabla_{\alpha}C^{\gamma\delta\beta\alpha} = {} & \frac{1}{4}\varepsilon^{\mu\nu\lambda\beta}Q_{\mu\nu\lambda\sigma\rho}\varepsilon^{\sigma\rho\gamma\delta} + \\ & + \frac{3}{2}g^{\beta[\delta}Q^{\gamma]\mu\nu}{}_{\mu\nu} + \nabla^{[\delta}R^{\gamma]\beta} - \frac{1}{6}g^{\beta[\gamma}\nabla^{\delta]}R\,.\end{aligned} \tag{A.10}$$

Appendix B

Computation of the limit of R^2R' in Szekeres space-times

In the proof of Lemma 5.1 we have used the result that, when $\tau_c(r)$ exists, then $\lim_{\tau\to\tau_c(r)} R^2R' = 0$. Since this result plays a key role in that proof, here we summarize the solutions of the generalized Friedmann equation, Eq. (5.33) (also Eq. (5.41)) and show how they can be used in the proof.

The solutions to Eq. (5.33) can be separated in various sub-families depending on the combination of the signs of the functions $M(r)$ and $f(r)$. It can be easily seen that not all solutions are real (e.g. the case of $M(r),\ f(r) < 0$). The solutions of interest are then [136]:

for $f(r) < 0$ and $M(r) > 0$,

$$R(\tau,r) = -2\frac{M(r)}{f(r)}\cos^2(\eta)\,,\ \eta + \frac{1}{2}\sin(2\eta) = \frac{\sqrt{-f(r)^3}}{2M(r)}(\tau - \tau_0(r))\ ; \tag{B.1}$$

for $f(r) > 0$ and $M(r) < 0$,

$$R(\tau,r) = -2\frac{M(r)}{f(r)}\cosh^2(\eta)\,,\ \eta + \frac{1}{2}\sinh(2\eta) = \frac{f(r)^{\frac{3}{2}}}{-2M(r)}(\tau - \tau_0(r))\ ; \tag{B.2}$$

for $f(r) > 0$ and $M(r) > 0$,

$$R(\tau,r) = 2\frac{M(r)}{f(r)}\sinh^2(\eta)\,,\ \frac{1}{2}\sinh(2\eta) - \eta = \frac{f(r)^{\frac{3}{2}}}{2M(r)}(\tau - \tau_c(r))\ ; \tag{B.3}$$

for $f(r) > 0$ and $M(r) = 0$,

$$R(\tau, r) = \sqrt{f(r)}\,(\tau - \tau_c(r)) \; ; \tag{B.4}$$

for $f(r) = 0$ and $M(r) > 0$,

$$R(\tau, r) = \left(\frac{9M(r)}{2}\right)^{\frac{1}{3}} (\tau - \tau_c(r))^{\frac{2}{3}} \; ; \tag{B.5}$$

for $f(r) = 0$ and $M(r) = 0$,

$$R(\tau, r) = \text{const.} \; ; \tag{B.6}$$

where some functional dependencies were omitted to simplify the notation.

Now, for the cases represented by Eqs. (B.4) – (B.5), the computation of $\lim_{\tau \to \tau_c(r)} R^2 R'$ can be done directly and verified to be zero.

Let us consider the case where $f(r) < 0$ and $M(r) > 0$, Eq. (B.1). Taking the derivative with respect to r of the parametric equation for $\eta(\tau, r)$ we find

$$\frac{\partial \eta}{\partial r} = \frac{1}{1 + \cos(2\eta)} \frac{\partial}{\partial r} \left(\frac{\sqrt{-f(r)^3}}{2M(r)} (\tau - \tau_0(r)) \right) . \tag{B.7}$$

On the other hand,

$$R' = -\frac{\partial}{\partial r}\left(\frac{2M(r)}{f(r)}\right) \cos^2(\eta) + 4\frac{M(r)}{f(r)} \cos(\eta) \sin(\eta) \frac{\partial \eta}{\partial r} \, . \tag{B.8}$$

Then, using Eqs. (B.1), (B.7) and (B.8), we find

$$
\begin{aligned}
\lim_{\tau\to\tau_c(r)} R^2R' &= \lim_{\eta\to\frac{\pi}{2}} \left(\frac{2M(r)}{f(r)}\right)^2 \cos^4(\eta) \times \\
&\times \left[4\frac{M(r)}{f(r)}\frac{\partial}{\partial r}\left(\frac{\sqrt{-f(r)^3}}{2M(r)}(\tau-\tau_0(r))\right)\frac{\cos(\eta)\sin(\eta)}{1+\cos(2\eta)} - \frac{\partial}{\partial r}\left(\frac{2M(r)}{f(r)}\right)\cos^2(\eta)\right] \\
&= -8\left(\frac{M(r)}{f(r)}\right)^3 \lim_{\eta\to\frac{\pi}{2}} \cos^3(\eta)\sin(\eta)\ \frac{\partial}{\partial r}\left(\frac{\sqrt{-f(r)^3}}{2M(r)}(\tau-\tau_0(r))\right) \\
&= 0\,,
\end{aligned}
\tag{B.9}
$$

if the limit exists.

For the cases of $f(r)>0$ and $M(r)<0$, and $f(r)=M(r)=0$ we see, from Eqs. (B.2) and (B.6), that the function $R(\tau,r)$ is never zero, hence $\tau_c(r)$ does not exist.

In the case of $f(r)>0$ and $M(r)>0$, Eq. (B.3), the considered limit can be computed similarly to the case of $f(r)<0$ and $M(r)>0$ and, as in Eq. (B.9), the result is also zero.

Appendix C

1+1+2 covariant quantities and Buchdahl limit

C.1 Covariantly defined quantities for the derivatives of the tangent vectors

Using the definitions of the projector operators onto the hypersurfaces $\mathcal{V}$ and $\mathcal{W}$ defined in Chapter 6 let us show how the covariant derivatives of the tangent vector fields u and e can be uniquely decomposed in their components along u, e and $\mathcal{W}$.

C.1.1 Decomposition on the sheet $\mathcal{W}$

Let us first consider the projected covariant derivatives of the tensors u and e on the sheet. These can be uniquely decomposed as

$$\delta_\alpha u_\beta \equiv N_\alpha{}^\sigma N_\beta{}^\gamma \nabla_\sigma u_\gamma = \frac{1}{2}N_{\alpha\beta}\tilde{\theta} + \Sigma_{\alpha\beta} + \varepsilon_{\alpha\beta}\Omega\,, \tag{C.1}$$

where

$$\begin{aligned} \tilde{\theta} &= \delta_\alpha u^\alpha\,, \\ \Sigma_{\alpha\beta} &= \delta_{\{\alpha}u_{\beta\}}\,, \\ \Omega &= \frac{1}{2}\varepsilon^{\sigma\gamma}\delta_\sigma u_\gamma\,, \end{aligned} \tag{C.2}$$

and

$$\delta_\alpha e_\beta = \frac{1}{2}N_{\alpha\beta}\phi + \zeta_{\alpha\beta} + \varepsilon_{\alpha\beta}\xi\,, \tag{C.3}$$

with

$$
\begin{aligned}
\phi &= \delta_\alpha e^\alpha \, , \\
\zeta_{\alpha\beta} &= \delta_{\{\alpha} e_{\beta\}} \, , \\
\xi &= \frac{1}{2} \varepsilon^{\sigma\gamma} \delta_\sigma e_\gamma \, ,
\end{aligned}
\tag{C.4}
$$

where the curly brackets represent the projected symmetric part without trace of a tensor in $\mathcal{W}$, that is, for a tensor $\psi_{\alpha\beta}$,

$$
\psi_{\{\alpha\beta\}} = \left[N^\mu{}_{(\alpha} N_{\beta)}{}^\nu - \frac{N_{\alpha\beta}}{2} N^{\mu\nu} \right] \psi_{\mu\nu} \, . \tag{C.5}
$$

and we used the volume 2-form $\varepsilon_{\alpha\beta}$, Eq. (6.4), to write a completely anti-symmetric tensor defined on the sheet, $\psi_{[\alpha\beta]}$, as

$$
\psi_{[\alpha\beta]} = \varepsilon_{\alpha\beta} \left(\frac{1}{2} \epsilon^{\gamma\sigma} \psi_{\gamma\sigma} \right) \, . \tag{C.6}
$$

