Universidade do Minho
Escola de Ciências

Irene Vitória Ribeiro de Brito

ADVANCES IN GENERAL RELATIVISTIC ELASTICITY – A MATHEMATICAL APPROACH

Abril de 2008

Universidade do Minho
Escola de Ciências

Irene Vitória Ribeiro de Brito

ADVANCES IN GENERAL RELATIVISTIC ELASTICITY – A MATHEMATICAL APPROACH

Doutoramento em Ciências
Área de Conhecimento:Matemática

Trabalho efectuado sob a orientação da

Professora Doutora Estelita da Graça Lopes Rodrigues Vaz

Abril de 2008

DECLARAÇÃO

Irene Vitória Ribeiro de Brito

Endereço electrónico: ireneb@mct.uminho.pt

Número do Bilhete de Identidade: 12261835

Título da Tese:

ADVANCES IN GENERAL RELATIVISTIC ELASTICITY – A MATHEMATICAL APPROACH

Orientador:

Professora Doutora Estelita da Graça Lopes Rodrigues Vaz

Ano de conclusão: 2008

Área de Conhecimento do Doutoramento:

Matemática

É AUTORIZADA A REPRODUÇAÕ INTEGRAL DESTA TESE/TRABALHO APENAS PARA EFEITOS DE INVESTIGAÇÃO, MEDIANTE DECLARAÇÃO ESCRITA DO INTERESSADO, QUE A TAL SE COMPROMETE

Universidade do Minho, 16 de Junho de 2008

Irene Vitória Ribeiro de Brito

Acknowledgements

I present my many thanks to Professor Estelita Vaz, who supervised my work, for the guidance, help, advice and encouragement during this project.
I thank Professor Jaume Carot, Professor Lars Samuelsson, Professor Maria Piedade Ramos, Professor Brian Edgar and Professor Graham Hall for helpful and constructive discussions and suggestions.
I also thank Simone Calogero for being available for my questions.
I thank the colleagues of the DMCT (Departamento de Matemática para a Ciência e Tecnologia), who helped me in various ways and surrounded me with a pleasant atmosphere.
I express my thanks to my parents to whom I dedicate this work.

“Das schönste Glück des denkenden Menschen ist, das Erforschliche erforscht zu haben und das Unerforschliche ruhig zu verehren.”

Johann Wolfgang von Goethe (1749-1832)

ADVANCES IN GENERAL RELATIVISTIC ELASTICITY - A MATHEMATICAL APPROACH

Abstract

In recent years there has been increasing consideration of and interest in general relativistic elasticity. In this framework, the elasticity difference tensor has been introduced in the literature by Karlovini and Samuelsson (2003) [35]. This tensor contains information about the space-time connection and the material metric.

In this thesis, a mathematical analysis is presented for the elasticity difference tensor. Some of its properties are investigated and a tetrad formulation is given for this tensor. Furthermore, the elasticity difference tensor is decomposed along the eigenvectors of the pulled-back material metric, thereby obtaining three second order tensors. The following eigenvalue-eigenvector problem is carried out: It is studied under which conditions the eigenvectors of the pulled-back material metric remain also eigenvectors for those three second order tensors. The corresponding eigenvalues are also presented.

Another topic which is investigated in this thesis is to consider two conformally related material metrics and study the consequences on relativistic elastic quantities, such as the constant volume shear tensor, the energy-momentum tensor and the elasticity difference tensor. Relations between these objects associated with both material metrics are obtained and the previously mentioned eigenvalue-eigenvector problem is studied in this context.

Due to the fact that neutron stars are the objects of study in astrophysical problems in general relativistic elasticity, and since neutron stars can be modelled by spherically and axially symmetric metrics, the results are applied to spherically symmetric space-times and to a particular class of axially symmetric space-times.

Moreover, existing results for non-static spherically symmetric space-times with a flat material metric are generalized by considering a non-flat material metric conformally related to the flat one. Thereby the Einstein field equations are rewritten for the new configuration.

Desenvolvimentos na elasticidade em relatividade geral - uma abordagem matemática

Resumo

A área de elasticidade em relatividade geral tem, recentemente, despertado interesse na comunidade científica, traduzido no aparecimento de trabalhos publicados. Neste contexto, o “elasticity difference tensor” foi introduzido na literatura por Karlovini e Samuelsson (2003) [35]. Este tensor contém em si informação sobre a conexão do espaço-tempo e sobre a métrica material.

Na presente tese apresenta-se um estudo matemático sobre o “elasticity difference tensor”: são exploradas algumas propriedades; obtém-se uma formulação para este tensor em termos de um tetrado; o “elasticity difference tensor” é decomposto ao longo dos vectores próprios do “pull-back” da métrica material, obtendo-se desta forma três tensores simétricos de segunda ordem. Para estes tensores são estudadas as condições para que os vectores próprios da métrica material permaneçam como vectores próprios para os mesmos tensores. Também se apresentam os valores próprios correspondentes.

Um outro tema abordado nesta tese é o seguinte: Considerando duas métricas materiais conformemente relacionadas, são estudadas as consequências em quantidades elásticas, sendo as mais relevantes o “constant volume shear tensor”, o tensor de impulsão-energia e o “elasticity difference tensor”. Neste contexto são obtidas relações entre estes objectos associados às duas métricas e é explorado o problema de valores e vectores próprios, descrito anteriormente.

Devido ao facto de as estrelas de neutrões serem objecto de estudo na elasticidade relativista e estas serem modeladas por métricas esfericamente simétricas e métricas simétricas em relação a um eixo, os resultados são aplicados a espaços-tempo representados por estas métricas.

Nesta tese também se expõem os resultados obtidos por generalização de resultados existentes para espaços-tempo não-estáticos e esfericamente simétricos com métrica material plana, considerando uma métrica material não plana. Também são reescritas as equações de Einstein para esta nova configuração.

Contents

Chapter 1

Introduction

1.1 State of the art

In recent years there has been a growing interest in the theory of elasticity to the theory of general relativity. Based on the classical Newtonian elasticity theory going back to the 17th century and Hooke's law, some authors began to adapt the theory of elasticity to relativity due to the necessity to study many astrophysical problems such as the interaction between the gravitational field and an elastic solid body in the description of stellar matter, as well as to understand the interaction of gravitational waves and gravitational radiation and to study deformations of neutron star crusts. One of the first elastic phenomenon considered in the relativistic context was Weber's observation of the elastic response of an aluminium cylinder to gravitational radiation and the detection of gravitational waves: Weber (1960,1961,1969) [68],[69],[70]. Neutron stars have attracted attention since it has been argued, Pines (1971) [56], that the crusts of neutron stars are in elastic states and since the existence of a solid crust has been established, and there has been speculation on the possibility of solid cores in neutron stars, Shapiro and Teukoplsky (1983) [61], McDermott et al. (1988) [50], Haensel (1995) [30]. There were many attempts to formulate a relativistic version of elasticity theory whereby laws of non relativistic continuum mechanics had to be reformulated in a relativistic way. The study of elastic media in special relativity was first carried out by Noether

(1910) [52], Born (1911) [8], Herglotz (1911) [33] and Nordström (1911) [53]. The discussion of elasticity theory in general relativity started with Synge (1959) [64], De Witt (1962) [71], Rayner (1963) [57] and Bennoun (1964,1965) [6],[7]. De Witt (1962) [71] worked on the development of a fully relativistic theory of perfect elasticity by reformulating the theory of Herglotz (1911) [33], who developed the formal mathematics to place the theory of the elastic medium in the context of special relativity. In classical elasticity, the strain (or deformation) of an elastic body is measured relative to a "natural" (unstrained) state, and the basic stress-strain relation is a linear equation (Hooke's law) connecting stress and strain. Synge's and Bennoun's presentations are based on a modified Hooke's law, which states that the rate of stress is proportional to the rate of strain. These authors avoided defining an absolute state of strain. In their opinion it was impossible to carry over the classical concept of strain into general relativity, because the unstrained or "natural" state of an elastic body is unattainable since gravity cannot be turned off. Also based on this point of view was the work done by Papapetrou (1972) [54], who investigated vibrations of an elastic body induced by a gravitational wave. Rayner (1963) [57] provided a Hookean perfect elasticity theory with a linear stress-strain relationship and Bennoun (1965) [7], a general, but not necessarily linear elasticity theory. As pointed out by Hernandez (1970) [34], Rayner's measure of strain was not well defined and was somewhat arbitrary. Hernandez emphasized that there is no problem in considering the concept of absolute strain in general relativity because strain is a microscopic quantity in elasticity theory; and for each microscopic portion of the body, one can imagine removing that small portion of the body to a distant point, where it is free from all stresses, and where the natural state of the infinitesimal piece of the elastic material can be seen. There is no natural state for the body, but there is for the material of the body. Bressan (1964) [9] studied wave fronts in nonlinear elastic solids. Glass and Winicour (1972) [29] extended Rayner's theory by basing it upon a generalized Hooke's law for prestressed materials, which states that stress minus equilibrium stress is proportional to strain. Roy et al. (1973) [60] presented an attempt to apply Rayner's theory of elasticity in general relativity for the construction of realistic models by obtaining a general solution of the field equations of general relativity theory for an elastic sphere of constant density. In 1973, the authors Carter and Quintana (1972) [19] developed a relativistic formulation of the concept of a perfectly elastic solid and constructed a quasi-Hookean perfect elasticity theory suitable for applications to high-pressure neutron star matter, based on linearization with respect to a shear tensor instead of a strain tensor. These authors presented a nonlinear theory of elasticity adapted to general relativity. Carter (1980) [18] noted later that the basic theoretical framework of their theory had already been given by Souriau (1965) [62]. The theory developed by Carter and Quintana served as a starting point for further studies and applications in this field some of which are listed below. Carter and Quintana (1975) [20] also showed how to calculate stationary elastic rotational deformations of a relativistic neutron star model in the sense of Carter and Quintana (1972) [19]. Carter (1973) [17] derived characteristic equations for sound wave fronts in an elastic solid in terms of the formalism given in Carter and Quintana (1972) [19]. Carter (1073) [16] applied perturbation analysis to the theory of a general relativistic perfectly elastic medium as developed by Carter and Quintana (1972) [19]. Recently, Karlovini and Samuelsson (2003) [35] extended the presentation given by Carter and Quintana including new methods, results and modifications. Karlovini et al. (2004) [37] studied radial perturbations of general relativistic stars with elastic matter sources and Karlovini and Samuelsson (2004) [36] presented a recipe for obtaining stationary rigid motion exact solutions to the Einstein equations with elastic matter source.

Maugin (1971,1978) [47], [49] dealt with a nonlinear elastic medium in interaction with the gravitational field and with electromagnetic fields; he obtained field equations for a nonlinear elastic magnetized homogeneous solid in the frame of general relativity and he developed a theory of general relativistic magnetoelasticity valid under conditions of extremely high pressure. Maugin (1977) [48] also studied wave propagation speeds in initially stressed nonlinear relativistic elastic solids.

Kijowski and Magli (1992,1997) [39],[41] presented a gauge-type theory of relativistic elastic media and a generalization in Kijowski and Magli (1998) [42]. The theory is free of any assumption about the existence of a global relaxation state of the material. Kijowski and Magli (1992) [46] and Magli (1993) [45], [44] also studied interior solutions of the Einstein field equations in elastic media. A similar approach to that of Kijowski and Magli (1992) [39] was formulated by Cattaneo and Gerardi (1975) [23] and Cattaneo (1980) [22] based on the assumption that there exists a global relaxation state of the material when the gravitational interaction is hypothetically "switched off"; this, however, is poorly justified from the physical point of view and it is not relativistically invariant as remarked by Kijowski and Magli. Tahvildar-Zadeh (1998) [65] presented relativistic elastodynamics with small shear strains using a variational formulation. Recently, Beig and Schmidt (2003) [1] obtained existence and uniqueness theorems for elasticity in the setting of Einstein's gravity. A general existence theorem covering the case of elasticity has also been announced by Christodoulou (2000) [24]. Beig and Schmidt (2003) [2] showed the existence of static solutions describing elastic bodies deformed by their own Newtonian gravitational field and, later, Beig and Schmidt (2005) [3] established the existence of elastic bodies deformed under rigid rotation. Park (2000) [55] established and proved existence theorems for the case of spherically symmetric static solutions for elastic bodies. Calogero and Heinzle (2007) [11] studied the dynamics of Bianchi type I elastic space-times.

Most of the relativistic reformulations identify a material body with a three-dimensional material space manifold. Four-dimensional material space manifolds have been used by Kijowski et al. (1990) [43] and Kijowski and Magli (1998) [42] to describe thermoelastic continua where the extra dimension is associated with temperature rather than time.

1.2 Objectives

The state of the art reveals that, in recent years, there has been an increasing consideration of general relativistic elasticity. Due to the development in astrophysics and applications to neutron stars it is of considerable interest to extend and enlarge information in the area of general relativistic elasticity.

The main goal of this thesis is to provide advances in the area of general relativistic elasticity by contributing to it with this present work. The area of general relativistic elasticity is approached mathematically in the considered study topics.

The main contributions of this thesis are given to the following topics. The recognition and importance of general relativistic elasticity motivate a detailed study of quantities used in this context; in this thesis the mathematical investigation is concentrated on the elasticity difference tensor defined in Karlovini and Samuelsson (2003) [35]. The research area also motivates us to consider important subjects in general relativity and to adapt them to general relativistic elasticity in order to study them in this framework. The conformal transformations represent one of these topics. A first step of an approach in this field is given in this thesis. Furthermore, applications are carried out for spherically symmetric and axially symmetric space-times, since these space-times are used to model neutron stars. Also, existing results about non-static spherically symmetric space-time metrics with flat material metric are generalized by working with a non-flat material metric.

Some of the contributions, in particular the results presented in Chapter 3 and Chapter 5, namely the analysis of the elasticity difference tensor and the examples, appear in the publication Vaz and Brito (2008) [66]. Another article Brito, Vaz and Carot (2008) [10], which contains results obtained for non-static spherically symmetric metrics, given in Chapter 5 and Chapter 6, is in advanced preparation. Also a third article about conformal transformations is being developed, to which the results of Chapter 4 will contribute.

1.3 Outline

This thesis is structured as follows. Chapter 1 contains the state of the art, a description of the objectives of this thesis and its outline.

Chapter 2 introduces terminology and reviews basic concepts of the theory of relativistic elasticity which are relevant for the work presented in this thesis. The configuration map linking the space-time and the material space is introduced. The relativistic deformation gradient, associated with the configuration mapping, is an important element to pull-back tensors from the material space to the space-time. The material metric and the projection tensor are defined. These tensors are fundamental for the construction of the elasticity difference tensor. Two measures of strain and shear appearing in the context of relativistic elasticity are presented: the relativistic strain tensor and the constant volume shear tensor. A formulation based on the eigenvalues and the eigenvectors of the pulled-back material metric is described, which later plays an important role for the analysis of the elasticity difference tensor. The expression of the energy-momentum tensor for elastic matter is given. This chapter ends with the exposition of the elasticity difference tensor.

Chapter 3 is entirely devoted to the elasticity difference tensor by providing a mathematical analysis of this challenging object. First of all, basic properties are explained. Then it is shown how the elasticity difference tensor arises from the difference tensor when one considers two specific metrics. In this context an interesting expression is obtained, which indicates how to write the difference of the projected Riemann tensors associated with the two metrics entirely in terms of the elasticity difference tensor. These first results motivate the further study of this tensor in the future. A tetrad formulation of the elasticity difference tensor is presented and its traces are calculated. The elasticity difference tensor is decomposed along the eigenvectors of the pulled-back material metric into three second order tensors. The following eigenvalue-eigenvector problem is carried out for these tensors. It is studied under which conditions the eigenvectors of the pulled-back material metric remain also eigenvectors for the three second order tensors. The corresponding eigenvalues are then presented.

Chapter 4 provides a first step in the field of conformal transformations and conformally related metrics in general relativistic elasticity. It concerns the simplest case. Two conformally related material metrics are considered in the same space-time. Relations between some relativistic quantities are found. Also in this chapter, attention is focussed on the elasticity difference tensor and, again, the eigenvalue-eigenvector problem is analysed in this case.

Chapter 5 contains applications of the results obtained in the previous chapters to concrete examples. Spherically symmetric and axially symmetric space-times are chosen, since these space-times are used in the framework of relativistic elasticity to model neutron stars. The software *Maple GRTensor* was used to perform some calculations.

Another topic in general relativistic elasticity is investigated, namely, existing results for a given space-time and a flat material metric are generalized by considering a non-flat material metric conformally related with the flat one. Thereby the Einstein field equations are rewritten for this new configuration. These results appear in Chapter 6.

Finally, in Chapter 7, general conclusions of the work contained in this thesis are drawn and some attractive study topics are mentioned, which arise immediately from the ideas and methods which are presented here. These new problems reinforce the motivation and the interest for continuing this investigation which has been started in general relativistic elasticity.

Chapter 2

Elasticity in General Relativity - General Results

This chapter provides a review of the theory of general relativistic elasticity by explaining its basic concepts. Thereby it is mainly concentrated on presenting elements - definitions and formulations - which are relevant for the work presented here.
When searching for information about the theory of general relativistic elasticity in published articles one is faced with the problem of finding different notations, designations and definitions occasionally for the same relativistic elastic object, for example the strain tensor is defined in various manners.
From the disposal of various formalisms of general relativistic elasticity, definitions and formulations have to be selected according to the requirements of the work. Here, the main objective is to enlarge information and results about a specific tensor, the elasticity difference tensor, defined by Karlovini and Samuelsson (2003) [35] in the framework of general relativistic elasticity; to approach the problem of having two conformally related material metrics by studying consequences on some relativistic elastic quantities; and to reconsider a case studied by Magli (1993) [45] in order to generalize it. Therefore, the exposition given in this chapter follows the orientations proposed by Karlovini and Samuelsson (2003) [35] on the one hand, who pursued partly the work of Carter and Quintana (1972) [19], and by Magli (1993) [45] on the other hand.

This chapter starts with the set-up of the theory based on the existence of a configuration mapping linking the space-time and the material space. An important element is the relativistic deformation gradient used to pull-back tensors from the material space to the space-time. Fundamental tensors like the material metric and the projection tensor are defined needed to construct later the elasticity difference tensor. A formulation based on the eigenvalues and eigenvectors of the pulled-back material metric is presented. This formulation plays an important role for the analysis of the elasticity difference tensor in Chapter 3. The definition of the energy-momentum tensor in the context of general relativistic elasticity is given. Finally, the construction of the elasticity difference tensor is explained, one of the central points in this presentation, to which attention is focused in the next chapter.

2.1 Basic concepts

Let M be the general relativistic space-time, a four-dimensional manifold endowed with a Lorentz metric g of signature $(-,+,+,+)$, assumed to be time-orientable. Suppose that the space-time is filled with a continuous material. Let $\{\omega^a\}$, $a = 0,1,2,3$, be a coordinate system defined on M.

In order to distinguish tensors defined on M from tensors defined on the material space, which is the next topic to be presented, the following coordinate index convention is used throughout this work: lowercase Latin indices a,b,... take the values 0,1,2,3 and denote space-time indices; capital Latin indices A,B,... range from 1 to 3 and denote material indices.

Thus, tensor components written with lowercase Latin indices represent space-time tensors and tensors with capital Latin indices are called material tensors. The material tensors are defined on the material space X, which is described in the next subsection. Later, in order to distinguish tetrad indices from coordinate indices, the following index convention consisting of Greek letters is introduced: Orthonormal frame space-time indices are represented by letters from the second half of the Greek alphabet ($\mu, \nu, \rho ... =$

0, 1, 2, 3) and orthonormal frame spatial indices are denoted by letters from the first half of the Greek alphabet ($\alpha, \beta, \gamma ... = 1, 2, 3$). The Einstein summation convention and the notation for the symmetric part of tensors will be applied to coordinate indices only, unless otherwise stated.

