Filomat 35:14 (2021), 4613–4636
https://doi.org/10.2298/FIL2114613P

Published by Faculty of Sciences and Mathematics,
University of Nivs, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

From Euclid to Corner Sums – a Trail of Telescoping Tricks

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Abstract. Euclid's algorithm is extended to binomials, geometric sums and corner sums. Two-sided non-commuting, non-constant linear difference equations will be solved, and the solution is applied to corner sums, thereby presenting an explicit formula for the generator of the bi-module spanned by the two starting corner sums.

1. Introduction

Euclid's Algorithm is, without question, one of the most important "super algorithms" in mathematics. It is fast and can be executed efficiently via recurrence relations. In this paper we shall extend this algorithm to binomials, geometric sums and corner sums.

"Telescoping sums" appear in many branches of mathematics, from block matrices to convergence and from iteration to polynomial division.

The most common telescoping sum is the "corner sum"

$$\Gamma_k(x, c, y) = x^{k-1}c + x^{k-2}cy + .. + xcy^{k-2} + cy^{k-1}, \tag{1}$$

in which the parameters *need not commute*! Indeed, we may consider elements from an arbitrary (non necessarily abelian) ring R with unity 1. These sums arise naturally when powering a triangular matrix: the corner element is precisely $\Gamma_k(x, c, y)$. We shorten $\Gamma_k(x, 1, y)$ to $\Gamma_k(x, y)$.

These sums are a generalization of the Difference Quotient

$$\frac{x^n - y^n}{x - y} = x^{n-1} + x^{n-2}y + \cdots + y^{n-1},$$

for two commuting variables x and y.

2020 *Mathematics Subject Classification*. Primary 39A06
Keywords. Telescoping Sum, Corner Sum, Polynomials, Recurrence Relation
Received: 30 October 2020; Accepted: 07 February 2021
Communicated by Dragan S. Djordjević
Research partially financed by Portuguese Funds through FCT (Fundação para a Ciência e a Tecnologia) within the Projects UIDB/00013/2020 and UIDP/00013/2020.
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Corner sums can be expressed in terms of Hankel matrices which have the form

$$H = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ a_2 & & a_n & 0 \\ \vdots & \iddots & & \\ a_n & 0 & & \end{bmatrix} = \sum_{i=1}^{n} a_i H_i \text{ where } H_i = \begin{bmatrix} 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \iddots & & & \vdots \\ 1 & & \iddots & & \\ \vdots & \iddots & & & \\ 0 & & \cdots & & 0 \end{bmatrix} \tag{2}$$

has its i-th counter diagonal filled with ones.

Since $\Gamma_k(x, c, y)$ is a bilinear form we may express it as

$$\Gamma_k(x, c, y) = [1, x, .., x^{n-1}](cH_k) \begin{bmatrix} 1 \\ y \\ \vdots \\ y^{n-1} \end{bmatrix} = \mathbf{x}^T (cH_k)\mathbf{y}. \tag{3}$$

On the other hand, if we introduce a polynomial form $f(x) = a_1 + a_2 x + .. + a_n x^{n-1}$, then we also have the bilinear form

$$W_f(x, y) = \mathbf{x}^T H_f \mathbf{y} = [1, x, .., x^{n-1}] \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ a_2 & & a_n & 0 \\ \vdots & \iddots & & \\ a_n & 0 & & \end{bmatrix} \begin{bmatrix} 1 \\ y \\ \vdots \\ y^{n-1} \end{bmatrix} = \sum_{k=1}^{n} \Gamma_k(x, a_k, y) \tag{4}$$

where $H_f = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ a_2 & & a_n & 0 \\ \vdots & \iddots & & \\ a_n & 0 & & \end{bmatrix}$. As such we see that $H_{k+1} = H_{x^k}$.

Corner sums appear naturally when we examine matrix equations and matrix powering. For example, if $AX - XB = C$, then $A^k X - XB^k = \Gamma_k(A, C, B)$. Likewise when we power a block triangular matrix $M = \begin{bmatrix} x & c \\ 0 & y \end{bmatrix}$ it follows that

$$M^k = \begin{bmatrix} x & c \\ 0 & y \end{bmatrix}^k = \begin{bmatrix} x^k & \Gamma_k(x, c, y) \\ 0 & y^k \end{bmatrix}. \tag{5}$$

As such we see that corner sums appear whenever we block diagonalize matrices to obtain canonical forms, as for example, in the cyclic decomposition theorem [6].

Corner sums do not just generalize difference quotients, they actually act very much like "a derivative". Indeed, consider a given polynomial form $f(x) = a_1 + a_2 x + \cdots + a_n x^{n-1}$ for which we define its right and left evaluations by

$$f^{(r)}(x) = a_1 + a_2 x + \cdots + a_n x^{n-1}$$

and

$$f^{(\ell)}(x) = a_1 + x a_2 + \cdots + x^{n-1} a_n.$$

These lead to the left and right corner sums. Indeed, for M as above,

$$f^{(r)}(M) = \begin{bmatrix} f^{(r)}(x) & \Gamma_f^{(r)}(x, c, y) \\ 0 & f^{(r)}(y) \end{bmatrix}, \tag{6}$$

where the right and left corner sums are defined by

$$\Gamma_f^{(r)}(x,c,y) = \sum_{i=1}^{n} a_i \Gamma_i(x,c,y) \text{ and } \Gamma_f^{(\ell)}(x,c,y) = \sum_{i=1}^{n} \Gamma_i(x,c,y) a_i. \tag{7}$$

These clearly ensure that

$$\Gamma_{x^k}(x,c,y) = \Gamma_k(x,c,y). \tag{8}$$

Now if $x = y$, $c = 1$, and $a_i x = x a_i$ then

$$f\left(\begin{bmatrix} x & 1 \\ 0 & x \end{bmatrix}\right) = \begin{bmatrix} f(x) & f'(x) \\ 0 & f(x) \end{bmatrix},$$

from which we see that $\Gamma_f(x,1,x) = f'(x)$.

Without assuming any commutivity we may state the following generalizations of the "difference quotient"

$$xW_f(x,y) - W_f(x,y)y = f^{(\ell)}(x) - f^{(r)}(y), \tag{9}$$

which uses

$$x\Gamma_k(x,c,y) - \Gamma_k(x,c,y)y = x^k c - c y^k. \tag{10}$$

Since the vector $\mathbf{x}$ commutes with the scalar x we see that $xW_f = x\mathbf{x}^T H_f \mathbf{y} = \mathbf{x}^T(xH_f)\mathbf{y}$ and so

$$f^{(\ell)}(x) - f^{(r)}(y) = xW_f - W_f y = \mathbf{x}^T(xH_f - H_f y)\mathbf{y}.$$

Replacing x by A and y by B, we see that $X = W_f(A,B)$ is a solution to $AX - XB = C_f(A,B)$, provided a_i commutes with A.

We may go one step further and use two polynomials, extending the Bezoutian concept for two commuting variables

$$\frac{f(x)g(y) - f(y)g(x)}{x-y} = \sum_{i,j=1}^{n} b_{ij} x^i y^j = \mathbf{x}^T B \mathbf{y}.$$

Consider

$$\begin{aligned} f^{(\ell)}(x)g^{(r)}(y) - f^{(r)}(x)g^{(\ell)}(x) &= f^{(\ell)}(x)[g^{(r)}(y) - g^{(\ell)}(x)] + [f^{(\ell)}(x) - f^{(r)}(y)]g^{(\ell)}(x) \\ &= (xW_f - W_f y)g^{(\ell)}(x) - f^{(\ell)}(x)(xW_g - W_g y) \\ &= \mathbf{x}^T[xH_f - H_f y]\mathbf{y} g^{(\ell)}(x) - f^{(\ell)}(x)[\mathbf{x}^T(xH_g - H_g y)\mathbf{y}]. \end{aligned} \tag{11}$$

An important special case is when $c = 1 = y$. In this case the corner sum $\Gamma_n(x,1,1)$ reduces to the **geometric progression** (GP) (also called geometric sum)

$$G_n(x) = 1 + x + x^2 + \cdots + x^{n-1}. \tag{12}$$

Needless to say, if just $y = 1$ then we obtain

$$\Gamma_k(x,c,1) = G_k(x)c. \tag{13}$$

When x, y and c commute then

$$\Gamma_k(x,c,y) = y^{k-1} G_k(x/y)c. \tag{14}$$

This shows that each identity involving $G(x)$ generates a corresponding identity for $\Gamma_k(x,c,y)$ with commuting variables, and conversely.

Applications of GPs are even more numerous, and can be found in:

(i) the study of nilpotent elements, including matrices;
(ii) the study of convergence, such as in the ratio test in Calculus;
(iii) in Euclid's Division algorithm applied to special polynomials;
(iv) in powers and generalized inverses of the unit shifts $1 + ab$ and $1 + ba$;
(v) in the "inversion" of the telescoping process;
(vi) in many iterative schemes, such as in the Picard iteration $X_{k+1} = Ax_k + B$, with $X_0 = C$. In fact, its solution takes the form [11]

$$X_k = G_k(A)B + A^k C. \tag{15}$$

Likewise the Cesaro-Neumann iteration makes repeated use of telescoping identities [8].

2. More Properties of Corner sums

Let us next examine some of the basic properties of corner sums. When there is no risk of confusion, we shall write Γ_k for $\Gamma_k(x, c, y)$.

Proposition 2.1. *The corner sum $\Gamma_k(x, c, y)$ has the following properties:*

1. *It is "self reciprocal" i.e.*

$$\Gamma_k(x, c, y) = x^{k-1}\Gamma_k(\frac{1}{x}, c, \frac{1}{y})y^{k-1}, \tag{16}$$

2.

$$\Gamma_{k+1}(x, c, y) = x^k c + \Gamma_k(x, c, y).y = x\Gamma_k(x, c, y) + cy^k, \tag{17}$$

3. *The "internal" addition law*

$$\Gamma_k(x, c + d, y) = \Gamma_k(x, c, y) + \Gamma_k(x, d, y) \tag{18}$$

and the "external" addition law

$$\Gamma_{r+s}(x, c, y) = x^r\Gamma_s(x, c, y) + \Gamma_r(x, c, y)y^s, \tag{19}$$

hold, which for $y = 1 = c$ the latter reduces to

$$G_{r+s}(x) = x^r G_s(x) + G_r(x) = x^s G_r(x) + G_s(x). \tag{20}$$

4. *The **homogeneity conditions** are*

$$\Gamma_k(x, xd, y) = x\Gamma_k(x, d, y) \text{ and } \Gamma_k(x, dy, y) = \Gamma_k(x, d, y)y. \tag{21}$$

Being self reciprocal implies that a geometric progression $G_k(x)$ is also self reciprocal, i.e. $x^{k-1}G_k(1/x) = G_k(x)$.

Setting $y = 1 = c$ in (17) gives the fundamental telescoping identity

$$(1 - x)G_n(x) = 1 - x^n. \tag{22}$$

As such, we may write $G_n(x) = \frac{1 - x^n}{1 - x}$ with the understanding that $x \neq 1$. We shall refer to $x^n - 1$ as the "binomial of the GP".