C.1.2 Decomposition on $\mathcal{V}$

The decomposition of the projected covariant derivatives of u onto $\mathcal{V}$, is given by

$$
D_\alpha u_\beta = h^\sigma_\alpha h^\gamma_\beta \nabla_\sigma u_\gamma = \frac{1}{3} h_{\alpha\beta} \boldsymbol{\theta} + \boldsymbol{\sigma}_{\alpha\beta} + \boldsymbol{\omega}_{\alpha\beta} \, , \tag{C.7}
$$

with

$$
\boldsymbol{\theta} = h^{\alpha\beta} D_\alpha u_\beta \, , \tag{C.8}
$$

$$
\boldsymbol{\sigma}_{\alpha\beta} = D_{\langle\alpha} u_{\beta\rangle} \, , \tag{C.9}
$$

$$
\boldsymbol{\omega}_{\alpha\beta} = h^\sigma{}_{[\alpha} h_{\beta]}{}^\gamma D_\sigma u_\gamma \, , \tag{C.10}
$$

where we used the angular brackets to represent the projected symmetric part without trace of a tensor on $\mathcal{V}$, Eq. (2.52), that is, for a tensor, $\psi_{\alpha\beta}$,

$$
\psi_{\langle\alpha\beta\rangle} = \left[h^\mu{}_{(\alpha} h_{\beta)}{}^\nu - \frac{h_{\alpha\beta}}{3} h^{\mu\nu} \right] \psi_{\mu\nu} \, . \tag{C.11}
$$

Notice that in the presence of a torsion tensor field characterized by a non-null $W_{\alpha\beta}$ component, Eq. (2.43), the quantities $\boldsymbol{\theta}$, $\boldsymbol{\sigma}_{\alpha\beta}$ and $\boldsymbol{\omega}_{\alpha\beta}$ do not represent the kinematical quantities defined in Eq. (2.36).

The scalar and tensor quantities in Eqs. (C.8) – (C.10) can themselves be further decomposed in their contributions exclusively on $\mathcal{W}$ and along e, such that

$$
\boldsymbol{\theta} = \tilde{\theta} + \bar{\theta} \, , \tag{C.12}
$$

where $\tilde{\theta}$ is defined in Eq. (C.2) and

$$\bar{\theta} = -u_\beta \left(e^\alpha D_\alpha e^\beta\right) = -u_\beta \hat{e}^\beta\,; \tag{C.13}$$

$$\boldsymbol{\sigma}_{\alpha\beta} = \Sigma_{\alpha\beta} + 2\Sigma_{(\alpha}e_{\beta)} + \Sigma\left(e_\alpha e_\beta - \frac{1}{2}N_{\alpha\beta}\right)\,, \tag{C.14}$$

with

$$\begin{aligned}\Sigma_{\alpha\beta} &= \boldsymbol{\sigma}_{\{\alpha\beta\}}\,,\\ \Sigma_\alpha &= N_\alpha^\gamma e^\beta \boldsymbol{\sigma}_{\gamma\beta}\,,\\ \Sigma &= e^\alpha e^\beta \boldsymbol{\sigma}_{\alpha\beta} = -N^{\alpha\beta}\boldsymbol{\sigma}_{\alpha\beta}\,,\end{aligned} \tag{C.15}$$

and

$$\boldsymbol{\omega}_{\alpha\beta} = \varepsilon_{\alpha\beta}\Omega - \varepsilon_{\alpha\lambda}\omega^\lambda e_\beta + e_\alpha \varepsilon_{\beta\lambda}\omega^\lambda\,, \tag{C.16}$$

where Ω is given in Eq. (C.2) and

$$\omega^\lambda = \frac{1}{2}\varepsilon^{\mu\nu\lambda}D_\mu u_\nu\,, \tag{C.17}$$

which can be itself decomposed as

$$\omega^\lambda = \Omega e^\lambda + \Omega^\lambda \quad \text{, with } \Omega^\lambda = N_\alpha^\lambda \omega^\alpha = \frac{1}{2}N_\alpha^\lambda \varepsilon^{\mu\nu\alpha}D_\mu u_\nu\,, \tag{C.18}$$

therefore, equivalently,

$$\boldsymbol{\omega}_{\alpha\beta} = \varepsilon_{\alpha\beta\gamma}\left(\Omega e^\gamma + \Omega^\gamma\right)\,. \tag{C.19}$$

The quantities $\boldsymbol{\theta}$, Σ, $\tilde{\theta}$ and $\bar{\theta}$ are not independent, in fact:

$$\bar{\theta} = \frac{1}{3}\boldsymbol{\theta} + \Sigma\,, \tag{C.20}$$

$$\tilde{\theta} = \frac{2}{3}\boldsymbol{\theta} - \Sigma\,; \tag{C.21}$$

as such, when setting up the 1+1+2 formalism only two are chosen. The convention followed here uses the variables $\boldsymbol{\theta}$ and Σ.

For the projected covariant derivative of the vector field e on $\mathcal{V}$ we have

$$D_\alpha e_\beta = h_\alpha{}^\sigma h_\beta{}^\gamma \nabla_\sigma e_\gamma = \delta_\alpha e_\beta + e_\alpha \bar{a}_\beta\,, \tag{C.22}$$

where $\delta_\alpha e_\beta$ is given by Eq. (C.3) and

$$\bar{a}_\alpha = e^\mu D_\mu e_\alpha = \hat{e}_\alpha\,. \tag{C.23}$$

C.1.3 Decomposition on the full manifold

Finally, we can decompose the total covariant derivatives of u and e, such that

$$\nabla_\alpha u_\beta = -u_\alpha \left(\mathcal{A} e_\beta + \mathcal{A}_\beta\right) + D_\alpha u_\beta \,, \tag{C.24}$$

with

$$\begin{aligned} \mathcal{A} &= -u_\gamma u^\mu \nabla_\mu e^\gamma = -u_\gamma \dot{e}^\gamma \,, \\ \mathcal{A}_\alpha &= N_{\alpha\beta} \dot{u}^\beta \,, \end{aligned} \tag{C.25}$$

and

$$\begin{aligned} \nabla_\alpha e_\beta = D_\alpha e_\beta - u_\alpha \alpha_\beta - \mathcal{A} u_\alpha u_\beta + \left[\frac{1}{3}\boldsymbol{\theta} + \Sigma\right] e_\alpha u_\beta + \\ + \left[\Sigma_\alpha - \varepsilon_{\alpha\sigma}\Omega^\sigma\right] u_\beta \,, \end{aligned} \tag{C.26}$$

where

$$\alpha_\mu = h^\sigma_\mu \dot{e}_\sigma \,. \tag{C.27}$$

C.2 Gaussian curvature of the sheet

Here we will define and find the relation between the Gaussian curvature of the 2-hypersurface $\mathcal{W}$ and the covariantly defined quantities of the 1+1+2 formalism in the presence of a torsion tensor field in the context of the ECSK theory. For this, we will need first to find the definition of the Gaussian curvature and its relation with the Riemann curvature tensor of a manifold of dimension 2. Moreover, we will also need to find the projected components of the Einstein tensor onto an hypersurface. This results will be presented in a completely general setting; then we will apply them to the particular case of an LRS II space-time, permeated by a Weyssenhoff fluid in the context of the ECSK theory.