Other notation conventions used in this thesis are the following: ε_{ABC} and ε^{abcd} represent permutation symbols, respectively defined by

$$\varepsilon_{ABC} = \begin{cases} +1 & \text{if } ABC \text{ is an even permutation of } 123 \\ -1 & \text{if } ABC \text{ is an odd permutation of } 123 \\ 0 & \text{if two or more indices are equal} \end{cases}$$

and

$$\varepsilon^{abcd} = \begin{cases} +1 & \text{if } abcd \text{ is an even permutation of } 0123 \\ -1 & \text{if } abcd \text{ is an odd permutation of } 0123 \\ 0 & \text{if two or more indices are equal.} \end{cases}$$

The symbol δ^a_b denotes the Kronecker delta.

2.1.1 Material Space

The **material space** X is a three-dimensional manifold, representing an abstract collection of idealized "molecules", or particles, of the material. The manifold X is also called the "body" or "body manifold". Each point in X represents a particle of the material. The coordinates on X are denoted by $\{\xi^A\}$, $A = 1, 2, 3$. The material space can be equipped with various types of tensor fields which characterize the structure of the material in a reference state. The most important of these tensor fields is certainly the material metric K_{AB}, measuring distances between particles in the locally relaxed state of matter. Another fundamental tensor defined on X is the particle density form n_{ABC}. Integrating this three-form over a certain volume in X yields the number of particles contained in that volume.

2.1.2 Configuration mapping

The space-time configuration of the material is described by a C^k ($k > 1$) mapping

$$\Psi : M \longrightarrow X,$$

the **configuration mapping**, which associates to each point p of the space-time M the particle $\bar{p} = \Psi(p) \in X$ of the material coinciding with p at a given time.
Using the coordinate systems defined previously, the configuration of the material can be described by the fields $\xi^A = \xi^A(\omega^a)$.
Associated with the configuration mapping, there are two important tools to work out the theory, namely the operators push-forward Ψ_* and pull-back Ψ^*.

The operator

$$\Psi_* : T_pM \longrightarrow T_{\Psi(p)}X$$

maps a vector $v \in T_pM$ to a vector $\Psi_*v \in T_{\Psi(p)}X$. The pushforward Ψ_*v of the vector v by Ψ is defined by[1]

$$(\Psi_*v)(f) = v(f \circ \Psi),$$

where f is a function $f : X \longrightarrow \mathbb{R}$ and $f \circ \Psi = \Psi^* f$.
The operator

$$\Psi^* : T^*_{\Psi(p)}X \longrightarrow T^*_pM$$

takes one-forms at $\Psi(p)$ to one-forms at p. The pullback $\Psi^*\varpi \in T^*_pM$ of the one-form $\varpi \in T^*_{\Psi(p)}X$ is defined by its action on a vector v:

$$(\Psi^*\varpi)(v) = \varpi(\Psi_*v).$$

These definitions can be extended to higher rank tensors. The pushforward of contravariant tensors S of type $(k, 0)$ is defined by

$$(\Psi_*S)(\varpi_1, ..., \varpi_k) = S(\Psi^*\varpi_1, ..., \Psi^*\varpi_k)$$

[1] See e.g. Wald (1984) [67] for a more detailed description of these standard definitions.

and the pullback of covariant tensors T of type $(0,l)$, by

$$(\Psi^*T)(v_1,...,v_l) = T(\Psi_* v_1,...,\Psi_* v_l).$$

2.1.3 Relativistic deformation gradient

The mapping

$$\Psi_* : T_pM \longrightarrow T_{\Psi(p)}X$$

gives rise to a (3×4) matrix in the chosen coordinate charts ω^a and ξ^A. This matrix is called the **relativistic deformation gradient**, whose entries are

$$\xi^A_a = \frac{\partial \xi^A}{\partial \omega^a},$$

representing the derivatives of the spatial coordinates with respect to the space-time coordinates.

The pushforward and the pullback operators can be described with the use of this matrix:

$$\Psi_* v^a = \frac{\partial \xi^A}{\partial \omega^a} v^a = v^A$$

and

$$\Psi^* \varpi_A = \frac{\partial \xi^A}{\partial \omega^a} \varpi_A = \varpi_a.$$

The relativistic deformation gradient is applied to material tensors to perform the pull-back operation and to space-time tensors to perform the pushforward operation, according to the previous definitions, in the following way.

The operator push-forward Ψ_* takes contravariant tensors from M to X:

$$\Psi_* t^{ab...} = \frac{\partial \xi^A}{\partial \omega^a}\frac{\partial \xi^B}{\partial \omega^b} \cdots t^{ab...} = t^{AB...},$$

and the operator pull-back Ψ^* takes covariant tensors from X to M:

$$\Psi^* t_{AB...} = \frac{\partial \xi^A}{\partial \omega^a}\frac{\partial \xi^B}{\partial \omega^b} \cdots t_{AB...} = t_{ab...}.$$

It is required that the relativistic deformation gradient has maximal rank, $\dim(\mathrm{Im}\Psi_*) = 3$, and that its Kernel is an one-dimensional timelike subspace of T_pM, $\forall p \in M$:

$\dim(\mathrm{Ker}\Psi_*) = 1$. Since the space-time is time orientable, one can choose a generator u^a of the Kernel such that:

$$u^0 > 0,\ u^a u_a = -1 \text{ and } u^a \xi_a^B = 0. \tag{2.1}$$

Notice that for each $\bar{p} \in X$, $\Psi^{-1}(\bar{p})$ is an integral curve of u, the worldline of the particle $\bar{p}$.

2.1.4 Matter velocity field

The vector field u^a, whose components are uniquely determined by the conditions (2.1), namely

$$u^0 > 0$$
$$u^a u_a = -1$$
$$u^a \xi_a^B = 0,$$

is called the **velocity field of the matter** or the matter four-velocity.

It is well known that the covariant derivative of a timelike unit vector field u can be decomposed as follows[2]

$$u_{a;b} = -\dot{u}_a u_b + u_{a;c} h^c{}_b = -\dot{u}_a u_b + \frac{1}{3}\Theta h_{ab} + \sigma_{ab} + \omega_{ab}, \tag{2.2}$$

where $\dot{u}^a$ denotes the acceleration vector, $h_{ab} = g_{ab} + u_a u_b$, σ_{ab} is the shear tensor field, ω_{ab}, the antisymmetric vorticity tensor field and Θ, the expansion scalar field for the congruence associated with u. These kinematical quantities are defined by

$$\dot{u}^a = u^a{}_{;b} u^b \tag{2.3}$$

$$\Theta = u^a{}_{;a} \tag{2.4}$$

$$\sigma_{ab} = u_{(a;b)} + \dot{u}_{(a} u_{b)} - \frac{1}{3}\Theta h_{ab} \tag{2.5}$$

$$\omega_{ab} = u_{[a;b]} + \dot{u}_{[a} u_{b]}, \tag{2.6}$$

[2]See Ehlers (1993) [25] or Stephani *et al.* (2003) [63].

where round brackets and square brackets enclosing indices denote, as usual, symmetrization and anti-symmetrization, respectively.

2.1.5 Particle density

Following Karlovini and Samuelsson (2003) [35], let $n_{abc} = \Psi^* n_{ABC}$ be the pull-back of the particle density form and ϵ_{abcd} be the space-time volume form associated with g_{ab}. The flowline tangential particle current is defined by

$$n^a = \frac{1}{3!}\epsilon^{abcd} n_{bcd}. \tag{2.7}$$

The particle current satisfies the continuity equation $\nabla_a n^a = 0$ and is proportional to u, i.e. $n^a = nu^a$. The quantity n, satisfying $n = \sqrt{-n_a n^a}$, is the **particle density**. Defining the spatial volume form by $\epsilon_{abc} = \epsilon_{abcd} u^d$, from (2.7) it follows that

$$n_{abc} = n\epsilon_{abc}. \tag{2.8}$$

The relation between the particle density and the material metric is shown later in Section 2.1.11.

2.1.6 Material metric

As stated before, the **material metric** K_{AB} is a second order symmetric tensor with signature (+,+,+) defined on the material space. This Riemannian metric describes the 'would be" rest-frame space distances between adjacent particles measured in the locally relaxed state of the material[3]. This state can be understood as follows. Considering an infinitesimal portion of the material, this portion will tend spontaneously to a relaxed state, where no external forces act on it. At the locally relaxed state the influence of the rest of the material, possibly prestressed, is eliminated. A material is said to be prestressed if it corresponds to a curved material metric. The flat, Euclidean metric corresponds to non prestressed materials and is the simplest example of a material metric.

[3] See e.g. Kijowski and Magli (1997) [41], Kijowski and Magli (2005) [38].

2.1.7 Pulled-back material metric

The pull-back of the material metric

$$k_{ab} = \Psi^* K_{AB} = \xi^A_a \xi^B_b K_{AB} \tag{2.9}$$

is a Riemannian metric tensor on the subspaces of T_pM orthogonal to u^a. The pulled-back material metric is symmetric and satisfies

$$k_{ab}u^a = 0 \tag{2.10}$$

and

$$\mathcal{L}_u k_{ab} = 0, \tag{2.11}$$

so that k is Lie dragged by u.

For an arbitrary tensor field $T^{a_1 \dots a_k}{}_{b_1 \dots b_l}$ the Lie derivative is defined by

$$\mathcal{L}_u T^{a_1 \dots a_k}{}_{b_1 \dots b_l} = u^c T^{a_1 \dots a_k}{}_{b_1 \dots b_l ; c} - \sum_{i=1}^{k} T^{a_1 \dots c \dots a_k}{}_{b_1 \dots b_l} u^{a_i}{}_{;c} + \sum_{j=1}^{l} T^{a_1 \dots a_k}{}_{b_1 \dots c \dots b_l} u^c{}_{;b_j}.$$

The two conditions (2.10) and (2.11) are consequences of (2.9). Since $u^a \xi^A_a = 0$, the condition (2.10) follows immediately. To prove the second condition, first, rewrite (2.11), using the definition of the Lie derivative and (2.10), as

$$\mathcal{L}_u k_{ab} = u^c (k_{ab,c} - k_{ac,b} - k_{bc,a}),$$

where a comma denotes a partial derivative. Then, substituting (2.9) into the last expression and from $u^c \xi^C_c = 0$, $u^c \dfrac{\partial K_{AB}}{\partial \omega^c} = u^c \xi^C_c \dfrac{\partial K_{AB}}{\partial \xi^C} = 0$ and from the fact that second partial derivatives commute, one concludes that $\mathcal{L}_u k_{ab} = 0$.

The condition (2.11) means that the material distance between particles remains constant along the matter flow. In this case, the material is said to have no memory[4]. More precisely, it means that extracting an infinitesimal portion of the material and letting it relax, in order to achieve a description of the particles' distance in the locally relaxed state where the material is free from external forces, leads to the same distance

[4] See Kijowski and Magli (1994) [40].

between the particles, independently of the moment at which the portion has been extracted.

2.1.8 Projection tensor

The **projection tensor** is the second order symmetric tensor defined by

$$h_{ab} = g_{ab} + u_a u_b, \tag{2.12}$$

associated with the matter velocity field u. It is a Riemannian metric tensor on the subspaces of T_pM orthogonal to u^a and clearly satisfies $h_{ab}u^a = 0$. A relevant task of the projection tensor consists in decomposing tensor fields $t^{a\ldots}{}_{b\ldots}$, defined on M, into components perpendicular to the matter velocity field by projecting them orthogonally to u:

$$T^{c\ldots}{}_{d\ldots} = h^c{}_a \cdots h^b{}_d \, t^{a\ldots}{}_{b\ldots}.$$

In this way, the tensor field $T^{c\ldots}{}_{d\ldots}$ is such that $T^{c\ldots}{}_{d\ldots}u_c = \cdots = T^{c\ldots}{}_{d\ldots}u^d = 0$.

2.1.9 Relativistic strain tensor

In the relativistic literature different definitions are proposed for the strain tensor - a tensor which measures the state of strain of the material. Most of them are based on the idea that the material is locally relaxed, or unstrained, at a point $x \in M$ if and only if the material metric coincides with the physical metric on the subspace orthogonal to u^a, in which case

$$h_{ab} = k_{ab}, \tag{2.13}$$

i.e. the pulled-back material metric is equal to the projection tensor.
Following Cattaneo (1973) [21], Maugin (1978) [49], Magli (1993) [45] and [44], the **strain tensor** is defined by

$$s_{ab} = \frac{1}{2}(h_{ab} - k_{ab}). \tag{2.14}$$

This tensor measures the deviation of the material from the unstrained state, or locally relaxed state, by comparing the value of h_{ab} with the value of the pulled-back material metric k_{ab}. Therefore, the material is said to be in an unstrained state, or locally relaxed, if the strain tensor vanishes.

The strain tensor can be rewritten as

$$s_{ab} = \frac{1}{2}(g_{ab} - \tilde{g}_{ab}), \tag{2.15}$$

where

$$\tilde{g}^a{}_b = g^{ac}(k_{cb} - u_c u_b) \tag{2.16}$$

is called the symmetric strain operator[5].

The operator $\tilde{g}^a{}_b$ and three invariants of the strain tensor defined by

$$\begin{aligned} I_1 &= \frac{1}{2}\left(\mathrm{Tr}\tilde{g} - 4\right) \\ I_2 &= \frac{1}{4}\left[\mathrm{Tr}\tilde{g}^2 - \left(\mathrm{Tr}\tilde{g}\right)^2\right] + 3 \\ I_3 &= \frac{1}{2}\left(\det\tilde{g} - 1\right) \end{aligned} \tag{2.17}$$

have been used by Magli (1993) [45] to rewrite the energy-momentum tensor as will be seen in Section 2.1.12.2.

These invariants are related to the coefficients of the characteristic polynomial of $\tilde{g}^a{}_b$ in the following way:

$$\begin{aligned} \det(\tilde{g}^a{}_b - \lambda\delta^a{}_b) =& (1-\lambda)\det(k^a{}_b - \lambda\delta^a{}_b) = \\ & \lambda^4 - (2I_1 + 4)\lambda^3 - 2I_2\lambda^2 - (2I_3 - 2I_2 - 2I_1 + 4)\lambda + 2I_3 + 1. \end{aligned}$$

Carter and Quintana (1972) [19] give a more detailed description of the unstrained state, by explaining that this state, which the Hookean elasticity theory is based on, is a locally relaxed state of the material in the sense that at each point in X there is a particular value κ_{ab} of the projection tensor h_{ab} for which the energy per particle ϵ is

[5] See e.g. Magli (1993) [44].

minimum. This tensor, κ_{ab}, is designated as unstrained reference tensor.

Together with h_{ab}, κ_{ab} builds the Lagrangian strain tensor

$$e_{ab} = \frac{1}{2}(h_{ab} - \kappa_{ab}). \tag{2.18}$$

However, the same authors avoided using a strain tensor and introduced a shear tensor for the reasons explained in the next section.

2.1.10 Constant volume shear tensor

Carter and Quintana claimed that the theory using the unstrained reference tensor, the value of the projection tensor for which the energy per particle takes a minimum value, and the Lagrangian strain tensor (2.18), based on the Hookean elasticity theory is inadequate for the description of solid matter at high pressures occurring in the interior of neutron stars, because a fully relaxed state may not exist. For example the crystalline structure in neutron star crusts would break down by relaxing that crystal. Under these circumstances it is impossible to define the unstrained reference tensor and the Lagrangian strain tensor. To solve this problem, Carter and Quintana (1972) [19], followed by Karlovini and Samuelsson (2003) [35], suggested considering states in which the material has not an absolute minimum of energy per particle ϵ_0, but a minimum by restricting the constant particle number density n. This state is defined as the unsheared state.

Consider a family of positive definite tensor fields $\eta_{AB}(n)$ parameterized by n. For the unsheared state with particle number density n, the value of $\eta_{AB}(n)$ represents the value of h_{AB} in which ϵ has the minimum value $\check{\epsilon}$ for that particular value of n. The tensor η_{AB} is such that $g^{AC}\eta_{CB} = \delta^A{}_B$ for the unsheared state and $g^{AC} = \eta^{-1AC}$.

In this case, the **constant volume shear tensor** s_{ab} defined as[6]

$$s_{ab} = \frac{1}{2}(h_{ab} - \eta_{ab}) \tag{2.19}$$

[6]See Carter and Quintana (1972) [19], Karlovini and Samuelsson (2003) [35].

is used to measure deviations from the state of minimum energy at fixed particle number density n. It gives the difference between the actual value of h_{ab} and the corresponding value $\eta_{ab} = \Psi^* \eta_{AB}$ for the unsheared state at the same volume, where η_{ab} describes the most relaxed state at a given fixed particle density n. The constant volume shear tensor vanishes for the unsheared state.

Let ϵ_{ABC} be the volume form of η_{AB}, its pull-back coincides with the spatial volume form $\Psi^* \epsilon_{ABC} = \epsilon_{abc}$. From (2.8), it turns out that the relation between the particle density form n_{ABC} and ϵ_{ABC} is given by

$$n_{ABC} = n \epsilon_{ABC}. \tag{2.20}$$

The authors Karlovini and Samuelsson (2003) [35] defined for reasons of convenience the tensor K_{AB} as being conformal to η_{AB} and having n_{ABC} as its volume form.

Let $K_{AB} = f(n) \eta_{AB}$, then the relation between the determinants $\det K$ of K_{AB} and $\det \eta$ of η_{AB} is given by

$$\det K = f^3(n) \det \eta.$$

On the other hand, due to (2.20) and the definitions of the volume forms

$$n_{ABC} = \sqrt{\det K}\, \varepsilon_{ABC}$$

and

$$\epsilon_{ABC} = \sqrt{\det \eta}\, \varepsilon_{ABC},$$

it follows that

$$\det K = n^2 \det \eta.$$

One concludes that $f(n)$ must be of the form

$$f(n) = n^{\frac{2}{3}}.$$

Thus, the tensor K_{AB} is defined by

$$K_{AB} = n^{\frac{2}{3}} \eta_{AB}. \tag{2.21}$$

In this thesis, the following expression for the constant volume shear tensor is considered

$$s_{ab} = \frac{1}{2}(h_{ab} - n^{-2/3}k_{ab}). \tag{2.22}$$

It is used in Chapter 4 to obtain a formula relating the constant volume shear tensors for two conformally related metrics and in Chapter 5 to investigate whether a material is in an unsheared state for a given space-time configuration.

2.1.11 Eigenvalue and eigenvector formulation

In this section, the eigenvalue-eigenvector formulation for the pulled-back material metric ${k^a}_b$ is presented. The eigendirections of ${k^a}_b$ are used to construct an orthonormal tetrad that substitutes the space-time metric whenever one uses the tetrad formalism. This tetrad has the special property that it contains information about the material metric. The relationship between the eigenvalues of ${k^a}_b$ and the particle density is also obtained. The tetrad and the associated eigenvalue-eigenvector formulation, which was proposed by Karlovini and Samuelsson (2003) [35], enable to write other tensors, like the pressure tensor and the energy-momentum tensor, in terms of the eigenvectors or also in terms of the eigenvalues of ${k^a}_b$, as will be seen later. The formulation and the tetrad also play a main role in the analysis of the elasticity difference tensor in Chapter 3.

Having pointed out the highlights of this section, now they are described in more detail. From (2.10), it is clear that u^b is an eigenvector of ${k^a}_b$ associated with the eigenvalue 0. Let the non zero eigenvalues of ${k^a}_b$ be denoted by n_1^2, n_2^2 and n_3^2. It is well known that the eigenvalues are determined from the standard equation

$$|{k^a}_b - \lambda {\delta^a}_b| = 0.$$

These eigenvalues are related with the determinant of ${k^a}_b$, which in turn can be used to define the particle density n, introduced in Section 2.1.5. One has

$$n^2 = n_1^2 n_2^2 n_3^2 = \det({k^a}_b). \tag{2.23}$$

It follows that the particle density is the product of three positive quantities

$$n = n_1 n_2 n_3 = \sqrt{\det(k^a{}_b)}. \tag{2.24}$$

These quantities n_1, n_2 and n_3, the positive square roots of the eigenvalues of k, are called linear particle densities[7], since their product equals the volume particle density n.

