Filomat 34:14 (2020), 4847–4860
https://doi.org/10.2298/FIL2014847M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Characterizations of Special Clean Elements and Applications

Xavier Mary[a], Pedro Patrício[b]

[a] *Laboratoire Modal'X, Université Paris Nanterre, France*
[b] *CMAT – Centro de Matemática and Departamento de Matemática, Universidade do Minho, Portugal*

Abstract. We prove that special clean decompositions of a given element of a ring are in one-to-one correspondence with the set of solutions of a simple equation in a corner ring. We then derive "constructive" proofs that in many rings, regular elements are special clean by solving this equation in specific cases. Other applications, such as uniqueness of decompositions, are given. Many examples of special clean decompositions of 2-2 matrices found by this methodology are also presented.

1. Introduction

In this paper, all rings are assumed unital, R denotes a (unital) ring, $J(R)$ its Jacobson radical, $E(R)$ its set of idempotents, and $U(R)$ its set of units. If $e \in E(R)$, then $\bar{e} = 1 - e$ denotes its complementary idempotent. We will use (Peirce) corner rings eRe ($e \in E(R)$) and the Peirce decomposition, which exhibits R as a Morita context ring (given by the two rings eRe and $\bar{e}R\bar{e}$, the bimodules $eR\bar{e}$ and $\bar{e}Re$, and multiplication as bimodule homomorphisms). Given $e \in E(R)$ the Peirce decomposition (equivalently the Peirce isomorphism) relative to the idempotent e is given by the canonical isomorphism between the ring R and the matrix ring $\begin{pmatrix} eRe & eR\bar{e} \\ \bar{e}Re & \bar{e}R\bar{e} \end{pmatrix}$. The isomorphism sends an element $a = \underbrace{eae}_{a_1} + \underbrace{ea\bar{e}}_{a_2} + \underbrace{\bar{e}ae}_{a_3} + \underbrace{\bar{e}a\bar{e}}_{a_4}$ to $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ (We will use upper letters for images under the isomorphism, a.k.a elements written in matrix form).

We say a is *(von Neumann) regular* (resp. *unit-regular*) if $a \in aRa$ (resp. $a \in aU(R)a$). By a result of Hartwig and Luh [16], a is unit-regular if and only if it is the product of an idempotent and a unit. A particular solution to $axa = a$ is called a *Von Neumann (or inner) inverse* of a. A solution to $xax = a$ is called a *weak (or outer) inverse*. Finally, an element that satisfies $axa = a$ and $xax = x$ is called an *inverse (or reflexive inverse, or relative inverse)* of a. A commuting inverse, if it exists, is unique and denoted by $a^{\#}$. It is the the unique solution to:

$$ax = xa, axa = a, xax = x.$$

2010 *Mathematics Subject Classification*. 16U99, 16E50, 15A09
Keywords. (special) clean elements; unit-regular elements; strongly regular (group) elements; stable range 1.
Received: 24 February 2020; Accepted: 01 May 2020
Communicated by Dijana Mosić
The research at CMAT was partially financed by Portuguese Funds through FCT (Fundação para a Ciência e a Tecnologia) within the Projects UIDB/00013/2020 and UIDP/00013/2020. It has also been conducted as part of the project Labex MME-DII (ANR11-LBX-0023-01).
Email addresses: xavier.mary@parisnanterre.fr (Xavier Mary), pedro@math.uminho.pt (Pedro Patrício)

It is usually called the *group inverse* of a. We let $R^{\#}$ denote the set of group invertible elements. These are exactly the *strongly regular* elements a such that $a \in a^2R \cap Ra^2$.

Finally, we recall the following results about rings with stable range 1 and unit-regular rings.

A ring R has *stable range 1* (and we note $sr(R) = 1$) if for all $a, b \in R$, $aR + bR = R$ implies that $(a + bc)R = R$ for some $c \in R$. Vaserstein proved in [33] that the property is actually left-right symmetric, and in [34] that right invertibility implies left invertibility in stable range 1 rings. Thus we will rather use the implication: for all $a, b \in R$, $ax + by \in U(R)$ for some $x, y \in R$ implies that $a + bc \in U(R)$ for some $c \in R$. In [21] Definition 3.1, this definition is localized as follows: an element $a \in R$ has right (resp. left) stable range 1, and we note $sr_r(a) = 1$ (resp. $sr_l(a) = 1$) if for all $b \in R$, $ax + by \in U(R)$ (for some $x, y \in R$) implies that $a + bc \in U(R)$ (for some $c \in R$) (resp. $xa + yb \in U(R)$ implies that $a + cb \in U(R)$ for some $c \in R$).

By [34] Theorem 2.8, the property of having stable range 1 is inherited by corner rings (and the proof is ring theoretical and constructive). So does the property of being unit-regular by [17], Proposition 8: Corner rings of a unit-regular rings are unit-regular. A proof based on module cancellation is due to Ehrlich[12] and Handelman[15]. The first ring theoritical (and elementwise) proof is probably due to Kaplansky, as explained in [17]. The link between unit-regularity in a ring and unit-regularity in a corner ring is then precisely studied in [26].

Many classes of rings have stable range 1: for instance, semi-local rings, right self-injective rings that are Dedekind finite or 0-dimensional commutative rings have stable range 1. Also unit-regular rings have stable range 1. More precisely, a result of Fuchs[13] and Kaplansky[19] (see also [14] Proposition 4.12) characterizes unit-regular rings as regular rings with stable range 1. Once again, an elementwise version is given in [21], where the authors prove (Theorem 3.5) by purely ring theoretical arguments that a regular element a has left (resp. right) stable range 1 if and only if it is unit-regular.

In [4], Camillo and Khurana proved that unit-regular rings are clean. In [20] the authors give an example of a unit-regular element that was not clean, thus showing that the link between the two notions was intricate (on the other hand, clean elements need not be regular). Many other examples were then given by Wu et. al. [37].

Actually, Camillo and Khuruna proved in [4] that elements of a unit-regular ring hold a stronger form of cleanness called special cleanness in [1]. Special cleanness is indeed both a refinement of cleanness but also of unit-regularity, as shown in [28] Theorem 4.1: a is special clean with decomposition $a = \bar{e} + u, aR \cap \bar{e}R = \{0\}$ iff it satisfies $a = \bar{e} + u = au^{-1}a$.

The paper is organized as follows. Section 2 is devoted the our main theorem, that gives an equational characterization of special cleanness for unit-regular elements. Then local (Section 3) and global conditions (Section 4) are given wich ensure special cleanness of regular elements. We then consider unique special decompositions in Section 5. In Section 6, we study further the class of rings with skew corner rings in the Jacoson radical ($eR\bar{e} \subset J(R)(\forall e \in E(R))$). While in Section 4 it was proved that regular elements in such rings are special clean, we prove that they are indeed strongly regular. Finally Section 7 is devoted to other examples, notably of unimodular row matrices.

2. The main theorem: constructing all special clean decompostions of unit-regular elements

Theorem 2.1. *Let $a \in R$ be unit-regular with inverse $v^{-1} \in U(R)$. Pose $f = av^{-1}$ and let v have Peirce decomposition $V = \begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix}$. Define function $\varphi : fR\bar{f} \times \bar{f}Rf \to \bar{f}R\bar{f}$ by $\varphi : (x, y) \mapsto yv_1x + yv_2 + v_3x + v_4$.*
Then the set of special clean decompositions of a is in one-to-one correspondence with the set of solutions to $\varphi(x, y) \in U(\bar{f}R\bar{f})$. Precisely, any invertible u such that $a - u$ is idempotent and $au^{-1}a = a$ has Peirce decomposition

$$U^{-1} = V^{-1}\begin{pmatrix} 1 & -x \\ -y & -\varphi(x, y) + yx \end{pmatrix} = V^{-1}\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \varphi(x, y) \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

with $\varphi(x,y) \in U(\bar{f}R\bar{f})$, and conversely.
In this case, the idempotent $\bar{e} = a - u$ has the form

$$\bar{E} = \begin{pmatrix} 0 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \varphi(x,y)^{-1} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ y & 1 \end{pmatrix} V.$$

Proof. Our methodoly goes as follows.