C.2.1 Uniqueness of the Riemann tensor

Given a Lorentzian manifold $(\mathcal{M}, g, S)$ and a point $q \in \mathcal{M}$, let Υ be a sub-space of dimension 2 of $T_q\mathcal{M}$. Then, we define the sectional curvature of Υ, $K_S\left(\Upsilon\right)$, as

$$K_S\left(\Upsilon\right) := \frac{R_{\alpha\beta\gamma\delta} Y^\alpha Z^\beta Y^\gamma Z^\delta}{Y_\mu Y^\mu Z_\nu Z^\nu - \left(Y_\mu Z^\mu\right)^2} \,, \tag{C.28}$$

where $Y, Z \in T_q\mathcal{M}$ are two linearly independent vectors and $R_{\alpha\beta\gamma\delta}$ are the components of the Riemann tensor of $(\mathcal{M}, g, S)$ in a coordinate system.

Now, if $(\mathcal{M}, g, S)$ is of dimension 2, we have that the Riemann tensor verifies the following relations

$$\begin{aligned} R_{\alpha\beta\gamma\delta} &= -R_{\beta\alpha\gamma\delta}\,, \\ R_{\alpha\beta\gamma\delta} &= -R_{\alpha\beta\delta\gamma}\,, \\ R_{\alpha\beta\gamma\delta} &= R_{\gamma\delta\alpha\beta}\,, \\ {R_{[\alpha\beta\gamma]}}^{\delta} &= 0\,. \end{aligned} \tag{C.29}$$

In particular, we see that the first Bianchi identity reduces to the familiar relation for torsionless space-times (compare with Eq. (2.12)). This a consequence of the manifold being assumed to be of dimension 2 and is valid whether the torsion tensor is null or not. This, then, allows us to find the familiar result (cf., e.g., Ref. [211])

Proposition C.1. *Let $(\mathcal{M}, g, S)$ be a Lorentzian manifold of dimension 2. The Riemann curvature tensor at a point $q \in \mathcal{M}$ is uniquely determined by the values of the sectional curvature of 2-dimensional sub-spaces of $T_q\mathcal{M}$.*

Notice that in the presence of torsion, the result in Proposition C.1 is valid only for manifolds of dimension 2, hence it is a weaker result than the case of torsionless manifolds.

In the case where $\mathcal{M}$ is of dimension 2, the sectional curvature (C.28) is called the Gaussian curvature which we will represent by K_G, and, at the light of Proposition C.1, it is related with the Riemann curvature tensor by

$$R_{\alpha\beta\gamma\delta} = K_G \left(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}\right)\,. \tag{C.30}$$

C.2.2 Projected components of the Einstein tensor onto an hypersurface

Let $(\mathcal{M}, g, S)$ be a Lorentzian manifold of dimension 4 admitting the existence of an hypersurface orthogonal congruence of affinely parameterized curves with tangent vector n, such that $n_\alpha n^\alpha = \epsilon$ and $\epsilon \in \{-1, 1\}$, that is, the congruence might be time-like or space-like. Following the results of Chapter 3, let us find some of the projected components of the Einstein tensor

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R\,, \tag{C.31}$$

along n and onto the hypersurface $\mathcal{N}$, to which the congruence is orthogonal to.[1]

Assuming $\{e^a\}$ to be the tangent vectors to $\mathcal{N}$ at a point and h_{ab} be the components of the induced

[1]Here we will call the sub-manifold to which the congruence is orthogonal to, at each point, as $\mathcal{N}$ in contrast with the notation of Chapter 3 to avoid confusion with the 1+1+2 case, where $\mathcal{V}$ refers to the hypersurface to which the time-like congruence with tangent vector u is orthogonal to.

metric, from Eqs. (3.30) and (3.31) we find[2]

$$G_{\alpha\beta}n^{\alpha}n^{\beta} = -\frac{\epsilon}{2}h^{ab}h^{mn}e^{\alpha}_{a}e^{\mu}_{m}e^{\beta}_{b}e^{\nu}_{n}R_{\alpha\mu\beta\nu}\,. \tag{C.32}$$

Then, using the Gauss embedding equation (3.26)

$$G_{\alpha\beta}n^{\alpha}n^{\beta} = \frac{\epsilon}{2}\left\{\epsilon K^{2} - \epsilon K_{ab}K^{ba} - {}^{3}R\right\}\,, \tag{C.33}$$

where K_{ab} represents the extrinsic curvature of $\mathcal{N}$, $K \equiv K_{a}{}^{a}$ and ${}^{3}R$ is the induced Ricci scalar of $\mathcal{N}$.

On the other hand,

$$G_{\alpha\beta}e^{\alpha}_{b}n^{\beta} = h^{ac}e^{\alpha}_{a}e^{\beta}_{b}e^{\gamma}_{c}R_{\alpha\beta\gamma\delta}n^{\delta}\,, \tag{C.34}$$

or, using the Codazzi equation (3.27),

$$G_{\alpha\beta}e^{\alpha}_{b}n^{\beta} = D_{a}K_{b}{}^{a} - D_{b}K - 2S_{b}{}^{cd}K_{dc}\,, \tag{C.35}$$

where $S_{ab}{}^{d}$ are the components of induced torsion tensor on $\mathcal{N}$ and D_{a} is the induced covariant derivative, associated with h_{ab} and $S_{ab}{}^{d}$.

C.2.3 Gaussian curvature in the 1+1+2 formalism

We are now in position to find the relation between the Gaussian curvature of the 2-sheet to which the congruences associated with u and e are orthogonal. We remark a slight change of notation in this sub-section: here lower case Latin letters run from 1 to 3 and will be used to indicate the components of a tensor field in a coordinate system defined in an open neighborhood of a point on $\mathcal{V}$; e^{a} represents the components of the vector field e on $\mathcal{V}$; we will refer to the tangent vectors to $\mathcal{W}$, orthogonal to both u and e, as $\{\mathsf{e}_{2}, \mathsf{e}_{3}\}$ and capital Latin letters $\{A, B\}$ run from 2 to 3.

Taking into account the 1+1+2 decomposition of the covariant derivative of u, Eq. (6.6), and the definition of extrinsic curvature, Eq. (3.20), we can write the extrinsic curvature of $\mathcal{V}$, as

$${}^{3}K_{ab} = N_{ab}\left(\frac{1}{3}\boldsymbol{\theta} - \frac{1}{2}\Sigma\right) + \left(\frac{1}{3}\boldsymbol{\theta} + \Sigma\right)e_{a}e_{b} + \Sigma_{ab} + 2\Sigma_{(a}\,e_{b)} + \varepsilon_{ab}\Omega - 2e_{[a}\,\varepsilon_{b]d}\Omega^{d}\,, \tag{C.36}$$

and

$${}^{3}K = \boldsymbol{\theta}\,, \tag{C.37}$$

where N_{ab} are the components of the induced projector tensor $N_{\alpha\beta}$ onto $\mathcal{V}$. Substituting Eqs. (C.36) and (C.37) in Eq. (C.33) — identifying $n \equiv u$ in that equation — and using the field equations of the ECSK theory

[2] We highlight that in this case, Greek indices run from 0 to 3 and Latin indices run from 1 to 3.

for a vanishing cosmological constant, Eq. (6.17), we find

$$ {}^{3}R = 16\pi\mu - \frac{2}{3}\boldsymbol{\theta}^2 + \frac{3}{2}\Sigma^2 + \Sigma_{ab}\Sigma^{ab} + (\Sigma_a\Sigma^a)^2 - 2\Omega^2 - 2\Omega^a\Omega_a \,. \tag{C.38} $$

Let us pause here and emphasize that Eqs. (C.36) – (C.38) are well defined only if u is hypersurface orthogonal, that is, when it verifies Lemma 4.1.

Continuing, let us now relate the extrinsic curvature of the 2-sheet $\mathcal{W}$ with the covariantly defined quantities of the 1+1+2 decomposition formalism. From Eq. (6.7) we find

$$ {}^{2}K_{AB} = \frac{1}{2}N_{AB}\phi + \zeta_{AB} + \varepsilon_{AB}\xi \,, \tag{C.39} $$

and

$$ {}^{2}K = \phi \,, \tag{C.40} $$

where N_{AB} and ε_{AB} are, respectively, the components of the induced metric and the 2-volume form on $\mathcal{W}$. We shall highlight that we have not yet imposed the space-time to be LRS II nor have we imposed any restrictions on the torsion tensor, other than assuming implicitly the conditions of the Frobenius' theorem. Hence, in Eqs. (C.39) we have not set ξ to zero as that follows as a consequence of the space-time being imposed to be LRS II and endowed with a Weyssenhoff-like torsion.