The next step to set up the tetrad is to find the three spacelike eigendirections of $k^a{}_b$ associated with the three eigenvalues, according to the equation

$$(k^a{}_b - \lambda \delta^a{}_b) v^b = 0,$$

and to join the timelike vector u^a, the velocity field of matter, to them. The spacelike eigenvectors x^a, y^a and z^a for the tetrad, which are orthogonal to u^a, are determined to satisfy the orthonormality conditions

$$-u_a u^a = x_a x^a = y_a y^a = z_a z^a = 1,$$

all other inner products being zero.

The constructed tetrad $\{u^a, x^a, y^a, z^a\}$ in M consists of the three spacelike eigenvectors of $k^a{}_b$ and of the velocity field of matter u^a.

Using this tetrad, the pulled-back material metric can be written as

$$k_{ab} = n_1^2 \, x_a x_b + n_2^2 \, y_a y_b + n_3^2 \, z_a z_b, \tag{2.25}$$

and the space-time metric takes the form

$$g_{ab} = -u_a u_b + h_{ab} = -u_a u_b + x_a x_b + y_a y_b + z_a z_b, \tag{2.26}$$

where

$$h_{ab} = x_a x_b + y_a y_b + z_a z_b. \tag{2.27}$$

It should be noticed that the eigenvectors x, y, z are automatically orthogonal whenever the corresponding eigenvalues are distinct. However, if the eigenvalues are not all distinct, the eigendirections associated with the same eigenvalue can be chosen to be orthogonal.

[7] See Karlovini and Samuelsson (2003) [35].

2.1.12 Relativistic energy-momentum tensor

In the context of relativistic elasticity, one can find two different expressions for the energy-momentum tensor: the symmetric and the canonical energy-momentum tensor. The Einstein equations $G_{ab} = 8\pi T_{ab}$ describe, by means of the energy-momentum tensor for elastic matter, the interaction of the elastic material with the gravitational field. In this thesis, the Einstein equations with the canonical energy-momentum tensor are considered in the last Chapter, where they are "re-obtained" for a generalized existing case.

The equations of motion for elastic matter $\nabla_b T^{ab} = 0$, using the symmetric energy-momentum tensor, appear in Section 2.2, where it is shown that they can be written in terms of the elasticity difference tensor.

It is worth mentioning that, up to now, the dominant energy conditions have been studied by Calogero and Heinzle (2007) [11] for Bianchi type I elastic space-times. They showed that the dominant energy conditions are violated by a particular class of constitutive equations as the singularity is approached.

2.1.12.1 Symmetric energy-momentum tensor

The expression for the **symmetric energy-momentum tensor** is[8]

$$T_{ab} = -\rho g_{ab} + 2\frac{\partial \rho}{\partial g^{ab}} = \rho u_a u_b + p_{ab}, \tag{2.28}$$

where the pressure tensor

$$p_{ab} = 2\frac{\partial \rho}{\partial g^{ab}} - \rho h_{ab} \tag{2.29}$$

satisfies

$$u^a p_{ab} = 0. \tag{2.30}$$

The energy density ρ can be rewritten as

$$\rho = n\epsilon, \tag{2.31}$$

[8] See e.g. Karlovini and Samuelsson (2003) [35].

where n denotes the particle density and ϵ is the energy per particle.

Karlovini and Samuelsson (2003) [35] used the tetrad $\{u, x, y, z\}$, defined previously, to write the pressure tensor as

$$p_{ab} = p_1 x_a x_b + p_2 y_a y_b + p_3 z_a z_b, \tag{2.32}$$

where

$$p_i = n n_i \frac{\partial \epsilon}{\partial n_i} \tag{2.33}$$

are the eigenvalues of $p^a{}_b$, called principal pressures.

This expression for the pressure tensor is obtained from (2.29) in the following way. In order to simplify the calculations, the notation[9] $\{e_\alpha\}$, $\alpha = 1, 2, 3$, is introduced for the spacelike eigenvectors, where $\{e_1^a, e_2^a, e_3^a\} = \{x^a, y^a, z^a\}$.

To begin with, rewrite the operator $\frac{\partial}{\partial g^{ab}}$ as

$$\frac{\partial}{\partial g^{ab}} = \frac{\partial k^c{}_d}{\partial g^{ab}} \frac{\partial}{\partial k^c{}_d}. \tag{2.34}$$

Then, since

$$\frac{\partial k^c{}_d}{\partial g^{ab}} = \frac{\partial (g^{cm} k_{md})}{\partial g^{ab}} = \delta^c{}_{(a} k_{b)d},$$

it follows from (2.34) that

$$\frac{\partial}{\partial g^{ab}} = \frac{1}{2}\left(k_{ca} \frac{\partial}{\partial k^b{}_c} + k_{cb} \frac{\partial}{\partial k^a{}_c} \right), \tag{2.35}$$

which can further be written as

$$\frac{\partial}{\partial g^{ab}} = \frac{1}{2}\left(k_{ca} \sum_{\tau=1}^{3} \frac{\partial n_\tau}{\partial k^b{}_c} \frac{\partial}{\partial n_\tau} + k_{cb} \sum_{\sigma=1}^{3} \frac{\partial n_\sigma}{\partial k^a{}_c} \frac{\partial}{\partial n_\sigma} \right). \tag{2.36}$$

To proceed, a useful relation for $\dfrac{\partial n_\sigma}{\partial k^a{}_c}$ is obtained.

Writing the pulled-back material metric as

$$k^a{}_b = \sum_{\alpha=1}^{3} n_\alpha^2 e_\alpha^a e_{\alpha b}$$

[9] Please see the explanation given on page 10 for the index convention for Greek indices.

and contracting this equation with $e_{\alpha a}e^b_\alpha$, yields the eigenvalue

$$n_\alpha^2 = k^a{}_b e_{\alpha a} e^b_\alpha,$$

so that the linear particle density is given by

$$n_\alpha = \sqrt{k^a{}_b e_{\alpha a} e^b_\alpha}.$$

Therefore, calculating $\dfrac{\partial n_\alpha}{\partial k^a{}_b}$ leads to

$$\frac{\partial n_\alpha}{\partial k^a{}_b} = \frac{1}{2n_\alpha} e_{\alpha a} e^b_\alpha. \tag{2.37}$$

Returning to expression (2.36), using (2.37) and $k_{ab} = \displaystyle\sum_{\alpha=1}^{3} n_\alpha^2 e_{\alpha a} e_{\alpha b}$, it turns out that the operator $\frac{\partial}{\partial g^{ab}}$ takes the form

$$\frac{\partial}{\partial g^{ab}} = \frac{1}{2}\sum_{\alpha=1}^{3} n_\alpha e_{\alpha a} e_{\alpha b} \frac{\partial}{\partial n_\alpha}. \tag{2.38}$$

This expression allows to write the pressure tensor (2.29) in the form

$$p_{ab} = \sum_{\alpha=1}^{3} n_\alpha \frac{\partial \rho}{\partial n_\alpha} e_{\alpha a} e_{\alpha b} - \rho h_{ab}. \tag{2.39}$$

Continuing the calculations and taking into account (2.31), $n = n_1 n_2 n_3$ and $h_{ab} = \displaystyle\sum_{\alpha=1}^{3} e_{\alpha a} e_{\alpha b}$ one gets

$$p_{ab} = \sum_{\alpha=1}^{3} n n_\alpha \frac{\partial \epsilon}{\partial n_\alpha} e_{\alpha a} e_{\alpha b}, \tag{2.40}$$

which is equivalent to writing (2.32) together with (2.33).

One concludes that the pressure tensor $p^a{}_b$ and the pulled-back material metric $k^a{}_b$ have the same eigenvectors.

The energy-momentum tensor (2.28) becomes

$$T_{ab} = n\epsilon u_a u_b + n n_1 \frac{\partial \epsilon}{\partial n_1} x_a x_b + n n_2 \frac{\partial \epsilon}{\partial n_2} y_a y_b + n n_3 \frac{\partial \epsilon}{\partial n_3} z_a z_b, \tag{2.41}$$

when the relations (2.31) and (2.32) with (2.33) are used.

The definitions for the energy-momentum tensor and the pressure tensor presented here are considered in Chapter 4, where they are rewritten for two conformally related material metrics.

2.1.12.2 Canonical energy-momentum tensor

Another expression for the energy momentum tensor that appears in the general relativistic elastic literature is the **canonical energy-momentum tensor** given by

$$\mathcal{T}^a_{\ b} = \frac{1}{\sqrt{-g}} \left(\frac{\partial \Lambda}{\partial \xi^A_a} \xi^A_b - \delta^a{}_b \Lambda \right) = \delta^a{}_b \rho - \frac{\partial \rho}{\partial \xi^A_a} \xi^A_b, \tag{2.42}$$

where ρ denotes the energy density[10], ξ^A_a represents the relativistic deformation gradient and Λ is the Lagrangian density defined by

$$\Lambda = -\sqrt{-g}\rho. \tag{2.43}$$

Kijowski and Magli (1994) [40] showed, using the Belinfante-Rosenfeld theorem[11], that this definition coincides with the definition of the symmetric energy-momentum tensor (2.28)

$$T_{ab} = -\frac{2}{\sqrt{-g}} \frac{\partial \Lambda}{\partial g^{ab}} = 2 \frac{\partial \rho}{\partial g^{ab}} - \rho g_{ab}, \tag{2.44}$$

used e.g. by Karlovini and Samuelsson (2003) [35], Beig and Schmidt (2003) [1], Beig and Wernig-Pichler (2007) [4], up to a sign:

$$\mathcal{T}_{ab} = -T_{ab}. \tag{2.45}$$

In order to prove this result, consider the push-forward of the space-time metric $\Psi^* g^{cd} = G^{CD}$ given by

$$G^{CD} = \xi^C_c \xi^D_d g^{cd}. \tag{2.46}$$

The canonical energy-momentum tensor (2.42) can be rewritten as

$$\mathcal{T}^a_{\ b} = -\frac{\partial \rho}{\partial G^{CD}} \frac{\partial G^{CD}}{\partial \xi^A_a} \xi^A_b + \rho \delta^a{}_b. \tag{2.47}$$

[10] Some authors, e.g. Magli (1993) [45], use the letter ϵ to represent the energy density.

[11] See Belinfante (1994) [5] and Rosenfeld (1940) [59].

Calculating $\dfrac{\partial G^{CD}}{\partial \xi^A_a}$ from (2.46) leads to

$$\frac{\partial G^{CD}}{\partial \xi^A_a} = g^{ad}\delta^C{}_A \xi^D_d + g^{ac}\delta^D{}_A \xi^C_c . \tag{2.48}$$

Substituting this result in (2.47) gives

$$\mathcal{T}^a_{\ b} = -2\frac{\partial \rho}{\partial G^{AD}} g^{ad}\xi^A_b\xi^D_d + \rho\delta^a{}_b . \tag{2.49}$$

On the other hand, the symmetric energy-momentum tensor (2.28) can be expressed as

$$T_{ab} = 2\frac{\partial \rho}{\partial G^{CD}}\frac{\partial G^{CD}}{\partial g^{ab}} - \rho g_{ab}. \tag{2.50}$$

From (2.46) it follows that

$$\frac{\partial G^{CD}}{\partial g^{ab}} = \frac{1}{2}\left(\xi^C_a\xi^D_b + \xi^C_b\xi^D_a\right). \tag{2.51}$$

Consequently, (2.50) transforms in the following way, when one rises the first index and uses the last result:

$$T^a_{\ b} = 2\frac{\partial \rho}{\partial G^{CD}} g^{am}\xi^C_m\xi^D_b - \rho g^a{}_b . \tag{2.52}$$

Comparing this expression with (2.49), one obtains the expected result: the expressions for the energy-momentum tensor differ in a sign.

The energy-momentum tensor (2.42) can be rewritten as[12]

$$\mathcal{T}^a_{\ b} = \rho\delta^a{}_b - \frac{\partial \rho}{\partial I_3}\det\tilde{g}\; h^a{}_b + \left(\mathrm{Tr}\tilde{g}\frac{\partial \rho}{\partial I_2} - \frac{\partial \rho}{\partial I_1}\right)k^a{}_b - \frac{\partial \rho}{\partial I_2}k^a{}_c k^c{}_b , \tag{2.53}$$

where the operator $\tilde{g}$, already presented in (2.16), is defined by $\tilde{g}^a{}_b = g^{ac}(k_{cb} - u_c u_b)$, k being the pulled-back material metric, and $h^a{}_b$ represents the projection tensor $h^a{}_b = g^a{}_b + u^a u_b$. The quantities I_1, I_2 and I_3 are the invariants of the strain tensor given in (2.17), chosen by Magli (1993) [45] to parameterize the equation of state by writing the energy density as a function of these invariants.

[12] See Magli (1993) [45].

To obtain (2.53) from (2.42), first note that[13]:

$$\frac{\partial\rho}{\partial\xi_a^A}\xi_b^A = \frac{\partial\rho}{\partial I_C}\frac{\partial I_C}{\partial\tilde{g}_{de}}\frac{\partial\tilde{g}_{de}}{\partial\xi_a^A}\xi_b^A. \tag{2.54}$$

This expression shows how to relate the energy density with the relativistic deformation gradient using the dependence on the invariants and on $\tilde{g}$.

Applying $\frac{\partial}{\partial\tilde{g}_{de}}$ to (2.17), the expressions $\frac{\partial I_C}{\partial\tilde{g}_{de}}$ for $C = 1, 2, 3$ become:

$$\begin{aligned}
\frac{\partial I_1}{\partial\tilde{g}_{de}} &= \frac{1}{2}\frac{\partial\mathrm{Tr}\tilde{g}}{\partial\tilde{g}_{de}} = \frac{1}{2}g^{de} \\
\frac{\partial I_2}{\partial\tilde{g}_{de}} &= \frac{1}{4}\left(\frac{\partial\mathrm{Tr}\tilde{g}^2}{\partial\tilde{g}_{de}} - \frac{\partial(\mathrm{Tr}\tilde{g})^2}{\partial\tilde{g}_{de}}\right) = \frac{1}{2}\left(\tilde{g}^{de} - \tilde{g}^m{}_m g^{de}\right) \\
\frac{\partial I_3}{\partial\tilde{g}_{de}} &= \frac{1}{2}\frac{\partial\mathrm{det}\tilde{g}}{\partial\tilde{g}_{de}} = \frac{1}{2}\left(\mathrm{det}\tilde{g}\right)\tilde{g}^{-1de},
\end{aligned} \tag{2.55}$$

where

$$\begin{aligned}
\mathrm{Tr}\tilde{g} &= g^{mf}\,\tilde{g}_{fm} \\
\mathrm{Tr}\tilde{g}^2 &= g^{ab}\,\tilde{g}_{bm}\,g^{mp}\,\tilde{g}_{pa} \\
\mathrm{det}\tilde{g} &= \varepsilon^{abcd}\,\tilde{g}_{a0}\,\tilde{g}_{b1}\,\tilde{g}_{c2}\,\tilde{g}_{d3}
\end{aligned} \tag{2.56}$$

$$\tilde{g}^{-1de}\tilde{g}_{ef} = \delta^d{}_f. \tag{2.57}$$

Further, since $\tilde{g}_{de} = k_{de} - u_d u_e$ and $k_{de} = \xi_d^D\xi_e^E K_{DE}$, calculating $\frac{\partial\tilde{g}_{de}}{\partial\xi_a^A}\xi_b^A$ yields

$$\frac{\partial\tilde{g}_{de}}{\partial\xi_a^A}\xi_b^A = k_{be}\delta^a{}_d + k_{bd}\delta^a{}_e - u_d\frac{\partial u_e}{\partial\xi_a^A}\xi_b^A - u_e\frac{\partial u_d}{\partial\xi_a^A}\xi_b^A. \tag{2.58}$$

Hence, multiplying $\frac{\partial\tilde{g}_{de}}{\partial\xi_a^A}\xi_b^A$ with each expression given in (2.55), one obtains

$$\begin{aligned}
\frac{\partial I_1}{\partial\tilde{g}_{de}}\frac{\partial\tilde{g}_{de}}{\partial\xi_a^A}\xi_b^A &= k^a{}_b \\
\frac{\partial I_2}{\partial\tilde{g}_{de}}\frac{\partial\tilde{g}_{de}}{\partial\xi_a^A}\xi_b^A &= k^{ad}k_{db} - \tilde{g}^m{}_m k^a{}_b \\
\frac{\partial I_3}{\partial\tilde{g}_{de}}\frac{\partial\tilde{g}_{de}}{\partial\xi_a^A}\xi_b^A &= \mathrm{det}\tilde{g}h^a{}_b,
\end{aligned} \tag{2.59}$$

where the following relations are used: $u^d\frac{\partial u_d}{\partial\xi_a^A} = 0$, $\tilde{g}^a{}_b u^b = u^a$, $(\tilde{g}^{-1})^a{}_b u^b = u^a$ and $\tilde{g}^{ad}k_{db} = k^{ad}k_{db}$.

[13] See the appendix given in Magli (1993) [45].

Inserting (2.59) in

$$\mathcal{T}^a_{\ b} = \delta^a{}_b \rho - \frac{\partial \rho}{\partial I_C} \frac{\partial I_C}{\partial \tilde{g}_{de}} \frac{\partial \tilde{g}_{de}}{\partial \xi^A_a} \xi^A_b,$$

gives the desired formula (2.53).

2.2 Elasticity difference tensor

The elasticity difference tensor, one of the main topics in this thesis, was introduced in the literature by Karlovini and Samuelsson (2003) [35].

After having defined basic concepts, like the material metric, its pull-back and the projection tensor, needed to construct the elasticity difference tensor, the way is prepared to devote this section to the elasticity difference tensor.

In Section 2.2.1, it is explained how the elasticity difference tensor was defined by Karlovini and Samuelsson (2003) [35]. Further, in Section 2.2.2, it is shown how the equations of motion can be written in terms of this tensor.

Here, it becomes clear that the elasticity difference tensor is an important tensor in general relativistic elasticity, since, on the one hand, it depends on the pulled-back material metric and on the projected space-time connection and, on the other hand, it gives a contribution to the equations of motion by entering in its expression.

The next chapter is entirely focussed on the elasticity difference tensor, where it is studied in more detail.

2.2.1 Definition

On the way to the presentation of the elasticity difference tensor, another operator is introduced: the spatially projected connection. This operator is also needed to construct the elasticity difference tensor.

Let ∇ represent the connection associated with the space-time metric g_{ab}. The spatially projected connection D_a is defined by acting on an arbitrary tensor field $t^{b\dots}{}_{c\dots}$ as follows:

$$D_a t^{b\ldots}{}_{c\ldots} = h^d{}_a h^b{}_e \ldots h^f{}_c \ldots \nabla_d t^{e\ldots}{}_{f\ldots}, \tag{2.60}$$

and it satisfies $D_a h_{bc} = 0$.

Before proceeding, the definition of the convected derivative is provided. The convected derivative $[T^{b\ldots}{}_{a\ldots}]^{\cdot}$ of a general mixed space-time tensor field $T^{b\ldots}{}_{a\ldots}$ is defined by[14]

$$[T^{b\ldots}{}_{a\ldots}]^{\cdot} = \dot{T}^{b\ldots}{}_{a\ldots} - T^{c\ldots}{}_{a\ldots}(u^b{}_{;c} + u^b \dot{u}_c) - \cdots + T^{b\ldots}{}_{c\ldots}(u^c{}_{;a} + u^c \dot{u}_a) + \cdots ,$$

where

$$\dot{T}^{b\ldots}{}_{a\ldots} = T^{b\ldots}{}_{a\ldots;c} u^c$$

and

$$\dot{u}_c = u_{c;m} u^m .$$

When applied to covariant orthogonal space-time tensors, the convected derivative coincides with the Lie derivative. The condition of having zero convected derivative is necessary for the pushforward of a space-time vector field to be a well defined vector field on the material space. Orthogonal tensors with zero convected derivative are said to be materially constant.