1. First step: Construct from v **all** unit-inverses of a;
2. Second step: Among these unit-inverses u^{-1}, find those such that $a - u$ is idempotent (if any).

So let $a \in R$ be unit-regular with inverse v^{-1}, and pose $f = av^{-1}$. By a result of Hartwig and Luh [16], the set $\mathcal{UI}_a$ of unit-inverses of a has the following form:

$$\mathcal{UI}_a = \{u^{-1} = v^{-1}(f + k - fkf) | k \in R, f + k - fkf \text{ is invertible}\}.$$

In matrix form, we get that $U^{-1} = V^{-1}\begin{pmatrix} 1 & -x \\ -y & z \end{pmatrix}$, where by the Schur complement theorem $z - yx = \zeta$ is a unit in the corner ring $\bar{f}R\bar{f}$. Hence equivalently, unit-inverses are of the form $U^{-1} = V^{-1}\begin{pmatrix} 1 & -x \\ -y & \zeta + yx \end{pmatrix}$ with ζ a unit (in $\bar{f}R\bar{f}$). So let U^{-1} be of this form. Then

$$\begin{aligned} U &= \begin{pmatrix} 1 & -x \\ -y & \zeta + yx \end{pmatrix}^{-1} V \\ &= \begin{pmatrix} 1 + x\zeta^{-1}y & +x\zeta^{-1} \\ \zeta^{-1}y & \zeta^{-1} \end{pmatrix} \begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix} \\ &= \begin{pmatrix} v_1 + x\zeta^{-1}yv_1 + x\zeta^{-1}v_3 & v_2 + x\zeta^{-1}yv_2 + x\zeta^{-1}v_4 \\ \zeta^{-1}yv_1 + \zeta^{-1}v_3 & \zeta^{-1}yv_2 + \zeta^{-1}v_4 \end{pmatrix} \end{aligned}$$

As before $fa = a = fv$ gives $A = \begin{pmatrix} v_1 & v_2 \\ 0 & 0 \end{pmatrix}$. Pose $\bar{e} = a - u$. Then

$$\bar{E} = \begin{pmatrix} -x\zeta^{-1}yv_1 - x\zeta^{-1}v_3 & -x\zeta^{-1}yv_2 - x\zeta^{-1}v_4 \\ -\zeta^{-1}yv_1 - \zeta^{-1}v_3 & -\zeta^{-1}yv_2 - \zeta^{-1}v_4 \end{pmatrix}.$$

Put $P = \begin{pmatrix} 0 & x \\ 0 & 1 \end{pmatrix}$, $Z = \begin{pmatrix} 0 & 0 \\ 0 & -\zeta^{-1} \end{pmatrix}$, $Q = \begin{pmatrix} 0 & 0 \\ yv_1 + v_3 & yv_2 + v_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ y & 1 \end{pmatrix} V$ and $Z^\# = \begin{pmatrix} 0 & 0 \\ 0 & -\zeta \end{pmatrix}$.
Then $\bar{E} = PZQ$ and $QP = \begin{pmatrix} 0 & 0 \\ 0 & \varphi(x,y) \end{pmatrix}$, so that $\bar{E}^2 - \bar{E} = (PZQ)^2 - PZQ = P(ZQPZ - Z)Q = PZ(QP - Z^\#)ZQ$.

Let $M = \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}$. Then $MP = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $QV^{-1} = \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$. Thus $\bar{E}^2 - \bar{E} = 0$ iff $MPZ(QP - Z^\#)ZQV^{-1} = 0$ since M and V are invertible, which in turn is equivalent with

$$\begin{pmatrix} 0 & 0 \\ 0 & -\zeta^{-1}\varphi(x,y) - 1 \end{pmatrix} \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -\zeta^{-1}\varphi(x,y) - 1 \end{pmatrix} = 0.$$

Finally $\varphi(x,y) = -\zeta$ is a necessary and sufficient condition for $\bar{e} = a - u$ to be idempotent, or equivalently since u^{-1} is an inner inverse of a, for a to be special clean with $a = \bar{e} + u = au^{-1}a$. □

Function φ can also be expressed as the matrix product $\varphi(x,y) = \begin{bmatrix} y & 1 \end{bmatrix} V \begin{bmatrix} x \\ 1 \end{bmatrix}$, or equivalently $\varphi(x,y) = (y + \bar{f})v(x + \bar{f})$ (for $x \in fR\bar{f}$ and $y \in \bar{f}Rf$).

Example 2.2. *Let D be a division ring and $A \in \mathcal{M}_2(D)$ of rank 1. Then A can be decomposed in Smith's normal form as*

$$A = P\begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} Q = P\left(\begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} QP\right) P^{-1}$$

for some invertible P, Q and non-zero $\alpha \in D$. Thus A is similar to the matrix $A' = \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} QP$ which has the form $A' = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ (for some $a, b \in D$) and special clean decompositions of A are in one-to-one correspondence with special clean decompositions of A'. We now find the special clean decompositions of A'.
Asssume first that $a \neq 0$ and pose $V^{-1} = \begin{pmatrix} a^{-1} & -a^{-1}b \\ 0 & 1 \end{pmatrix}$. Then $A'V^{-1}A' = A'$ and $F = A'V^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. By Theorem 2.1, the special clean decompositions of A' (hence of A) are in one-to-correspondence with $D^2 \backslash H_{a,b}$, where $H_{a,b}$ is the hyperbola of equation $yax + yb + 1 = 0$ (since $V = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ and any non-zero element is a unit in D).
Second, assume that $a = 0$. Then $b \neq 0$ since $rank(A) = 1$ and posing $V^{-1} = \begin{pmatrix} 0 & 1 \\ b^{-1} & 0 \end{pmatrix}$ then again $F = A'V^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Once again by Theorem 2.1, the special clean decompositions of A are in one-to-correspondence with $D^2 \backslash L_b$, where L_b is the line of equation $yb + x = 0$ ($V = \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}$).

3. Sufficiency for special cleanness: local conditions

Unless otherwise stated, in the following $a \in R$ is a given unit-regular element with unit inverse v^{-1}, $f = av^{-1}$ and we use the notations of Theorem 2.1.

Our next two results build special clean decompositions from specific units in $\bar{f}R\bar{f}$ (resp. fRf). They are direct consequences of Theorem 2.1.

Lemma 3.1. *If $yv_2 + v_4$ (resp. $v_3x + v_4$) is a unit in $\bar{f}R\bar{f}$ for some $y \in \bar{f}Rf$ (resp. $x \in fR\bar{f}$), then a is special clean.*

Proof. Consider the first case and assume that $yv_2 + v_4$ is a unit. Then $\varphi(0, y)$ is a unit, and by Theorem 2.1 a is special clean. The second case is dual. □

Lemma 3.2. *Pose $V^{-1} = \begin{pmatrix} \mu_1 & \mu_2 \\ \mu_3 & \mu_4 \end{pmatrix}$. If there exists $y \in \bar{f}Rf$ (resp. $x \in fR\bar{f}$) such that $\mu_1 - \mu_2 y$ (resp. $\mu_1 - x\mu_3$) is a unit in fRf, then a is special clean.*

Proof. Assume that $\mu_1 - \mu_2 y$ is a unit (in fRf) and pose $\tilde{Y} = \begin{pmatrix} \mu_1 & \mu_2 \\ y & 1 \end{pmatrix}$. Then $Y/1 = \mu_1 - \mu_2 y$ is the Schur complement of 1 and is invertible, hence $\tilde{Y}$ is invertible. Thus $\tilde{Y}V = \begin{pmatrix} 1 & 0 \\ yv_1 + v_3 & yv_2 + v_4 \end{pmatrix}$ is invertible, and $\tilde{Y}V/1 = yv_2 + v_4 = \varphi(0, y)$ is invertible in $\bar{f}R\bar{f}$. Finally, as $\varphi(0, y)$ is a unit, then by Theorem 2.1 a is special clean.
The second case is dual. □