From Eq. (3.33) — identifying $n \equiv e$ in that equation — and Eq. (C.30), we then find the following relation between the Ricci scalar of the hypersurface $\mathcal{V}$ and the Gaussian curvature of $\mathcal{W}$,

$$ {}^{3}R = 2K_G - \frac{3}{2}\phi^2 + \zeta_{AB}\zeta^{AB} + 2\xi^2 + 2D_a a^a - 2\hat{\phi} - 4S_c{}^{ab}e^c D_b e_a \,, \tag{C.41} $$

Equations (C.38) and (C.41) then allows us to find a relation between the Gaussian curvature of the 2-sheet and the covariantly defined quantities of the 1+1+2 formalism. In the particular case of an LRS II space-time permeated by a Weyssenhoff fluid, in the context of the ECSK theory we find

$$ K_G = \frac{8\pi}{3}\mu - \mathcal{E} + \frac{1}{4}\phi^2 - \tau^2 \,, \tag{C.42} $$

which, matches Eq. (6.57) in the case when the congruence associated with e is hypersurface orthogonal, that is, when $\Omega = \tau$.

C.3 Buchdahl limit

For ease of comparison with the classical result for the Buchdahl limit, in this appendix we will make the computations in a particular coordinate system. Let us start by finding relations between the covariantly defined quantities ϕ and $\mathcal{A}$ and the metric components of the space-time.

Given a static, spherically symmetric space-time, it admits a line element of the form

$$ds^2 = -f(r)\,dt^2 + g(r)\,dr^2 + r^2\left(d\theta^2 + \sin^2(\theta)\,d\varphi^2\right) . \tag{C.43}$$

In such coordinate system the components of the unit vector field u, tangent to the curves of constant coordinates r, θ and φ, are given by

$$u^\alpha = \frac{1}{\sqrt{f(r)}}\delta_t^\alpha , \tag{C.44}$$

whereas, the components of the unit vector field e, tangent to the curves with coordinates t, θ and φ constant, are given by

$$e^\alpha = \frac{1}{\sqrt{g(r)}}\delta_r^\alpha . \tag{C.45}$$

On the other hand, given that K, Eq. (6.57), represents the Gaussian curvature of the 2-surface orthogonal to both u and e, we have that in this coordinate system

$$K(r) = \frac{1}{r^2} . \tag{C.46}$$

From the definitions and Eqs. (C.43) – (C.45), the tt and rr components of the metric, $f(r)$ and $g(r)$, are related with the functions $\mathcal{A}$ and ϕ by

$$\mathcal{A}(r) = \frac{1}{2f(r)\sqrt{g(r)}}\frac{d\,f(r)}{dr} , \tag{C.47}$$

$$\phi(r) = \frac{2}{r\sqrt{g(r)}} . \tag{C.48}$$

We are now in position to derive the generalized Buchdahl limit.

Consider two static, spherically symmetric space-times: $\mathcal{M}^-$ and $\mathcal{M}^+$, smoothly matched at a common boundary $\mathcal{S}$. The interior space-time, $\mathcal{M}^-$, is assumed to be a solution of the structure equations (6.84) – (6.92), for a matter fluid with corrected energy density function

$$\tilde{\mu}(r) \equiv 8\pi\mu(r) - \delta^2(r) , \tag{C.49}$$

such that $\tilde{\mu}(r) \geq 0$ and $d\tilde{\mu}/dr \leq 0$. The exterior space-time, $\mathcal{M}^+$, is described by the Schwarzschild vacuum solution.

Let us start by finding a formal general expression for the covariant quantity $\phi(r)$, Eq. (C.4). Integrating Eqs. (6.60) and (6.87) we find,

$$\phi(r) = \frac{2}{r}\sqrt{1 - \frac{2m(r)}{r}} , \tag{C.50}$$

where we have chosen the positive sign for $\phi\left(r\right)$ and

$$m\left(r\right)=\frac{1}{2}\int_{0}^{r}\tilde{\mu}\left(x\right)\,x^{2}\,dx\,, \tag{C.51}$$

is usually dubbed mass function.

Now, in the static case and using the circumferential radius as parameter for the integral curves of e, Eqs. (6.46) and (6.86) read

$$\begin{aligned}\frac{r}{2}\phi\left(r\right)\tilde{p}_{,r}&=-\mathcal{A}\left(\tilde{\mu}+\tilde{p}\right)\,,\\ \tilde{p}&=\mathcal{A}\phi-K+\frac{1}{4}\phi^{2}\,,\end{aligned} \tag{C.52}$$

where $\tilde{p}\left(r\right)\equiv 8\pi p\left(r\right)-\delta\left(r\right)^{2}$, represents the corrected pressure. Let us remark that Eqs. (C.52) have the same form as those used in Ref. [1] so, all results in that reference are also valid for the corrected energy density and pressure. Notwithstanding, for completeness we shall make the derivation for the maximum amount of mass a star with constant radius and decreasing corrected energy density can hold, here.

From Eqs. (C.46) – (C.52), we have

$$\frac{d}{dr}\left[\frac{1}{r\sqrt{g}}\frac{d}{dr}\sqrt{f}\right]=\sqrt{g\,f}\frac{d}{dr}\left(\frac{m\left(r\right)}{r^{3}}\right)\,, \tag{C.53}$$

where we have omitted some functional dependencies to avoid saturating the expression. Now, assuming that $d\tilde{\mu}/dr\leq 0$, the quantity $m\left(r\right)/r^{3}$ must also be a monotonically decreasing function, hence, from Eq. (C.53)

$$\frac{d}{dr}\left[\frac{1}{r\sqrt{g}}\frac{d}{dr}\sqrt{f}\right]\leq 0\,. \tag{C.54}$$

Therefore, we can write

$$\frac{1}{r\sqrt{g\left(r\right)}}\frac{d\sqrt{f}}{dr}\left(r\right)\geq\frac{1}{r_{0}\sqrt{g\left(r_{0}\right)}}\frac{d\sqrt{f}}{dr}\left(r_{0}\right)\,, \tag{C.55}$$

for any $r<r_{0}$. Now, let the matching surface $\mathcal{S}$ to be orthogonal to the integral curves of e and r_{0} represent the value of the circumferential radius of $\mathcal{S}$. Assuming that the interior and exterior space-times are smoothly matched, we can then use the line element of the exterior Schwarzschild space-time

$$ds_{+}^{2}=-\left(1-\frac{2M}{R}\right)dT^{2}+\left(1-\frac{2M}{R}\right)dR^{2}+R^{2}\left(d\Theta^{2}+\sin^{2}\left(\Theta\right)d\Phi^{2}\right)\,, \tag{C.56}$$

to compute the RHS of Eq. (C.55). Let us remark that Eqs. (3.52) – (3.54) not only guarantee that observers in $\mathcal{M}^{-}$ and $\mathcal{M}^{+}$ agree with the induced metric at $\mathcal{S}$ but also that the jumps of the derivatives of the metric components of $\mathcal{M}^-$ or $\mathcal{M}^+$ are zero (see proof of Proposition 3.2). Therefore, Eq. (C.55) reads

$$\frac{1}{r\sqrt{g(r)}}\frac{d\sqrt{f}}{dr}(r) \geq \frac{M}{r_0^3}\,, \tag{C.57}$$

where the Schwarzschild parameter agrees with the value of the mass function, Eq. (C.51), at r_0, that is: $M = m(r_0)$. Using Eqs. (C.48), (C.50) and (C.57) we have

$$\sqrt{f(0)} \leq \left(1 - \frac{2M}{r_0}\right)^{\frac{1}{2}} - \frac{M}{r_0^3}\int_0^{r_0} \frac{r}{\sqrt{1 - \frac{2m(r)}{r}}}\,dr\,, \tag{C.58}$$

where to find the first term of the RHS we have used the line element (C.56) and, again, the fact that the induced metric at $\mathcal{S}$ is the same. Now, since we assume $d\tilde{\mu}/dr \leq 0$, the value of the mass function, Eq. (C.51), must be greater or equal to the value it would have if $\tilde{\mu}$ was constant, hence,

$$m(r) \geq \frac{M}{r_0^3}r^3\,. \tag{C.59}$$

Then, using the inequality (C.59) in (C.58), we find

$$\sqrt{f(0)} \leq \left(1 - \frac{2M}{r_0}\right)^{\frac{1}{2}} - \frac{M}{r_0^3}\int_0^{r_0} \frac{r}{\sqrt{1 - \frac{M}{r_0^3}r^3}}\,dr = \frac{3}{2}\sqrt{1 - \frac{2M}{r_0}} - \frac{1}{2}\,. \tag{C.60}$$

Since $\sqrt{f(0)}$ is a non-negative function, the RHS can not be negative, hence,

$$\frac{M}{r_0} \leq \frac{4}{9}\,, \tag{C.61}$$

which, formally, matches the expression found in Ref. [1].