We also have

$$(1 - x^2)G_n(x) = 1 + x - x^n - x^{n+1}. \tag{23}$$

We may use (20) to obtain $G_n(x) = x^2 G_{n-2}(x) + (1 + x)$, since $n = (n - 2) + 2$.

As an application of the homogeneity conditions, we consider the case where $c = ax - xb$. Then $\Gamma_k(a, c, b) = \Gamma_k(a, ax - xb, b) = a\Gamma_k(a, x, b) - \Gamma_k(a, x, b)b = a^k x - xb^k$.

Proposition 2.2. *For polynomials f and g and M as in (5),*

1. *the external addition law is extended to*

$$\Gamma_{gh}^{(t)}(x,c,y) = g^{(t)}(x)\Gamma_h^{(t)}(x,c,y) + \Gamma_g^{(t)}(x,c,y)h^{(t)}(y). \tag{24}$$

where t is either ℓ or r (left or right), using the fact that $(gh)(M) = g(M)h(M)$.

2. *the **composition law***

$$\begin{aligned} f^{(r)}(g^{(r)}(M)) &= f^{(r)}\left(\begin{bmatrix} g^{(r)}(x) & \Gamma_g^{(r)}(x,c,y) \\ 0 & g^{(r)}(y)\end{bmatrix}\right) \\ &= \begin{bmatrix} f^{(r)}(g^{(r)}(x)) & \Gamma_f^{(r)}(g^{(r)}(x),\Gamma_g^{(r)}(x,c,y),g^{(r)}(y)) \\ 0 & f^{(r)}(g^{(r)}(y))\end{bmatrix}\end{aligned}$$

holds.

A similar result holds for the left *evaluations. Consequently*

$$\Gamma_{f^{(r)}\circ g^{(r)}}(x,c,y) = \Gamma_f^{(r)}(g^{(r)}(x),\Gamma_g^{(r)}(x,c,y),g^{(r)}(y)). \tag{25}$$

Note that the composition law contains a corner sum *within* a corner sum!

Taking $f(x) = x^r$ and $g(x) = x^s$ in the composition law, we arrive at the **product rule**:

Proposition 2.3 (Product rule).

$$\Gamma_{rs}(x,c,y) = \Gamma_r(x^s,\Gamma_s(x,c,y),y^s) = \Gamma_s(x^r,\Gamma_r(x,c,y),y^r). \tag{26}$$

The product rule may also be written as

$$\Gamma_{rs}(x,c,y) = \Gamma_r(x,\Gamma_s(x^r,c,y^r),y) = \Gamma_s(x,\Gamma_r(x^s,c,y^s),y), \tag{27}$$

which follows by directly computation.

For example, $\Gamma_{10}(x,c,y) = \Gamma_5(x^2,\Gamma_2(x,c,y),y^2) = \Gamma_2(x^5,\Gamma_5(x,c,y),y^5)$.

Related to these is the identity

$$\Gamma_r(x^b,\Gamma_s(x^a,c,y^a),y^b) = \Gamma_s(x^a,\Gamma_r(x^b,c,y^b),y^a), \tag{28}$$

which is easily verified directly.

Combining the external addition law with the homogeneity condition, we also have

$$\Gamma_t(x^a,\Gamma_{r+s}(x,c,y),y^a) = \Gamma_t(x^a,x^r\Gamma_s(x,c,y) + \Gamma_r(x,c,y)y^s,y^a)$$

which reduces to

$$\Gamma_t(x^a,\Gamma_{r+s}(x,c,y),y^a) = x^r\Gamma_t(x^a,\Gamma_s(x,c,y),y^a) + \Gamma_t(x^a,\Gamma_r(x,c,y),y^a)y^s. \tag{29}$$

For example,

$$\Gamma_p(x^r,\Gamma_{r+s}(x,c,y),y^r) = \Gamma_{p+1}(x^r,\Gamma_s(x,c,y),y^r) - \Gamma_s(x,c,y)y^{pr} + \Gamma_p(x^r,\Gamma_r(x,c,y),y^r)y^s. \tag{30}$$

In particular when $p = r = 3$ and $s = 8$ this reduces to

$$\Gamma_3(x^3,\Gamma_{11}(x,c,y),y^3) = \Gamma_4(x^3,\Gamma_8(x,c,y),y^3) - \Gamma_8(x,c,y)y^9 + \Gamma_9(x,c,y)y^8, \tag{31}$$

in which $\Gamma_9(x,c,y)y^8 - \Gamma_8(x,c,y)y^9 = x^8cy^8$.

Using the homogeneity condition the product rule takes the form

$$\Gamma_{rs}(x,c,y) = \sum_{i=0}^{s-1}(x^r)^{s-1-i}\Gamma_r(x,c,y)(y^r)^i. \tag{32}$$

Now if $n = mq + r$ with $0 \le r < m \le n$, then we may combine the addition and multiplicative laws, (19) and (27) to give the non-commutative "division algorithm" for corner sums:

Proposition 2.4. *Given $n = mq + r$ with $0 \le r < m \le n$, then*

$$\Gamma_{r+mq}(x, c, y) = x^r \Gamma_m(x, \Gamma_q(x^m, c, y^m), y) + \Gamma_r(x, c, y) y^{mq}. \tag{33}$$

Setting $y = 1 = c$, gives the GP division algorithm

$$G_n(x) = x^r G_q(x^m) G_m(x) + G_r(x) = x^r G_m(x^q) G_q(x) + G_r(x). \tag{34}$$

For $n = mq$,

$$G_{mq}(x) = G_m(x) G_q(x^m). \tag{35}$$

For example, $G_{2m} = G_2(x^m) G_m(x)$.

If x is nilpotent, say $x^N = 0$, then

$$x\Gamma_N(x, c, y) - \Gamma_N(x, c, y) y = -c y^N, \tag{36}$$

which for $y = 1 = c$ reduces to $(1 - x)G_N(x) = 1$ and $(1 - x)^{-1} = G_N(x)$. For $n \le N$ we then arrive at

$$(1 - x^n)^{-1} = G_k(x^n) = 1 + x^n + \cdots + x^{n(k-1)}, \quad \text{where } k = \lfloor N/n \rfloor + 1. \tag{37}$$

Moreover we have $G_n(x) = (1 - x)^{-1}(1 - x^n)$ and hence

$$G_n(x)^{-1} = (1 - x)(1 - x^n)^{-1} = (1 - x)(1 + x^n + \cdots + x^{n(k-1)}). \tag{38}$$

Another application of the geometric sum can be found in the study of generalized inverses [13]. For example

Lemma 2.5. *For elements in an associative ring with unity,*

1. *If $ba = 0$ then $(a + b)^n = \Gamma_{n+1}(a, b)$.*
2. *If $b^2 = b$ then $\Gamma_{n+1}(a, b) = a^n + G_n(a)b$.*
3. *If $eb = 0 = be$ and $e^2 = e$ then $(b + e)^n = b^n + e$.*
   Of particular interest is the case $b = a^2 a^-, e = aa^-$, where a is a (von Neumann) invertible element and a^- denotes an inner inverse of a (i.e. $aa^-a = a$).

A GP can also be obtained from its binomial, using the idea of a Drazin inverse. Indeed ([9]) if A is a matrix with minimal polynomial $\psi_A(\lambda) = (\lambda - 1)^s f(\lambda)$ such that $gcd(\lambda - 1, f) = 1$, then

$$G_n(A) = \sum_{i=0}^{n-1} A^i = (I - A)^D (I - A^n) + \sum_{i=0}^{s-1} \binom{n}{i+1} (A - I)^i Z_i^0, \tag{39}$$

where the Z_i^0 are the principal idempotents of A.

The Cesaro sum is defined as $C_n(A) = G_n(A)/n$ and is used in iteration, Probability Theory, Markov Chains and Non-Negative matrices.

It can be shown that $PA^N Q \longrightarrow 0$ as $N \to \infty$ iff $PG_N(A)Q$ converges as $N \to \infty$ ([11]) where P and Q are invertible matrices.

3. Polynomials

Much of polynomial theory deals with the Division Algorithm and in particular with Euclid's Algorithm. In the special case of a linear divisor, we recall the Bezout Theorem, which heavily depends on the telescoping trick. Indeed, much of matrix theory uses the divisor $\lambda I - A$, leading up to the study of annihilating polynomials, adjoints and elementary divisors. All use telescoping repeatedly.

To study polynomials in one variable we often have to study polynomials in two variables. The catch however, is that for polynomials in two (possibly non-commuting) variables, there is no unique division algorithm (but we can use Groebner bases) and the set of such polynomials is not a PID and there is no gcd!

We shall now show that Euclid's construction for the gcd of two integers, induces parallel gcd algorithms for binomials and Geometric sums as well as "gcd-like" construction for the gcd of two corner sums in non-commuting variables x and y.

For a given m and n, say $n = mq + r$, with $0 \leq r < m \leq n$, Euclid's Algorithm gives a sequence of integer quotients and remainders (q_i, r_i). We shall now show that for three special classes of polynomials, the sequences (q_i, r_i), will induce the corresponding quotient and remainder sequences (Q_i, R_i), and we give explicit expressions for them.

These sequences are of the form

(i) $x^k - 1$ (binomial)

(ii) $G_k(x)$ (geometric progression)

(iii) $\Gamma_k(x, c, y)$ (corner sums).

The story of Euclid's algorithm is really one of finding the generator for the principal ideal generated by the starting elements.

Indeed,

(i) for integers if $r_{N+1} = gcd(r_0, r_1)$ then $r_{N+1}R = r_0R + r_1R$, where R is the ring of integers $\mathbb{Z}$.

(ii) for binomials, if $x^{r_{N+1}} - 1 = gcd(x^{r_0} - 1, x^{r_1} - 1)$ then $(x^{r_{N+1}} - 1)R = (x^{r_0} - 1)R + (x^{r_1} - 1)R$ where $R = \mathbb{Z}[x]$.

(iii) For geometric sums, if $G_{r_{N+1}}(x) = gcd(G_{r_0}(x), G_{r_1}(x))$ then $G_{r_{N+1}}(x)R = G_{r_0}(x)R + G_{r_1}(x))R$, where $R = \mathbb{Z}[x]$. We shall also solve the recurrence relation used in Euclid's algorithm and use it to give explicit formula, in the first two cases, for the coefficients of the generator equation of the principle ideal.

(iv) For corner sums, when x and y do not commute, the ideal has to be replaced by the bi-module generated by Γ_{r_0} and Γ_{r_0}, which we address shortly.

3.1. *Finding the gcd*

We now show that there is a precise parallel between Euclid's Algorithm for two integers m and n, and the algorithm for the corresponding binomials $x^m - 1$ and $x^n - 1$.

We recall (34) and start by multiplying (34) by $x - 1$, to obtain the following pivotal binomial identity, valid over any ring R with 1.

$$\boxed{x^n - 1 = x^r(x^{mq} - 1) + (x^r - 1) = (x^m - 1)x^r[x^{m(q-1)} + x^{m(q-2)} + ... + 1] + (x^r - 1)} \tag{40}$$

or more compactly

$$\boxed{n = mq + r \Leftrightarrow (x^n - 1) = (x^m - 1)Q(x) + (x^r - 1)}, \tag{41}$$

where $Q(x) = x^r[x^{m(q-1)} + x^{m(q-2)} + ... + 1] = x^rG_q(x^m)$.