Then we deduce:

Corollary 3.3. *If $sr_l(v_4) = 1$ in $\bar{f}R\bar{f}$ (in particular if v_4 is unit-regular in $\bar{f}R\bar{f}$), then a is special clean.*

Proof. Assume that $sr_l(v_4) = 1$. As $V^{-1}V = I$ then $\mu_3 v_2 + \mu_4 v_4 = \bar{f}$. By left stable range 1, there exists t such that $t\mu_3 v_2 + v_4 \in U(\bar{f}R\bar{f})$. Pose $y = t\mu_3$. Then $yv_2 + v_4$ is a unit in $\bar{f}R\bar{f}$ and by Lemma 3.2 a is special clean. □

Corollary 3.4. *If $sr_r(\mu_4) = 1$ in fRf (in particular if μ_4 is unit-regular in fRf), then a is special clean.*

Proof. Assume that $sr_r(\mu_4) = 1$. As $V^{-1}V = I$ then $\mu_1 v_1 + \mu_2 v_3 = f$. By right stable range 1, there exists t such that $\mu_1 + \mu_2 v_3 t \in U(fRf)$. Pose $y = -v_3 t$. Then $\mu_1 - \mu_2 y$ is a unit in fRf and by Lemma 3.2 a is special clean. □

One may wonder what happens if one considers the element v_1 instead of v_4 or μ_1. In this case, we are able to prove special cleaness under the assumption of unit-regularity of v_1 in fRf. But actually more can be said in this specific case. Indeed, in [31] Theorem 3.14., the authors prove special cleanness of a under the hypothesis that a is merely regular, but also unit-regular in a special corner ring. Moreover, in this case they prove that a^2 is also unit-regular. As an application of the main theorem, we recover Nielsen and Ster's result (in a slightly different form). Note that our proof relies heavily on their computations in the first step (construction of the unit-inverse w^{-1}). The case v_1 unit-regular will then be deduced as a special case.

Proposition 3.5. *Let $a \in R$ be regular and b such that $aba = a$. Pose $f = ab \in E(R)$. If the element $a^2b = faf$ is unit-regular in fRf, then a and a^2 are special clean.*

Proof. We use the Peirce decomposition relative to f. Let ν be a unit-inverse of $a_1 = a^2b = faf$ in fRf. Then using the associativity of the product in

$$\begin{pmatrix} 1-a_1\nu & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a_1 & a_2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} b_1 & 0 \\ b_3 & 0 \end{pmatrix}$$

gives $(1 - a_1\nu) = (1 - a_1\nu)a_2b_3 = (1 - a_1\nu)a_2b_3(1 - a_1\nu)$. Pose

$$W^{-1} = \begin{pmatrix} \nu & \nu a_2 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -b_3(1-a_1\nu) & 1 \end{pmatrix}\begin{pmatrix} 1 & (1-a_1\nu)a_2 \\ 0 & 1 \end{pmatrix}$$

which is invertible as a product of units, then it holds that $AW^{-1}A = A$ and $AW^{-1} = F$. The special clean equation relative to

$$W = \begin{pmatrix} a_1 & a_2 \\ b_3(\nu^{-1}-a_1) & b_3(1-a_1\nu)a_2 - 1 \end{pmatrix}$$

is

$$\varphi(x, y) = ya_1x + ya_2 + b_3(\nu^{-1} - a_1)x + b_3(1 - a_1\nu)a_2 - 1.$$

Pose $x = -\nu a_2$ and $y = 0$ (resp. $y = -b_3(1 - a_1\nu)$ and $x = 0$). Then $\varphi(x, 0) = -1$ (resp. $\varphi(0, y) = -1$) and a is special clean.

We consider now a^2 and pose $c = b\nu$. As $(a^2b)\nu(a^2b) = a^2b$ then $a^2ca^2 = a^2b\nu a^2 = a^2b\nu a^2ba = a^2ba = a^2$ and a^2 is regular. Pose $z^{-1} = w^{-1}(\nu + \bar{f})$ which is invertible as a product of units. Then $a^2z^{-1}a^2 = a^2w^{-1}(\nu + \bar{f})a^2 = a^2w^{-1}\nu a^2 = af\nu a^2 = a^2b\nu a^2ba = a^2$ since $\bar{f}a = \bar{f}aba = \bar{f}fa = 0$ and $aw^{-1} = f = ab$, and a^2 is unit-regular with unit-inverse z^{-1}. In matrix form,

$$Z = \begin{pmatrix} \nu^{-1} & 0 \\ 0 & 1 \end{pmatrix} W = \begin{pmatrix} \nu^{-1}a_1 & \nu^{-1}a_2 \\ b_3(\nu^{-1}-a_1) & b_3(1-a_1\nu)a_2 - 1 \end{pmatrix}.$$

Since the last line is equal to the last line of W, $z_3x + z_4 = w_3x + w_4 = -1$ for $x = -\nu a_2$, and a^2 is special clean. □

In matrix form, $A = \bar{E} + U = AU^{-1}A$ with

$$U = \begin{pmatrix} a_1 - \nu a_2b_3(\nu^{-1}-a_1) & a_2 - \nu a_2\,(b_3(1-a_1\nu)a_2 - 1) \\ b_3(\nu^{-1}-a_1) & b_3(1-a_1\nu)a_2 - 1 \end{pmatrix}$$

and

$$\bar{E} = \begin{pmatrix} \nu a_2 b_3(\nu^{-1} - a_1) & \nu a_2\,(b_3(1 - a_1\nu)a_2 - 1) \\ -b_3(\nu^{-1} - a_1) & -b_3(1 - a_1\nu)a_2 + 1 \end{pmatrix}.$$

Not every special clean element is of this form. Consider for instance the integer matrix $A = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \in \mathcal{M}_2(\mathbb{Z})$. Then $a_1 = 2$ is not regular in $\mathbb{Z}$ but A is special clean. Indeed $AV^{-1}A = A$ for $V^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}$, $AV^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and since $V = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$ the special clean equation is $\varphi(x, y) = 2yx+y+x \in U(R)$, which admits for instance the solution $x = -1, y = 0$. Computations give $U = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $AU^{-1}A = A$, $A - U = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} \in E(R)$ as required.

If we start directly with a unit-regular with inverse v^{-1} as before, then letting $b = v^{-1}$ and $f = ab = av^{-1}$ we get $a^2b = faf = a_1 = v_1$ and we deduce directly:

Corollary 3.6. *If v_1 is unit-regular in fRf then a and a^2 are special clean.*

4. Sufficiency for special cleanness: global conditions

By its very definition, function φ relies on the two skew corner rings $fR\bar{f}$ and $\bar{f}Rf$, and the corner ring $\bar{f}R\bar{f}$. Thereofore, its seems reasonable to expect that global conditions on the ring inherited by corner rings, or global conditions on (skew) corner rings will help solve the equation $\varphi(x, y) \in U(\bar{f}R\bar{f})$. This is indeed the case, as shown below.

Unless otherwise stated, in the following $a \in R$ is a given unit-regular element with unit inverse v^{-1}, $f = av^{-1}$ and we use the notations of Theorem 2.1.

4.1. Special cleanness under stable range 1

A direct application of Corollaries 3.3 and 3.4 gives:

Corollary 4.1. *Let a be unit-regular with unit-inverse v^{-1}, and pose $f = av^{-1}$. If $sr(\bar{f}R\bar{f}) = 1$ or $sr(fRf) = 1$ then a is special clean.*

As recorded in the introduction, the classical result of Camillo and Khuruna ([4] Theorem 1) states that a ring is unit-regular if and only if it is special clean. Also, it is known ([14] Proposition 4.12) that unit-regular ring have stable range 1. In [5], Theorem 3 Camillo and Yu proved that an exchange ring has stable range 1 if and only if its regular elements are unit-regular. Then Chen ([6], Theorem 2.1) extended the result and proved that an exchange ring has stable range 1 if and only if its regular elements are special clean. And in [35] Theorem 3.3, Wang et. al. gave an elementwise version of the statement: if $a \in R$ is regular and $sr(R) = 1$ then a is special clean. But the proof of Camillo and Khuruna, as well as the ones of Chen and Wang et. al., highly rely on module operations such as internal cancellation. Here we deduce their equivalences from our previous results that use only ring theoretical arguments.