Appendix D

Kruskal-Szekeres coordinates and extrinsic curvature of static shells in the Reissner-Nordström space-time

D.1 Kruskal-Szekeres coordinates of the Reissner-Nordström space-time

D.1.1 General formalism for a static spherically symmetric space-time

In this appendix we shall construct two coordinate systems for the Reissner-Nordström space-time, each well behaved in a neighborhood of the event horizon, $r = r_+$, or in a neighborhood of the Cauchy horizon, $r = r_-$, which, together, cover the Reissner-Nordström space-time. To find these new coordinate systems we shall use the formalism introduced in Ref. [201] which we shall quickly present.

Given a static, spherically symmetric space-time whose line element can be written in the form

$$ds^2 = -\Phi\left(r\right) dt^2 + \Phi^{-1}\left(r\right) dr^2 + r^2 d\Omega^2 \,, \tag{D.1}$$

where the function $\Phi\left(r\right)$ is assumed to have zeros or poles representing pseudo-singularities, which can be removed by a change of coordinates, let us determine a simultaneous transformation of the coordinates r and t to new coordinates $X\left(r,t\right)$ and $T\left(r,t\right)$ such that the line element can be written as

$$ds^2 = f^2\left(X,T\right)\left(dX^2 - dT^2\right) + r^2\left(X,T\right) d\Omega^2 \,, \tag{D.2}$$

where $f^2\left(X,T\right)$ is to be regular in a sub-region covered by the coordinates X and T.

Comparing Eqs.(D.1) and (D.2) it is found that [201]

$$\begin{aligned} X &= h\left(r^* + t\right) + g\left(r^* - t\right) \ , \\ T &= h\left(r^* + t\right) - g\left(r^* - t\right) \ , \end{aligned} \tag{D.3}$$

with

$$dr^* = \Phi^{-1}\left(r\right) dr \, , \tag{D.4}$$

h and g are arbitrary functions of one variable and

$$f^2 = \frac{\Phi\left(r\right)}{4h'\left(r^* + t\right) g'\left(r^* - t\right)} \, , \tag{D.5}$$

where prime denotes differentiation with respect to the functions variable.

In order to f^2, in Eq. (D.5), be non singular, any singularity in the numerator $\Phi\left(r\right)$ must be canceled by the denominator, for all t. Assuming Φ to have only poles of order 1, setting

$$\begin{aligned} h\left(r^* + t\right) &= A\, e^{\gamma\left(r^* + t\right)} \, , \\ g\left(r^* - t\right) &= B\, e^{\gamma\left(r^* - t\right)} \, , \end{aligned} \tag{D.6}$$

where the scale factors $A, B \in \mathbb{C}$, it is possible to choose a value for the constant γ such that f^2 is regular and positive throughout the region covered by the coordinate patch. Substituting Eq. (D.6) in Eqs. (D.3) and (D.5) we find

$$f^2 = \frac{\Phi\left(r\right)\, e^{-2\gamma r^*}}{4AB\gamma^2} \, , \tag{D.7}$$

and

$$\begin{aligned} X\left(r, t\right) &= A\, e^{\gamma\left(r^* + t\right)} + B\, e^{\gamma\left(r^* - t\right)} \, , \\ T\left(r, t\right) &= A\, e^{\gamma\left(r^* + t\right)} - B\, e^{\gamma\left(r^* - t\right)} \, , \end{aligned} \tag{D.8}$$

in terms of the coordinates r and t. From Eq. (D.8) we can find the inverse transformation and define the coordinate r, implicitly, in terms of the coordinates X and T, such that

$$X^2 - T^2 = 4AB\, e^{2\gamma r^*} \, . \tag{D.9}$$

Lastly, since f^2 in Eq. (D.7) depends on the values of A and B, the scale-factors have to be chosen in such a way that f^2 is positive. Moreover, given that the transformation between the coordinates $\{t, r\}$ and $\{T, X\}$ depend on A and B, these must be chosen such that the coordinates X and T take only real values.

D.1.2 Removal of the coordinate singularity at the event horizon

Having introduced the general formalism, we can apply it to the specific case of the Reissner-Nordström space-time. In this case, the line element in terms of the coordinates $\{t, r\}$ is given by:

$$ds^2 = -\Phi(r)\, dt^2 + \Phi^{-1}(r)\, dr^2 + r^2 d\Omega^2 \,, \tag{D.10}$$

with

$$\Phi(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} = \frac{(r - r_-)(r - r_+)}{r^2} \,, \tag{D.11}$$

and

$$r_\pm = M \pm \sqrt{M^2 - Q^2} \,. \tag{D.12}$$

From Eq. (D.11), in these coordinates we see that the line element for the non-extremal case Reissner-Nordström space-time $(M^2 > Q^2)$ contains two coordinate singularities at $r = r_-$ and at $r = r_+$. Then, using the formalism of subsection D.1.1, two coordinate patches need to be found, each well defined in the neighborhood of each of the pseudo-singularities. Notice, however, that there is a common region where both coordinate patches overlap.

Let us first find a coordinate patch that covers a neighborhood of the pseudo-singularity at $r = r_+$. Using Eq. (D.4), in the case of the Reissner-Nordström model, r^* is given by:

$$r^* = r + R_+ \log\left(\frac{r - r_+}{2M}\right) - R_- \log\left(\frac{r - r_-}{2M}\right) \,, \tag{D.13}$$

where we have set the value of the integration constant to $2M$ and

$$R_\pm = \frac{r_\pm^2}{r_+ - r_-} \,. \tag{D.14}$$

To remove the coordinate singularity at $r = r_+$ we will impose the constant γ that appears in Eqs. (D.6) – (D.9) to take the following value

$$\gamma = \frac{1}{2R_+} \,. \tag{D.15}$$

Substituting Eqs. (D.13) and (D.15) in Eq. (D.7) we find

$$f^2 = \frac{4M^2 R_+^2}{AB\, r^2} e^{-\frac{r}{R_+}} \left(\frac{r - r_-}{2M}\right)^{1 + \left({r_-}/{r_+}\right)^2} \,, \tag{D.16}$$

which is well behaved near the event horizon at $r = r_+$.