This show that the geometric sum does indeed enter naturally into the division algorithm!

An immediate consequence is that

$$m \mid n \text{ iff } (x^m - 1) \mid (x^n - 1) \text{ iff } G_m(x) | G_n(x). \tag{42}$$

This first part of this chain is used with $x = 2$ in the construction of Mersenne primes and the construction of Fermat and Miller pseudo primes as used in cryptography [14].

Alternatively we could use the fact that $a|b$ iff $a|(b - a)$.

For the corner sum, the fact that $m|n$ will result in a compact functional equation.

A second by-product is the following result:

Theorem 3.1. *Over a Euclidean domain,*

$$\gcd(x^m - 1, x^n - 1) = x^{\gcd(m,n)} - 1 = (x - 1) \cdot \gcd[G_m(x), G_n(x)]. \tag{43}$$

*P*roof. We may use this "parallel division" to obtain three parallel Euclid-chains starting with $r_0 = n$ and $r_1 = m$ or with $x^{r_0} - 1$ and $x^{r_1} - 1$ or with G_{r_0} and G_{r_1}.

$$\begin{array}{l|l|l}
r_0 = r_1 q_1 + r_2 & x^{r_0} - 1 = (x^{r_1} - 1)Q_1 + (x^{r_2} - 1) & G_{r_0}(x) = G_{r_1}(x)Q_1 + G_{r_2}(x) \\
r_1 = r_2 q_2 + r_3 & & \\
\vdots & \vdots & \vdots \\
r_{i-1} = r_i q_i + r_{i+1} & x^{r_{i-1}} - 1 = (x^{r_i} - 1)Q_i + (x^{r_{i+1}} - 1) & G_{r_{i-1}}(x) = G_{r_i}(x)Q_i + G_{r_{i+1}}(x) \\
\vdots & \vdots & \vdots \\
r_{N-1} = r_N q_N + \boxed{r_{N+1}} & x^{r_{N-1}} - 1 = (x^{r_N} - 1)Q_N + \boxed{(x^{r_{N+1}} - 1)} & G_{r_{N-1}}(x) = G_{r_N}(x)Q_i + G_{r_{N+1}}(x) \\
r_N = r_{N+1} q_{N+1} + 0 & x^{r_N} - 1 = (x^{r_{N+1}} - 1)Q_{N+1} + 0(x) & G_{r_N}(x) = G_{r_{N+1}}(x)Q_{N+1} + 0(x)
\end{array}.$$

Note that $r_{N+2} = 0$.

Since $Q_i = x^{r_{i+1}} G_{q_i}(x^{r_i})$ we see how the geometric sums are related via

$$G_{r_{i-1}}(x) = x^{r_{i+1}} G_{r_i}(x) \cdot G_{q_i}(x^{r_i}) + G_{r_{i+1}}(x) = x^{r_{i+1}} G_{q_i r_i}(x) + G_{r_{i+1}}(x). \tag{44}$$

At the last stage, when $r_N = r_{N+1} q_{N+1}$, this gives

$$G_{r_N}(x) = x^0 \cdot G_{r_{N+1}}(x) \cdot G_{q_{N+1}}(x^{r_N+1}),$$

which checks (35).

The result corresponding to (43) is false for lcms , i.e. $\mathrm{lcm}(x^m - 1, x^n - 1) \neq x^{\mathrm{lcm}(m,n)} - 1$, as seen from the case $n = 3$ and $m = 2$. Since $\gcd(m, n) = 1$, we know that $\gcd((x^2 - 1), (x^3 - 1)) = x - 1$ and hence $\mathrm{lcm}[(x^2 - 1), (x^3 - 1)] = (x^2 - 1)(x^3 - 1)/(x - 1) = (x + 1)(x^3 - 1)$. This clearly divides $(x^6 - 1)$ but will not equal it. The quotient equals $x^2 - x + 1$.

If we set $r_i = u_i r_0 + v_i r_1$, $i = 0, \ldots, k + 1$, then the u_i and v_i satisfy *the same* recurrence as the r_i, i.e. $r_{i+1} = r_{i-1} - q_i r_i$, except with different initial conditions. Indeed

$$u_{i+1} = u_{i-1} - q_i u_i, \;\; u_0 = 1, u_1 = 0$$

and

$$v_{i+1} = v_{i-1} - q_i v_i, \;\; v_0 = 0, v_1 = 1$$

We may do exactly the same for the polynomials $x^{r_i} - 1$ and $G_{r_{i-1}}(x)$ with recurrences

$$x^{r_{i+1}} - 1 = (x^{r_{i-1}} - 1) - Q_i(x)(x^{r_i} - 1) \text{ and } G_{r_{i+1}}(x) = G_{r_{i-1}} - Q_i \cdot G_{r_i}(x)$$

to give

$$x^{r_i} - 1 = U_i(x)[x^{r_0} - 1] + V_i(x)[x^{r_1} - 1] \text{ and } G_{r_i} = U_i(x)G_{r_0} + V_i(x)G_{r_1} \tag{45}$$

The $U_i(x)$ and $V_i(x)$ satisfy

$$U_{i+1}(x) = U_{i-1}(x) - Q_i(x)U_i(x), \; U_0(x) = 1, \; U_1(x) = 0$$

and

$$V_{i+1}(x) = V_{i-1}(x) - Q_i(x)V_i(x), \; V_0(x) = 0, \; V_1(x) = 1.$$

At the final stage, where $i = N + 1$, we arrive at the "ideal equation"

$$\begin{array}{lcl} r_{N+1} & = & \gcd(r_o, r_1) & = & u_{N+1}r_0 + v_{N+1}r_1, \\ x^{r_{N+1}} - 1 & = & \gcd[x^{r_0} - 1, x^{r_1} - 1] & = & U_{N+1}(x)(x^{r_0} - 1) + V_{N+1}(x)(x^{r_1} - 1) \\ G_{r_{N+1}}(x) & = & \gcd[G_{r_0}(x), G_{r_1}(x)] & = & U_{N+1}(x)G_{r_0}(x) + V_{N+1}(x)G_{r_1}(x). \end{array}$$

The corresponding result for corner sums is more complicated, and the expression for the coefficients will be given in the section on Sandwich Recurrence Relations.

Consequently, recalling that $r_0 = n$ and $r_1 = m$, we see that

$$(m, n) = 1 = (r_0, r_1) \Leftrightarrow r_{N+1} = 1 \Leftrightarrow (x - 1) = \gcd[x^{r_0} - 1, x^{r_1} - 1] \Leftrightarrow 1 = \gcd[G_{r_0}(x), G_{r_1}(x)] \tag{46}$$

In which case $1 = u_{N+1}n + v_{N+1}m$ as well as

$$x - 1 = U_{N+1}(x)(x^n - 1) + V_{N+1}(x)(x^m - 1) \quad \text{and} \quad 1 = U_{N+1}(x)G_n(x) + V_{N+1}(x)G_m(x). \tag{47}$$

For convenience we shall write $U(x)$ for U_{N+1} and $V(x)$ for V_{N+1}.

Note that in the above we had assumed that $n \geq m$. If the opposite holds, then we have to interchange U and V.

We shall next see that there is a parallel algorithm for corner sums.

4. The Corner Recurrence

Recall by the addition law and the product rule, that

$$\Gamma_{r+mq}(x, c, y) = x^r\,\Gamma_m(x, \Gamma_q(x^m, c, y^m), y) + \Gamma_r(x, c, y)y^m. \tag{48}$$

Parallel to the sequences (r_k), $(x^{r_k} - 1)$ and (G_{r_k}) we may construct the corresponding sequence of corner sums as

$$\Gamma_{r_0}(x, c, y) = x^{r_2}\Gamma_{r_1}(x, \Gamma_{q_1}(x^{r_1}, c, y^{r_1}), y) + \Gamma_{r_2}(x, c, y)y^{r_1 q_1} \tag{49}$$

$$\Gamma_{r_1}(x, c, y) = x^{r_3}\Gamma_{r_2}(x, \Gamma_{q_2}(x^{r_2}, c, y^{r_2}), y) + \Gamma_{r_3}(x, c, y)y^{r_2 q_2} \tag{50}$$

$$\Gamma_{r_2}(x, c, y) = x^{r_4}\Gamma_{r_3}(x, \Gamma_{q_3}(x^{r_3}, c, y^{r_3}), y) + \Gamma_{r_4}(x, c, y)y^{r_3 q_3} \tag{51}$$

$$\vdots \tag{52}$$

$$\Gamma_{r_{k-1}}(x, c, y) = x^{r_{k+1}}\Gamma_{r_k}(x, \Gamma_{q_k}(x^{r_k}, c, y^{r_k}), y) + \Gamma_{r_{k+1}}(x, c, y)y^{r_k q_k}. \tag{53}$$

At the final stage, when $k = N + 1$ and $r_{N+2} = 0$, we have

$$\Gamma_{r_N}(x, c, y) = \Gamma_{r_{N+1}}(x, \Gamma_{q_{N+1}}(x^{r_{N+1}}, c, y^{r_{N+1}}), y). \tag{54}$$

The process of "back-substituting" the Γ_{r_i} to obtain $\Gamma_{r_{N+1}}$ as an "expression" in terms of the initial values Γ_{r_0} and Γ_{r_1}, can be done by hand for small cases. For larger cases it is best done by setting up a "difference equation" that is satisfied by a "weighted version" of the Γ_is. This we now pursue.

Let us first recall (32), and introduce the following abbreviation:
$x^{r_k} \longrightarrow x_k$, $y^{r_k} \longrightarrow y_k$, $\Gamma_{r_k}(x, c, y) \longrightarrow \Gamma_k$ and define the product $P_k = y_k^{q_k} \cdots y_1^{q_1} = y^{q_k r_k + \cdots + q_1 r_1}$. We also set $P_0 = 1 = P_{-1}$.

We may rewrite the above steps as

$$\Gamma_0 = x_2 \sum_{i=0}^{q_1 - 1} (x_1)^{q_1 - 1 - i}\Gamma_1 y_1^i + \Gamma_2 y_1^{q_1}. \tag{55}$$

$$\Gamma_1 = x_3 \sum_{i=0}^{q_2-1} (x_2)^{q_2-1-i} \Gamma_2 y_2^i + \Gamma_3 y_2^{q_2} \tag{56}$$

and generally

$$\Gamma_{k-1} = x_{k+1} \sum_{i=0}^{q_k-1} (x_k)^{q_k-1-i} \Gamma_k y_k^i + \Gamma_{k+1} y_k^{q_k}. \tag{57}$$

If we now define $F_k = \Gamma_k P_{k-1}$ and multiply through on the right by P_{k-1}, then we arrive at

$$\Gamma_{k-1} P_{k-1} = x_{k+1} \sum_{i=0}^{q_k-1} (x_k)^{q_k-1-i} (\Gamma_k P_{k-1}) y_k^i + \Gamma_{k+1} (y_k)^{q_k} P_{k-1}. \tag{58}$$

This gives the recurrence

$$F_{k+1} = -x_{k+1} \sum_{i=0}^{q_k-1} (x_k)^{q_k-1-i} F_k y_k^i + F_{k-1} (y_{k-1}^{q_{k-1}}) \tag{59}$$

We thus have a "sandwich" recurrence relation of the form

$$w_{k+1} = \sum_{i=0}^{q_k-1} a_i^{(k)} w_k \alpha_i^{(k)} + w_{k-1} \beta_k.$$

where $a_i^{(k)} = x^{r_{k+1}+r_k(q_k-1-i)}$, $\alpha_i^{(k)} = y^{r_k i}$ and $\beta_k = y_{k-1}^{q_{k-1}}$. It is clear that the latter two commute. We shall for convenience drop the brackets in the exponents.