Corollary 4.2. *Let R be a ring with $sr(R) = 1$ and $a \in R$. Then a is regular iff it is special clean.*

Proof. As special clean elements are unit-regular, the condition is necessary. We prove that it is also sufficient. So let $a \in R$ be regular with inverse $x \in R$. Then $ax + (1 - ax) = 1$ and by stable range 1, exists y such that $a + (1 - ax)y = v$ is a unit. It follows that $(ax)v = (ax)(a + (1 - ax)y) = axa = a$ and a is unit-regular as a product of an idempotent and a unit (or directly $av^{-1}a = (axv)v^{-1}a = axa = a$) with inverse v^{-1}, and pose $f = av^{-1}(= ax)$. Then, as recorded in the introduction, by [34] the corner ring fRf has stable range 1. We conclude by Corollary 4.1. □

This in turns provides alternative proofs of the results of Camillo, Khurana and Chen:

Corollary 4.3. *Let R be a ring.*

1. *R is unit-regular iff all elements of R are special clean;*
2. *R is exchange with stable range 1 iff R is exchange and regular elements are special clean.*

4.2. Special cleanness and the Jacobson radical

We now turn our attention to the skew-corner rings.

Corollary 4.4. *Let $a \in R$ be a regular element with inverse $x \in R$, and pose $f = ax$. If $fR\bar{f} \subseteq J(R)$ or $\bar{f}Rf \subseteq J(R)$, then a is special clean.*

Proof. We first prove unit-regularity, and then we deduce special cleanness.

Step 1: Unit-regularity.
We consider Peirce decompositions of a and x relative to the idempotent f. As $a_1x_1 + a_2x_3 = f$ and $a_2x_3 = f(a_2x_3)f \in fJ(R)f = J(fRf)$ (by assumption, one of the terms a_2, x_3 is in $J(R)$, and the equality $fJ(R)f = fRf \cap J(R) = J(fRf)$ for any non-zero idempotent f can by found for instance in [24]), then a_1x_1 is a unit in fRf and x_1 is left invertible in fRf. But dually $x_1a_1 + x_2a_3 = f$ and x_1 is right invertible, so that x_1 is a unit in fRf. Symetrically, a_1 is a unit in fRf. Pose $\tilde{A} = \begin{pmatrix} a_1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\tilde{X} = \begin{pmatrix} x_1 & 0 \\ 0 & 1 \end{pmatrix}$. Pose $V = \begin{pmatrix} a_1 & 0 \\ -x_3a_1 & 1 \end{pmatrix}$. Then V is invertible with inverse $V^{-1} = \begin{pmatrix} x_1 & 0 \\ x_3 & 1 \end{pmatrix}$ and $AVA = A$ so that A is unit-regular.

Step 2: Special cleanness.
We now consider Peirce decompositions relative to $g = av^{-1}$. As above from $a_1\mu_1 + a_2\mu_3 = g$ and $a_2\mu_3 = g(a_2\mu_3)g \in gJ(R)g = J(gRg)$ we deduce that $a_1\mu_1$ is a unit in gRg and by symmetry we deduce that μ_1 is a unit in gRg. We then conclude by Lemma 3.2.

□

Thus, if all skew-corner rings are contained in $J(R)$, then any regular element is special clean. We will see in Section 6 that in this case, a stronger result actually holds.

5. Uniquely special clean elements and uniquely special clean rings

Uniqueness of certain inverses or decompositions of elements is sometimes a key issue. Therefore, scholars look for necessary and sufficient conditions for uniqueness, regarding either a specific element or all the elements of the ring.

Regarding unicity of inverses, Hartwig and Luh [16] Theorem 4 proved in 1977 that a unit-regular ring whose non-zero elements have a unique unit-inverse is either Boolean or a division ring . Recently, Danchev [10] extended this result to regular rings and proved that regular rings whose non-zero elements have a unique Von Neumann inverse are exactly division rings (in the same article, Danchev also studies other uniqueness conditions).

On the other hand, uniquely (strongly) clean rings have notably been studied in [2], [8], [9], [7], [30], [38], and uniquely clean elements by Khuruna et al [23]. It is notably known that a ring is uniquely clean iff it is uniquely strongly clean and abelian, but that there exists uniquely strongly clean rings that are not abelian (hence not uniquely clean). Also, uniquely clean elements need not be strongly clean in general, but this is the case if every corner ring is clean.

Here we consider unicity of special clean decompositions. We say that an element $a \in R$ is *uniquely special clean* if there exists a unique $u \in U(R)$ such that $a - u \in E(R)$ and $au^{-1}a = a$. A ring is uniquely special clean if all its elements are. We start by some specific cases.

Lemma 5.1. *Let R be a ring.*

1. *A unit $u \in U(R)$ is uniquely special clean;*
2. *An idempotent $e \in E(R)$ is uniquely special clean iff it is central;*
3. *A strongly regular element $z \in R^\#$ is uniquely special clean iff $e = zz^\#$ is central.*

Proof.

1. A unit u admits a unique inner inverse u^{-1} and $u - u = 0 \in E(R)$.
2. Let $e \in E(R)$, and assume that e is central. Then $e = e1e$ and we can consider Peirce decomposition relative to $e = e1$. But as e is central then $eR\bar{e} = \bar{e}Re = \{0\}$. By Theorem 2.1 special clean decompositions are in bijective correspondence with solutions of $\varphi(x, y) = \bar{e} \in U(\bar{e}R\bar{e})$, with $x \in \bar{e}Re = \{0\}$ and $y \in eR\bar{e} = \{0\}$. Thus e is uniquely special clean with unique decomposition $e = \bar{e} + u = eu^{-1}e$ with $u = e - \bar{e} = 2e - 1 = u^{-1}$.
For the converse, assume that e is uniquely special clean and let $b \in R$. Pose $y = \bar{e}be = be - eb$. Then $y \in \bar{e}Re$ and $\varphi(0, y) = \bar{e} \in U(\bar{e}R\bar{e})$. Thus e admits two special clean decompositions, one with $u = 2e - 1$ and a second one with $v = 2e - 1 - y$ ($V = \begin{pmatrix} 1 & 0 \\ -y & -1 \end{pmatrix}$). By unicity of the decomposition $v = u$ whence $y = be - eb = 0$, and since b was arbitrary, e is central.
3. Let $z \in R^\#$, and assume that $e = zz^\#$ is central. We pose $v = z + \bar{e}$. As e is central then v is invertible with $v^{-1} = z^\# + \bar{e}$. Then $zv^{-1}z = z$ and $zv^{-1} = e$. Thus we consider Peirce decomposition relative to e. As e is central then $eR\bar{e} = \bar{e}Re = \{0\}$, and by Theorem 2.1 special clean decompositions are in bijective correspondence with solutions of $\varphi(x, y) = \bar{e} \in U(\bar{e}R\bar{e})$, with $x \in \bar{e}Re = \{0\}$ and $y \in eR\bar{e} = \{0\}$. Thus e is uniquely special clean with unique decomposition $z = \bar{e} + u = eu^{-1}e$ with $u = z - \bar{e}, u^{-1} = z^\# - \bar{e}$.
For the converse, assume that $z \in R^\#$ is uniquely special clean and let $b \in R$. Pose $e = zz^\#$ and $y = \bar{e}be = be - ebe$. Then $y \in \bar{e}Re$ and $\varphi(0, y) = \bar{e} \in U(\bar{e}R\bar{e})$. Once again by unicity of the decomposition $y = be - ebe = 0$ and $be = ebe$. Dually $eb = ebe$ and as b is arbitrary then e is central.