As was stated in the previous section, the choice of the scale-factors A and B is quite arbitrary and we shall impose their values to be such that in the limit when the electric charge Q goes to zero we recover the Kruskal-Szekeres coordinates defined in Ref. [65] for the Schwarzschild space-time. So, the values for the scale-factors for the various regions are

$$\text{I}\begin{cases}A= & {}^{1}\!/_{2}\\ B= & {}^{1}\!/_{2}\end{cases},\qquad \text{I}'\quad\begin{cases}A= & -{}^{1}\!/_{2}\\ B= & -{}^{1}\!/_{2}\end{cases},$$
$$\text{II}\begin{cases}A= & -{}^{i}\!/_{2}\\ B= & {}^{i}\!/_{2}\end{cases},\qquad \text{II}'\quad\begin{cases}A= & {}^{i}\!/_{2}\\ B= & -{}^{i}\!/_{2}\end{cases}. \tag{D.17}$$

We see that our choice for the scale-factors differs for each region. This is a consequence of the behavior of the coordinates $\{t, r\}$, nonetheless, obviously, the geometry of the space-time is unaltered since, different choices for the scale factor that obey the restrictions imposed in the previous section, give the same expression for the metric, aside a conformal constant factor. Our choice, though, leaves the metric completely unaltered, hence, substituting the various values for the scale-factors listed in Eq. (D.17) in Eq. (D.16) we get, for every region covered by the coordinate patch,

$$f^2 = \frac{16M^2}{r^2} R_+^2 e^{-\frac{r}{R_+}} \left(\frac{r-r_-}{2M}\right)^{1+\left({}^{r_-}\!/_{r_+}\right)^2} . \tag{D.18}$$

Substituting Eq. (D.17) in Eq. (D.9) allow us to write the inverse transformation for the coordinate r in terms of the coordinates X and T as

$$X^2 - T^2 = e^{\frac{r}{R_+}} \left(\frac{r-r_+}{2M}\right)\left(\frac{r-r_-}{2M}\right)^{-\left({}^{r_-}\!/_{r_+}\right)^2} . \tag{D.19}$$

For completeness, we shall also define the transformations for the coordinates $\{T, X\}$ in terms of the coordinates $\{t, r\}$ for the various regions. Substituting Eqs. (D.15) and (D.17) in Eq. (D.8) we find

$$\begin{aligned}
&\text{I} && \begin{cases}X= & e^{r/2R_+}\left(\frac{r-r_+}{2M}\right)^{1/2}\left(\frac{r-r_-}{2M}\right)^{-R_-/2R_+}\cosh\left(\frac{t}{2R_+}\right)\\ T= & e^{r/2R_+}\left(\frac{r-r_+}{2M}\right)^{1/2}\left(\frac{r-r_-}{2M}\right)^{-R_-/2R_+}\sinh\left(\frac{t}{2R_+}\right)\end{cases},\\
&\text{I}' && \begin{cases}X= & -e^{r/2R_+}\left(\frac{r-r_+}{2M}\right)^{1/2}\left(\frac{r-r_-}{2M}\right)^{-R_-/2R_+}\cosh\left(\frac{t}{2R_+}\right)\\ T= & -e^{r/2R_+}\left(\frac{r-r_+}{2M}\right)^{1/2}\left(\frac{r-r_-}{2M}\right)^{-R_-/2R_+}\sinh\left(\frac{t}{2R_+}\right)\end{cases},\\
&\text{II} && \begin{cases}X= & e^{r/2R_+}\left(\frac{r_+-r}{2M}\right)^{1/2}\left(\frac{r-r_-}{2M}\right)^{-R_-/2R_+}\sinh\left(\frac{t}{2R_+}\right)\\ T= & e^{r/2R_+}\left(\frac{r_+-r}{2M}\right)^{1/2}\left(\frac{r-r_-}{2M}\right)^{-R_-/2R_+}\cosh\left(\frac{t}{2R_+}\right)\end{cases},\\
&\text{II}' && \begin{cases}X= & -e^{r/2R_+}\left(\frac{r_+-r}{2M}\right)^{1/2}\left(\frac{r-r_-}{2M}\right)^{-R_-/2R_+}\sinh\left(\frac{t}{2R_+}\right)\\ T= & -e^{r/2R_+}\left(\frac{r_+-r}{2M}\right)^{1/2}\left(\frac{r-r_-}{2M}\right)^{-R_-/2R_+}\cosh\left(\frac{t}{2R_+}\right)\end{cases}.
\end{aligned} \tag{D.20}$$

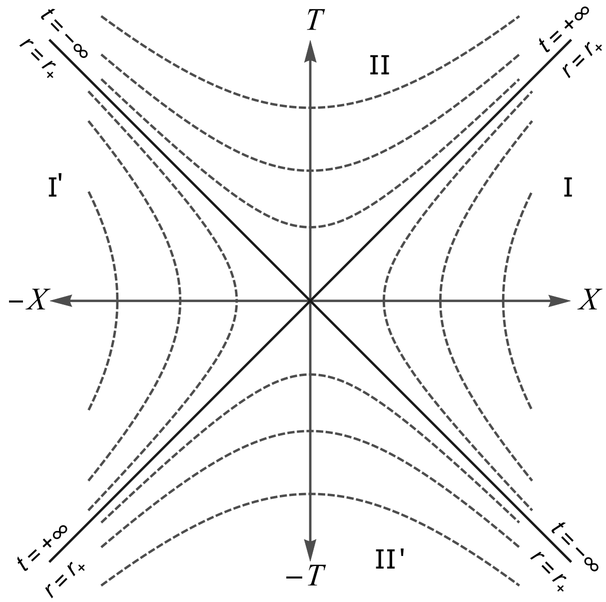


Figure 20: Relation between the Kruskal-Szekeres coordinates $\{T, X\}$ that cover a neighborhood of the event horizon and the Schwarzschild coordinates $\{t, r\}$. The hyperbolas represent curves of constant r coordinate while curves of constant t are straight lines through the origin.

This relations for X and T can be used to find the coordinate t as a function of these coordinates, such that:

$$\begin{aligned} t &= 2R_+ \text{ arctanh}\left(\frac{T}{X}\right) \qquad &&\text{, in regions I and I}', \\ t &= 2R_+ \text{ arctanh}\left(\frac{X}{T}\right) \qquad &&\text{, in regions II and II}'. \end{aligned} \tag{D.21}$$

These coordinate transformations are exhibited graphically in Figure 20.

D.1.3 Removal of the coordinate singularity at the Cauchy horizon

Let us now define a second coordinate patch where the coordinate singularity at $r = r_-$ is removed. In this case the function r^* is given by

$$r^* = r + R_+ \log\left(\frac{r_+ - r}{2M}\right) - R_- \log\left(\frac{r_- - r}{2M}\right) . \tag{D.22}$$

This change in the expression for the function r^* — compared with Eq. (D.13) — should not come with surprise since, as was stated in the previous subsection, the function r^* is defined up to an integration constant which shall be chosen such that the metric is real for each coordinate patch.

Now, since we want to remove the coordinate singularity at $r = r_-$, we shall impose

$$\gamma = -\frac{1}{2R_-}\,. \tag{D.23}$$

Substituting Eqs. (D.22) and (D.23) in Eq. (D.7) we find

$$f^2 = \frac{4M^2}{AB\,r^2} R_-^2 e^{\frac{r}{R_-}} \left(\frac{r_+ - r}{2M}\right)^{1+\left({}^{r_+}\!/{}_{r_-}\right)^2} \,. \tag{D.24}$$

As in the previous subsection, we have now to choose the scale-factors A and B for the various sub-regions covered by the second coordinate patch hence,

$$\begin{aligned} &\text{III}\begin{cases} A = & {}^1\!/{}_2 \\ B = & {}^1\!/{}_2 \end{cases}, \qquad &&\text{III}'\ \begin{cases} A = & -{}^1\!/{}_2 \\ B = & -{}^1\!/{}_2 \end{cases}, \\ &\text{II}\begin{cases} A = & {}^i\!/{}_2 \\ B = & -{}^i\!/{}_2 \end{cases}, \qquad &&\text{II}'\ \begin{cases} A = & -{}^i\!/{}_2 \\ B = & {}^i\!/{}_2 \end{cases}. \end{aligned} \tag{D.25}$$

This choice for the scale-factors leaves the expression for the metric unaltered for the various sub-regions covered by the coordinate patch, such that,

$$f^2 = \frac{16M^2}{r^2} R_-^2 e^{\frac{r}{R_-}} \left(\frac{r_+ - r}{2M}\right)^{1+\left({}^{r_+}\!/{}_{r_-}\right)^2} \,. \tag{D.26}$$