Now, consider a differential operator $\tilde{D}_a$ acting on space-time tensors obtained from the pull-back of the Levi-Civita connection $\tilde{D}_A$ of k_{AB} under the following hypothesis: (i) there exists a torsion-free connection $\tilde{\nabla}$ on M such that

$$\tilde{D}_a t^{b\ldots}{}_{c\ldots} = h^d{}_a h^b{}_e \ldots h^f{}_c \ldots \tilde{\nabla}_d t^{e\ldots}{}_{f\ldots}; \tag{2.61}$$

(ii) for all space-time vector fields V^b and Z^a, which have zero convected derivative

$$\Psi_*(V^b \tilde{D}_b Z^a) = V^B \tilde{D}_B Z^A, \qquad V^B = \Psi_*(V^b), \qquad Z^A = \Psi_*(Z^a).$$

The operator $\tilde{D}_a$ satisfies $\tilde{D}_a k_{bc} = 0$.

[14] A comprehensive description of the convected derivative can be found in Carter and Quintana (1972) [19].

It follows that

$$\tilde{D}_b X^a - D_b X^a = h^m{}_b h^a{}_n (\tilde{\nabla}_m X^n - \nabla_m X^n) = S^a{}_{bc} X^c, \tag{2.62}$$

for any space-time vector field X.

The tensor field $S^a{}_{bc}$ is the **elasticity difference tensor**.

Using hypothesis (ii), this third order tensor can be written as

$$S^a{}_{bc} = \frac{1}{2} k^{-1am} (D_b k_{mc} + D_c k_{mb} - D_m k_{bc}), \tag{2.63}$$

where k^{-1am} is such that $k^{-1am} k_{mb} = h^a{}_b$.

As can be seen, the elasticity difference tensor depends on the pulled-back material metric and on the spatially projected connection associated with the space-time metric.

2.2.2 Equations of motion for elastic matter

The energy-momentum tensor T_{ab} satisfies the equations of motion

$$\nabla_b T^{ab} = 0, \tag{2.64}$$

which are also called energy and momentum conservation equations.

Applying the equations of motion to the energy-momentum tensor given by (2.28):

$$T_{ab} = \rho u_a u_b + p_{ab}, \tag{2.65}$$

where

$$p_{ab} = 2 \frac{\partial \rho}{\partial g^{ab}} - \rho h_{ab}, \tag{2.66}$$

and projecting the equations along u,

$$u_a \nabla_b T^{ab}, \tag{2.67}$$

and orthogonal to u,

$$h^a{}_c \nabla_b T^{cb}, \tag{2.68}$$

one finds, respectively,

$$\rho_{,b}u^b + (\rho h^{ab} + p^{ab})D_{(a}u_{b)} = 0 \tag{2.69}$$

$$(\rho h^{ab} + p^{ab})\dot{u}_b + D_b p^{ab} = 0, \tag{2.70}$$

where D_a denotes the spatially projected connection defined in (2.60).

Karlovini and Samuelsson (2003) [35] re-expressed the term $D_b p^{ab}$ occuring in the Euler equations (2.70) as

$$D_b p^{ab} = A^{ab}{}_{cd} S^{cd}{}_b, \tag{2.71}$$

where $A^{ab}{}_{cd}$ is the relativistic Hadamard elasticity tensor defined by

$$A^{ab}{}_{cd} = 2\frac{\partial p^{ab}}{\partial g_{cd}} - p^{ab}h_{cd} - h^a{}_c p^b{}_d \tag{2.72}$$

and S represents the elasticity difference tensor.

Using (2.71), they showed that the Euler equations (2.70) can be rewritten as

$$(\rho h^{ab} + p^{ab})\dot{u}_b + A^{abcd}S_{cdb} = 0, \tag{2.73}$$

emphasizing the dependence of the Euler equations on the elasticity difference tensor.

Chapter 3

A mathematical study of the Elasticity Difference Tensor

This chapter provides a mathematical analysis of the elasticity difference tensor. Properties of the elasticity difference tensor are investigated. Using an orthonormal tetrad, a general expression for the elasticity difference tensor is obtained which brings in Ricci rotation coefficients and the linear particle densities. Moreover, the elasticity difference tensor is decomposed along the eigenvectors of the pulled-back material metric into three second order tensors, for which it is studied if the eigenvectors of the pulled-back material metric remain eigenvectors for the three second order tensors.

3.1 Motivation and basic properties

The elasticity difference tensor has recently been introduced in the literature by Karlovini and Samuelsson (2003) [35] in the context of general relativistic elasticity. As shown in the previous chapter, in Section 2.2, the elasticity difference tensor is defined by

$$S^a{}_{bc} = \frac{1}{2}k^{-1am}(D_b k_{mc} + D_c k_{mb} - D_m k_{bc}), \tag{3.1}$$

where k_{ab} is the pulled-back material metric

$$k_{ab} = n_1^2\, x_a x_b + n_2^2\, y_a y_b + n_3^2\, z_a z_b, \tag{3.2}$$

k^{-1am} is such that $k^{-1am}k_{mb} = h^a{}_b$ and the operator D_a, which satisfies $D_a h_{bc} = 0$, represents the spatially projected connection obtained through the projection of the connection ∇ associated with the space-time metric g, according to (2.60).

Observing the definition of the elasticity difference tensor, one can see that it is related with the space-time connection, emphasizing therefore the geometric significance of the elasticity difference tensor.

Moreover, the elasticity difference tensor occurs contracted with the relativistic Hadamard elasticity tensor in the Euler equations for elastic matter, already presented in (2.73) in Chapter 2.

These aspects together with the fact that, due to the recent increasing consideration of relativistic elasticity in the literature, it becomes interesting and important to study in detail quantities appearing in this context, motivate the study of the elasticity difference tensor.

The following two properties of the elasticity difference tensor are straightforward:

(i) it is symmetric in the two covariant indices,

$$S^a{}_{bc} = S^a{}_{cb}; \tag{3.3}$$

(ii) it is a completely flowline orthogonal tensor field,

$$S^a{}_{bc}u_a = 0 = S^a{}_{bc}u^b = S^a{}_{bc}u^c. \tag{3.4}$$

3.2 Interpretative construction of the elasticity difference tensor

This section provides an alternative mathematical construction for the elasticity difference tensor, which requires a second metric defined on M and its associated Levi-Civita connection. This construction is interpretative in the sense that here the elasticity difference tensor arises in such a way that its origin is attributed directly to the existence

of two space-time connections associated with two specific metrics.
The foundation for the construction is given by the difference tensor. The difference tensor appears in the general relativistic literature, when two different space-time metrics are considered. And associated with it one can find expressions for the difference of the Riemann and the Ricci tensors, which are written directly in terms of the difference tensor. Continuing the construction by specifying a second required metric and using the projection tensor one arrives at the expression of the elasticity difference tensor. It is shown, how the difference of the Riemann and the Ricci tensors can be written in terms of the elasticity difference tensor.

3.2.1 Difference tensor

Given a space-time manifold M with metric tensor g, assume that another different metric tensor $\tilde{g}$ is defined on M. These metrics naturally determine two unique derivative operators ∇ and $\tilde{\nabla}$, respectively. The metric connections satisfy $\nabla g = 0$ and $\tilde{\nabla}\tilde{g} = 0$. Having chosen a coordinate system, one can write $g_{ab;c} = 0$ and $\tilde{g}_{ab||c} = 0$, where ; denotes the covariant derivative relative to g_{ab} and $||$ the covariant derivative relative to $\tilde{g}_{ab}$. It is a well known result[1] that the difference between two connections $\tilde{\nabla} - \nabla$ defines a tensor of type $(1,2)$ with components

$$C^{n}{}_{mc} = \tilde{\Gamma}^{n}{}_{mc} - \Gamma^{n}{}_{mc}, \tag{3.5}$$

where $\Gamma^{n}{}_{mc}$ and $\tilde{\Gamma}^{n}{}_{mc}$ are the Christoffel symbols associated with the referred two metrics. One can write this tensor in terms of the metric $\tilde{g}$ and its covariant derivative with respect to the metric g, yielding

$$C^{n}{}_{ml} = \frac{1}{2}\tilde{g}^{np}(\tilde{g}_{pm;l} + \tilde{g}_{pl;m} - \tilde{g}_{ml;p}), \tag{3.6}$$

where $\tilde{g}^{np}$ is such that $\tilde{g}^{np}\tilde{g}_{pr} = \delta^{n}{}_{r}$.

[1]See e.g. Rosen (1963) [58], Misner et al. (1970) [51], Wald (1984) [67], Carroll (2004) [15].

The difference tensor $C^n{}_{ml}$ can be used to write the difference of the Riemann and the Ricci tensors associated with the two metrics in the following form:

$$\tilde{R}^a{}_{bcd} - R^a{}_{bcd} = -C^a{}_{bd;c} + C^a{}_{bc;d} - C^a{}_{lc}C^l{}_{bd} + C^a{}_{ld}C^l{}_{bc} \tag{3.7}$$

and

$$\tilde{R}_{bd} - R_{bd} = -C^a{}_{bd;a} + C^a{}_{ba;d} - C^a{}_{la}C^l{}_{bd} + C^a{}_{ld}C^l{}_{ba}. \tag{3.8}$$

The expressions for the difference tensor and for the difference of the Riemann and Ricci tensors appear in the literature, for example in Misner et al. (1970) [51], in Wald (1984) [67], where the case of two conformally related metrics is considered, or in Rosen (1963) [58], where one of the two metric tensors is flat.

3.2.2 Elasticity difference tensor

Now, assume that the two metric tensors g and $\tilde{g}$ are specified by $g_{ab} = -u_a u_b + h_{ab}$ and $\tilde{g}_{ab} = -u_a u_b + k_{ab}$, so that ∇ and $\tilde{\nabla}$ are their associated Levi-Civita connections. The difference tensor of these two connections can be expressed as in (3.6). Projecting the difference tensor orthogonally to u according to

$$h^a{}_n\, h^m_b\, h^l{}_c\, C^n{}_{ml}, \tag{3.9}$$

and using the definition of the spatially projected connection[2], one obtains

$$S^a{}_{bc} = h^a{}_n\, h^m_b\, h^l{}_c\, C^n{}_{ml} = \frac{1}{2}k^{-1am}(D_b k_{mc} + D_c k_{mb} - D_m k_{bc}). \tag{3.10}$$

One can see that the expression on the right hand side of (3.10) is the elasticity difference tensor given in (2.63).

Therefore, under this approach, the elasticity difference tensor can be viewed as the projection, orthogonally to u, of the difference between two Levi-Civita connections: the connection associated with the space-time metric g and the connection associated with the metric $\tilde{g}_{ab} = -u_a u_b + k_{ab}$, where k_{ab} is the pull-back of the material metric K_{AB} and u is the velocity field of matter.

[2]See (2.60) on page 30.

3.2.3 Difference of the projected Riemann and Ricci tensors

Calculating the spatial projection of equation (3.7) using (2.60) and (3.9), yields the following expression for the difference of the Riemann tensors:

$$\begin{aligned} &h^f{}_m\, h^n{}_g\, h^p{}_e\, h^q{}_h \,[\, h^m{}_a\, h^b{}_n\, h^c{}_p\, h^d{}_q\, (\tilde{R}^a{}_{bcd} - R^a{}_{bcd})] \\ &= -D_e S^f{}_{gh} + D_h S^f{}_{ge} - S^f{}_{ke} S^k{}_{gh} + S^f{}_{kh} S^k{}_{ge}. \end{aligned} \tag{3.11}$$

The spatial projection of (3.8) expressing the difference of the Ricci tensors can be obtained analogously by equating the indices $a = c$ $(e = f)$ in (3.11):

$$\begin{aligned} &h^e{}_m\, h^n{}_g\, h^p{}_e\, h^q{}_h \,[\, h^m{}_a\, h^b{}_n\, h^a{}_p\, h^d{}_q\, (\tilde{R}^a{}_{bad} - R^a{}_{bad})] = \\ &- D_e S^e{}_{gh} + D_h S^e{}_{ge} - S^e{}_{ke} S^k{}_{gh} + S^e{}_{kh} S^k{}_{ge}. \end{aligned} \tag{3.12}$$

Therefore, these expressions, which contain the elasticity difference tensor, give the difference between the projected Riemann and Ricci tensors associated with the metrics referred to in Section 3.2.2.

The reason for projecting all indices twice on the left hand side of equation (3.11) is the following:

First, projecting all indices of equation (3.7) one obtains:

$$h^m{}_a h^b{}_n h^c{}_p h^d{}_q (\tilde{R}^a{}_{bcd} - R^a{}_{bcd}) = -D_p C^m{}_{nq} + D_q C^m{}_{np} - S^m{}_{lp} S^l{}_{nq} + S^m{}_{lq} S^l{}_{np}, \tag{3.13}$$

where $h^m{}_a h^b{}_n h^d{}_q h^c{}_p C^a{}_{bd;c} = D_p C^m{}_{nq}$.

Now, from the relation

$$D_p S^f{}_{gh} = D_p (h^f{}_m h^n{}_g h^q{}_h C^m{}_{nq}) = (D_p C^m{}_{nq}) h^f{}_m h^n{}_g h^q{}_h, \tag{3.14}$$

it follows that one must project all indices m, n, p, and q in (3.13) in order to obtain the tensor S in all expressions on the right hand side of equation (3.13). So, this leads to equation (3.11).

3.3 The elasticity difference tensor in tetrad notation

3.3.1 General expression

In order to obtain the tetrad components of the elasticity difference tensor, consider the following notation for the orthonormal tetrad:

$$e^a_\mu = (e^a_0, e^a_1, e^a_2, e^a_3) = (u^a, x^a, y^a, z^a).$$

It is important to stress that $\{u, x, y, z\}$ is the tetrad defined in Section 2.1.11, constructed by taking the eigendirections of the material metric and the velocity field of matter u.

Tetrad indices can be raised or lowered with the metric

$$\eta_{\mu\nu} = \eta^{\mu\nu} = diag(-1, 1, 1, 1).$$

The following relation between the metric η and the metric g is valid:

$$g_{ab} = \sum_{\mu,\nu=0}^{3} e_{\mu a} e_{\nu b} \eta^{\mu\nu}.$$

The triad components of the elasticity difference tensor can be calculated by the standard definition from

$$S^\alpha{}_{\beta\gamma} = S^a{}_{bc} e^\alpha_a e^b_\beta e^c_\gamma, \tag{3.15}$$

the result being

$$\begin{aligned} S^\alpha{}_{\beta\gamma} = \frac{1}{2n^2_\alpha}[&\left(n^2_\alpha - n^2_\gamma\right)\gamma^\alpha{}_{\gamma\beta} + \left(n^2_\alpha - n^2_\beta\right)\gamma^\alpha{}_{\beta\gamma} + \left(n^2_\gamma - n^2_\beta\right)\gamma_{\beta\gamma}{}^\alpha \\ &+ D_n(n^2_\alpha) e^n_\beta \delta^\alpha{}_\gamma + D_p(n^2_\alpha) e^p_\gamma \delta^\alpha{}_\beta - D_l(n^2_\beta) e^{l\alpha} \delta_{\beta\gamma}]. \end{aligned} \tag{3.16}$$

Here, the following notation is used for the Ricci rotation coefficients:

$$\gamma_{\mu\nu\rho} = e_{\mu a;b} e^a_\nu e^b_\rho,$$

where the spatial Ricci rotation coefficients are denoted by

$$\gamma_{\alpha\beta\gamma} = e_{\alpha a;b} e^a_\beta e^b_\gamma.$$

An alternative form for (3.16) is:

$$\begin{aligned} S^{\alpha}{}_{\beta\gamma} = \frac{1}{2}[&(1-\epsilon_{\gamma\alpha})\gamma^{\alpha}{}_{\gamma\beta} + (1-\epsilon_{\beta\alpha})\gamma^{\alpha}{}_{\beta\gamma} + (\epsilon_{\gamma\alpha} - \epsilon_{\beta\alpha})\gamma_{\beta\gamma}{}^{\alpha} \\ &+ m_{\beta\alpha}\delta^{\alpha}{}_{\gamma} + m_{\gamma\alpha}\delta^{\alpha}{}_{\beta} - m^{\alpha}{}_{\beta}\delta_{\beta\gamma}\epsilon_{\beta\alpha}], \end{aligned} \tag{3.17}$$

where $\epsilon_{\gamma\alpha} = \left(\dfrac{n^2_\gamma}{n^2_\alpha}\right)$ and $m^{\alpha}{}_{\beta} = D_a(\ln n^2_\beta)e^{a\alpha}$.

The Ricci rotation coefficients, when related to the quantities used in the decomposition (2.2), can be split into the set[3]:

$$\gamma_{0\alpha 0} = \dot{u}_\alpha \tag{3.18}$$

$$\gamma_{0\alpha\beta} = \frac{1}{3}\Theta\delta_{\alpha\beta} + \sigma_{\alpha\beta} - \epsilon_{\alpha\beta\gamma}\omega^{\gamma} \tag{3.19}$$

$$\gamma_{\alpha\beta 0} = -\epsilon_{\alpha\beta\gamma}\Omega^{\gamma} \tag{3.20}$$

$$\gamma_{\alpha\beta\gamma} = -A_\alpha\delta_{\beta\gamma} + A_\beta\delta_{\alpha\gamma} - \frac{1}{2}(\epsilon_{\gamma\delta\alpha}N^{\delta}{}_{\beta} - \epsilon_{\gamma\delta\beta}N^{\delta}{}_{\alpha} + \epsilon_{\alpha\beta\delta}N^{\delta}{}_{\gamma}). \tag{3.21}$$

The quantity ω^a represents the vorticity vector defined by

$$\omega^a = \frac{1}{2}\epsilon^{abc}\omega_{bc}.$$

The quantity Ω is defined by

$$\Omega^{\mu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}u_\nu e_\rho \dot{e}_\sigma,$$

where $\dot{e}_\sigma = e_{\sigma;m}u^m$, and represents the rate of rotation of the spatial frame $\{e_\alpha\}$ with respect to a Fermi propagated basis.

The quantities A and N appear in the decomposition of the spatial commutation functions[4] $\Gamma^{\alpha}{}_{\beta\gamma} = \gamma^{\alpha}{}_{\gamma\beta} - \gamma^{\alpha}{}_{\beta\gamma}$, where N is a symmetric object.

[3]See e.g. Ellis and Elst (1998) [26], [28].

[4]See Ellis and McCallum (1969) [27].

3.3.2 Traces

For the elasticity difference tensor it is possible to define two independent traces. Since $S^a{}_{bc}$ is symmetric in the last two indices, the traces are obtained by contracting the first with the second index, $S^a{}_{ac}$, and by contracting the last two indices, $S^a{}_b{}^b$. Here are given their expressions in the orthonormal tetrad already chosen:

$$\sum_{\alpha=1}^{3} S^{\alpha}{}_{\alpha\gamma} = \sum_{\alpha=1}^{3} \frac{1}{2} m_{\gamma\alpha} = \sum_{\alpha=1}^{3} \frac{1}{n_\alpha} D_a(n_\alpha) e^a_\gamma \tag{3.22}$$

and

$$\sum_{\beta=1}^{3} S^{\alpha}{}_{\beta\beta} = \sum_{\beta=1}^{3} \left[(1-\epsilon_{\beta\alpha})\gamma^{\alpha}{}_{\beta\beta} + m_{\beta\alpha}\delta^{\alpha}{}_{\beta} - \frac{1}{2} m^{\alpha}{}_{\beta}\epsilon_{\beta\alpha} \right]. \tag{3.23}$$

3.4 A decomposition for the elasticity difference tensor

The elasticity difference tensor can be expressed using three second order symmetric tensors, denoted by $\underset{\alpha}{M}_{bc}$, $\alpha = 1, 2, 3$, as follows:

$$S^a{}_{bc} = \underset{1}{M}_{bc} x^a + \underset{2}{M}_{bc} y^a + \underset{3}{M}_{bc} z^a = \sum_{\alpha=1}^{3} \underset{\alpha}{M}_{bc} e^a_\alpha. \tag{3.24}$$

The three tensors building up the elasticity difference tensor are defined by

$$\underset{1}{M}_{bc} = S^a{}_{bc} x_a \tag{3.25}$$

$$\underset{2}{M}_{bc} = S^a{}_{bc}\, y_a \tag{3.26}$$

$$\underset{3}{M}_{bc} = S^a{}_{bc}\, z_a. \tag{3.27}$$

The three tensors $\underset{\alpha}{M}_{bc}$ are analysed in order to understand to what extent the principal directions of the pulled back material metric remain privileged directions of the elasticity difference tensor through the tensors $\underset{\alpha}{M}_{bc}$, following the eigenvalue-eigenvector approach for these second order tensors.