□

We deduce the following characterization of uniquely special clean rings:

Corollary 5.2. *Let R be a ring. Then it is uniquely special clean if and only if it is a strongly regular ring.*

Proof. Let R be a uniquely special clean ring. Then it is regular and idempotents are central, hence it is strongly regular (see for instance [27]).
Consider now a strongly regular ring R. Then it is regular and idempotents are central. Thus any element $z \in R$ is strongly regular with $e = zz^\#$ central, thus uniquely special clean by Lemma 5.1. □

6. A proof that special clean elements are reflexive inverses of strongly regular elements via Peirce decomposition, and consequences

In [29] it is proved that unit-regularity can be weakened to group-regularity: $a \in R$ is unit-regular iff it admits a Von Neumann inverse $z \in R^\#$. In [28] this result is sharpened as follows: $a \in R$ is special clean iff it admits a reflexive inverse $z \in R^\#$. However, the proof is not direct and relies on the (b, c)-inverse of Drazin[11]. Therefore, we propose in this section a direct proof (we use that strongly regular elements are precisely units in Peirce corner rings). This decomposition, together with the previous results, will in turn have interesting consequences.

Theorem 6.1. *Let R be a ring and $a \in R, e \in E(R)$. Then the following statement are equivalent:*

1. *There exists $z \in U(eRe)$ such that $aza = a, zaz = z$;*
2. *The Peirce decomposition of a relative to the idempotent e is of the form $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ with $a_1 \in U(eRe)$ (with inverse z) and $a_4 = a_3 z a_2$;*
3. *$u = a - \bar{e} \in U(R)$ and $au^{-1}a = a$ (a is special clean).*

Proof.

(1) ⇒ (2) Let $z \in U(eRe)$ such that $aza = a, zaz = z$. Under the Peirce isomorphim relative to e, $E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ and $Z = \begin{pmatrix} z & 0 \\ 0 & 0 \end{pmatrix}$. As $ZAZ = Z$ with z a unit in eRe we deduce first that $za_1z = z$ in eRe, hence a_1 is the inverse of z in eRe. Second from $AZA = A$ we the get $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_3za_2 \end{pmatrix}$.

(2) ⇒ (3) Assume (2) and pose $U = A - \bar{E} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_3za_2 - 1 \end{pmatrix}$. As a_1 is a unit in eRe with inverse z then we can compute the Schur complement of a_1 in U, which is -1. Schur's theorem then asserts that U is invertible with inverse $U^{-1} = \begin{pmatrix} z - za_2a_3z & za_2 \\ a_3z & -1 \end{pmatrix}$. But then also $AU^{-1}A = A$ and a is special clean with $a = \bar{e} + u = au^{-1}a$.

(3) ⇒ (1) Finally assume (3), that is a is special clean with decomposition $a = \bar{e} + u = au^{-1}a$, and pose $z = u^{-1}au^{-1}$, $z' = eae$ and $f = au^{-1}$. By construction z is a reflexive inverse of a. We now prove that it is strongly regular with group inverse $z^\# = z'$ and that $zz^\# = e$. Under the Peirce isomorphism relative to f, $A = FA$ gives $A = \begin{pmatrix} a_1 & a_2 \\ 0 & 0 \end{pmatrix}$. But also $FU = A$ so that $a_1 = u_1$ and $a_2 = u_2$. From this we deduce that $\bar{E} = A - U = \begin{pmatrix} 0 & 0 \\ -u_3 & -u_4 \end{pmatrix}$ (and we recover in particular that $aR \cap \bar{e}R = \{0\}$). As $\bar{E}$ is idempotent, we also get $\bar{E}^2 - \bar{E} = \begin{pmatrix} 0 & 0 \\ u_4u_3 + u_3 & u_4^2 + u_4 \end{pmatrix} = 0$. Right multiplication by $U^{-1} = \begin{pmatrix} \mu_1 & \mu_2 \\ \mu_3 & \mu_4 \end{pmatrix}$ which is invertible gives $(\bar{E}^2 - \bar{E})U^{-1} = \begin{pmatrix} 0 & 0 \\ * & (u_4u_3 + u_3)\mu_2 + (u_4^2 + u_4)\mu_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ * & u_4 + 1 \end{pmatrix} = 0$ (as from $UU^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ we get $u_3\mu_2 + u_4\mu_4 = 1$ in $\bar{f}R\bar{f}$). Thus $\bar{E} = \begin{pmatrix} 0 & 0 \\ -u_3 & 1 \end{pmatrix}$ and $U = \begin{pmatrix} u_1 & u_2 \\ u_3 & -1 \end{pmatrix}$. As the matrix U is invertible, then the Schur complement $\zeta = u_1 + u_2u_3$ is a unit in fRf and $U^{-1} = \begin{pmatrix} \zeta^{-1} & -\zeta^{-1}u_2 \\ -u_3\zeta^{-1} & -1 + u_3\zeta^{-1}u_2 \end{pmatrix}$. Finally $Z = U^{-1}F = \begin{pmatrix} \mu_1 & 0 \\ \mu_3 & 0 \end{pmatrix} = \begin{pmatrix} \zeta^{-1} & 0 \\ u_3\zeta^{-1} & 0 \end{pmatrix}$, wich is group invertible with group inverse $Z^\# = \begin{pmatrix} \zeta & 0 \\ u_3\zeta & 0 \end{pmatrix} = EAE$ and

$$ZZ^\# = Z^\#Z = E = \begin{pmatrix} 1 & 0 \\ u_3 & 0 \end{pmatrix}.$$

□

By looking at the proof of the theorem, we see that special clean decompositions are in bijective correspondance with completely regular reflexive inverses. Precisely:

Corollary 6.2. *Let R be a ring and $a \in R$ be a special clean element. Then there is a bijective correspondance between special clean decompositions and strongly regular reflexive inverses given by $(e, u) \mapsto z = u^{-1}au^{-1}$ with reciprocal $z \mapsto (e = zz^\#, u = a - \bar{e})$, where $a = \bar{e} + u = au^{-1}a$ denotes the special clean decomposition.*
In particular a is uniquely special clean if and only if it admits a unique reflexive inverse which is also strongly regular.

We are in position to prove the following theorem:

Theorem 6.3. *Let R be a ring such that all skew corner rings $eR(1 - e), e \in E(R)$ are contained in $J(R)$, and let $a \in R$. Then a is regular iff it is strongly regular.*

Proof. We have only to prove the implication. So let R be such a ring, and $a \in R$ be regular. We make the following steps.

Step 1: First, by Corollary 4.4, a is special clean, and $a = \bar{e} + u = au^{-1}a$ for some $e \in E(R), u \in U(R)$.

Step 2: By Theorem 6.1, a admits a Peirce decomposition relative to e of the form $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ with a_1 a unit in eRe (with inverse z) and $a_4 = a_3 z a_2$. As skew corner rings are in $J(R)$ by hypothesis then $a_2, a_3 \in J(R)$ hence $a_4 = a_3 z a_2 \in J(R)$ since $J(R)$ is an ideal. It follows that $a = z^\# + j$, with $z^\# = a_1 \in R^\#$ and $j = a_2 + a_3 + a_4 \in J(R)$.

Step 3: We conclude by arguments of [18]. By citehuylebrouck1986generalized Proposition 1, as $a = z^\# + j$ is regular and $z^\#$ is regular with reflexive inverse z, then

$$(1 - zz^\#) j (1 + zj)^{-1} (1 - zz^\#) = 0.$$

But it then follows from Proposition 3 therein that $a = z^\# + j$ is actually group invertible, which ends the proof.

□

It is straightforward to see that the class of rings such that all skew corner rings are in $J(R)$ contains the *weakly normal rings* (rings such that $Rea(1-e)$ is a nil left ideal for any $a \in R, e \in E(R)$ by [36] Theorem 2.1.), since any nil ideal is contained in the $J(R)$. It also contains those rings such that $R/J(R)$ is abelian. Thus, Theorem 6.3 appears as a common generalization of the following two results, whose proof rely on very distinct arguments and are therefore recalled for convenience.