The initial conditions are $w_0 = F_0 = \Gamma_0(x, c, y) = \Gamma_{r_0}$ and $w_1 = \Gamma_1 = \Gamma_{r_1}(x, c, y)$.

This recurrence is a special case of the more general two-sided sandwich recurrence

$$w_{k+1} = \sum_{i=0}^{q_k-1} a_i^k w_k \alpha_i^k + \sum_{i=0}^{p_k-1} b_i^k w_{k-1} \beta_i^k, \quad w_0 = \lambda, w_1 = \mu, \tag{60}$$

where the coefficients are not constant and do not (necessarily) commute. On account of the linearity, this may be split as $w_k = X_k + Y_k$, where (X_k) and (Y_k) satisfy the same recurrence but with initial conditions $X_0 = \lambda, X_1 = 0$ and $Y_0 = 0, Y_1 = \mu$ respectively.

We next address the solution process.

5. The Matrix Recurrence

We start by writing the sandwich recurrence relation (60) in matrix from as

$$w_{k+1} = A_k w_k \alpha_k + B_k w_{k-1} \beta_k, \text{ with } w_0 = \lambda,\ w_1 = \mu, \tag{61}$$

where $A_k = [a_0^k, .., a_{q_k-1}^k]$, $B_k = [b_0^k, .., b_{q_k-1}^k]$ and $\boldsymbol{\alpha}_k = \begin{bmatrix} \alpha_0^k \\ \vdots \\ \alpha_{q_k-1}^k \end{bmatrix}$, $\boldsymbol{\beta}_k = \begin{bmatrix} \beta_0^k \\ \vdots \\ \beta_{q_k-1}^k \end{bmatrix}$.

Likewise we consider the associated recurrences

$$X_{k+1} = A_k X_k \alpha_k + B_k X_{k-1}\beta_k, \ \text{ with } \ X_0 = \lambda, \ X_1 = 0, \tag{62}$$

and

$$Y_{k+1} = A_k Y_k \alpha_k + B_k Y_{k-1}\beta_k, \ \text{ with } \ Y_0 = 0, \ Y_1 = \mu. \tag{63}$$

Following Euclid, we back substitute at each step and write

$$w_k = U_k w_0 U'_k + V_k w_1 V'_k, \tag{64}$$

where the matrix coefficients satisfy the following row/columns recurrence relations

$$U_{k+1} = [A_k U_k, B_k U_{k-1}] \ \text{ (as rows) } U_0 = 1, U_1 = 0 \tag{65}$$

and

$$U'_{k+1} = \begin{bmatrix} U'_k \alpha_k \\ U'_{k-1}\beta_k \end{bmatrix} \ \text{ (as columns) } U'_0 = 1, U'_1 = 0 \tag{66}$$

with similar recurrences for V_k and V'_k and associated initial conditions $V_0 = V'_0 = 0, V_1 = V'_1 = 1$.

Comparing the two settings we see that

$$X_k = U_k \lambda U'_k, \ \ Y_k = V_k \mu V'_k, \text{ with } X_0 = \lambda, \ X_1 = 0, \ Y_0 = 0, \ Y_1 = \mu. \tag{67}$$

Let us now generate the first few terms of these recurrences.

$$\begin{aligned} w_2 &= A_1\mu\alpha_1 + B_1\lambda\beta_1 \\ w_3 &= (A_2A_1)\mu(\alpha_1\alpha_2) + A_2(B_1\lambda\beta_1)\alpha_2 + B_2\mu\beta_2 = [A_2A_1, B_2]\mu \begin{bmatrix} \alpha_1\alpha_2 \\ \beta_2 \end{bmatrix} + (A_2B_1)\lambda(\beta_1\alpha_2) \end{aligned}$$

From this, or from the recurrence, we see that

$$V_2 = [A_1, 0], V'_2 = \begin{bmatrix} \alpha_1 \\ 0 \end{bmatrix}, U_2 = [0, B_1], U'_2 = \begin{bmatrix} 0 \\ \beta_1 \end{bmatrix}$$

as well as

$$V_3 = [A_2A_1, B_2], V'_3 = \begin{bmatrix} \alpha_1\alpha_2 \\ \beta_2 \end{bmatrix}$$

and

$$U_3 = [0, A_2B_1, 0], U'_3 = \begin{bmatrix} 0 \\ \beta_1\alpha_2 \\ 0 \end{bmatrix}.$$

Likewise

$$V_4 = [A_3A_2A_1, A_3B_2, B_3A_1, 0] \text{ and } V'_4 = \begin{bmatrix} \alpha_1\alpha_2\alpha_3 \\ \beta_2\alpha_3 \\ \alpha_1\beta_3 \\ 0 \end{bmatrix}$$

in addition to

$$U_4 = [0, A_3A_2B_1, 0, 0, B_3B_1] \text{ and } U'_4 = \begin{bmatrix} 0 \\ \beta_1\alpha_2\alpha_3 \\ 0 \\ 0 \\ \beta_1\beta_3 \end{bmatrix}.$$

We now make two important observations:

(1) It suffices to only compute the left-hand coefficients V_k and U_k, because the right-hand coefficients follow immediately by symmetry (with entries in reversed order).

(ii) The V_k rows have a much simpler pattern than the U_k, with a last entry that vanishes.

(iii) If we set all $\alpha_1 = 1 = \beta_i$, then $V_k' = \begin{bmatrix} \mathbf{e} \\ 0 \end{bmatrix}$ and since we multiply $V_k \mu V_k'$ we obtain the terms in the left-handed recurrence $v_{k+1} = A_k v_k + B_k v_{k-1}$, by *adding* the terms in the row V_k! (the zero term drops out!) Indeed, $v_2 = a_1, v_3 = a_2 a_1 + b_2$, and $v_4 = a_3 a_2 a_1 + a_3 b_2 + b_3 a_1$ etc.

Needless to say we may reverse this argument, and use the one-sided (say left) recurrence to generate the vectors V_k, and V_k' for the sandwich recurrence.

Before we shall do this let us first digress and complete the gcd story.

5.1. *The gcd for corner sums*

The "gcd story" for corner sums is more complicated because of the non-commutativity of the variables. The "ideal" structure that is associated with the gcd concept in the first three cases, (those of the integer, binomial and geometric sum cases) has to be replaced by the corresponding bi-module structure.

Trying to express the terminal "gcd" $\Gamma_{r_{N+1}}(x, c, y)$ in terms of the initial corner sums Γ_{r_0} and Γ_{r_1} amounts to "solving" the sandwich recurrence (59). We need both the product rule (26) as well as the actual solution form (64).

We begin by defining the proper setting.

Given two rings R and S, with common elements a and b. The bi-module generated by a and b, relative to R, S is defined and denoted by

Definition 5.1. *$M(a, b) = \langle a, b\rangle_{R,S} = \sum_{i=1}^{K} r_i a s_i + \sum_{j=1}^{N} \rho_j b \sigma_j$, for all $K, N = 1, 2, \ldots$, and all $r_i, \rho_j \in R$ and all $s_i, \sigma_j \in S$.*

It is clear that

(i) $M(a, b) + M(a, b) \subseteq M(a, b)$ (ii) $RM(a, b) \subseteq M(a, b)$ and (iii) $M(a, b)S \subseteq S$.

These show that $M(a, b)$ is a "two-sided" *bi-module*, which generalizes the ideal concept. We shall call $M(a, b)$ the bi-module generated by a and b – relative to the rings R and S.

We shall refer to $M(a, b)$ a *principal* bi-module, if there exists a generator $d \in R \cap S$, such that $M(a, b) = M(d)$. In other words,

$$x \in M(a, b) \text{ iff } x = \sum_{i=1}^{L} \pi_i d \lambda_i, \tag{68}$$

for some $\pi_i \in R$ and $\lambda_i \in S$. In particular this means that

$$d = \sum_{i=1}^{K} r_i a s_i + \sum_{j=1}^{N} r_j' b s_j', \tag{69}$$

for some K and N, and $r_i, r_j' \in R$ and $s_i, s_j' \in S$, as well as

$$a = \sum_{i=1}^{T} \pi_i d \lambda_i \text{ and } b = \sum_{i=1}^{L} \pi_i' d \lambda_i', \tag{70}$$

for some T and L and $\pi_i, \pi_i' \in R$ and $\lambda_i, \lambda_i' \in S$.

In order to complete the gcd parallel, we define "division" in the bi-module as

Definition 5.2. *$x|a$ if $a = \sum_{i=1}^{K} r_i x s_i$, for some K and $r_i \in R$ and $s_i \in S$.*

It follows immediately from (68) and (70) that d generates $M(a,b)$ iff $d|a, d|b$ and $x|a, x|b$ implies $x|d$.

The division defined above will be a *partial order* provided $a = \sum_{i=1}^{K} r_i a s_i$ forces all $r_i = 1 = s_i$. This will be the case for corner sums with integer polynomial rings $R = \mathbb{Z}[x]$ and $S = \mathbb{Z}[y]$.

For the terminal corner sum, the solution

$$\Gamma_{r_{N+1}} = U_{r_{N+1}} \Gamma_{r_0} U'_{r_{N+1}} + V_{r_{N+1}} \Gamma_{r_1} V'_{r_{N+1}} \tag{71}$$

shows that $\Gamma_{r_{N+1}} \in \langle \Gamma_{r_0}, \Gamma_{r_1} \rangle_{\mathbb{Z}[x],\mathbb{Z}[x]}$, while the product rule and the fact that $r_{N+1}|r_0$ and $r_{N+1}|r_1$ ensure that

$$\Gamma_{r_0} \in \langle \Gamma_{r_{N+1}} \rangle, \ \ \Gamma_{r_1} \in \langle \Gamma_{r_{N+1}} \rangle.$$

As such we see that $\Gamma_{r_{N+1}}$ indeed generates the bi-module $M(\Gamma_{r_0}, \Gamma_{r_1})$.

6. The NC^2 Case

We next focus on the NON-commutative, NON-constant (i.e. NC^2) (left-handed) linear Recurrence Relation

$$w_{k+1} = a_k w_k + b_k w_{k-1}, \quad w_0 = \lambda, \ w_1 = \mu. \tag{72}$$

where a_k and b_k need NOT commute.

It is clear that the special case where all $b_i = 1$, reduces to Euclid's integer recurrence relation.