Substituting Eq. (D.25) in Eq. (D.9) we find

$$X^2 - T^2 = e^{-\frac{r}{R_-}} \left(\frac{r_- - r}{2M}\right) \left(\frac{r_+ - r}{2M}\right)^{-\left({}^{r_+}\!/{}_{r_-}\right)^2} \,, \tag{D.27}$$

which defines, implicitly, the coordinate r in terms of the coordinates X and T. As in the previous subsection, for completeness we shall define the transformations for the various sub-regions covered by the coordinate patch that relate the coordinates $\{T, X\}$ with the coordinates $\{t, r\}$. Substituting Eqs. (D.23) and (D.25) in

Eq. (D.8) we find

$$
\begin{aligned}
&\text{III} \quad \begin{cases} X = & e^{-r/2R_-}\left(\frac{r_--r}{2M}\right)^{1/2}\left(\frac{r_+-r}{2M}\right)^{-R_+/2R_-}\cosh\left(-\frac{t}{2R_-}\right) \\ T = & e^{-r/2R_-}\left(\frac{r_--r}{2M}\right)^{1/2}\left(\frac{r_+-r}{2M}\right)^{-R_+/2R_-}\sinh\left(-\frac{t}{2R_-}\right) \end{cases}, \\
&\text{III}' \quad \begin{cases} X = & -e^{-r/2R_-}\left(\frac{r_--r}{2M}\right)^{1/2}\left(\frac{r_+-r}{2M}\right)^{-R_+/2R_-}\cosh\left(-\frac{t}{2R_-}\right) \\ T = & -e^{-r/2R_-}\left(\frac{r_--r}{2M}\right)^{1/2}\left(\frac{r_+-r}{2M}\right)^{-R_+/2R_-}\sinh\left(-\frac{t}{2R_-}\right) \end{cases}, \\
&\text{II}' \quad \begin{cases} X = & e^{-r/2R_-}\left(\frac{r-r_-}{2M}\right)^{1/2}\left(\frac{r_+-r}{2M}\right)^{-R_+/2R_-}\sinh\left(-\frac{t}{2R_-}\right) \\ T = & e^{-r/2R_-}\left(\frac{r-r_-}{2M}\right)^{1/2}\left(\frac{r_+-r}{2M}\right)^{-R_+/2R_-}\cosh\left(-\frac{t}{2R_-}\right) \end{cases}, \\
&\text{II} \quad \begin{cases} X = & -e^{-r/2R_-}\left(\frac{r-r_-}{2M}\right)^{1/2}\left(\frac{r_+-r}{2M}\right)^{-R_+/2R_-}\sinh\left(-\frac{t}{2R_-}\right) \\ T = & -e^{-r/2R_-}\left(\frac{r-r_-}{2M}\right)^{1/2}\left(\frac{r_+-r}{2M}\right)^{-R_+/2R_-}\cosh\left(-\frac{t}{2R_-}\right) \end{cases}.
\end{aligned}
\tag{D.28}
$$

These relations can then be used to find the transformation that gives the coordinate t in terms of the coordinates X and T, such that:

$$
\begin{aligned}
t &= -2R_- \operatorname{arctanh}\left(\frac{T}{X}\right) \qquad \text{, in regions III and III}', \\
t &= -2R_- \operatorname{arctanh}\left(\frac{X}{T}\right) \qquad \text{, in regions II and II}'.
\end{aligned}
\tag{D.29}
$$

These coordinate transformations are exhibited graphically in Figure 21.

D.2 Extrinsic curvature as seen from $\mathcal{M}^+$ for the non-extremal Reissner-Nordström space-time

Here we will make the derivation of the non-null components of the extrinsic curvature of the matching surface $\mathcal{S}$ as seen form the exterior non-extremal Reissner-Nordström space-time. We shall present the cases where the shell is outside the event horizon, with $R > r_+$, and inside the Cauchy horizon, with $R < r_-$, separately.

D.2.1 Thin shells outside the event horizon

As was stated in subsection 7.3.1.1, assuming the matching surface to be time-like, static and spherically symmetric, the non-null components of the extrinsic curvature of the matching hypersurface $\mathcal{S}$ are given by Eqs. (7.13) – (7.15). Anticipating some of the intermediate results, we shall first find the derivative of the radial coordinate r in order to the coordinates $\{X, T\}$. Taking the derivative of Eq. (7.22) in order to X and T

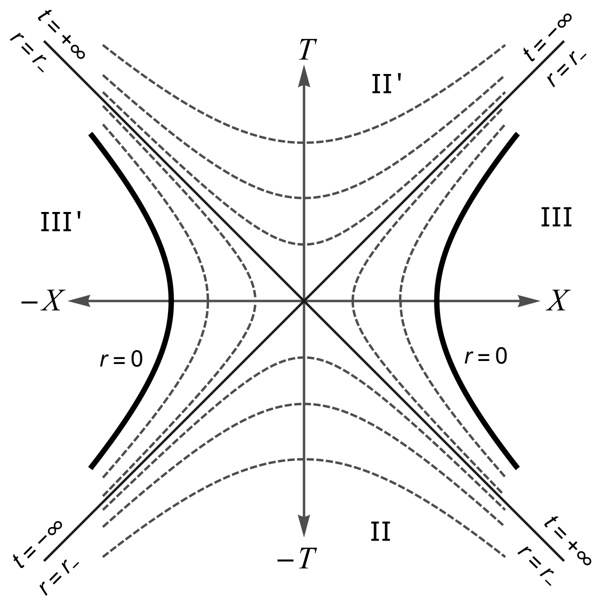


Figure 21: Relation between the Kruskal-Szekeres coordinates $\{T, X\}$ that cover a neighborhood of the Cauchy horizon and the Schwarzschild coordinates $\{t, r\}$. The hyperbolas represent curves of constant r coordinate while curves of constant t are straight lines through the origin. The thick black line represent the singularity at $r = 0$.

independently, we find

$$\frac{\partial r}{\partial X} = \frac{g_{XX}}{2R_+} X \,, \tag{D.30}$$

$$\frac{\partial r}{\partial T} = -\frac{g_{XX}}{2R_+} T \,. \tag{D.31}$$

From which, in particular, we get the following relation:

$$\frac{\partial r}{\partial T} = -\frac{T}{X}\frac{\partial r}{\partial X} \,. \tag{D.32}$$

To compute the Christoffel symbols associated with the metric (7.20) we need to find the derivatives of the metric components. Using Eqs. (D.30) – (D.32) we find

$$\partial_X g_{XX} = \frac{(g_{XX})^2}{r-r_-}\frac{r_+ - r_-}{2r_+^4}\left[2\frac{r_+^2 r_-}{r} - (r_+ - r_-)(r_+ + r)\right] X \,, \tag{D.33}$$

$$\partial_X g_{TT} = -\partial_X g_{XX} \,, \tag{D.34}$$

$$\partial_T g_{XX} = -\frac{T}{X}\partial_X g_{XX} \,, \tag{D.35}$$

$$\partial_T g_{TT} = \frac{T}{X}\partial_X g_{XX} \,, \tag{D.36}$$

where Eqs. (D.34) – (D.36) resulted only from Eq. (D.32) and the fact that the metric components in Eq. (7.20) are related by

$$g_{TT} = -g_{XX} \, . \tag{D.37}$$

Equations (D.33) – (D.36) can now be used to find the Christoffel symbols needed to compute the component $K_{\tau\tau}$ of the extrinsic curvature, Eq. (7.13). So,

$$\begin{aligned}
\Gamma^{X}_{XX} &= \frac{g_{XX}}{4r_+^4}\frac{r_+ - r_-}{r - r_-}\left[2\frac{r_+^2 r_-}{r} - (r_+ - r_-)(r_+ + r)\right] X \, , &\text{(D.38)}\\
\Gamma^{X}_{TT} &= \Gamma^{X}_{XX} \, , &\text{(D.39)}\\
\Gamma^{X}_{XT} &= \Gamma^{X}_{TX} = -\frac{T}{X}\Gamma^{X}_{XX} \, , &\text{(D.40)}\\
\Gamma^{T}_{TT} &= -\frac{T}{X}\Gamma^{X}_{XX} \, , &\text{(D.41)}\\
\Gamma^{T}_{XX} &= -\frac{T}{X}\Gamma^{X}_{XX} \, , &\text{(D.42)}\\
\Gamma^{T}_{TX} &= \Gamma^{T}_{XT} = \Gamma^{X}_{XX} \, , &\text{(D.43)}
\end{aligned}$$