Proposition 6.4 (Lemma 3.1 (6) in [36]). *Let R be a weakly normal ring, and $a \in R$. Then a is regular iff it is strongly regular.*

Proof. Let $a, x \in R$ such that $axa = a$. Then $e = 1 - xa$ is idempotent and $xa - x^2a^2 = x(1-xa)a(xa) \in Rea(1-e)$ is nilpotent, $(xa - x(xa)a)^n = 0$ for some n. As $a(xa) = a$ then there exists b such that $a(xa - x(xa)a)^n = a(xa - b(xa)a) = a - abxa^2 = 0$ and $a \in Ra^2$. We conclude by duality. □

Note that this proof is specific to weakly normal rings as it relies on the nilpotency of some elements (and not their belonging to the Jacobson radical, that may contain non-nilpotent elements).

Proposition 6.5 (From [25]). *Let R be a ring such that $R/J(R)$ is abelian and exchange, and $a \in R$. Then a is regular iff it is strongly regular.*

Proof. By [25] Theorem 4.6., such a ring is quasi-duo. Note that this follows also from the combinations of the following two results: an abelian exchange ring (here $R/J(R)$) is quasi-duo [3]; And R is quasi-duo iff $R/J(R)$ is [39]).
But in a quasi-duo ring, regular elements are strongly regular (see for instance the proof of Corollary 4.9. (2) in [25]). This ends the proof. □

In particular, Theorem 6.3 shows that the exchange property in Proposition 6.5 is superfluous.

Finally, we observe from Theorem 6.1 and Corollary 6.2 that, whereas cleanness relies obviously on both the additive structure of the ring (via the sum) and the multiplicative structure (via the unit), special cleanness does not. In particular we deduce the "apparently" intriguing result:

Corollary 6.6. *Let $(T, +, .)$ and $(R, +, .)$ be two rings and let $\phi : (T, .) \to (R, .)$ be a* ***semigroup morphism*** *between their multiplicative monoids. If a is special clean in T, then $\phi(a)$ is special clean in R.*

Proof. Let $a \in T$ be special clean, with decomposition $a = \bar{e} + u = au^{-1}a$. Then $z = u^{-1}au^{-1}$ is group invertible by Corollary 6.2 (with group inverse $z^\# = eae$) and a reflexive inverse of a. As ϕ is a semigroup morphism, then $\phi(z)$ is group invertible with inverse $\phi(z^\#)$, and a reflexive inverse of $\phi(a)$. Thus $\phi(a)$ is special clean by Theorem 6.1, with decomposition $\phi(a) = \bar{g} + v = \phi(a)v^{-1}\phi(a)$, where $\bar{g} = 1 - \phi(e) \in E(R)$ and $v = \phi(a) - \bar{g} \in U(R)$. □

Example 6.7. *Let R be a commutative ring and $T = \mathcal{M}_n(R)$ be the ring of square matrices of size n over R. Then $\phi : (T, .) \to (T, .)$ defined by $\phi(A) = \det(A).A$ is a monoid morphism, so that any special clean matrix $A \in T$ also satisfies that $\det(A).A$ is special clean in T.*

Example 6.8. *Let S, T be rings and $M = (T, .)$ the multiplicative monoid of T. Form $R = S[M]$ the monoid ring over M. Then $\phi : M = (T, .) \to (R, .)$ defined by $\phi : t \mapsto 1_S.t$ is a monoid morphism (that clearly does not respect the additive structures of T and R). It follows that $1_S.a \in R$ is special clean for any special clean element $a \in T$. But if $a = \bar{e} + u = au^{-1}a$ is some special clean decomposition of $a \in T$, then $1_S.\bar{e} + 1_S.u$ is **not** a special clean decomposition of $1_S.a$ in general. However,*

$$1_S.a = (1_S.1_T - 1_S.e) + V = (1_S.a)V^{-1}(1_S.a)$$

is a special clean decomposition of $1_S.a$ in R, with $(1_S.1_T - 1_S.e) \in E(R)$ and $V = (1_S.1_T - 1_S.e) - 1_S.a \in U(R)$.

7. More examples and counter-examples

In [37] Proposition 4.3 Wu et. al. proved that a matrix of the form $\begin{pmatrix} s+1 & 1 \\ 0 & 0 \end{pmatrix}$ is always unit-regular but never clean in $R_s = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ s^2\mathbb{Z} & \mathbb{Z} \end{pmatrix}$ for $s \geq 3$ (there is actually a typo in the statement of the their Proposition 4.3.: the matrix *in the statement* is $\begin{pmatrix} s+1 & s \\ 0 & 0 \end{pmatrix}$ but the matrix *in the proof* is indeed $\begin{pmatrix} s+1 & 1 \\ 0 & 0 \end{pmatrix}$). They also prove that the matrix $\begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix}$ ($s = 2$) is unit-regular but not clean in $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 2^3\mathbb{Z} & \mathbb{Z} \end{pmatrix}$. We give below a different proof that $A = \begin{pmatrix} s+1 & 1 \\ 0 & 0 \end{pmatrix}$ is not special clean in R_s for $s \geq 3$, and also prove that $\begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix}$ is uniquely special clean in $R_2 = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 2^2\mathbb{Z} & \mathbb{Z} \end{pmatrix}$ ($s = 2$), and that $\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}$ has infinitely many special clean decompositions in $R_1 = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}$ ($s = 1$)

Corollary 7.1. *Let $s \geq 1$ and consider the matrix $A = \begin{pmatrix} s+1 & 1 \\ 0 & 0 \end{pmatrix}$ in the ring $R_s = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ s^2\mathbb{Z} & \mathbb{Z} \end{pmatrix}$. Then*

1. *A is unit-regular but not special clean in R_s for $s \geq 3$;*
2. *$A = \begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix}$ has a unique special clean decomposition in R_2 given by $A = \bar{E} + U = AU^{-1}A$ with $U = \begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix}$ (and $\bar{E} = \begin{pmatrix} 0 & 0 \\ 4 & 1 \end{pmatrix}$).*
3. *$A = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}$ has 6 special clean decompositions in $R_1 = \mathcal{M}_2(\mathbb{Z})$.*

Proof. Pose $V = \begin{pmatrix} s+1 & 1 \\ -s^2 & -s+1 \end{pmatrix}$. Then V is invertible with inverse $V^{-1} = \begin{pmatrix} -s+1 & -1 \\ s^2 & s+1 \end{pmatrix}$ and $AV^{-1}A = A$ so that A is unit-regular with unit-inverse V^{-1}. Pose $F = AV^{-1}$. Then $F = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and the matrix form is precisely the Peirce decomposition associated to F. By Theorem 2.1, A is special clean if and only if the equation $\varphi(x, y) = y(s+1)x + y - s^2x - s + 1 \in \mathbb{Z}^{-1}$ is solvable. As $y \in s^2\mathbb{Z}$ we pose $y = s^2k$. The equation becomes $s[s[kx(s+1) - x + k] - 1] + 1 = -\zeta$ with $\zeta = \pm 1$.

Case 1: $\zeta = -1$. The equation becomes $s[s[kx(s+1) - x + k] - 1] = 0$ and since s is left-cancellable, $s[kx(s+1) - x + k] - 1 = 0$. Thus s is right-invertible, which is incompatible with $s \geq 2$. We conclude that the equation has no solution for $\zeta = 1$ and $s \geq 2$. For $s = 1$, $y = k$ and the set of solutions is $\{(x, y) \in ZZ^2 | 2yx - x + y - 1 = 0\} = \{(-1, 0), (0, 1)\}$.