First, expressions for $\underset{1}{M}$, $\underset{2}{M}$ and $\underset{3}{M}$ are given, which depend on the orthonormal tetrad vectors, the Ricci rotation coefficients and the linear particle densities. By contracting $S^a{}_{bc}$ in (3.1) with each one of the spatial tetrad one-forms, following (3.25), (3.26) and (3.27), and using then the relationships (3.2) and (2.60), after some appropriate simplifications, one obtains the subsequent expressions.

3.4.1 Expressions for $\underset{1}{M}$, $\underset{2}{M}$ and $\underset{3}{M}$

Expression for $\underset{1}{M}_{bc}$:

$$\begin{aligned}
\underset{1}{M}_{bc} &= u^m(x_{m;(b}u_{c)} + u_{(b}x_{c);m}) + x_{(b;c)} - x^m x_{(c}x_{b);m} + \ \gamma_{011}\ u_{(b}x_{c)} - \ \gamma_{010}\ u_b u_c \\
&+ \frac{1}{n_1}[2n_{1,(b}x_{c)} + 2n_{1,m}u^m u_{(b}x_{c)} + n_{1,m}x^m x_b x_c] \\
&+ \frac{1}{n_1^2}\{-x^m(z_b z_c n_3 n_{3,m} + y_b y_c n_2 n_{2,m}) \\
&+ n_2^2[(\gamma_{021} - \gamma_{120})u_{(b}y_{c)} + x^m(y_{m;(b}y_{c)} - y_{(b}y_{c);m})] \\
&+ n_3^2[(\gamma_{031} - \gamma_{130})u_{(b}z_{c)} + x^m(z_{m;(b}z_{c)} - z_{(b}z_{c);m})]\}
\end{aligned} \tag{3.28}$$

Expression for $\underset{2}{M}_{bc}$:

$$\begin{aligned}
\underset{2}{M}_{bc} &= [u^m(y_{m;(b}u_{c)} + u_{(b}y_{c);m}) + y_{(b;c)} - y^m y_{(c}y_{b);m} + \ \gamma_{022}\ u_{(b}y_{c)} - \ \gamma_{020}\ u_b u_c] \\
&+ \frac{1}{n_2}[2n_{2,(b}y_{c)} + 2n_{2,m}u^m u_{(b}y_{c)} + n_{2,m}y^m y_b y_c] \\
&+ \frac{1}{n_2^2}\{-y^m(z_b z_c n_3 n_{3,m} + x_b x_c n_1 n_{1,m}) \\
&+ n_1^2[(\gamma_{120} + \gamma_{012})u_{(b}x_{c)} + y^m(x_{m;(b}x_{c)} - x_{(b}x_{c);m})] \\
&+ n_3^2[(\gamma_{032} - \gamma_{230})u_{(b}z_{c)} + y^m(z_{m;(b}z_{c)} - z_{(b}z_{c);m})]\}
\end{aligned} \tag{3.29}$$

Expression for $\underset{3}{M}_{bc}$:

$$
\begin{aligned}
\underset{3}{M}_{bc} = & [u^m(z_{m;(b}u_{c)} + u_{(b}z_{c);m}) + z_{(b;c)} - z^m z_{(b}z_{c);m} + \ \gamma_{033}\ u_{(b}z_{c)} - \ \gamma_{030}\ u_b u_c] \\
& + \frac{1}{n_3}[2n_{3,(b}z_{c)} + 2n_{3,m}u^m u_{(b}z_{c)} + n_{3,m}z^m z_b z_c] \\
& + \frac{1}{n_3^2}\{-z^m(y_b y_c n_2 n_{2,m} + x_b x_c n_1 n_{1,m}) \\
& + n_1^2[(\gamma_{130} + \gamma_{013})u_{(b}x_{c)} + z^m(x_{m;(b}x_{c)} - x_{(b}x_{c);m})] \\
& + n_2^2[(\gamma_{023} + \gamma_{230})u_{(b}y_{c)} + z^m(y_{m;(b}y_{c)} - y_{(b}y_{c);m})]\}
\end{aligned} \tag{3.30}
$$

3.4.2 General expression for $\underset{\alpha}{M}$

The last expressions for the three second-order tensors can be represented by using just one general expression in the following way.

$$
\begin{aligned}
\underset{\alpha}{M}_{bc} = & u^m(e_{\alpha m;(b}u_{c)} + u_{(b}e_{\alpha c);m}) + e_{\alpha(b;c)} - \underset{\alpha}{e}^m e_{\alpha(c}e_{\alpha b);m} \\
& + \gamma_{0\alpha\alpha}\ u_{(b}e_{\alpha c)} - \ \gamma_{0\alpha 0}\ u_b u_c \\
& + \frac{1}{n_\alpha}[2n_{\alpha,(b}e_{\alpha c)} + 2n_{\alpha,m}u^m u_{(b}e_{\alpha c)} + n_{\alpha,m}\underset{\alpha}{e}^m e_{\alpha b}e_{\alpha c}] \\
& + \frac{1}{n_\alpha^2}\{-\underset{\alpha}{e}^m(e_{\beta b}e_{\beta c}n_\beta n_{\beta,m} + e_{\gamma b}e_{\gamma c}n_\gamma n_{\gamma,m}) \\
& + n_\gamma^2[(\gamma_{0\gamma\alpha} - \gamma_{\alpha\gamma 0})u_{(b}e_{\gamma c)} + \underset{\alpha}{e}^m(e_{\gamma m;(b}e_{\gamma c)} - e_{\gamma(b}e_{\gamma c);m})] \\
& + n_\beta^2[(\gamma_{0\beta\alpha} - \gamma_{\alpha\beta 0})u_{(b}e_{\beta c)} + \underset{\alpha}{e}^m(e_{\beta m;(b}e_{\beta c)} - e_{\beta(b}e_{\beta c);m})]\}.
\end{aligned} \tag{3.31}
$$

Here $\gamma \neq \beta \neq \alpha$ and a comma represents a partial derivative. To read (3.31) properly one must see that each value of $\alpha = 1, 2, 3$ fixes exactly one pair of values for (β, γ). For example, $\alpha = 1$ fixes (β, γ) as either $(2, 3)$ or $(3, 2)$, yielding the same result for both choices.

It should be noticed that this expression also contains the non-spatial Ricci rotation coefficients given in (3.18), (3.19) and (3.20).

3.5 Eigenvalue-eigenvector problem

In the next paragraphs, the eigenvalues and eigenvectors for the three tensors $\underset{\alpha}{M}_{bc}$ are investigated. The results are presented for each tensor $\underset{\alpha}{M}_{bc}$ separately, for reasons of clarity. However, all results can be presented in a condensed manner, as described in Section 3.5.5.

The expressions obtained for $\underset{\alpha}{M}_{bc}$ satisfy the conditions

$$\underset{\alpha}{M}_{bc} u^b = 0, \tag{3.32}$$

as a consequence of the orthonormality conditions of the tetrad together with (3.24) and (3.25-3.27).

Thus, all $\underset{\alpha}{M}_{bc}$ have u as a timelike eigenvector associated with a zero eigenvalue: $\underset{\alpha}{M}_b{}^c u^b = 0$. Therefore, their corresponding Segre type is $\{1, 111\}$ or one of its degeneracies.

Now, consider the following eigenvector-eigenvalue equation for $\underset{\alpha}{M}$:

$$\underset{\alpha}{M}_b{}^c \omega^b = \lambda \omega^c, \tag{3.33}$$

where $\omega^b = \delta_1 x^b + \delta_2 y^b + \delta_3 z^b$ is a vector defined on M.

This eigenvalue-eigenvector problem for $\underset{\alpha}{M}_{bc}$ is quite difficult to solve in general. However, one can investigate the conditions for the tetrad vectors to be eigenvectors of $\underset{\alpha}{M}_{bc}$. An interesting question is: do x, y and z - principal directions of $k^a{}_b$ - remain as principal directions of $\underset{1}{M}$, $\underset{2}{M}$ and $\underset{3}{M}$? The problem can also be formulated as follows: What conditions have to be satisfied for x or y or z to be a principal vector of $\underset{1}{M}$ or $\underset{2}{M}$ or $\underset{3}{M}$?

The results of this study are given in the next theorems.

On what follows, intrinsic derivatives of arbitrary scalar fields Φ, as derivatives along tetrad vectors, will be represented by Δ_{e_α} and defined as:

$$\Delta_{e_\alpha} \Phi = \Phi_{,m} e^m_\alpha,$$

where a comma stands again for a partial derivative.

3.5.1 Eigenvalue-eigenvector problem for $\underset{1}{M}$

The next three theorems, namely Theorem 1, Theorem 2 and Theorem 3, refer to the eigenvector-eigenvalue problem for $\underset{1}{M}$, considering $\omega^b = x^b$, $\omega^b = y^b$ and $\omega^b = z^b$, respectively, in (3.33).

Theorem 1 *x is an eigenvector of $\underset{1}{M}$ iff n_1 remains invariant along the directions of y and z, i.e. $\Delta_y(\ln n_1) = \Delta_z(\ln n_1) = 0$.*
The corresponding eigenvalue is $\lambda = \Delta_x(\ln n_1)$.

Proof: In order to solve this eigenvector-eigenvalue equation the following algebraic conditions are used

$$\underset{1}{M}_b{}^c x^b x_c = \lambda, \tag{3.34}$$

$$\underset{1}{M}_b{}^c x^b y_c = 0 \tag{3.35}$$

and

$$\underset{1}{M}_b{}^c x^b z_c = 0. \tag{3.36}$$

Using the orthogonality conditions satisfied by the tetrad vectors and the properties of the rotation coefficients, namely the fact that they are anti-symmetric on the first pair of indices, (3.35) and (3.36) yield $\Delta_y(\ln n_1) = 0 = \Delta_z(\ln n_1)$. On the other hand, from (3.34) one obtains $\lambda = \Delta_x(\ln n_1)$.
Conversely, if $\Delta_y(\ln n_1) = 0 = \Delta_z(\ln n_1)$, it follows that the conditions (3.35) and (3.36) are satisfied and that the eigenvalue is given by (3.34).

□

Theorem 2 *y is an eigenvector of $\underset{1}{M}$ iff n_1 remains invariant along the direction of y, i.e. $\Delta_y(\ln n_1) = 0$, and*
$-\frac{1}{2}\gamma_{132}[\frac{n_3^2}{n_1^2} - 1] + \frac{1}{2}\gamma_{123}[1 - \frac{n_2^2}{n_1^2}] + \frac{1}{2}\gamma_{231}[\frac{n_3^2}{n_1^2} - \frac{n_2^2}{n_1^2}] = 0.$
The corresponding eigenvalue is $\lambda = -\frac{n_2}{n_1^2}\Delta_x n_2 + \gamma_{122}(-\frac{n_2^2}{n_1^2} + 1)$.

Proof: Contracting $\underset{1}{M}{}_b{}^c y^b = \lambda y^c$ with x_c one obtains $\Delta_y(\ln n_1) = 0$. This condition is satisfied whenever $\Delta_y n_1 = 0$. The second condition results from $\underset{1}{M}{}_b{}^c y^b z_c = 0$. And contracting $\underset{1}{M}{}_b{}^c y^b = \lambda y^c$ with y_c yields the eigenvalue λ. The used simplifications are based on the orthogonality conditions of the tetrad vectors and on the properties of the rotation coefficients.

On the other hand, suppose that the conditions $\Delta_y(\ln n_1) = 0$ and $-\frac{1}{2}\gamma_{132}[\frac{n_3^2}{n_1^2} - 1] + \frac{1}{2}\gamma_{123}[1 - \frac{n_2^2}{n_1^2}] + \frac{1}{2}\gamma_{231}[\frac{n_3^2}{n_1^2} - \frac{n_2^2}{n_1^2}] = 0$ hold, then it can be shown that $\underset{1}{M}{}_b{}^c y^b x_c = 0$ and $\underset{1}{M}{}_b{}^c y^b z_c = 0$ are satisfied, so that y is eigenvector of $\underset{1}{M}$ associated with the presented eigenvalue.

□

Theorem 3 *z is an eigenvector of $\underset{1}{M}$ iff n_1 remains invariant along the direction of z, i.e. $\Delta_z(\ln n_1) = 0$, and*
$\frac{1}{2}\gamma_{123}[1 - \frac{n_2^2}{n_1^2}] + \frac{1}{2}\gamma_{132}[1 - \frac{n_3^2}{n_1^2}] + \frac{1}{2}\gamma_{231}[\frac{n_3^2}{n_1^2} - \frac{n_2^2}{n_1^2}] = 0.$
The corresponding eigenvalue is $\lambda = -\frac{n_3}{n_1^2}\Delta_x n_3 - \gamma_{133}(\frac{n_3^2}{n_1^2} - 1)$.

Proof: The first two conditions and the eigenvalue are obtained respectively from the following algebraic equations:

$$\underset{1}{M}{}_b{}^c z^b x_c = 0, \tag{3.37}$$

$$\underset{1}{M}{}_b{}^c z^b y_c = 0 \tag{3.38}$$

and

$$\underset{1}{M}{}_b{}^c z^b z_c = \lambda. \tag{3.39}$$

Here also the orthogonality conditions between the tetrad vectors and the properties of the rotation coefficients are used.

The converse is true as well. Suppose that the two conditions hold, then it follows that (3.37) and (3.38) are satisfied and (3.39) leads to the expression for the eigenvalue.

□

3.5.2 Eigenvalue-eigenvector problem for $\underset{2}{M}$

Solving the eigenvector-eigenvalue problem for $\underset{2}{M}$, considering $\omega^b = x^b$, $\omega^b = y^b$ and $\omega^b = z^b$, respectively, in (3.33), yields the results given in Theorem 4, Theorem 5 and Theorem 6.

Theorem 4 *x is an eigenvector of $\underset{2}{M}$ iff n_2 remains invariant along the direction of x, i.e. $\Delta_x(\ln n_2) = 0$, and*
$\frac{1}{2}\ \gamma_{123}\ [\frac{n_1^2}{n_2^2} - 1] + \frac{1}{2}\ \gamma_{132}\ [\frac{n_3^2}{n_2^2} - \frac{n_1^2}{n_2^2}] + \frac{1}{2}\ \gamma_{231}\ [1 - \frac{n_3^2}{n_2^2}] = 0.$
The corresponding eigenvalue is $\lambda = \gamma_{121}(\frac{n_1^2}{n_2^2} - 1) - \frac{n_1}{n_2^2}\Delta_y n_1$.

Proof: Starting with the following conditions:

$$\underset{2}{M}{}_b{}^c x^b y_c = 0 \tag{3.40}$$

$$\underset{2}{M}{}_b{}^c x^b z_c = 0 \tag{3.41}$$

$$\underset{2}{M}{}_b{}^c x^b x_c = \lambda \tag{3.42}$$

and applying simplification rules, attributed to the orthogonality conditions of the tetrad vectors and to the properties of the rotation coefficients, one obtains $\Delta_x(\ln n_2) = 0$ from (3.40). Equation (3.41) yields the second condition and (3.42), the eigenvalue λ.
Conversely, if $\Delta_x(\ln n_2) = 0$ and $\frac{1}{2}\ \gamma_{123}\ [\frac{n_1^2}{n_2^2} - 1] + \frac{1}{2}\ \gamma_{132}\ [\frac{n_3^2}{n_2^2} - \frac{n_1^2}{n_2^2}] + \frac{1}{2}\ \gamma_{231}\ [1 - \frac{n_3^2}{n_2^2}] = 0$ hold, then (3.40) and (3.41) are identically satisfied and the eigenvalue is obtained from (3.42).

□

Theorem 5 *y is an eigenvector of $\underset{2}{M}$ iff n_2 remains invariant along the directions of x and z, i.e. $\Delta_x(\ln n_2) = 0$ and $\Delta_z(\ln n_2) = 0$.*
The corresponding eigenvalue is $\lambda = \Delta_y(\ln n_2)$.

Proof: The first two conditions are consequences of

$$\underset{2}{M}_b{}^c y^b x_c = 0 \tag{3.43}$$

and

$$\underset{2}{M}_b{}^c y^b z_c = 0, \tag{3.44}$$

resulting in $\Delta_x(\ln n_2) = 0$ and $\Delta_z(\ln n_2) = 0$, respectively. The eigenvalue λ is obtained from the contraction of $\underset{2}{M}_b{}^c y^b = \lambda y^c$ with y_c:

$$\underset{2}{M}_b{}^c y^b y_c = \lambda. \tag{3.45}$$

On the other hand, supposing that $\Delta_x(\ln n_2) = 0$, $\Delta_z(\ln n_2) = 0$ and $\lambda = \Delta_y(\ln n_2)$, then (3.43), (3.44) and (3.45) follow directly.

$\square$

Theorem 6 *z is an eigenvector of $\underset{2}{M}$ iff n_2 remains invariant along the direction of z, i.e. $\Delta_z(\ln n_2) = 0$, and*
$\frac{1}{2}\gamma_{231}[1 - \frac{n_3^2}{n_2^2}] + \frac{1}{2}\gamma_{123}[-1 + \frac{n_1^2}{n_2^2}] + \frac{1}{2}\gamma_{132}[\frac{n_3^2}{n_2^2} - \frac{n_1^2}{n_2^2}] = 0.$
The corresponding eigenvalue is $\lambda = -\frac{n_3}{n_2^2}\Delta_y n_3 - \gamma_{233}(\frac{n_3^2}{n_2^2} - 1)$.

Proof: The conditions are obtained from the relations

$$\underset{2}{M}_b{}^c z^b y_c = 0 \tag{3.46}$$

and

$$\underset{2}{M}_b{}^c z^b x_c = 0. \tag{3.47}$$

The eigenvalue λ can be calculated using

$$\underset{2}{M}_b{}^c z^b z_c = \lambda. \tag{3.48}$$

Conversely, if $\Delta_z(\ln n_2) = 0$ and $\frac{1}{2}\gamma_{231}[1 - \frac{n_3^2}{n_2^2}] + \frac{1}{2}\gamma_{123}[-1 + \frac{n_1^2}{n_2^2}] + \frac{1}{2}\gamma_{132}[\frac{n_3^2}{n_2^2} - \frac{n_1^2}{n_2^2}] = 0$, then (3.46) and (3.47) are satisfied and it follows that z is an eigenvector for $\underset{2}{M}$ associated with the eigenvalue λ.

□

You can observe that (3.46) equals (3.44) of Theorem 5 and (3.47) equals (3.41) of Theorem 4, leading respectively to the same conditions. This is due to the symmetry property of $\underset{2}{M}$.

3.5.3 Eigenvalue-eigenvector problem for $\underset{3}{M}$

Now, the process is repeated for the third second-order symmetric tensor $\underset{3}{M}$. Setting $\omega^b = x^b$, $\omega^b = y^b$ and $\omega^b = z^b$ in (3.33) and solving the respective equations leads to Theorem 7, Theorem 8 and Theorem 9.