where, again, Eqs. (D.39) – (D.43) were found only by considering Eqs. (D.32) and (D.37). Substituting Eqs. (D.39) – (D.43) in Eq. (7.16) we find that components a^X and a^T of the acceleration vector field are given by

$$\begin{aligned}
a^X &= \frac{dU^X}{d\tau} + \frac{X^2 - T^2}{T^2}\Gamma^{X}_{XX} U^X U^X \, , &\text{(D.44)}\\
a^T &= \frac{dU^T}{d\tau} + \frac{X^2 - T^2}{X\,T}\Gamma^{X}_{XX} U^X U^X \, , &\text{(D.45)}
\end{aligned}$$

where the repetition of the indices does not mean summation but actual products of the components. Substituting Eq. (7.30) in Eqs. (D.44) and (D.45) we find:

$$\begin{aligned}
a^X &= \frac{dU^X}{d\tau} + g^{XX}\Gamma^{X}_{XX} \, , &\text{(D.46)}\\
a^T &= \frac{dU^T}{d\tau} + g^{XX}\Gamma^{X}_{XX}\frac{T}{X} \, . &\text{(D.47)}
\end{aligned}$$

We are now in position to compute the $K_{\tau\tau}$ component of the extrinsic curvature. Substituting Eqs. (7.34), (D.46) and (D.47) in Eq. (7.13) and using Eq. (7.25) yields

$$^{(+)}K_{\tau\tau} = -\mathrm{sign}\,(X)\sqrt{\frac{g_{XX}}{X^2 - T^2}}\left(g^{XX} + g^{XX}\Gamma^{X}_{XX}\frac{X^2 - T^2}{X}\right) \, . \tag{D.48}$$

Then, using Eqs. (7.20), (7.22) and (D.38) in Eq. (D.48) we get

$$ {}^{(+)}K^{\tau}{}_{\tau} = \frac{\text{sign}\left(X\right)}{2R^2\Upsilon}\left(r_+ + r_- - 2\frac{r_- r_+}{R}\right) , \tag{D.49} $$

where the induced metric in Eq. (7.9) was used to raise the indices and

$$ \Upsilon = \sqrt{\frac{\left(R-r_+\right)\left(R-r_-\right)}{R^2}} . \tag{D.50} $$

To find the other non-null components of the extrinsic curvature of Σ, $K_{\theta\theta}$ and $K_{\varphi\varphi}$, we have to compute the remaining entries of the Christoffel symbols. Equations. (7.20) and (D.32) yield

$$ \Gamma^X_{\theta\theta} = -rg^{XX}\frac{\partial r}{\partial X} , \tag{D.51} $$

$$ \Gamma^T_{\theta\theta} = -rg^{XX}\frac{T}{X}\frac{\partial r}{\partial X} , \tag{D.52} $$

$$ \Gamma^X_{\varphi\varphi} = -r\sin^2\left(\theta\right)g^{XX}\frac{\partial r}{\partial X} , \tag{D.53} $$

$$ \Gamma^T_{\varphi\varphi} = -r\sin^2\left(\theta\right)g^{XX}\frac{T}{X}\frac{\partial r}{\partial X} . \tag{D.54} $$

Substituting Eqs. (D.30) and (D.51) - (D.54) in Eqs. (7.14) and (7.15) we find

$$ {}^{(+)}K^{\theta}{}_{\theta} = {}^{(+)}K^{\varphi}{}_{\varphi} = \frac{\text{sign}\left(X\right)}{2R_+}\frac{\sqrt{g_{XX}\left(X^2-T^2\right)}}{R} . \tag{D.55} $$

D.2.2 Thin shells inside the Cauchy horizon

Let us now compute the non-null components of the extrinsic curvature of $\mathcal{S}$ for a thin shell inside the Cauchy horizon. Similarly to the previous subsection, we have first to compute some intermediate quantities. Taking the derivative to X and T, independently, of Eq. (7.48) we find

$$ \frac{\partial r}{\partial X} = -\frac{g_{XX}}{2R_-}X , \tag{D.56} $$

$$ \frac{\partial r}{\partial T} = \frac{g_{XX}}{2R_-}T . \tag{D.57} $$

Although, the expressions for $\partial r/\partial X$ and $\partial r/\partial T$ are different from those found in subsection D.2.1, Eqs. (D.30) and (D.31), are still related by Eq. (D.32).

Equations (7.47) and (D.56) yield

$$ \partial_X g_{XX} = \frac{\left(g_{XX}\right)^2}{2r_-^4}\frac{r_+-r_-}{r_+-r}\left[2\frac{r_-^2 r_+}{r} + \left(r_+-r_-\right)\left(r+r_-\right)\right]X . \tag{D.58} $$

Now, since in this coordinate patch Eqs. (D.32) and (D.37) are still verified, the other relevant derivatives of the metric components are also given by Eqs. (D.34) – (D.36), where g_{XX} refers, in this case, to the XX component of the metric in Eq. (7.47). From Eqs. (7.47) and (D.58) we find

$$\Gamma^X_{XX} = \frac{g_{XX}}{4r_-^4} \frac{r_+ - r_-}{r_+ - r} \left[2\frac{r_-^2 r_+}{r} + (r_+ - r_-)(r + r_-) \right] X \, . \tag{D.59}$$

Unsurprisingly, the other entries of the Christoffel symbols in the form Γ^k_{ij}, where $i, j, k \in \{X, T\}$, are given by Eqs. (D.39) – (D.43), which imply that the a^X and a^T components of the acceleration are also given by Eqs. (D.44) and (D.45). Then, using Eq. (7.50) we find

$$a^X = \frac{dU^X}{d\tau} + g^{XX} \Gamma^X_{XX} \, , \tag{D.60}$$

$$a^T = \frac{dU^T}{d\tau} + g^{XX} \Gamma^X_{XX} \frac{T}{X} \, . \tag{D.61}$$

Finally, substituting Eqs. (7.52), (D.60) and (D.61) in Eq. (7.13) we find

$${}^{(+)}K_{\tau\tau} = \text{sign}\,(X) \sqrt{\frac{g_{XX}}{X^2 - T^2}} \left[g^{XX} + g^{XX} \Gamma^X_{XX} \frac{X^2 - T^2}{X} \right] \, , \tag{D.62}$$

which, using Eqs. (7.47), (7.48) and (D.59), leads to

$${}^{(+)}K^\tau{}_\tau = \frac{\text{sign}\,(X)}{2R^2 \Upsilon} \left[r_+ + r_- - 2\frac{r_- r_+}{R} \right] \, , \tag{D.63}$$

where the induced metric on Σ, Eq. (7.9), was used to raise the indices and Υ is given by Eq. (D.50).

All is left now is to find the Christoffel symbols necessary to compute the components ${}^{(+)}K_{\theta\theta}$ and ${}^{(+)}K_{\varphi\varphi}$ of the extrinsic curvature of $\mathcal{S}$. However, since the angular part of the metric is the same for both coordinate patches, the Christoffel symbols $\Gamma^X_{\theta\theta}$, $\Gamma^X_{\varphi\varphi}$, $\Gamma^T_{\theta\theta}$ and $\Gamma^T_{\varphi\varphi}$ are also given by Eqs. (D.51) – (D.54), which, in conjunction with Eqs. (7.52), (D.56) and (D.57) yield

$${}^{(+)}K^\theta{}_\theta = {}^{(+)}K^\varphi{}_\varphi = \frac{\text{sign}\,(X)}{2R} \frac{\sqrt{g_{XX}\,(X^2 - T^2)}}{R_-} \, . \tag{D.64}$$

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