Case 2: $\zeta = 1$. The equation becomes $s\,[s\,[kx\,(s+1) - x + k] - 1] = -2$. In particular this equation is solvable only if s divides 2 and we can conclude that A is not special clean for $s \geq 3$.
Assume $s = 2$. Then $2\,[3kx - x + k] - 1 = -1$ hence $3kx - x + k = 0$. This equation has a trivial solution $(0,0)$. If $k \neq 0$ then $x \neq 0$ and conversely, and $x|k, k|x$ so that either $x = k$ or $x = -k$. In the first case the equation becomes $3x^2 = 0$ which has no solution in $\mathbb{Z}^*$. In the second case the equation becomes $-3x^2 - 2x = 0$ which has also no solution in $\mathbb{Z}^*$. There is a unique solution given by $U^{-1} = V^{-1}\begin{pmatrix} 1 & x \\ -y & -\varphi(x,y) - yx \end{pmatrix}$ with $x = 0, y = 0$ and $\varphi(x,y) = -\zeta = -1$, thus $U = V$.
Finally assume $s = 1$. Then the set of solutions is

$$\{(x,y) \in \mathbb{Z}^2 | 2yx - x + y + 1 = 0\} = \{(-2,1), (-1,2), (0,-1), (1,0)\}.$$

□

As $W = \begin{pmatrix} 3 & 1 \\ 8 & 3 \end{pmatrix} \neq U$ also satisfies $\begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix} W^{-1} \begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix}$ ($W^{-1} = \begin{pmatrix} 3 & -1 \\ -8 & 3 \end{pmatrix}$), we deduce that $A = \begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix}$ is uniquely special clean without being uniquely unit-regular (in R_2).

It is informative to consider what happens if one starts from W instead of V in the application of Theorem 2.1. We then get the function $\varphi(x,y) = 3yx + y + 8x + 3$ and we try to solve $\varphi(x,y) = -\zeta$ with $\zeta \in U(R)$, and $y = 4k$.

Case 1: $\zeta = -1$. The equation becomes $12kx + 4k + 8x = -2$ which has no solution.

Case 2: $\zeta = 1$. The equation becomes $12kx + 4k + 8x = -4$. The only solution is $x = 0, k = -1$ so that there is a unique special clean decomposition with $U^{-1} = W^{-1}\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 4 & 3 \end{pmatrix} = V^{-1}$ and we recover $U = V$.

As also $A = \begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix}$ is not strongly regular in R_2 ($1 + A - AW^{-1} = \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}$ is not a unit, see for instance [32]), we get that a uniquely special clean element need not be strongly regular. Also the idempotent e related to the special clean decomposition (equivalently $\bar{e} = a - u$) is not central. More precisely $eu = a$ but $ue \neq a$ ($UE = \begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -4 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$), and the (unique) special clean decomposition of a is not even a strongly clean decomposition.

We continue the investigation of such 2-by-2 matrices and consider the following result of Khuruna and Lam [20] Corollary 3.6. (a connected ring is a ring with only trivial idempotents).

Lemma 7.2. *Let K be a connected commutative ring, and consider the matrix $A = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \in \mathcal{M}_2(K)$.*

1. *If $a \in 1 + K^{-1}$, then A is always clean.*
2. *If otherwise, then A is clean iff there exist $x_0, y_0 \in K$ such that $ay_0 - bx_0 = 1$ and $y_0 + x_0K$ contains a unit of K. (In this case, A is also unit-regular).*

We build up upon this lemma and prove that for $K = \mathbb{Z}$:

Lemma 7.3. *Any unit-regular and clean matrix A of the form $A = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \in \mathcal{M}_2(\mathbb{Z})$ is special clean.*

Proof. We consider three cases: $a = 0, 2$ and otherwise.

Case 1: $a = 0$. Let $A = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \in \mathcal{M}_2(\mathbb{Z})$ and assume that A is unit regular with inverse $W^{-1} = \begin{pmatrix} \mu_1 & \mu_2 \\ \mu_3 & \mu_4 \end{pmatrix}$. Then $b\mu_3 = 1$, and $b = \mu_3 = \pm 1$. Then A is also unit-regular with unit-inverse $V^{-1} = \begin{pmatrix} 0 & -b \\ \mu_3 & 0 \end{pmatrix}$, and $V =$

$\begin{pmatrix} 0 & b \\ -\mu_3 & 0 \end{pmatrix}$. Pose $F = AV^{-1}$. Then $F = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and the matrix form is precisely the Peirce decomposition associated to F. By Theorem 2.1, A is special clean if and only if the equation $\varphi(x, y) = yb - \mu_3 x \in \mathbb{Z}^{-1}$ is solvable. Since $b = \mu_3 = \pm 1$ this equation as for set of solutions $\{(x, x \pm 1), x \in \mathbb{Z}\}$.

Case 2: $a = 2$. Let $A = \begin{pmatrix} 2 & b \\ 0 & 0 \end{pmatrix} \in \mathcal{M}_2(\mathbb{Z})$ and assume that A is unit regular with inverse $W^{-1} = \begin{pmatrix} \mu_1 & \mu_2 \\ \mu_3 & \mu_4 \end{pmatrix}$. Then $2\mu_1 + b\mu_3 = 1$, and $b = 2p+1$ is odd. Then A is also unit-regular with unit-inverse $V^{-1} = \begin{pmatrix} -p & -(2p+1) \\ 1 & 2 \end{pmatrix}$, and $V = \begin{pmatrix} 2 & 2p+1 \\ -1 & -p \end{pmatrix}$, $F = AV^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. By Theorem 2.1, A is special clean if and only if the equation $\varphi(x, y) = 2yx + y(2p + 1) - x - p \in \mathbb{Z}^{-1}$ is solvable. This equation has exactly 6 solutions: $\{(-p + 1, -1); (-p, -1); (-p, 1); (-p - 1, 0); (-p - 1, 2); (-p - 2, 1)\}$ (for $b = 1$ a.k.a. $p = 0$ we recover the 6 solutions of Corollary 7.1).

Case 3: $a \notin \{0, 2\}$. As $1 + a$ is not a unit, by Lemma 7.2, there exist $\mu_1, \mu_3 \in K$ such that $a\mu_1 + b\mu_3 = 1$ and $\mu_1 + \mu_3\mathbb{Z}$ contains a unit of $\mathbb{Z}$. Then A is unit-regular with unit-inverse $V^{-1} = \begin{pmatrix} \mu_1 & -b \\ \mu_3 & a \end{pmatrix}$, and $V = \begin{pmatrix} a & b \\ -\mu_3 & \mu_1 \end{pmatrix}$, $F = AV^{-1}$. Then $F = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. By Theorem 2.1, A is special clean if and only if the equation $\varphi(x, y) = yax + yb - \mu_3 x + \mu_1 \in \mathbb{Z}^{-1}$ is solvable. But $\mu_1 + \mu_3\mathbb{Z}$ contains a unit, and we can choose $x \in \mathbb{Z}$ such that $-\mu_3 x + \mu_1$ is a unit, so that $\varphi(x, 0) = -\mu_3 x + \mu_1 \in \mathbb{Z}^{-1}$.

□

Note that by Corollary 2.3 in [20], if $a = 2$ then $A = \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix} \in \mathcal{M}_2(\mathbb{Z})$ is clean but not unit-regular. The unit-regularity assumption can thus not be removed.

On the other hand, a straightforward application of Corollary 3.1 (or Corollary 3.2) gives that a diagonal matrix $A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{M}_2(R)$ is special clean iff a is regular in R for any ring R.