Theorem 7 *x is an eigenvector of $\underset{3}{M}$ iff n_3 remains invariant along the direction of x, i.e. $\Delta_x(\ln n_3) = 0$, and*
$\frac{1}{2}\ \gamma_{132}\ [\frac{n_1^2}{n_3^2} - 1] + \frac{1}{2}\ \gamma_{123}\ [\frac{n_2^2}{n_3^2} - \frac{n_1^2}{n_3^2}] + \frac{1}{2}\ \gamma_{231}\ [\frac{n_2^2}{n_3^2} - 1] = 0.$
The corresponding eigenvalue is $\lambda = \gamma_{131}(\frac{n_1^2}{n_3^2} - 1) - \frac{n_1}{n_3^2}\Delta_z n_1$.

Proof: In the same way as described in the other proofs you get these results from the contraction of the eigenvalue-eigenvector equation with z_c, y_c and x_c respectively:

$$\underset{3}{M}_b{}^c x^b z_c = 0, \tag{3.49}$$

$$\underset{3}{M}_b{}^c x^b y_c = 0 \tag{3.50}$$

and

$$\underset{3}{M}_b{}^c x^b x_c = \lambda. \tag{3.51}$$

The converse is true as well. Supposing that $\Delta_x(\ln n_3) = 0$ and $\frac{1}{2}\ \gamma_{132}\ [\frac{n_1^2}{n_3^2} - 1] + \frac{1}{2}\ \gamma_{123}\ [\frac{n_2^2}{n_3^2} - \frac{n_1^2}{n_3^2}] + \frac{1}{2}\ \gamma_{231}\ [\frac{n_2^2}{n_3^2} - 1] = 0$ hold and that the eigenvalue is given by $\lambda = \gamma_{131}(\frac{n_1^2}{n_3^2} - 1) - \frac{n_1}{n_3^2}\Delta_z n_1$, then it can be shown that (3.49), (3.50) and (3.51) are satisfied, so that x is an eigenvector of $\underset{3}{M}$ associated with the given eigenvalue.

□

Theorem 8 *y is an eigenvector of $\underset{3}{M}$ iff n_3 remains invariant along the direction of y, i.e. $\Delta_y(\ln n_3) = 0$ and*
$\frac{1}{2}\gamma_{132}[\frac{n_1^2}{n_3^2} - 1] + \frac{1}{2}\gamma_{231}[-1 + \frac{n_2^2}{n_3^2}] + \frac{1}{2}\gamma_{123}[\frac{n_2^2}{n_3^2} - \frac{n_1^2}{n_3^2}] = 0.$
The corresponding eigenvalue is $\lambda = \gamma_{232}(\frac{n_2^2}{n_3^2} - 1) - \frac{n_2}{n_3^2}\Delta_z n_2$.

Proof: The eigenvalue λ is found by contracting the equation $\underset{3}{M}_b{}^c y^b = \lambda y^c$ with y_c. The other two conditions are consequences of

$$\underset{3}{M}_b{}^c y^b z_c = 0 \tag{3.52}$$

and

$$\underset{3}{M}_b{}^c y^b x_c = 0, \tag{3.53}$$

where (3.53) yields the condition depending on the rotation coefficients.
Conversely, the conditions $\Delta_y(\ln n_3) = 0$ and $\frac{1}{2}\gamma_{132}[\frac{n_1^2}{n_3^2} - 1] + \frac{1}{2}\gamma_{231}[-1 + \frac{n_2^2}{n_3^2}] + \frac{1}{2}\gamma_{123}[\frac{n_2^2}{n_3^2} - \frac{n_1^2}{n_3^2}] = 0$ imply that y is an eigenvector of $\underset{3}{M}$ associated with the given eigenvalue.

□

Theorem 9 *z is an eigenvector of $\underset{3}{M}$ iff n_3 remains invariant along the directions of x and y, i.e. $\Delta_x(\ln n_3) = 0$ and $\Delta_y(\ln n_3) = 0$.*
The corresponding eigenvalue is $\lambda = \Delta_z(\ln n_3)$.

Proof: The first two conditions can be obtained by contracting $\underset{3}{M}_b{}^c z^b = \lambda z^c$ with x_c and y_c. The eigenvalue is found by contracting the same equation with z_c.
On the other hand, if $\Delta_x(\ln n_3) = 0$ and $\Delta_y(\ln n_3) = 0$, then one can show that z is an eigenvector of $\underset{3}{M}$ associated with the eigenvalue $\lambda = \Delta_z(\ln n_3)$.

□

3.5.4 Concluding remarks

In general, the previous theorems show that strong conditions have to be imposed on n_1, n_2 and n_3 and the metric in order to have x, y and z as principal directions of $\underset{1}{M}$,

$\underset{2}{M}$ and $\underset{3}{M}$.

However, the conditions to have x as eigenvector of $\underset{1}{M}$ seem less restrictive then the conditions for y and z to be eigenvectors of the same tensor, since the latter involve not only intrinsic derivatives of the scalar fields but also rotation coefficients of the metric. Furthermore, for x to be an eigenvector of $\underset{1}{M}$ only conditions on n_1 have to be satisfied, namely that n_1 remains constant along the directions of y and z, in which case the eigenvalue corresponding to x depends only on n_1. On the other hand, the conditions imposed for y and z to be eigenvectors of $\underset{1}{M}$ also involve n_2 and n_3. A similar interpretation is valid when one considers $\underset{2}{M}$ and $\underset{3}{M}$, where the role of x is now taken over by y and z, respectively. Thus, the significance and the role that x and n_1 play for $\underset{1}{M}$ are the same as y and n_2 play for $\underset{2}{M}$ and z and n_3 play for $\underset{3}{M}$.

Analysing the conditions of the theorems, one can say that they are particularly satisfied in the cases explained below.

Notice that the conditions for y and z to be eigenvectors of $\underset{1}{M}$ (see Theorem 2 and Theorem 3), for x and z to be eigenvectors of $\underset{2}{M}$ (see Theorem 4 and Theorem 6) and for x and y to be eigenvectors of $\underset{3}{M}$ (see Theorem 7 and Theorem 8), are satisfied whenever $n_1 = n_2 = n_3 = c$, where c is a constant. Consequently, the eigenvalues in the mentioned theorems result in $\lambda = 0$. In this case, k_{ab} takes the form $k_{ab} = c^2\, x_a x_b + c^2\, y_a y_b + c^2\, z_a z_b$.

Analysing the conditions of Theorem 1, Theorem 5 and Theorem 9, one can state the following.

1. Considering Theorem 1:

If $n_1 = c$, c being a constant, the two conditions in Theorem 1 are satisfied. This implies that $\lambda = 0$ and $k_{ab} = c^2\, x_a x_b + n_2^2\, y_a y_b + n_3^2\, z_a z_b$. In this case, x is an eigenvector for $\underset{1}{M}$.

2. Considering Theorem 5:

If $n_2 = c$, c being a constant, the two conditions in Theorem 5 are satisfied, then $\lambda = 0$

and $k_{ab} = n_1^2\, x_a x_b + c^2\, y_a y_b + n_3^2\, z_a z_b$. In this case, y is an eigenvector for $\underset{2}{M}$.

3. Considering Theorem 5:

If $n_3 = c$, c being a constant, the two conditions in Theorem 9 are satisfied, so that z is an eigenvector for $\underset{3}{M}$. In this case, one has $\lambda = 0$ and $k_{ab} = n_1^2\, x_a x_b + n_2^2\, y_a y_b + c^2\, z_a z_b$.

Next, the previous theorems are used to establish the conditions for x to be an eigenvector of $\underset{1}{M}$, $\underset{2}{M}$ and $\underset{3}{M}$ simultaneously (similar results are obtained when x is substituted by y or z):

(i) $\Delta_y(n_1) = 0$,

(ii) $\Delta_z(n_1) = 0$,

(iii) $\Delta_x(n_2) = 0$,

(vi) $\Delta_x(n_3) = 0$,

(v) $\frac{1}{2}\ \gamma_{123}\ [\frac{n_1^2}{n_2^2} - 1] + \frac{1}{2}\ \gamma_{132}\ [\frac{n_3^2}{n_2^2} - \frac{n_1^2}{n_2^2}] + \frac{1}{2}\ \gamma_{231}\ [1 - \frac{n_3^2}{n_2^2}] = 0$,

(vi) $\frac{1}{2}\ \gamma_{132}\ [\frac{n_1^2}{n_3^2} - 1] + \frac{1}{2}\ \gamma_{123}\ [\frac{n_2^2}{n_3^2} - \frac{n_1^2}{n_3^2}] + \frac{1}{2}\ \gamma_{231}\ [\frac{n_2^2}{n_3^2} - 1] = 0$.

These restrictions are, again, quite strong. If $n_1 = n_2 = n_3$ are constant scalar fields, x is a common eigenvector for $\underset{1}{M}$, $\underset{2}{M}$ and $\underset{3}{M}$. However this case is physically not interesting for the problem. Finding other solutions for the previous equations is not an easy task for the majority of metrics that one may consider.

Therefore, in general the principal directions of the pulled back material metric k are not principal directions for $\underset{1}{M}$, $\underset{2}{M}$ and $\underset{3}{M}$.

3.5.5 Summarizing the results

The results of the last theorems (Theorem 1-Theorem 9) can be reproduced and summarized in the two following theorems.

Theorem 10 *The tetrad vector* e_α *is an eigenvector for* $\underset{\alpha}{M}$ *iff* n_α *remains invariant along the two spatial tetrad vectors* e_β*, such that* $\beta \neq \alpha$*, i.e.* $\Delta_{e_\beta}(\ln n_\alpha) = 0$ *whenever* $\beta \neq \alpha$.
The corresponding eigenvalue is $\lambda = \Delta_{e_\alpha}(\ln n_\alpha)$.

Proof: In order to solve this eigenvector-eigenvalue equation the following algebraic conditions are used

$$\underset{\alpha}{M}{}_b{}^c e^b_\alpha e_{\alpha c} = \lambda, \tag{3.54}$$

$$\underset{\alpha}{M}{}_b{}^c e^b_\alpha e_{\beta c} = 0 \tag{3.55}$$

and

$$\underset{\alpha}{M}{}_b{}^c e^b_\alpha e_{\gamma c} = 0, \tag{3.56}$$

where $\gamma \neq \beta \neq \alpha$. Considering the orthogonality conditions satisfied by the tetrad vectors and the anti-symmetry of the Ricci rotation coefficients on the first pair of indices, expressions (3.55) and (3.56) yield $\Delta_{e_\beta}(\ln n_\alpha) = 0 = \Delta_{e_\gamma}(\ln n_\alpha)$. Therefore $\Delta_{e_\beta} n_\alpha = 0 = \Delta_{e_\gamma} n_\alpha$. On the other hand, from (3.54) one obtains, after some calculations, the eigenvalue $\lambda = \Delta_{e_\alpha}(\ln n_\alpha)$.
Conversely, suppose that $\Delta_{e_\beta}(\ln n_\alpha) = 0$ for $\beta \neq \alpha$ and $\lambda = \Delta_{e_\alpha}(\ln n_\alpha)$, then the conditions (3.54), (3.55) and (3.56) are identically satisfied, so that e_α is an eigenvector for $\underset{\alpha}{M}$.

□

For each value of α, the eigenvalue λ in Theorem 10 vanishes iff n_α remains constant along e_α. However this condition is satisfied whenever $n_\alpha = c$, with c as a constant. In this case, $k_{ab} = c^2\, e_{\alpha a}\, e_{\alpha b} + \sum_{\beta \neq \alpha} n^2_\beta\, e_{\beta a}\, e_{\beta b}$.

Theorem 11 *e_β is an eigenvector of $\underset{\alpha}{M}$ (with $\alpha \neq \beta$) iff the following conditions are satisfied:*

(i) $\Delta_{e_\beta}(\ln n_\alpha) = 0$*, i.e. n_α remains invariant along the direction of e_β;*

(ii) $\gamma_{\alpha\gamma\beta}[n_\alpha^2 - n_\gamma^2] + \gamma_{\alpha\beta\gamma}[n_\alpha^2 - n_\beta^2] + \gamma_{\beta\gamma\alpha}[n_\gamma^2 - n_\beta^2] = 0$*, where $\gamma \neq \beta \neq \alpha$ for one pair (β, γ).*

The corresponding eigenvalue is $\lambda = -\frac{n_\beta}{n_\alpha^2}\Delta_{e_\alpha} n_\beta + \gamma_{\alpha\beta\beta}(-\frac{n_\beta^2}{n_\alpha^2} + 1)$.

Proof: Contracting $\underset{\alpha}{M}{}_b{}^c e_\beta^b = \lambda e_\beta^c$ with $e_{\alpha c}$ one obtains $\Delta_{e_\beta}(\ln n_\alpha) = 0$. This condition is satisfied whenever $\Delta_{e_\beta} n_\alpha = 0$. The second condition is a consequence of $\underset{\alpha}{M}{}_b{}^c e_\beta^b e_{\gamma c} = 0$. Contracting $\underset{\alpha}{M}{}_b{}^c e_\beta^b = \lambda e_\beta^c$ with $e_{\beta c}$ yields the eigenvalue λ.
The simplifications performed are based on the orthogonality conditions of the tetrad vectors and on the properties of the rotation coefficients.
The converse is true as well. If the conditions (i) and (ii) hold, then it can be shown that e_β is an eigenvector of $\underset{\alpha}{M}$ associated with the eigenvalue $\lambda = -\frac{n_\beta}{n_\alpha^2}\Delta_{e_\alpha} n_\beta + \gamma_{\alpha\beta\beta}(-\frac{n_\beta^2}{n_\alpha^2} + 1)$.
□

Notice that the two conditions (i) and (ii) in Theorem 11 are satisfied simultaneously whenever $n_\alpha = n_\beta = n_\gamma = c$, with c a constant, in which case $\lambda = 0$ and $k_{ab} = c^2\, x_a x_b + c^2\, y_a y_b + c^2\, z_a z_b$.

The previous theorems show that strong conditions have to be imposed both on n_α ($\alpha = 1, 2, 3$) and the metric if one requires that the spatial tetrad vectors are principal directions of $\underset{\alpha}{M}$, for $\alpha = 1, 2, 3$.
However, the conditions for e_α to be an eigenvector of $\underset{\alpha}{M}$ are less restrictive then the conditions for e_β to be an eigenvector of the same tensor, for all values of $\beta \neq \alpha$: in the first case the conditions to be satisfied contain only intrinsic derivatives of the quantities n_α; in the second case, besides conditions on the intrinsic derivatives on the n_α, one also has conditions containing the Ricci rotation coefficients.
Furthermore, for e_α to be an eigenvector of $\underset{\alpha}{M}$, only conditions on n_α have to be satisfied: n_α must remain constant along the directions of e_β for all values of $\beta \neq \alpha$.

In this case the eigenvalue corresponding to e_α depends on n_α only. Moreover, the conditions for the vectors e_β, for all $\beta \neq \alpha$, to be eigenvectors of $\underset{\alpha}{M}$ depend explicitly on the three quantities n_1, n_2 and n_3.

Finally, the previous theorems are used to establish the conditions for each vector e_α, with $\alpha = 1, 2, 3$, to be an eigenvector of the three tensors $\underset{1}{M}$, $\underset{2}{M}$, $\underset{3}{M}$ simultaneously. One can show that those conditions are:

(i) $\Delta_{e_\beta}(\ln n_\alpha) = 0$,

(ii) $\Delta_{e_\alpha}(\ln n_\beta) = 0$,

(iii) $\gamma_{\alpha\beta\gamma}\ [n_\alpha^2 - n_\beta^2] + \gamma_{\alpha\gamma\beta}\ [n_\gamma^2 - n_\alpha^2] + \gamma_{\beta\gamma\alpha}[n_\beta^2 - n_\gamma^2] = 0$,

for all values of β and γ such that $\beta \neq \gamma \neq \alpha$.

Here conditions (i), (ii) and (iii) must be satisfied for all values of $\beta \neq \alpha$.

Ruling out the solution $n_1 = n_2 = n_3 =$ constant, which is not physically interesting, it is not easy to solve these last equations. However one can say again that, in general, the principal directions of the pulled back material metric k are not principal directions of the three tensors $\underset{1}{M}$, $\underset{2}{M}$ and $\underset{3}{M}$.

Chapter 4

Two conformally related material metrics

Conformal transformations have a number of uses in general relativity and reveal to be important in many applications. To mention some of them, conformal transformations can be used to obtain physically more interesting space-times and allow for an easier study of the new space-time's geometry by using special properties of the original space-time (the space-time which is undergone a conformal transformation). The conformally reducible 2+2 space-times studied by Carot and Tupper (2002) [14] serve as an example of these applications. These space-times include the class of warped space-times, which contain all spherically symmetric solutions and a wide variety of other space-times like Robertson-Walker, Bertotti-Robinson, de Sitter. The warped space-times also take the advantage of being conformally related with locally decomposable space-times, namely that they can be characterized by using properties of the locally decomposable ones[1]. Conformal transformations can also be used as a method for generating solutions[2]. Another important application can be found in the context of symmetries[3], where the conformal transformations are used to simplify their study, such that results can then be inferred about one metric and through the conformal relation about the other

[1]See Carot and Costa (1993) [12].

[2]See Carot and Mas (1986) [13]; Stephani et al. (2003) [63] page 45.

[3]See e.g. Hall (1989) [31] or Hall and Steele (1991) [32].

metric.

In the general context of conformal transformations one can find in the literature[4] results showing the relations between the Riemann tensors, the Ricci tensors and other relativistic objects associated with two metrics g_{ab} and $\bar{g}_{ab}$, which are conformally related: $g_{ab} = f^2\bar{g}_{ab}$, f being a smooth strictly positive function depending on the spacetime coordinates.

Guided by this topic arises a new idea in the context of general relativistic elasticity. Under the hypothesis of having two conformally related metrics, it is interesting to investigate the consequences on and the relations between relativistic elastic quantities, such as the elasticity difference tensor, associated with the two metrics. For further investigation, the obtained relations can then be advantageous to study elasticity for two specific space-times, which are conformally related, by transferring known results and properties for one space-time to the other space-time.

The intended problem branches into two possible study proposals.

One possibility is to consider two conformally related spacetime metrics g_{ab} and $\bar{g}_{ab}$: $g_{ab} = f^2\bar{g}_{ab}$, and to study the relations and the consequences for relativistic elastic quantities. This problem is interesting, but since the elastic quantities depend on material tensors, essentially on the material metrics, originally defined on the material space which are then pulled back to the space-time, it seems to be more straightforward and technically feasible to begin with the other study proposal.

Instead of having two conformally related space-time metrics, this other study possibility consists in considering two conformally related material metrics K_{AB}, $\bar{K}_{AB}$: $K_{AB} = f^2\bar{K}_{AB}$, and in studying the consequences on relativistic elastic quantities and the relations between the quantities associated with K_{AB} and $\bar{K}_{AB}$. Having in view the study of this problem, a starting point for a new problem is opened. Assuming that the pulled-back conformally related material metrics k_{ab} and $\bar{k}_{ab}$ belong to the space-times (M, g) and $(M, \bar{g})$, respectively, how are the space-time metrics g_{ab} and $\bar{g}_{ab}$ related? An additional question is: Can k_{ab} and $\bar{k}_{ab}$ belong to the same space-time, i.e.

[4]See e.g. Wald (1984) [67], Carroll (2004) [15].

$g_{ab} = \bar{g}_{ab}$?

Regarding applications of the elastic relativistic theory, one can observe in the literature, that it is quite common to work with a flat material metric. However, there is no apparent strong reason for such an assumption. Physically, one can assume that the material metric K has non-zero curvature and is obtained from another material metric $\bar{K}$ through a conformal transformation: $K = f^2\bar{K}$, the metric $\bar{K}$ being possibly flat or also non-flat. Further, concerning the space-time metrics, one can suppose that the pulled-back material metrics belong to the same space-time, that is: $g_{ab} = \bar{g}_{ab}$.