The solution may of course be expressed in terms of companion matrices. In fact if we let $\mathbf{w}_i = \begin{bmatrix} w_{i+1} \\ w_i \end{bmatrix}$, $\mathbf{w}_0 = \begin{bmatrix} \mu \\ \lambda \end{bmatrix}$ and set $L_i = \begin{bmatrix} a_i & b_i \\ 1 & 0 \end{bmatrix}$. Then

$$\mathbf{w}_i = L_i \mathbf{w}_{i-1} = (L_i L_{i-1}..L_1)\mathbf{w}_0, \tag{73}$$

however this tells us nothing about the representation of the w_k.

For example, $\begin{bmatrix} w_4 \\ w_3 \end{bmatrix} = \begin{bmatrix} a_3 & b_3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_2 & b_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 & b_1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \mu \\ \lambda \end{bmatrix}$.

The use of companion matrices was also presented in [12].

We begin with the special case where $v_0 = 0, v_1 = 1$. The general solution to the case where $v_0 = 0, v_1 = \mu$ is obtained by post multiplication of μ.

When $v_0 = 0, v_1 = 1$ then the first few links are

$v_2 = a_1, v_3 = a_2 a_1 + b_2,$
$v_4 = a_3 a_2 a_1 + a_3 b_2 + b_3 a_1,$
$v_5 = a_4 a_3 a_2 a_1 + a_4 a_3 b_2 + a_4 b_3 a_1 + b_4 a_2 a_1 + b_4 b_2$
$v_6 = a_5 a_4 a_3 a_2 a_1 + a_5 a_4 a_3 b_2 + a_5 a_4 b_3 a_1 + a_5 b_4 a_2 a_1 + a_5 b_4 b_2 + b_5 a_3 a_2 a_1 + b_5 a_3 b_2 + b_5 b_3 a_1,$
$v_7 = a_6 a_5 a_4 a_3 a_2 a_1 + (a_6 a_5 a_4 a_3 b_2 + a_6 a_5 a_4 b_3 a_1 + a_6 a_5 b_4 a_2 a_1 + a_6 b_5 a_3 a_2 a_1 + b_6 a_4 a_3 a_2 a_1) +$
$(a_6 a_5 b_4 b_2 + a_6 b_5 a_3 b_2 + a_6 b_5 b_3 a_1 + b_6 a_4 a_3 b_2 + b_6 a_4 b_3 a_1 + b_6 b_4 a_2 a_1) + b_6 b_4 b_2.$

To keep track of the pattern, the solutions can best be expressed in terms of **blocks**!

Consider v_{k+1} and observe the following facts about its words (i.e. terms or products):

1. the starting subscript is k;

2. the subscripts decrease from left to right;
3. each word contains at most two types of letters, a_i and b_j;
4. after an a_i the subscript goes down by ONE; after a b_j it goes down by TWO;
5. the word length L ranges from 1 to k;
6. if t is the number of b_j, then $t + L = k$ and $t \leq L$; hence $t \leq \lfloor \frac{k}{2} \rfloor$;
7. the words come in blocks $E_t(\ell)$ of length $L = k - t$, and cardinality $\binom{L}{r}$;
8. we have to allocate t slots for the b_i out of L slots in $\binom{L}{t}$ ways.

For notational convenience we replace a_r by r and b_s by $\bar{s}$ and have

$$v_{k+1} = \sum_{t=0}^{\lfloor \frac{k}{2} \rfloor} E_t(k-t). \tag{74}$$

A block of words is written in **matrix form**, and by the "addition" of these arrays we mean the addition of each of its rows to the total.

The solution for the IC $v_0 = 0$ and $v_1 = \mu$ is obtained by post multiplying by μ the solution when $v_0 = 0$ and $v_1 = 1$.

Examples.

(i) $k = 1$. $v_2 = E_0(1) = [1] = a_1$.

(ii) $k = 2$. $v_3 = E_0(2) + E_1(1) = [2, 1] + \overline{2} = a_2a_1 + b_2$.

(iii) $k = 3$. $v_4 = E_0(3) + E_1(2) = [3, 2, 1] + \begin{bmatrix} 3 & \overline{2} \\ \overline{3} & 1 \end{bmatrix} = a_3a_2a_1 + (a_3b_2 + b_3a_1)$.

(iv) $k = 4$. $v_5 = E_0(4) + E_1(3) + E_2(2) = a_4a_3a_2a_1 + (a_4a_3b_2 + a_4b_3a_1 + b_4a_2a_1) + b_4b_2$, i.e. $v_5 = [4, 3, 2, 1] + \begin{bmatrix} 4 & 3 & \overline{2} \\ 4 & \overline{3} & 1 \\ \overline{4} & 2 & 1 \end{bmatrix} + [\overline{4}, 2]$.

(v) $k = 5$. $v_6 = E_0(5) + E_1(4) + E_2(3) = a_5a_4a_3a_2a_1 + (a_5a_4a_3b_2 + a_5a_4b_3a_1 + a_5b_4a_2a_1 + b_5a_3a_2a_1) + (a_5b_4b_2 + b_5a_3b_2 + b_5b_3a_1)$

(vi) $k = 6$. $\lfloor \frac{k}{2} \rfloor = 3$ and $v_7 = E_0(6) + E_1(5) + E_2(4) + E_3(3)$ in which

$$E_0(6) = [6, 5, 4, 3, 2, 1], E_1(5) = \begin{bmatrix} 6 & 5 & 4 & 3 & \overline{2} \\ 6 & 5 & 4 & \overline{3} & 1 \\ 6 & 5 & \overline{4} & 2 & 1 \\ 6 & \overline{5} & 3 & 2 & 1 \\ \overline{6} & 4 & 3 & 2 & 1 \end{bmatrix}, E_2(4) = \begin{bmatrix} 6 & 5 & \overline{4} & \overline{2} \\ 6 & \overline{5} & 3 & \overline{2} \\ 6 & \overline{5} & \overline{3} & 1 \\ \overline{6} & 4 & 3 & \overline{2} \\ \overline{6} & 4 & \overline{3} & 1 \\ \overline{6} & \overline{4} & 2 & 1 \end{bmatrix},$$

and $E_3(3) = [\overline{6}, \overline{4}, \overline{2}]$. The cardinality of v_7 is $\#(v_7) = \binom{6}{0} + \binom{5}{1} + \binom{4}{2} + \binom{3}{3} = 1 + 5 + 6 + 1 = 13$ elements.

(vii) $k = 7$. $\lfloor \frac{k}{2} \rfloor = 3$. $v_8 = E_0(7) + E_1(6) + E_2(5) + E_3(4)$ in which

$$E_0(7) = [7, 6, 5, 4, 3, 2, 1], E_1(6) = \begin{bmatrix} 7 & 6 & 5 & 4 & 3 & \overline{2} \\ 7 & 6 & 5 & 4 & \overline{3} & 1 \\ 7 & 6 & 5 & \overline{4} & 2 & 1 \\ 7 & 6 & \overline{5} & 3 & 2 & 1 \\ 7 & \overline{6} & 4 & 3 & 2 & 1 \\ \overline{7} & 5 & 4 & 3 & 2 & 1 \end{bmatrix},$$

$$E_2(5) = \begin{bmatrix} 7 & 6 & 5 & \bar{4} & \bar{2} \\ 7 & 6 & \bar{5} & 3 & \bar{2} \\ 7 & 6 & \bar{5} & \bar{3} & 1 \\ 7 & \bar{6} & 4 & 3 & \bar{2} \\ 7 & \bar{6} & 4 & \bar{3} & 1 \\ 7 & \bar{6} & \bar{4} & 2 & 1 \\ \bar{7} & 5 & 4 & 3 & \bar{2} \\ \bar{7} & 5 & 4 & \bar{3} & 1 \\ \bar{7} & 5 & \bar{4} & 2 & 1 \\ \bar{7} & \bar{5} & 3 & 2 & 1 \end{bmatrix} \text{ and } E_3(4) = \begin{bmatrix} 7 & \bar{6} & \bar{4} & \bar{2} \\ \bar{7} & 5 & \bar{4} & \bar{2} \\ \bar{7} & \bar{5} & 3 & \bar{2} \\ \bar{7} & \bar{5} & \bar{3} & 1 \end{bmatrix}.$$

Thus $\#(v_8) = \binom{7}{0} + \binom{6}{1} + \binom{5}{2} + \binom{4}{3} = 1 + 6 + 10 + 4 = 21$.

The proof follows by induction and the fact that

$$a_k E_t(k-t-1) + b_k E_{t-1}(k-t-1)) = E_t(k-t), \quad t = 0, 1, \ldots, [k/2]. \tag{75}$$

The latter follows from the fact that each term (i.e. row) from $a_k E_t(k-t-1)$ as well as each row from $b_k E_{t-1}(k-t-1)$ is contained in the set of rows from $E_t(k-t)$. Moreover both sides have the same cardinality, because of the identity

$$\binom{k}{t} + \binom{k}{t-1} = \binom{k+1}{t}. \tag{76}$$

As such we must have equality. Right multiplication by μ gives the solutions for the case where $v_0 = 0, v_1 = \mu$.

Recalling (74) we may write

$$v_{k+1} = \sum_{\omega \in E_0(k)} \omega + \sum_{\omega \in E_1(k-1)} \omega + \sum_{\omega \in E_2(k-2)} \omega + \cdots \tag{77}$$

where ω is a "word" appearing in the sum, and can at the same time obtain the sandwich solution

$$V_{k+1} = \sum_{\omega \in E_0(k)} \omega\mu\omega^{op} + \sum_{\omega \in E_1(k-1)} \omega\mu\omega^{op} + \sum_{\omega \in E_2(k-2)} \omega\mu\omega^{op} + \cdots, \tag{78}$$

where ω^{op} is the reversed word associated with ω. Needless to say, this has the same number of terms as the one-sided solution v_k.

Let us next examine the first few terms of the sequence (u_k). The links are :
$u_2 = b_1,\ u_3 = a_2 b_1,\ u_4 = (a_3 a_2 + b_3) b_1$,
$u_5 = (a_4 a_3 a_2 + a_4 b_3 + b_4 a_2) b_1$,
$u_6 = [a_5 a_4 a_3 a_2 + (a_5 a_4 b_3 + a_5 b_4 a_2 + b_5 a_3 a_2) + b_5 b_3] b_1$, etc.

The u_i may be obtained from the v_i as follows.

1. In each term in v_{k+1} i.e. in each row of $E_t(k-t)$ for $t = 0, 1, \ldots, \lfloor \frac{k}{2} \rfloor$, replace a_i by a_{i+1} and b_j by b_{j+1}. That is, we replace $E_t(k-t)$ by $C_t(k-t)$ for $t = 0, 1, \ldots,\ lfloor\frac{k}{2}\rfloor$.
2. Multiply each row in $C_t(k-1)$ on the right by b_1 giving $D_{t+1}(k+1-t)$.
3. This gives u_{k+2}.
   We note that the blocks $C_t(k-t)$ are sub-matrices of $B_t(k+1-t)$.

The validity of this construction can be seen from the companion matrix product expression as given in (73). Indeed, recall that

$$\begin{bmatrix} u_{k+2} \\ u_{k+1} \end{bmatrix} = (L_{k+1}L_{k-1}..L_1)\begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{and} \begin{bmatrix} v_{k+1} \\ v_k \end{bmatrix} = (L_kL_{k-1}..L_1)\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and observe that $L_2\begin{bmatrix} 1 \\ 0 \end{bmatrix}b_1 = L_2\begin{bmatrix} b_1 \\ 0 \end{bmatrix} = L_2L_1\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Alternatively, we can construct the u_k from v_k as follows:
(i) drop all words (i.e. rows) not ending in 1.
(ii) replace 1 by $\bar{1}$.
This gives u_k.