Lemma 7.4. *Let R be a ring and $a \in R$. Then $A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{M}_2(R)$ is special clean iff a is regular in R.*

Proof. Consider such a matrix A, and pose $B = \begin{pmatrix} b & c \\ d & e \end{pmatrix} \in \mathcal{M}_2(R)$. Then $ABA = \begin{pmatrix} aba & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{M}_2(R)$ and A is regular iff a is regular. Then assume that $aba = a$ and pose $V = \begin{pmatrix} 1 & 1 \\ b-1 & b \end{pmatrix} \in \mathcal{M}_2(R)$. Then V is invertible with inverse $V^{-1} = \begin{pmatrix} b & -1 \\ 1-b & 1 \end{pmatrix} \in \mathcal{M}_2(R)$, and $AV^{-1}A = A$. It follows that for any $y \in R$, $yv_2 + v_4 = \mu_1 - \mu_2 y = y + b$ and choosing $y = 1 - b$ gives a unit in R. We conclude by either Corollary 3.1 or Corollary 3.2. □

We finally produce an example of a unit-regular and clean element that is not special clean. Consider the ring $R = \begin{pmatrix} \mathbb{Z} & 7\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}$, and $A = \begin{pmatrix} 2 & 7 \\ 0 & 0 \end{pmatrix} \in R$. Then $A = A\begin{pmatrix} 18 & -7 \\ -5 & 2 \end{pmatrix}A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 7 \\ 0 & -1 \end{pmatrix}$ is both unit-regular and clean in R. Let $V = \begin{pmatrix} 2 & 7 \\ 5 & 18 \end{pmatrix}$ be the inverse of $V^{-1} = \begin{pmatrix} 18 & -7 \\ -5 & 2 \end{pmatrix}$. Then by Theorem 2.1 A is special clean if and only if the equation $\varphi(7k, y) = 14yk + 7y + 5 \times 7k + 18 = \pm 1$ is solvable. But 17 and 19 are prime twins so that the equation $-7(2yk + y + 5k) = 18 \pm 1$ has no solution in $\mathbb{Z}^2$. Thus A is unit-regular, clean but not special clean in $R = \begin{pmatrix} \mathbb{Z} & 7\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}$.

Aknowledgements

The authors have recently be informed by Professor T.Y. Lam that Lemma 7.4 had also been obtained (but by different means) by him and his coauthors in [22] Theorem 7.13. In the same article, the authors provide another example of a unit-regular and clean element that is not special clean (Example 4.1), namely the matrix

$$A = \begin{pmatrix} 2 & 3 \\ 0 & 0 \end{pmatrix} \in \begin{pmatrix} \mathbb{Z} & 3\mathbb{Z} \\ 3\mathbb{Z} & \mathbb{Z} \end{pmatrix}.$$

They actually prove that this matrix is even uniquely clean in $\begin{pmatrix} \mathbb{Z} & 3\mathbb{Z} \\ 3\mathbb{Z} & \mathbb{Z} \end{pmatrix}$.

References

[1] E. Akalan and L. Vaš. Classes of almost clean rings. *Algebr. Representat. Theory*, 16(3):843–857, 2013.
[2] D. D. Anderson and V. P. Camillo. Commutative rings whose elements are a sum of a unit and idempotent. *Comm. Algebra*, 30(7):3327–3336, 2002.
[3] W.D. Burgess and W. Stephenson. Rings all of whose pierce stalks are local. *Canad. Math. Bull.*, 22(2):159–164, 1979.
[4] V.P. Camillo and D. Khurana. A characterization of unit regular rings. *Comm. Algebra*, 29(5):2293–2295, 2001.
[5] V.P. Camillo and H-P. Yu. Stable range one for rings with many idempotents. *Trans. Amer. Math. Soc.*, 347(8):3141–3147, 1995.
[6] H. Chen. Exchange rings with stable range one. *Czechoslovak Math. J.*, 57(2):579–590, 2007.
[7] H. Chen. On uniquely clean rings. *Comm. Algebra*, 39(1):189–198, 2010.
[8] J. Chen, W.K. Nicholson, and Y. Zhou. Group rings in which every element is uniquely the sum of a unit and an idempotent. *J. Algebra*, 306(2):453–460, 2006.
[9] J. Chen, Z. Wang, and Y. Zhou. Rings in which elements are uniquely the sum of an idempotent and a unit that commute. *J. Pure Appl. Algebra*, 213(2):215–223, 2009.
[10] P. V. Danchev. Uniqueness in Von Neumann regular unital rings. *Palest. J. Math.*, 7(1), 2018.
[11] M.P. Drazin. A class of outer generalized inverses. *Linear Algebra Appl.*, 436(7):1909–1923, 2012.
[12] G. Ehrlich. Units and one-sided units in regular rings. *Trans. Amer. Math. Soc.*, 216:81–90, 1976.
[13] L. Fuchs. On a substitution property of modules. *Monatshefte für Mathematik*, 75(3):198–204, 1971.
[14] K.R. Goodearl. *Von Neumann regular rings*. Krieger Pub Co, 1991.
[15] D. Handelman. Perspectivity and cancellation in regular rings. *J. Algebra*, 48(1):1–16, 1977.
[16] R.E. Hartwig and J. Luh. A note on the group structure of unit regular ring elements. *Pacific J. Math.*, 71(2):449–461, 1977.
[17] R.E. Hartwig and J. Luh. On finite regular rings. *Pacific J. Math.*, 69(1):73–95, 1977.
[18] D. Huylebrouck and R. Puystjens. Generalized inverses of a sum with a radical element. *Linear Algebra Appl.*, 84:289–300, 1986.
[19] I. Kaplansky. Bass's first stable range condition. *Mimeographed notes*, 1971.
[20] D. Khurana and T.Y. Lam. Clean matrices and unit-regular matrices. *J. Algebra*, 280(2):683–698, 2004.
[21] D. Khurana and T.Y. Lam. Rings with internal cancellation. *J. Algebra*, 284(1):203–235, 2005.
[22] D. Khurana, T.Y. Lam, P.P. Nielsen, and J. Šter. Special clean elements in rings. *J. Algebra Appl., online ready*, 2019.
[23] D. Khurana, T.Y. Lam, P.P. Nielsen, and Y. Zhou. Uniquely clean elements in rings. *Comm. Algebra*, 43(5):1742–1751, 2015.
[24] T.Y. Lam. *A first course in noncommutative rings*, volume 131. Springer Science & Business Media, 2013.
[25] T.Y. Lam and A.S. Dugas. Quasi-duo rings and stable range descent. *J. Pure Appl. Algebra*, 195(3):243–259, 2005.
[26] T.Y. Lam and W. Murray. Unit regular elements in corner rings. *Bull. HongKong Math. Soc.*, 1(1):61–65, 1997.
[27] L. Li and B.M. Schein. Strongly regular rings. In *Semigroup Forum*, volume 32, pages 145–161. Springer, 1985.
[28] X. Mary. Characterizations of clean elements by means of outer inverses in rings and applications. *J. Algebra Appl., online ready*, 2019.
[29] X. Mary and P. Patrício. Group-regular rings. *Filomat*, 33(11):3551–3560, 2019.
[30] W.K. Nicholson and Y. Zhou. Clean rings: a survey. In *Advances in ring theory*, pages 181–198. World Scientific, 2005.
[31] P.P. Nielsen and J. Šter. Connections between unit-regularity, regularity, cleanness, and strong cleanness of elements and rings. *Trans. Amer. Math. Soc.*, 370(3):1759–1782, 2018.
[32] R. Puystjens and R.E. Hartwig. The group inverse of a companion matrix. *Linear Multilinear Algebra*, 43(1-3):137–150, 1997.
[33] L.N. Vasershtein. Stable rank of rings and dimensionality of topological spaces. *Functional Analysis and its Applications*, 5(2):102–110, 1971.
[34] L.N. Vaserstein. Bass's first stable range condition. *J. Pure Appl. Algebra*, 34(2-3):319–330, 1984.
[35] Z. Wang, J. Chen, D. Khurana, and T.Y. Lam. Rings of idempotent stable range one. *Algebras and representation theory*, 15(1):195–200, 2012.
[36] J. Wei and L. Li. Weakly normal rings. *Turk. J. Math.*, 36(1):47–57, 2012.
[37] Y. Wu, G. Tang, G. Deng, and Y. Zhou. Nil-clean and unit-regular elements in certain subrings of $\mathbb{M}_2(\mathbb{Z})$. *Czechoslovak Math. J.*, 69(1):197–205, 2019.
[38] X. Yang. A survey of strongly clean rings. *Acta Appl. Math.*, 108(1):157, 2009.
[39] H-P. Yu. On quasi-duo rings. *Glasg. Math. J.*, 37(1):21–31, 1995.