Thus, as a first step and attempt to approach this field of problems, here it is assumed that the pulled-back conformally related material metrics belong to the same space-time, i.e. $g_{ab} = \bar{g}_{ab}$. Consequences like how some relativistic elastic quantities change under a conformal transformation of the material metric are studied. Among other topics the relationship between the elasticity difference tensors associated with the two material metrics is analysed. Furthermore the eigenvalue-eigenvector problem considered in Chapter 3 is investigated from this point of view.

4.1 Problem set-up

Let (M, g) be a space-time equipped with coordinates w^a and (X, K), a material space with coordinates ξ^A.

The configuration map $\Psi : M \longrightarrow X$ leads to the following presentation of the material coordinates: $\xi^A = \xi^A(w^a)$.

Consider two conformally related material metrics K_{AB} and $\bar{K}_{AB}$ defined on the material space such that

$$K_{AB} = f^2\bar{K}_{AB}, \tag{4.1}$$

where f is a smooth, strictly positive function depending on the material coordinates (consequently on the space-time coordinates): $f = f(\xi^A(w^a))$.

The corresponding pulled-back material metrics k_{ab} and $\bar{k}_{ab}$ are in the same way conformally related:

$$k_{ab} = f^2 \bar{k}_{ab}. \tag{4.2}$$

In fact, using the relativistic deformation gradient $\xi^A_a = \frac{\partial \xi^A}{\partial w^a}$ to perform the pull-back operation: $\Psi^* K_{AB} = f^2 \Psi^* \bar{K}_{AB}$, one obtains (4.2) from (4.1).
Working with the orthonormal tetrad $\{u, x, y, z\}$, where x, y and z are the spatial eigenvectors of $k^a{}_b$ and $\bar{k}^a{}_b$, these metrics can respectively be written as

$$k_{ab} = n_1^2 x_a x_b + n_2^2 y_a y_b + n_3^2 z_a z_b, \tag{4.3}$$

and

$$\bar{k}_{ab} = \bar{n}_1^2 x_a x_b + \bar{n}_2^2 y_a y_b + \bar{n}_3^2 z_a z_b, \tag{4.4}$$

where n_i^2 and $\bar{n}_i^2$, $i = 1, 2, 3$, denote the eigenvalues of the respective metrics.

4.2 Consequences

4.2.1 Relations between the eigenvalues and between the particle number densities

Since $k = f^2 \bar{k}$, one concludes that the eigenvalues of the metric k, given in (4.3), are related with those of the metric $\bar{k}$, given in (4.4), in the following way:

$$n_1^2 = f^2 \bar{n}_1^2 \tag{4.5}$$

$$n_2^2 = f^2 \bar{n}_2^2 \tag{4.6}$$

$$n_3^2 = f^2 \bar{n}_3^2. \tag{4.7}$$

The determinants n^2 of k and $\bar{n}^2$ of $\bar{k}$ satisfy

$$n^2 = (n_1 n_2 n_3)^2 = f^6 (\bar{n}_1 \bar{n}_2 \bar{n}_3)^2 = f^6 \bar{n}^2. \tag{4.8}$$

Consequently, for the particle number densities n and $\bar{n}$ one has:

$$n = f^3 \bar{n}. \tag{4.9}$$

4.2.2 The energy-momentum tensor and further relations

Consider the following expression of the energy-momentum tensor[5]

$$T_{ab} = -\rho g_{ab} + 2\frac{\partial \rho}{\partial g^{ab}} = \rho u_a u_b + p_{ab}, \tag{4.10}$$

$\rho = n\epsilon$ being the energy density and p_{ab}, the pressure tensor.

The pressure tensor can be written as

$$p_{ab} = p_1 x_a x_b + p_2 y_a y_b + p_3 z_a z_b, \tag{4.11}$$

where

$$p_i = n n_i \frac{\partial \epsilon}{\partial n_i}, \tag{4.12}$$

for $i = 1, 2, 3$.

The tensors $p^a{}_b$, $k^a{}_b$ and $\bar{k}^a{}_b$ have the same eigenvectors: x^a, y^a and z^a.

The energy-momentum tensor T_{ab} depends on the metric k through its eigenvalues, more precisely, through the square roots n_1, n_2 and n_3 of the eigenvalues. However, taking into account the metric $\bar{k}$, the energy-momentum tensor also depends on $\bar{k}$, this time through the square roots $\bar{n}_1$, $\bar{n}_2$ and $\bar{n}_3$ of the eigenvalues. Let the energy-momentum tensor associated with $\bar{k}$ be denoted by $\bar{T}_{ab}$.

From the fact that the energy-momentum tensors coincide

$$T_{ab} = \bar{T}_{ab}, \tag{4.13}$$

[5]See Section 2.1.12.1 of Chapter 2.

one obtains the conditions presented below.

In effect, writing the energy-momentum tensor considering the metric k,

$$T_{ab} = n\epsilon u_a u_b + nn_1 \frac{\partial \epsilon}{\partial n_1} x_a x_b + nn_2 \frac{\partial \epsilon}{\partial n_2} y_a y_b + nn_3 \frac{\partial \epsilon}{\partial n_3} z_a z_b, \tag{4.14}$$

equating it to the energy-momentum tensor associated with the metric $\bar{k}$,

$$\bar{T}_{ab} = \bar{n}\bar{\epsilon} u_a u_b + \bar{n}\bar{n}_1 \frac{\partial \bar{\epsilon}}{\partial \bar{n}_1} x_a x_b + \bar{n}\bar{n}_2 \frac{\partial \bar{\epsilon}}{\partial \bar{n}_2} y_a y_b + \bar{n}\bar{n}_3 \frac{\partial \bar{\epsilon}}{\partial \bar{n}_3} z_a z_b, \tag{4.15}$$

and using the relations (4.5-4.7) and (4.9), leads to the following conditions

$$\epsilon = \frac{1}{f^3} \bar{\epsilon} \tag{4.16}$$

$$\frac{\partial \epsilon}{\partial n_1} = \frac{1}{f^4} \frac{\partial \bar{\epsilon}}{\partial \bar{n}_1} \tag{4.17}$$

$$\frac{\partial \epsilon}{\partial n_2} = \frac{1}{f^4} \frac{\partial \bar{\epsilon}}{\partial \bar{n}_2} \tag{4.18}$$

$$\frac{\partial \epsilon}{\partial n_3} = \frac{1}{f^4} \frac{\partial \bar{\epsilon}}{\partial \bar{n}_3} \tag{4.19}$$

The next paragraphs show how the quantities depending on the metric $\bar{k}$ are related to those depending on the metric k.

4.2.3 Relation between the constant volume shear tensors

For two conformally related material metrics defined on the same space-time (M, g), satisfying $k_{ab} = f^2 \bar{k}_{ab}$, the constant volume shear tensor associated with the metric k,

$$s_{ab} = \frac{1}{2} \left(h_{ab} - n^{-2/3} k_{ab} \right), \tag{4.20}$$

coincides with the constant volume shear tensor associated with the metric $\bar{k}$, given by

$$\bar{s}_{ab} = \frac{1}{2} \left(h_{ab} - \bar{n}^{-2/3} \bar{k}_{ab} \right), \tag{4.21}$$

so that

$$s_{ab} = \bar{s}_{ab}. \tag{4.22}$$

This result can be proved by substituting $k_{ab} = f^2 \bar{k}_{ab}$ and $n = f^3 \bar{n}$ in equation (4.20).

Therefore, two conformally related material metrics belonging to the same space-time have the same constant volume shear tensor. In other words, the conformal transformation of the material metric leaves the constant volume shear tensor invariant. In this case, the state of shear of a material characterized by a conformal material metric is independent of the conformal factor.

4.2.4 Relation between the elasticity difference tensors

Consider the elasticity difference tensor corresponding to the pulled-back material metric k:

$$S^a{}_{bc} = \frac{1}{2} k^{-1am} \left(D_b k_{mc} + D_c k_{mb} - D_m k_{bc} \right) \tag{4.23}$$

and the elasticity difference tensor corresponding to the pulled-back material metric $\bar{k}$:

$$\bar{S}^a{}_{bc} = \frac{1}{2} \bar{k}^{-1am} \left(D_b \bar{k}_{mc} + D_c \bar{k}_{mb} - D_m \bar{k}_{bc} \right). \tag{4.24}$$

Note that D, the spatially projected connection[6], is equal for both expressions, $D = \bar{D}$, since it is obtained from the connection associated with the space-time metric g, which is supposed to be equal for k and $\bar{k}$, i.e. $g = \bar{g}$ and $\nabla = \bar{\nabla}$.
Introducing $k = f^2 \bar{k}$ in (4.23) and making use of $k^{-1am} k_{mb} = h^a{}_b$ leads to the following relation between S and $\bar{S}$

$$S^a{}_{bc} = \bar{S}^a{}_{bc} + \frac{1}{f} \left(h^a{}_c D_b f + h^a{}_b D_c f - \bar{k}^{-1am} \bar{k}_{bc} D_m f \right), \tag{4.25}$$

where $\bar{k}^{-1am} \bar{k}_{bc} = k^{-1am} k_{bc}$ is valid.

4.2.4.1 Tetrad expression and traces

Using the orthonormal tetrad

$$e^a_\mu = (e^a_0, e^a_1, e^a_2, e^a_3) = (u^a, x^a, y^a, z^a)$$

the relationship (4.25) in tetrad components takes the form

$$S^\alpha{}_{\beta\gamma} = \bar{S}^\alpha{}_{\beta\gamma} + \frac{1}{f} \left(\delta^\alpha{}_\gamma D_b(f) e^b_\beta + \delta^\alpha{}_\beta D_c(f) e^c_\gamma - \frac{\bar{n}^2_\beta}{\bar{n}^2_\alpha} \delta_{\beta\gamma} D_m(f) e^m_\alpha \right), \tag{4.26}$$

[6]See (2.60) in Chapter 2.

where the tetrad expression for the elasticity difference tensor $S^{\alpha}{}_{\beta\gamma}$ is given in (3.17) and

$$\begin{aligned}\bar{S}^{\alpha}{}_{\beta\gamma} = \frac{1}{2}[&(1-\bar{\epsilon}_{\gamma\alpha})\gamma^{\alpha}{}_{\gamma\beta} + (1-\bar{\epsilon}_{\beta\alpha})\gamma^{\alpha}{}_{\beta\gamma} + (\bar{\epsilon}_{\gamma\alpha} - \bar{\epsilon}_{\beta\alpha})\gamma_{\beta\gamma}{}^{\alpha} \\ &+ \bar{m}_{\beta\alpha}\delta^{\alpha}{}_{\gamma} + \bar{m}_{\gamma\alpha}\delta^{\alpha}{}_{\beta} - \bar{m}^{\alpha}{}_{\beta}\delta_{\beta\gamma}\bar{\epsilon}_{\beta\alpha}],\end{aligned} \tag{4.27}$$

with $\bar{\epsilon}_{\gamma\alpha} = \left(\frac{\bar{n}_{\gamma}^2}{\bar{n}_{\alpha}^2}\right)$ and $\bar{m}^{\alpha}{}_{\beta} = D_a(\ln \bar{n}_{\beta}^2)e^{a\alpha}$.

The relationships between the traces of the elasticity difference tensors in tetrad components are given by

$$\sum_{\alpha=1}^{3} S^{\alpha}{}_{\alpha\gamma} = \sum_{\alpha=1}^{3}\left[\bar{S}^{\alpha}{}_{\alpha\gamma} + \frac{1}{f}\left(D_b(f)\, e^{b}_{\gamma}\right)\right] \tag{4.28}$$

and

$$\sum_{\beta=1}^{3} S^{\alpha}{}_{\beta\beta} = \sum_{\beta=1}^{3}\left[\bar{S}^{\alpha}{}_{\beta\beta} + \frac{1}{f}\left(2\delta^{\alpha}{}_{\beta}D_b(f)\, e^{b}_{\beta} - \frac{\bar{n}_{\beta}^2}{\bar{n}_{\alpha}^2}D_m(f)e^{m}_{\alpha}\right)\right]. \tag{4.29}$$

The expression for $S^{\alpha}{}_{\alpha\gamma}$ and $S^{\alpha}{}_{\beta\beta}$ can be found in (3.22) and (3.23), respectively, and the traces for $\bar{S}^{\alpha}{}_{\beta\gamma}$ are

$$\sum_{\alpha=1}^{3} \bar{S}^{\alpha}{}_{\alpha\gamma} = \sum_{\alpha=1}^{3}\frac{1}{2}\bar{m}_{\gamma\alpha} = \sum_{\alpha=1}^{3}\frac{1}{\bar{n}_{\alpha}}D_a(\bar{n}_{\alpha})e^{a}_{\gamma} \tag{4.30}$$

and

$$\sum_{\beta=1}^{3} \bar{S}^{\alpha}{}_{\beta\beta} = \sum_{\beta=1}^{3}\left[(1-\bar{\epsilon}_{\beta\alpha})\gamma^{\alpha}{}_{\beta\beta} + \bar{m}_{\beta\alpha}\delta^{\alpha}{}_{\beta} - \frac{1}{2}\bar{m}^{\alpha}{}_{\beta}\bar{\epsilon}_{\beta\alpha}\right]. \tag{4.31}$$

4.2.5 Relations between the second-order tensors $\underset{1}{M}$, $\underset{2}{M}$, $\underset{3}{M}$ and $\underset{1}{\bar{M}}$, $\underset{2}{\bar{M}}$, $\underset{3}{\bar{M}}$

According to (3.24) of Chapter 3, the decomposition of the elasticity difference tensor for the metric k can be written as

$$S^{a}{}_{bc} = \underset{1}{M}_{bc}x^{a} + \underset{2}{M}_{bc}y^{a} + \underset{3}{M}_{bc}z^{a} = \sum_{\alpha=1}^{3}\underset{\alpha}{M}_{bc}e^{a}_{\alpha} \tag{4.32}$$

and the decomposition of the elasticity difference tensor for the metric $\bar{k}$, as

$$\bar{S}^a{}_{bc} = \underset{1}{\bar{M}}_{bc} x^a + \underset{2}{\bar{M}}_{bc} y^a + \underset{3}{\bar{M}}_{bc} z^a = \sum_{\alpha=1}^{3} \underset{\alpha}{\bar{M}}_{bc} e^a_\alpha. \tag{4.33}$$

The second order tensors are defined by

$$\underset{1}{M}_{bc} = S^a{}_{bc} x_a \qquad \underset{2}{M}_{bc} = S^a{}_{bc} y_a \qquad \underset{1}{M}_{bc} = S^a{}_{bc} z_a \tag{4.34}$$

and

$$\underset{1}{\bar{M}}_{bc} = \bar{S}^a{}_{bc} x_a \qquad \underset{2}{\bar{M}}_{bc} = \bar{S}^a{}_{bc} y_a \qquad \underset{1}{\bar{M}}_{bc} = \bar{S}^a{}_{bc} z_a. \tag{4.35}$$

Inserting (4.25) into (4.34) and using (4.35), the calculations reveal that the relations between the tensors $\underset{1}{M}$, $\underset{2}{M}$, $\underset{3}{M}$ and $\underset{1}{\bar{M}}$, $\underset{2}{\bar{M}}$, $\underset{3}{\bar{M}}$ are:

$$\underset{1}{M}_{bc} = \underset{1}{\bar{M}}_{bc} + \frac{1}{f}\left(x_c D_b f + x_b D_c f - \frac{1}{n_1^2} k_{bc} x^m D_m f\right) \tag{4.36}$$

$$\underset{2}{M}_{bc} = \underset{2}{\bar{M}}_{bc} + \frac{1}{f}\left(y_c D_b f + y_b D_c f - \frac{1}{n_2^2} k_{bc} y^m D_m f\right) \tag{4.37}$$

$$\underset{3}{M}_{bc} = \underset{3}{\bar{M}}_{bc} + \frac{1}{f}\left(z_c D_b f + z_b D_c f - \frac{1}{n_3^2} k_{bc} z^m D_m f\right) \tag{4.38}$$

Rewriting the last expressions using the orthonormal tetrad

$$e^a_\mu = (e^a_0, e^a_1, e^a_2, e^a_3) = (u^a, x^a, y^a, z^a),$$

employed in Section 3.3 of Chapter 3, leads to

$$\underset{\alpha}{M}_{bc} \;\; = \underset{\alpha}{\bar{M}}_{bc} + \tfrac{1}{f}\left(e^\alpha_c D_b f + e^\alpha_b D_c f - \tfrac{1}{n_\alpha^2} k_{bc} e^{\alpha m} D_m f\right). \tag{4.39}$$

Recall that letters from the first half of the Greek alphabet denote spatial tetrad indices.

4.3 Eigenvalue-eigenvector problem

In this section, the eigenvalue-eigenvector problem, studied in Chapter 3, is here reconsidered, now for the case of having two conformally related pulled-back material metrics $k_{ab} = f^2\bar{k}_{ab}$ belonging to the same space-time.

The first purpose is to investigate conditions for x, y and z, the eigendirections of k and $\bar{k}$, to be eigenvectors for $\underset{\alpha}{M}$, $\alpha = 1, 2, 3$, knowing that (4.39) holds. That means that the equation

$$\underset{\alpha}{M}{}_b{}^c \omega^b = \lambda \omega^c \tag{4.40}$$

is solved particularly for $\omega^b = x^b$, $\omega^b = y^b$ and $\omega^b = z^b$.

Since the tensors $\underset{\alpha}{M}$ are related with $\underset{\alpha}{\bar{M}}$, $\alpha = 1, 2, 3$, as shown in the previous section, it is also interesting to continue and extend the analysis by finding conditions for x, y and z to be also eigenvectors for $\underset{\alpha}{\bar{M}}$. This problem corresponds to solve

$$\underset{\alpha}{\bar{M}}{}_b{}^c \omega^b = \bar{\lambda} \omega^c \tag{4.41}$$

for $\omega^b = x^b$, $\omega^b = y^b$ and $\omega^b = z^b$, in addition to (4.40). The last step enables then to establish the relation between the corresponding eigenvalues λ and $\bar{\lambda}$ to which each of the eigenvectors is associated.

The developed analysis is summarized in the following theorems. The results obtained from the first problem considering the eigenvalue-eigenvector equation (4.40) appear in the theorems in item a) and the results concerned with the problem of solving (4.40) together with (4.41) appear in item b).

4.3.1 Eigenvalue-eigenvector problem for $\underset{1}{M}$ and $\underset{1}{\bar{M}}$

The first three theorems, Theorem 12, Theorem 13 and Theorem 14, are devoted to investigate the stated problem for $\underset{1}{M}$ which is related with $\underset{1}{\bar{M}}$ through equation (4.36).

Theorem 12 *a) x is an eigenvector for $\underset{1}{M}$ iff $\underset{1}{\bar{M}}{}_b{}^c x^b y_c + \Delta_y(\ln f) = 0$ and $\underset{1}{\bar{M}}{}_b{}^c x^b z_c + \Delta_z(\ln f) = 0$.*
The corresponding eigenvalue is $\lambda = \underset{1}{\bar{M}}{}_b{}^c x^b x_c + \Delta_x(\ln f)$.
b) x is an eigenvector for $\underset{1}{M}$ and $\underset{1}{\bar{M}}$ iff f remains invariant along the directions of y and z, i.e. $\Delta_y(\ln f) = \Delta_z(\ln f) = 0$.

The relation between the corresponding eigenvalues is given by

$$\lambda = \bar{\lambda} + \Delta_x(\ln f). \tag{4.42}$$

Proof: Consider the conditions

$$\underset{1}{M}{}_b{}^c x^b x_c = \lambda, \tag{4.43}$$

$$\underset{1}{M}{}_b{}^c x^b y_c = 0, \tag{4.44}$$

$$\underset{1}{M}{}_b{}^c x^b z_c = 0, \tag{4.45}$$

which must be satisfied for x to be an eigenvector of $\underset{1}{M}$. Introducing the expression (4.36) for $\underset{1}{M}$ in these equations and performing simplifications leads to the results given in a).