This may also be seen from the companion product representation.

Examples.

1. $v_3 = [2,1] + \bar{2} \longrightarrow [3,2] + \bar{3} \longrightarrow [3,2,\bar{1}] + [\bar{3},\bar{1}] = a_3a_2b_1 + b_3b_1 = u_4$.
2. $v_4 = [3,2,1] + \begin{bmatrix} 3 & \bar{2} \\ \bar{3} & 1 \end{bmatrix} \to [4,3,2] + \begin{bmatrix} 4 & \bar{3} \\ \bar{4} & 2 \end{bmatrix} \to [4,3,2,\bar{1}] + \begin{bmatrix} 4 & \bar{3} & \bar{1} \\ \bar{4} & 2 & \bar{1} \end{bmatrix} = D_1(4) + D_2(3) = (a_4a_3a_2 + a_4b_3 + b_4a_2)b_1 = u_5$.
3. $v_7 = E_0(6) + E_1(5) + E_2(4) + E_3(3) \to$

$$[7,6,5,4,3,2] + \begin{bmatrix} 7 & 6 & 5 & 4 & \bar{3} \\ 7 & 6 & 5 & \bar{4} & 2 \\ 7 & 6 & \bar{5} & 3 & 2 \\ 7 & \bar{6} & 4 & 3 & 2 \\ \bar{7} & 5 & 4 & 3 & 2 \end{bmatrix} + \begin{bmatrix} 7 & 6 & \bar{5} & \bar{3} \\ 7 & \bar{6} & 4 & \bar{3} \\ 7 & \bar{6} & \bar{4} & 2 \\ \bar{7} & 5 & 4 & \bar{3} \\ \bar{7} & 5 & \bar{4} & 2 \\ \bar{7} & \bar{5} & 3 & 2 \end{bmatrix} + [\bar{7},\bar{5},\bar{3}] = C_0(6) + C_1(5) + C_2(4) + C_3(3).$$

We next add in the terms b_1 to give

$$D_2(6) + D_3(5) + D_4(4) = [7,6,5,4,3,2,\bar{1}] + \begin{bmatrix} 7 & 6 & 5 & 4 & \bar{3} & \bar{1} \\ 7 & 6 & 5 & \bar{4} & 2 & \bar{1} \\ 7 & 6 & \bar{5} & 3 & 2 & \bar{1} \\ 7 & \bar{6} & 4 & 3 & 2 & \bar{1} \\ \bar{7} & 5 & 4 & 3 & 2 & \bar{1} \end{bmatrix} + \begin{bmatrix} 7 & 6 & \bar{5} & \bar{3} & \bar{1} \\ 7 & \bar{6} & 4 & \bar{3} & \bar{1} \\ 7 & \bar{6} & \bar{4} & 2 & \bar{1} \\ \bar{7} & 5 & 4 & \bar{3} & \bar{1} \\ \bar{7} & 5 & \bar{4} & 2 & \bar{1} \\ \bar{7} & \bar{5} & 3 & 2 & \bar{1} \end{bmatrix} + [\bar{7},\bar{5},\bar{3},\bar{1}] =$$

$a_7a_6a_5a_4a_3a_2b_1 + (a_7a_6a_5a_4b_3b_1 + a_7a_6a_5b_4a_2b_1 + a_7a_6b_5a_3a_2b_1 + a_7b_6a_4a_3a_2b_1 + b_7a_5a_4a_3a_2b_1) + (a_7a_6b_5b_3b_1 + a_7b_6a_4b_3b_1 + a_7b_6b_4a_2b_1 + b_7a_5a_4b_3b_1 + b_7a_5b_4a_2b_1 + b_7b_5a_3a_2b_1) + b_7b_5b_3b_1 = u_8$.
4. Using the alternative method we may construct u_6 as follows.

$$v_6 = [5,4,3,2,1] + \begin{bmatrix} 5 & 4 & 3 & \bar{2} \\ 5 & 4 & \bar{3} & 1 \\ 5 & \bar{4} & 2 & 1 \\ \bar{5} & 3 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 5 & \bar{4} & \bar{2} \\ \bar{5} & 3 & \bar{2} \\ \bar{5} & \bar{3} & 1 \end{bmatrix}.$$

We keep $[5,4,3,2,1] + \begin{bmatrix} 5 & 4 & \bar{3} & 1 \\ 5 & \bar{4} & 2 & 1 \\ \bar{5} & 3 & 2 & 1 \end{bmatrix} + [\bar{5},\bar{3},1]$ and replace 1 by $\bar{1}$ to give

$$[5,4,3,2,\bar{1}] + \begin{bmatrix} 5 & 4 & \bar{3} & \bar{1} \\ 5 & \bar{4} & 2 & \bar{1} \\ \bar{5} & 3 & 2 & \bar{1} \end{bmatrix} + [\bar{5},\bar{3},\bar{1}] = u_6$$

In general we have

$$u_{k+1} = D_1(k) + D_2(k-1) + D_3(k-2) + \cdots \tag{79}$$

Remarks
(i) An alternative approach using matrix products and continued fractions was given in [1].
(ii) It is not clear how the "master solution", as used in [2], and [5], comes into play in the non-constant case.

Right multiplication by λ gives the solution when $u_0 = \lambda$ and $u_1 = 0$.

Example $n = 5$ and $m = 3$.

Clearly $5 = 3 \cdot 1 + 2$ and $3 = 2 \cdot 1 + 1$, so that $r_0 = 5, r_1 = 3, r_2 = 2, r_3 = 1$ and $q_1 = 1, q_2 = 1$. Thus $N = 2$ and $r_3 = U_3 r_0 + V_3 r_1$.
Now recall that $V_3 = E_1(3) = [2, 1] + \overline{2} = a_2 a_1 + b_2$ in which $a_i = -q_i$ and $b_1 = 1$. So we get $V_3 = Q_2 Q_1 + 1$.
Also $V_2 = [1]$ so that $U_3 = [2, \overline{1}] = a_2 b_1 = -Q_2$. But we know that $Q_i = x^{r_{i+1}} G_{q_i}(x^{r_i})$ and $G_1(.) = 1$, so that we obtain
$U_3 = -Q_2 = -x^{r_3} G_{q_2}(x^{r_2}) = -x G_1(x^2) = -x$ and
$V_3 = 1 + Q_1 Q_2 = 1 + [x^{r_2} G_{q_1}(x^{r_1}][x^{r_3} G_{q_2}(x^{r_2}] = 1 + x^2 G_1(.) x G_1(.) = 1 + x^3$.
This gives the ideal equation

$$\boxed{-x(\mathbf{x^5 - 1}) + (1 + x^3)(\mathbf{x^3 - 1}) = \mathbf{x - 1}}. \tag{80}$$

We may use the **same** sequence of remainders (r_i), to obtain the corner sum iterates:

$\Gamma_5(x, c, y) = x^2 \Gamma_3(x, \Gamma_1(x^3, c, y^3), y) + \Gamma_2(x, c, y)$
$\Gamma_3(x, c, y) = x \Gamma_2(x, (\Gamma_1(x^2, c, y^2) + \Gamma_1(x, c, y) y^3$
$\Gamma_2(x, c, y) = x^0 \Gamma_1(x, \Gamma(x, c, y), y)$.

Since $\Gamma_1 = c$, and writing Γ_i for $\Gamma_i(x, c, y)$ we arrive at

$$\Gamma_1 y^5 = x^3 \Gamma_3 + \Gamma_3 y^3 - x \Gamma_5. \tag{81}$$

Example $n = 58$ and $m = 22$.

This time $58 = 2 \cdot 22 + 14$, $22 = 1 \cdot 14 + 8$, $14 = 1 \cdot 8 + 6$, $8 = 1 \cdot 6 + 2$ and $6 = 3 \cdot 2 + 0$. Thus $r_0 = 58, r_1 = 22, r_2 = 14, r_3 = 8, r_4 = 6, r_5 = 2$ and $q_1 = 2, q_2 = 1, q_3 = 1, q_4 = 1$. The terminal parameter is $N = 4$ and $R_5 = U_5 r_0 + V_5 r_1$.

Next we recall that $v_5 = E_0(4) + E_1(3) + E_2(2) = [4, 3, 2, 1] + ([4, 3, \overline{2}] + [4, \overline{3}, 1] + [\overline{4}, 2, 1]) + [\overline{4}, \overline{2}] = a_4 a_3 a_2 a_1 + (a_4 a_3 b_2 + a_4 b_3 a_1 + b_4 a_2 a_1) + b_4 b_2 = Q_4 Q_3 Q_2 Q_1 + Q_4 Q_3 + Q_4 Q_1 + Q_2 Q_1 + 1$.
On the other hand, because $v_4 = [3, 2, 1] + [3, \overline{2}] + [\overline{3}, 1]$ we see that $u_5 = [4, 3, 2, \overline{1}] + [4, \overline{3}, \overline{1}] + [\overline{4}, 2, \overline{1}] = -Q_4 Q_3 Q_2 - Q_4 - Q_2$.
Lastly, we compute $Q_1 = x^{14} G_2(x^{22}) = x^{14}(1 + x^{22})$ and $Q_2 = x^8 G_1(?) = x^8$ as well as $Q_3 = x^6$ and $Q_4 = x^2$. We then get
$U_5 = -(x^2 x^6 x^8 + x^2 + x^8) = -(x^{16} + x^8 + x^2)$ and
$V_5 = x^{30}(1 + x^{22}) + x^2 x^6 + x^2 x'^{14}(1 + x^{22}) + x^{22}(1 + x^{22}) + 1 = x^{52} + x^{44} + x^{38} + x^{30} + x^{22} + x^{16} + x^8 + 1$.

The ideal equation becomes

$$\boxed{-(x^{16} + x^8 + x^2)(\mathbf{x^{58} - 1}) + (x^{52} + x^{44} + x^{38} + x^{30} + x^{22} + x^{16} + x^8 + 1)(\mathbf{x^{22} - 1}) = \mathbf{x^2 - 1}}. \tag{82}$$

It goes without saying that we may divide by $x - 1$ and obtain the corresponding ideal equation for geometric sums

$$-(x^{16} + x^8 + x^2) G_{58}(x) + (x^{52} + x^{44} + x^{38} + x^{30} + x^{22} + x^{16} + x^8 + 1) G_{22}(x) = G_2(x)$$

Remark
As an example of the solution to the sandwich recurrence we return to the corner sums.

7. Sandwiched Corner Sums

As a special application let us examine the sandwich recurrence for the corner sums:

$$F_{k+1} = A_k F_k \alpha_k + B_k F_{k-1}\beta_k, \tag{83}$$

where $A_k = -x_{k+1}[x_k^{q_k-1},...,x_k,1] = -x^{r_{k+1}}[(x^{r_k})^{q_k-1}...,x^{r_k},1]$
$\alpha_k = \begin{bmatrix} 1 \\ y_k \\ \vdots \\ y_k^{q_k-1} \end{bmatrix} = \begin{bmatrix} 1 \\ y^{r_k} \\ \vdots \\ y^{r_k(q_k-1)} \end{bmatrix}$ and $\beta_k = y_{k-1}^{q_k-1} = y^{r_k(q_k-1)}$.

To illustrate the solution process we examine the case where $r_0 = 11$ and $r_1 = 8$.
We shall obtain the desired expansion for the "terminal" gcd corner sum $\Gamma_{r_{N+1}}$ by solving the sandwich recurrence and contrast it with the solution that is obtained by "back substitution". To follow Euclid, we shall do the latter first.

Let $r_0 = 11$ and $r_1 = 8$. Following Euclid we have $r_0 = 11 = 1 \cdot 8 + 2$, $r_1 = 8 = 2 \cdot 3 + 2$, $r_2 = 3 = 1 \cdot 2 + 1$, $r_3 = 2 = 2 \cdot 1 + 0$, $r_4 = 1, r_5 = 0$. Also, $q_1 = 1, q_2 = 2, q_3 = 1, q_4 = 2$.

Since $N + 1 = 4$, we shall need three steps of the iteration which are as follows:

(i) $\Gamma_3 \cdot y^8 = \Gamma_{11} - x^3\Gamma_8$
(ii) $\Gamma_2 \cdot y^6 = \Gamma_8 - x^2\Gamma_3(x, gC_2(x^3, c, y^3), y)$
(iii) $\Gamma_1 \cdot y^2 = \Gamma_3 - x\Gamma_2$.

To perform the back substitution we multiply the latter equation by y^6 giving $\Gamma_1 \cdot y^8 = (\Gamma_3 \cdot y^6) - x(\Gamma_2 \cdot y^6)$. Substituting from (ii) we get $\Gamma_1 \cdot y^8 = (\Gamma_3 \cdot y^6) - x[\Gamma_8 - x^2\Gamma_3(x, \Gamma_2(x^3, c, y^3), y)]$ and hence

$$\Gamma_1 \cdot y^8 = (\Gamma_3.y^6) - x\Gamma_8 + x^3\Gamma_3(x, \Gamma_2(x^3, c, y^3), y)].$$

We next multiply this by y^8 and use (i), to give

$$\Gamma_1 \cdot y^{16} = (\Gamma_3 \cdot y^8)y^6 - x\Gamma_8 \cdot y^8 + x^3\Gamma_3(x, \Gamma_2(x^3, c, y^3), y)y^8.$$

Substituting from (i) we arrive at:

$$\Gamma_1 \cdot y^{16} = (\Gamma_{11} - x^3\Gamma_8)y^6 - x\Gamma_8 \cdot y^8 + x^3[\Gamma_{11}(x, \Gamma_2(x^3, c, y^3), y) - x^3\Gamma_8(x, \Gamma_2(x^x, c, y^3), y)].$$

And hence we get

$$cy^{16} = [\Gamma_{11} \cdot y^6 + x^3[\Gamma_{11}(x, \Gamma_2(x^3, c, y^3), y)] - [x^3\Gamma_8 \cdot y^6 + x\Gamma_8 \cdot y^8 + x^6\Gamma_8(x, \Gamma_2(x^3, c, y^3), y)], \tag{84}$$

which may be checked by direct computation. We next have to express this just in terms of Γ_{11} and Γ_8. Recall that $\Gamma_{11}(x, \Gamma_2(x^3, c, y^3), y) = \Gamma_2(x^3, \Gamma_{11}, y^3)$ and that $\Gamma_8(x, \Gamma_2(x^3, c, y^3), y)] = \Gamma_2(x^3, \Gamma_8, y^3)] = \Gamma_{11}$. This shows that

$$cy^{16} = \Gamma_{11} \cdot y^6 + x^3[x^3\Gamma_{11} + \Gamma_{11}y^3] - x^3\Gamma_8 \cdot y^6 - x\Gamma_8 \cdot \mathbf{y}^8 - x^6[x^3\Gamma_8 + \Gamma_8 \cdot y^3] \tag{85}$$

or in compact form

$$cy^{16} = \Gamma_3(x^3, \Gamma_{11}, y^3) - \Gamma_4(x^3, \Gamma_8, y^3) + [\Gamma_8 \cdot y^6 - x\Gamma_8 \cdot y^8] \tag{86}$$

in which the last difference exactly equals $cy^{16} - x^8cy^8$. This gives

$$\boxed{\Gamma_3(x^3, \Gamma_{11}, y^3) = \Gamma_4(x^3, \Gamma_8, y^3) + x\Gamma_8 \cdot y^8.} \tag{87}$$

which we met earlier in (31).

Let us next use the sandwich recurrence to check this result.

In our example, $P_1 = 8, P_2 = y^{14}$ and $P_3 = y^{16}$. We must compute $F_4 = \Gamma_{r_4} P_3 = \Gamma_1.y^{16} = cy^{16}$. From the sandwich recurrence we have $F_4 = I + II$, where,

$$I = \{(A_3A_2A_1)F_1(\alpha_1\alpha_2\alpha_3) + (A_3B_2)F_1(\beta_2\alpha_3) + (B_3A_1)F_1(\alpha_1\beta_3)\}$$

and

$$II = (A_3A_2B_1)F_0(\beta_1\alpha_2\alpha_3) + (B_3B_1)F_0(\beta_1\beta_3).$$

We next compute the coefficients as:
(i) $A_1 = -x^3, A_2 = -x^2[x^3, 1], A_3 = -x$
(ii) $\alpha_1 = 1, \alpha_2 = \begin{bmatrix} 1 \\ y^3 \end{bmatrix}, \alpha_3 = 1$. (iii) $\beta_1 = 1, \beta_2 = y^8, \beta_3 = y^6$ and all $B_i = 1$. Hence $(A_3A_2A_1)F_1(\alpha_1\alpha_2\alpha_3) = (-x)\{-x^2[x^3, 1](x^3F_1.1)\begin{bmatrix} 1 \\ y^3 \end{bmatrix}\}1 = x^8F_1 + x^5F_1y^3$, in addition to $(A_3B_2)F_1(\beta_2\alpha_3) = (-x\{1F_1 \cdot y^8\} \cdot 1) = -xF_1y^8$ and $(B_3A_1)F_1(\alpha_1\beta_3)\} = x^3F_1y^6$.
For the second part we compute
(i) $(A_3A_2B_1)F_0(\beta_1\alpha_2\alpha_3) = -x\{-x^2[x^3, 1](1 \cdot F_0 \cdot 1)\begin{bmatrix} 1 \\ y^3 \end{bmatrix}\}1 = x^6F_0 + x^3F_0y^3 + F_0 \cdot y^6$.
(ii) $B_3B_1)F_0(\beta_1\beta_3) = 1(1 \cdot F_0 \cdot 1)y^6 = F_0 \cdot y^6$.
This shows that

$$F_4 = -x^9F_1 + x^6F_1y^3 - xF_1y^{11} - x^3F_1y^6 + x^6F_0 + x^3F_0y^3 + F_0y^6. \tag{88}$$

Lastly setting $F_4 = cy^{16}, F_1 = \Gamma_8$ and $F_0 = \Gamma_{11}$, again yields

$$cy^{16} = [x^6\Gamma_{11} + x^3\Gamma_{11}y^3 + \Gamma_{11} \cdot y^6] - [x^9\Gamma_8 - x^6\Gamma_8y^3 + x\Gamma_8y^8 + x^3\Gamma_8 \cdot y^6. \tag{89}$$

7.1. The third order case

The above block book-keeping method can be extended to higher order NC^2 recurrences. We shall restrict ourselves to the third order case.
When we have a third order NC^2 difference equation,

$$w_{k+1} = a_kw_k + b_kw_{k-1} + c_kw_{k-2}, \quad w_0 = \lambda,\ w_1 = \mu, w_2 = \nu, \tag{90}$$

we have three variable words in our solution and we must use **multinomial coefficients** to count the number of words. For the special case where $w_0 = 0,\ w_1 = 0, w_2 = 1$ we obtain the following links:

$$\begin{aligned} w_3 &= a_2, \\ w_4 &= a_3a_2 + b_3, \\ w_5 &= a_4a_3a_2 + (a_4b_3 + b_4a_2) + c_4, \\ w_6 &= a_5a_4a_3a_2 + (a_5a_4b_3 + a_5b_4a_2 + b_5a_3a_2) + (a_5c_4 + b_5b_3) + c_5a_2. \end{aligned}$$

We denote by $E^k_L(a, b, c)$ the set of all words ω of length L on a_i, b_i and c_i, where a, b and c denote de number of a_i, b_i and c_i, resp., in which the (positive) subscripts start in k and decrease from left to right and such that the subscript drops by 1 after an a_i, by two after a b_i and by three after a c_i.

We aim to show that

$$w_{k+1} = \sum_{\substack{a+2b+3c=k-1 \\ a+b+c=L \\ L=1}}^{k-1} E^k_L(a, b, c).$$

The proof will follow by (complete) induction.

Each word in

$$\sum_{\substack{a+2b+3c=k\\a+b+c=L\\L=1}}^{k} E_L^{k+1}(a,b,c) \tag{91}$$

either starts with a_{k+1}, with b_{k+1} or with c_{k+1}.

(i) Words that start in (91) with a_{k+1} are of the form

$$a_{k+1}\sum_{\substack{a+2b+3c=k-1\\a+b+c=L\\L=0}}^{k} E_L^{k}(a,b,c) = a_{k+1}\sum_{\substack{a+2b+3c=k-1\\a+b+c=L\\L=1}}^{k-1} E_L^{k}(a,b,c)$$

since there are no singleton words for $k \geq 6$.
By induction,

$$a_{k+1}\sum_{\substack{a+2b+3c=k-1\\a+b+c=L\\L=0}}^{k} E_L^{k}(a,b,c) = a_{k+1}\sum_{\substack{a+2b+3c=k-1\\a+b+c=L\\L=1}}^{k-1} E_L^{k}(a,b,c) = a_{k+1}w_{k+1}.$$

(ii) Words that start in (91) with b_{k+1} are of the form (recall the subscript drops by 2 after b_{k+1})

$$b_{k+1}\sum_{\substack{a+2b+3c=k-2\\a+b+c=L\\L=0}}^{k-1} E_L^{k-1}(a,b,c).$$

As in the previous case, the bounds for L can be rewritten since $L = 0$ cannot occur. Also, $L = k-1$ would mean $a+b+c = k-1$ and $a+2b+3c = k-2$, which in turn implies $b+2c = -1$. Therefore, L varies between 1 and $k-2$.
Therefore, such words are of the form

$$b_{k+1}\sum_{\substack{a+2b+3c=k-2\\a+b+c=L\\L=1}}^{k-2} E_L^{k-2}(a,b,c) = b_{k+1}w_k,$$

by the inductive step.

(iii) Words that start in (91) with c_{k+1} are of the form (recall the subscript drops by 3 after c_{k+1})

$$c_{k+1}\sum_{\substack{a+2b+3c=k-3\\a+b+c=L\\L=0}}^{k-1} E_L^{k-2}(a,b,c).$$

Again, the bounds for L can be rewritten since $L = 0$ can not occur. Also, $L > k-3$ would mean $a+b+c > k-2$ and $a+2b+3c = k-3$, which in turn implies $b+2c < 0$. Therefore, L varies between 1 and $k-3$.
Therefore, such words are of the form

$$c_{k+1}\sum_{\substack{a+2b+3c=k-3\\a+b+c=L\\L=1}}^{k-3} E_L^{k-2}(a,b,c) = c_{k+1}w_{k-1},$$

by the inductive step.

Using (i)–(iii) in (91), we obtain

$$\sum_{\substack{a+2b+3c=k\\ a+b+c=L\\ L=1}}^{k} E_L^{k+1}(a,b,c) = a_{k+1}w_{k+1} + b_{k+1}w_k + c_{k+1}w_{k-1} = w_{k+2}.$$

7.2. Examples

As a first example, take $k = 6$ so that we compute w_7. We will need $E_L^6(a,b,c)$, for $L = 1,\dots 5$. The possible sets of words are $E_5^6(5,0,0), E_4^6(3,1,0), E_3^6(2,0,1), E_3^6(1,2,0)$ and $E_2^6(0,1,1)$ to give

$$w_7 = E_5^6(5,0,0) + E_4^6(3,1,0) + E_3^6(2,0,1) + E_3^6(1,2,0) + E_2^6(0,1,1).$$

Again, we will simplify the notation by writing j for a_j, $\bar{j}$ for b_j and $\bar{\bar{j}}$ for c_j. We obtain

$$E_5^6(5,0,0) = \begin{bmatrix} 6 & 5 & 4 & 3 & 2 \end{bmatrix}, E_4^6(3,1,0) = \begin{bmatrix} 6 & 5 & 4 & \bar{3} \\ 6 & 5 & \bar{4} & 2 \\ 6 & \bar{5} & 3 & 2 \\ \bar{6} & 4 & 3 & 2 \end{bmatrix}, E_3^6(2,0,1) = \begin{bmatrix} 6 & 5 & \bar{\bar{4}} \\ 6 & \bar{\bar{5}} & 2 \\ \bar{\bar{6}} & 3 & 2 \end{bmatrix}$$

and

$$E_3^6(1,2,0) = \begin{bmatrix} 6 & \bar{5} & \bar{3} \\ \bar{6} & 4 & \bar{3} \\ \bar{6} & \bar{4} & 2 \end{bmatrix}, E_2^6(0,1,1) = \begin{bmatrix} \bar{6} & \bar{\bar{4}} \\ \bar{\bar{6}} & \bar{3} \end{bmatrix}.$$

The numbers of terms are

a	b	c	
5	–	–	$\binom{5}{5,0,0}$
3	1		$\binom{4}{3,1,0}$
1	2	–	$\binom{3}{2,0,1}$
2	–	1	$\binom{3}{1,2,0}$
–	1	1	$\binom{2}{0,1,1}$

.

Parallel to (w_k), we have the recurrence relations

$$u_{k+1} = a_k u_k + b_k u_{k-1} + c_k u_{k-2}, u_0 = 1, u_1 = u_2 = 0$$

and

$$v_{k+1} = a_k v_k + b_k v_{k-1} + c_k v_{k-2}, v_0 = 0, v_1 = 1, v_2 = 0.$$

Let us compute, for future purpose, the first terms.

$$\begin{aligned} u_3 &= c_2 \\ u_4 &= a_3c_2 \\ u_5 &= a_4a_3c_2 + b_4c_2 \\ u_6 &= a_5a_4a_3c_2 + a_5b_4c_2 + b_5a_3c_2 + c_5c_2 \end{aligned}$$

and

$$\begin{aligned} v_3 &= b_2 \\ v_4 &= a_3b_2 + c_3 \\ v_5 &= a_4a_3b_2 + a_4c_3 + b_4b_2 \\ v_6 &= a_5a_4a_3b_2 + a_5a_4c_3 + a_5b_4b_2 + b_5a_3b_2 + b_5c_3 + c_5b_2 \end{aligned}$$

In order to give an algorithm that would allow us to present the terms of (u_k) and (v_k), we note that

$$\begin{bmatrix} w_{k+1} \\ w_k \\ w_{k-1} \end{bmatrix} = L_k L_{k-1} \cdots L_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

where

$$L_k = \begin{bmatrix} a_k & b_k & c_k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{92}$$

(I) The sequence (u_k).
Since $\begin{bmatrix} u_{k+1} \\ u_k \\ u_{k-1} \end{bmatrix} = L_k L_{k-1} \cdots L_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, where the matrices L_i are as in (92) and $L_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} c_2$ then we get

$$\begin{bmatrix} u_{k+1} \\ u_k \\ u_{k-1} \end{bmatrix} = \prod_{i=3}^{k} L_i \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} c_2.$$

Note that $\prod_{i=3}^{k} L_i \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ gives essentially (w_k), with a index shift by one. This allows us to obtain u_{k+1} as follows:

1. Obtain w_k;
2. Replace x_i by x_{i+1}, where $x \in \{a, b, c\}$;
3. Multiply every summand on the right by c_2.

As an example, let us apply the above algorithm to compute u_6.

1. We are given $w_5 = a_4a_3a_2 + a_4b_3 + b_4a_2 + c_4$;
2. We change the indices to get $a_5a_4a_3 + a_5b_4 + b_5a_3 + c_5$;
3. Multiplying on the right by c_2, we obtain

$$u_6 = a_5a_4a_3c_2 + a_5b_4c_2 + b_5a_3c_2 + c_5c_2.$$

(II) The sequence (v_k).
From

$$\begin{aligned} \begin{bmatrix} v_{k+1} \\ v_k \\ v_{k-1} \end{bmatrix} &= L_k L_{k-1} \cdots L_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ &= L_k \cdots L_3 \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} b_2 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) \\ &= L_k \cdots L_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} b_2 + L_k \cdots L_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \end{aligned}$$

which shows that we may compute v_{k+1} by simultaneously using w_k and u_k as follows:

1. (a) obtain w_k
   (b) replace x_i by x_{i+1}, where $x \in \{a, b, c\}$,
   (c) multiply every summand on the right by b_2.

2. (a) obtain u_k
   (b) replace x_i by x_{i+1}, where $x \in \{a, b, c\}$
3. add all the expressions obtained.

Let us give an example by computing v_6:

1. We use w_5 to give $a_5a_4a_3b_2 + a_5b_4b_2 + b_5a_3b_2 + c_5b_2$
2. We use u_5 to give $a_5a_4c_3 + b_5c_3$
3. $v_6 = a_5a_4a_3b_2 + a_5b_4b_2 + b_5a_3b_2 + c_5b_2 + a_5a_4c_3 + b_5c_3$

Our second example uses the computer program SageMath [15] to check the solutions to the NC^2 three term Recurrence relation. We shall only use the initial conditions $w_0 = 0, w_1 = 0, w_1 = 1$, from which the other two can be derived. The code is available at http://w3.math.uminho.pt/ pedro/Telescoping/telescoping.html. For the case where $k = 15$, all words of length 7 must be of the form

$\#(a_i)$	$\#(b_j)$	$\#(c_k)$	*number of words*
0	7	0	$\binom{7}{0,7,0} = 1$
1	5	1	$\binom{7}{1,5,1} = 42$
2	3	2	$\binom{7}{2,3,2} = 210$
3	1	3	$\binom{7}{3,1,3} = 140$

The set of the 42 possible words with 1 a, 5 b's and 1 c is as follows

$$\begin{gathered}
a_{15}b_{14}b_{12}b_{10}b_8b_6c_4, a_{15}b_{14}b_{12}b_{10}b_8c_6b_3, a_{15}b_{14}b_{12}b_{10}c_8b_5b_3,\\
a_{15}b_{14}b_{12}c_{10}b_7b_5b_3, a_{15}b_{14}c_{12}b_9b_7b_5b_3, a_{15}c_{14}b_{11}b_9b_7b_5b_3,\\
b_{15}a_{13}b_{12}b_{10}b_8b_6c_4, b_{15}a_{13}b_{12}b_{10}b_8c_6b_3, b_{15}a_{13}b_{12}b_{10}c_8b_5b_3,\\
b_{15}a_{13}b_{12}c_{10}b_7b_5b_3, b_{15}a_{13}c_{12}b_9b_7b_5b_3, b_{15}b_{13}a_{11}b_{10}b_8b_6c_4,\\
b_{15}b_{13}a_{11}b_{10}b_8c_6b_3, b_{15}b_{13}a_{11}b_{10}c_8b_5b_3, b_{15}b_{13}a_{11}c_{10}b_7b_5b_3,\\
b_{15}b_{13}b_{11}a_9b_8b_6c_4, b_{15}b_{13}b_{11}a_9b_8c_6b_3, b_{15}b_{13}b_{11}a_9c_8b_5b_3,\\
b_{15}b_{13}b_{11}b_9a_7b_6c_4, b_{15}b_{13}b_{11}b_9a_7c_6b_3, b_{15}b_{13}b_{11}b_9b_7a_5c_4,\\
b_{15}b_{13}b_{11}b_9b_7c_5a_2, b_{15}b_{13}b_{11}b_9c_7a_4b_3, b_{15}b_{13}b_{11}b_9c_7b_4a_2,\\
b_{15}b_{13}b_{11}c_9a_6b_5b_3, b_{15}b_{13}b_{11}c_9b_6a_4b_3, b_{15}b_{13}b_{11}c_9b_6b_4a_2,\\
b_{15}b_{13}c_{11}a_8b_7b_5b_3, b_{15}b_{13}c_{11}b_8a_6b_5b_3, b_{15}b_{13}c_{11}b_8b_6a_4b_3,\\
b_{15}b_{13}c_{11}b_8b_6b_4a_2, b_{15}c_{13}a_{10}b_9b_7b_5b_3, b_{15}c_{13}b_{10}a_8b_7b_5b_3,\\
b_{15}c_{13}b_{10}b_8a_6b_5b_3, b_{15}c_{13}b_{10}b_8b_6a_4b_3, b_{15}c_{13}b_{10}b_8b_6b_4a_2,\\
c_{15}a_{12}b_{11}b_9b_7b_5b_3, c_{15}b_{12}a_{10}b_9b_7b_5b_3, c_{15}b_{12}b_{10}a_8b_7b_5b_3,\\
c_{15}b_{12}b_{10}b_8a_6b_5b_3, c_{15}b_{12}b_{10}b_8b_6a_4b_3, c_{15}b_{12}b_{10}b_8b_6b_4a_2.
\end{gathered}$$

8. Concluding remarks and questions

1. We may extend the two-sided recurrence to three or more terms.
2. We note that recurrence relation for corner sums correspond to annihilating polynomials for 2×2 matrices M, which leads to Division Algorithms.
3. How can we relate annihilating polynomials to the corner sums?
4. Can we find other uses or applications of GPs?
5. Are there any other classes of objects such as matroids for which we can apply the telescoping tricks, and mimic Euclid's construction?
6. How do Hankel matrices telescope?
7. Are there any other relations such as the switching identity $G_m(x^n)/G_m(x) = G_n(x^m)/G_n(x)$, using three powers $x^{mnk} - 1$?

Acknowledgements

The authors thank the anonymous referee for his/her valuable corrections.

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