Assume additionally that

$$\underset{1}{\bar{M}}{}_b{}^c x^b x_c = \bar{\lambda}, \tag{4.46}$$

$$\underset{1}{\bar{M}}{}_b{}^c x^b y_c = 0, \tag{4.47}$$

$$\underset{1}{\bar{M}}{}_b{}^c x^b z_c = 0. \tag{4.48}$$

These conditions must be verified in order to have x as eigenvector for $\underset{1}{\bar{M}}$. Substituting these expressions in a), one obtains the formula relating the eigenvalues λ and $\bar{\lambda}$ and the conditions appearing in b).

Also, supposing that the conditions given in a) and b), respectively, are satisfied, then it follows that x is an eigenvector for $\underset{1}{M}$, respectively, for $\underset{1}{M}$ and $\underset{1}{\bar{M}}$.

$\square$

Theorem 13 *a) y is an eigenvector for $\underset{1}{M}$ iff $\underset{1}{\bar{M}}{}_b{}^c y^b x_c + \Delta_y(\ln f) = 0$ and $\underset{1}{\bar{M}}{}_b{}^c y^b z_c = 0$. The corresponding eigenvalue is $\lambda = \underset{1}{\bar{M}}{}_b{}^c y^b y_c - \frac{n_2^2}{n_1^2}\Delta_x(\ln f)$.*
b) y is an eigenvector for $\underset{1}{M}$ and $\underset{1}{\bar{M}}$ iff f remains invariant along the direction of y, i.e. $\Delta_y(\ln f) = 0$, and $\underset{1}{\bar{M}}{}_b{}^c y^b z_c = 0$.
The relation between the corresponding eigenvalues is given by

$$\lambda = \bar{\lambda} - \frac{n_2^2}{n_1^2}\Delta_x(\ln f). \tag{4.49}$$

Proof: The results are obtained by contracting $\underset{1}{M}{}_b{}^c y^b = \lambda y^c$ and $\underset{1}{\bar{M}}{}_b{}^c y^b = \lambda y^c$ with x_c, y_c and z_c in the way explained below.
Solving

$$\underset{1}{M}{}_b{}^c y^b x_c = 0, \tag{4.50}$$

by using (4.36) gives the first condition in a), which together with

$$\underset{1}{\bar{M}}{}_b{}^c y^b x_c = 0, \tag{4.51}$$

results in $\Delta_y(\ln f) = 0$, the first condition in b).
Imposing

$$\underset{1}{M}{}_b{}^c y^b z_c = 0 \tag{4.52}$$

leads to the second condition in a)

$$\underset{1}{\bar{M}}{}_b{}^c y^b z_c = 0, \tag{4.53}$$

which must also be satisfied in case b). The eigenvalue λ in a) is calculated from

$$\underset{1}{M}{}_b{}^c y^b y_c = \lambda. \tag{4.54}$$

And taking into account

$$\underset{1}{\bar{M}}{}_b{}^c y^b y_c = \bar{\lambda} \tag{4.55}$$

one obtains the relation between the eigenvalues exposed in b).

Conversely, if the conditions presented in a) and b), respectively, hold, then it can be shown that y is an eigenvector for $\underset{1}{M}$, respectively, for $\underset{1}{M}$ and $\underset{1}{\bar{M}}$, associated with the corresponding eigenvalues.

$\square$

Theorem 14 *a) z is an eigenvector for $\underset{1}{M}$ iff $\underset{1}{\bar{M}}{}_b{}^c z^b x_c + \Delta_z(\ln f) = 0$ and $\underset{1}{\bar{M}}{}_b{}^c z^b y_c = 0$. The corresponding eigenvalue is $\lambda = \underset{1}{\bar{M}}{}_b{}^c z^b z_c - \frac{n_3^2}{n_1^2}\Delta_x(\ln f)$.*
b) z is an eigenvector for $\underset{1}{M}$ and $\underset{1}{\bar{M}}$ iff f remains invariant along the direction of z, i.e. $\Delta_z(\ln f) = 0$, and $\underset{1}{\bar{M}}{}_b{}^c z^b y_c = 0$.
The relation between the corresponding eigenvalues is given by

$$\lambda = \bar{\lambda} - \frac{n_3^2}{n_1^2}\Delta_x(\ln f). \tag{4.56}$$

Proof: Consider the eigenvalue-eigenvector equations $\underset{1}{M}{}_b{}^c z^b = \lambda z^c$ and $\underset{1}{\bar{M}}{}_b{}^c z^b = \bar{\lambda} z^c$. The first condition in a) results from calculating

$$\underset{1}{M}{}_b{}^c z^b x_c = 0, \tag{4.57}$$

where $\underset{1}{M}$ is substituted by the expression (4.36). One gets the condition $\Delta_z(\ln f) = 0$ in b) introducing

$$\underset{1}{\bar{M}}{}_b{}^c z^b x_c = 0 \tag{4.58}$$

in the expression obtained from (4.57).

Imposing

$$\underset{1}{M}{}_b{}^c z^b y_c = 0 \tag{4.59}$$

yields

$$\underset{1}{\bar{M}}{}_b{}^c z^b y_c = 0, \tag{4.60}$$

which must be satisfied in both cases a) and b). The expression for the eigenvalue given in a) is obtained from

$$\underset{1}{M}_b{}^c z^b z_c = \lambda \tag{4.61}$$

by using again (4.36). Inserting

$$\underset{1}{\bar{M}}_b{}^c z^b z_c = \bar{\lambda} \tag{4.62}$$

in that expression for λ implies

$$\lambda = \bar{\lambda} - \frac{n_3^2}{n_1^2}\Delta_x(\ln f). \tag{4.63}$$

On the other hand, the conditions given in a) and b), respectively, imply that z is an eigenvector for $\underset{1}{M}$, respectively, for $\underset{1}{M}$ and $\underset{1}{\bar{M}}$, associated with the given eigenvalues.

□

4.3.2 Eigenvalue-eigenvector problem for $\underset{2}{M}$ and $\underset{2}{\bar{M}}$

The eigenvalue-eigenvector problem is now studied for the tensors $\underset{2}{M}$ and $\underset{2}{\bar{M}}$. The results appear in Theorem 15, Theorem 16 and Theorem 17.

Theorem 15 *a) x is an eigenvector for $\underset{2}{M}$ iff $\underset{2}{\bar{M}}_b{}^c x^b y_c + \Delta_x(\ln f) = 0$ and $\underset{2}{\bar{M}}_b{}^c x^b z_c = 0$.*
The corresponding eigenvalue is $\lambda = \underset{2}{\bar{M}}_b{}^c x^b x_c - \frac{n_1^2}{n_2^2}\Delta_y(\ln f)$.
b) x is an eigenvector for $\underset{2}{M}$ and $\underset{2}{\bar{M}}$ iff f remains invariant along the direction of x, i.e. $\Delta_x(\ln f) = 0$, and $\underset{2}{\bar{M}}_b{}^c x^b z_c = 0$.
The relation between the corresponding eigenvalues is given by

$$\lambda = \bar{\lambda} - \frac{n_1^2}{n_2^2}\Delta_y(\ln f). \tag{4.64}$$

Proof: In order to solve the eigenvalue-eigenvector problems $\underset{2}{M}_b{}^c x^b = \lambda x^c$ and $\underset{2}{\bar{M}}_b{}^c x^b = \bar{\lambda} x^c$, one must contract each of these equations with x_c, y_c and z_c. The results are obtained as follows. Calculating

$$\underset{2}{M}_b{}^c x^b y_c = 0, \tag{4.65}$$

by using (4.37), gives the first condition in a), and introducing there the equation

$$\underset{2}{\bar{M}}_b{}^c x^b y_c = 0, \tag{4.66}$$

leads to the condition $\Delta_x(\ln f) = 0$ in b). The condition

$$\underset{2}{M}_b{}^c x^b z_c = 0 \tag{4.67}$$

yields

$$\underset{2}{\bar{M}}_b{}^c x^b z_c = 0, \tag{4.68}$$

appearing in a), which must also be imposed in case b). The expression for the eigenvalue λ in a) is obtained from

$$\underset{2}{M}_b{}^c z^b z_c = \lambda \tag{4.69}$$

together with (4.37). Substituting in that expression the condition

$$\underset{2}{\bar{M}}_b{}^c z^b z_c = \bar{\lambda}, \tag{4.70}$$

the relation between the eigenvalues λ and $\bar{\lambda}$ in b) is established.
Conversely, from the conditions established in a) and b), respectively, it follows that x is an eigenvector for $\underset{2}{M}$, respectively for $\underset{2}{M}$ and $\underset{2}{\bar{M}}$, associated with the given eigenvalues.

□

Theorem 16 *a) y is an eigenvector for $\underset{2}{M}$ iff $\underset{2}{\bar{M}}_b{}^c y^b x_c + \Delta_x(\ln f) = 0$ and $\underset{2}{\bar{M}}_b{}^c y^b z_c + \Delta_z(\ln f) = 0$.*
The corresponding eigenvalue is $\lambda = \underset{2}{\bar{M}}_b{}^c y^b y_c + \Delta_y(\ln f)$.
b) y is an eigenvector for $\underset{2}{M}$ and $\underset{2}{\bar{M}}$ iff f remains invariant along the directions of x and z, i.e. $\Delta_x(\ln f) = \Delta_z(\ln f) = 0$.
The relation between the corresponding eigenvalues is given by

$$\lambda = \bar{\lambda} + \Delta_y(\ln f). \tag{4.71}$$

Proof: Consider the conditions

$$\underset{2}{M}_b{}^c y^b x_c = 0, \tag{4.72}$$

$$\underset{2}{M}_b{}^c y^b y_c = \lambda, \tag{4.73}$$

$$\underset{2}{M}_b{}^c y^b z_c = 0, \tag{4.74}$$

which must be satisfied to have y^b as eigenvector for $\underset{2}{M}$, and the conditions

$$\underset{2}{\bar{M}}_b{}^c y^b x_c = 0, \tag{4.75}$$

$$\underset{2}{\bar{M}}_b{}^c y^b y_c = \bar{\lambda}, \tag{4.76}$$

$$\underset{2}{\bar{M}}_b{}^c y^b z_c = 0, \tag{4.77}$$

which correspond to y^b being an eigenvector for $\underset{2}{\bar{M}}$. You get the first condition in a) from (4.72) by considering (4.37). Together with (4.75) that condition results in $\Delta_x(\ln f) = 0$, presented in b). Calculating (4.74) and making use of (4.37) yields the second condition in a). Inserting there the expression (4.77) gives the condition $\Delta_z(\ln f) = 0$ in b). The eigenvalue λ in a) is obtained from (4.73), where (4.37) is used. Joining that equation for λ and (4.76) allows to calculate the relation between the eigenvalues given in b).

On the other hand, if the conditions given in a) and b) hold, then (4.72-4.77) are satisfied, so that y is an eigenvector for $\underset{2}{M}$, respectively for $\underset{2}{M}$ and $\underset{2}{\bar{M}}$.

□

Theorem 17 *a) z is an eigenvector for $\underset{2}{M}$ iff $\underset{2}{\bar{M}}_b{}^c z^b y_c + \Delta_z(\ln f) = 0$ and $\underset{2}{\bar{M}}_b{}^c z^b x_c = 0$. The corresponding eigenvalue is $\lambda = \underset{2}{\bar{M}}_b{}^c z^b z_c - \frac{n_3^2}{n_2^2}\Delta_y(\ln f)$.*
b) z is an eigenvector for $\underset{2}{M}$ and $\underset{2}{\bar{M}}$ iff f remains invariant along the direction of z, i.e. $\Delta_z(\ln f) = 0$, and $\underset{2}{\bar{M}}_b{}^c z^b x_c = 0$.
The relation between the corresponding eigenvalues is given by

$$\lambda = \bar{\lambda} - \frac{n_3^2}{n_2^2}\Delta_y(\ln f). \tag{4.78}$$

Proof: Consider the eigenvalue-eigenvector equations

$$\underset{2}{M}_b{}^c z^b = \lambda z^c \tag{4.79}$$

and

$$\underset{2}{\bar{M}}_b{}^c z^b = \bar{\lambda} z^c. \tag{4.80}$$

Solving

$$\underset{2}{M}_b{}^c z^b y_c = 0, \tag{4.81}$$

by using (4.37), one gets the first condition in a). Substituting there the expression

$$\underset{2}{\bar{M}}_b{}^c z^b y_c = 0, \tag{4.82}$$

leads to $\Delta_z(\ln f) = 0$ given in b). The condition

$$\underset{2}{M}_b{}^c z^b x_c = 0 \tag{4.83}$$

implies

$$\underset{2}{\bar{M}}_b{}^c z^b x_c = 0, \tag{4.84}$$

which must be satisfied for case a) and b). The eigenvalue λ in a) is obtained from

$$\underset{2}{M}_b{}^c z^b z_c = \lambda \tag{4.85}$$

by using (4.37). Substituting in the expression for λ the condition

$$\underset{2}{\bar{M}}_b{}^c z^b z_c = \bar{\lambda} \tag{4.86}$$

leads to the relation between the eigenvalues given in b).

Conversely, the conditions presented in a) and b), respectively, imply that (4.81-4.86) are satisfied, which mean that z is an eigenvector for $\underset{2}{M}$, respectively, for $\underset{2}{M}$ and $\underset{2}{\bar{M}}$ associated with the corresponding eigenvalues.

$\square$

4.3.3 Eigenvalue-eigenvector problem for $\underset{3}{M}$ and $\underset{3}{\bar{M}}$

The following three theorems, Theorem 18, Theorem 19 and Theorem 20, are concerned with the eigenvalue-eigenvector problem for the tensors $\underset{3}{M}$ and $\underset{3}{\bar{M}}$.

Theorem 18 *a) x is an eigenvector for $\underset{3}{M}$ iff $\underset{3}{\bar{M}}{}_b{}^c x^b z_c + \Delta_x(\ln f) = 0$ and $\underset{3}{\bar{M}}{}_b{}^c x^b y_c = 0$.*
The corresponding eigenvalue is $\lambda = \underset{3}{\bar{M}}{}_b{}^c x^b x_c - \frac{n_1^2}{n_3^2}\Delta_z(\ln f)$.
b) x is an eigenvector for $\underset{3}{M}$ and $\underset{3}{\bar{M}}$ iff f remains invariant along the direction of x, i.e. $\Delta_x(\ln f) = 0$, and $\underset{3}{\bar{M}}{}_b{}^c x^b y_c = 0$.
The relation between the corresponding eigenvalues is given by

$$\lambda = \bar{\lambda} - \frac{n_1^2}{n_3^2}\Delta_z(\ln f). \tag{4.87}$$

Proof: Contracting the eigenvalue-eigenvector equations $\underset{3}{M}{}_b{}^c x^b = \lambda x^c$ and $\underset{3}{\bar{M}}{}_b{}^c x^b = \bar{\lambda} x^c$ with x_c, y_c and z_c, one obtains the following results.
Calculating

$$\underset{3}{M}{}_b{}^c x^b z_c = 0 \tag{4.88}$$

together with (4.38) gives the first condition in a). Inserting

$$\underset{3}{\bar{M}}{}_b{}^c x^b z_c = 0 \tag{4.89}$$

in those condition results in $\Delta_x(\ln f) = 0$, the condition appearing in case b). Imposing

$$\underset{3}{M}{}_b{}^c x^b y_c = 0, \tag{4.90}$$

leads to

$$\underset{3}{\bar{M}}{}_b{}^c x^b y_c = 0, \tag{4.91}$$

which must be verified in case a) and in case b). Substituting (4.38) in

$$\underset{3}{M}{}_b{}^c x^b x_c = \lambda \tag{4.92}$$

reveals that the eigenvalue λ takes the form presented in a). And taking into account the condition

$$\underset{3}{\bar{M}}_b{}^c x^b x_c = \bar{\lambda} \tag{4.93}$$

implies that the eigenvalues λ and $\bar{\lambda}$ are related by $\lambda = \bar{\lambda} - \frac{n_1^2}{n_3^2}\Delta_z(\ln f)$.
On the other hand, if the conditions given in a) and b) hold, then one can prove that (4.88-4.91) are satisfied; and the expressions for the eigenvalues satisfy (4.92) and (4.93), respectively. Consequently, x is an eigenvector for $\underset{3}{M}$, respectively, for $\underset{3}{M}$ and $\underset{3}{\bar{M}}$.

□

Theorem 19 *a) y is an eigenvector for $\underset{3}{M}$ iff $\underset{3}{\bar{M}}_b{}^c y^b z_c + \Delta_y(\ln f) = 0$ and $\underset{3}{\bar{M}}_b{}^c y^b x_c = 0$.*
The corresponding eigenvalue is $\lambda = \underset{3}{\bar{M}}_b{}^c y^b y_c - \frac{n_2^2}{n_3^2}\Delta_z(\ln f)$.
b) y is an eigenvector for $\underset{3}{M}$ and $\underset{3}{\bar{M}}$ iff f remains invariant along the direction of y, i.e. $\Delta_y(\ln f) = 0$, and $\underset{3}{\bar{M}}_b{}^c y^b x_c = 0$.
The relation between the corresponding eigenvalues is given by

$$\lambda = \bar{\lambda} - \frac{n_2^2}{n_3^2}\Delta_z(\ln f). \tag{4.94}$$

Proof: Considering the eigenvalue-eigenvector equations $\underset{3}{M}_b{}^c y^b = \lambda y^c$ and $\underset{3}{\bar{M}}_b{}^c y^b = \bar{\lambda} y^c$, the results are found by contracting these equations with x_c, y_c and z_c in the way explained below.
Using (4.38) and calculating

$$\underset{3}{M}_b{}^c y^b z_c = 0 \tag{4.95}$$

yields the first equation in a). Inserting

$$\underset{3}{\bar{M}}_b{}^c y^b z_c = 0 \tag{4.96}$$

in those equation results in $\Delta_y(\ln f) = 0$, the condition given in b).
Imposing

$$\underset{3}{M}_b{}^c y^b x_c = 0 \tag{4.97}$$

yields

$$\underset{3}{\bar{M}}_b{}^c y^b x_c = 0, \tag{4.98}$$

so that (4.98) must be satisfied in case a) and b). Solving

$$\underset{3}{M}_b{}^c y^b y_c = \lambda \tag{4.99}$$

and considering (4.38) one obtains the expression for the eigenvalue presented in a). Introducing

$$\underset{3}{\bar{M}}_b{}^c y^b y_c = \bar{\lambda} \tag{4.100}$$

in those expression it follows that

$$\lambda = \bar{\lambda} - \frac{n_2^2}{n_3^2}\Delta_z(\ln f). \tag{4.101}$$

Conversely, the conditions presented in a) and b) imply that (4.95-4.98) are identically satisfied. The identities (4.99) and (4.100) are true for the eigenvalues given in a) and b), respectively. This proves that y is an eigenvector for $\underset{3}{M}$ in case a) and that y is an eigenvector for $\underset{3}{M}$ and $\underset{3}{\bar{M}}$ in case b).

□

Theorem 20 *a) z is an eigenvector for $\underset{3}{M}$ iff $\underset{3}{\bar{M}}_b{}^c z^b x_c + \Delta_x(\ln f) = 0$ and $\underset{3}{\bar{M}}_b{}^c z^b y_c + \Delta_y(\ln f) = 0$.*
The corresponding eigenvalue is $\lambda = \underset{3}{\bar{M}}_b{}^c z^b z_c + \Delta_z(\ln f)$.
b) z is an eigenvector for $\underset{3}{M}$ and $\underset{3}{\bar{M}}$ iff f remains invariant along the directions of x and y, i.e. $\Delta_x(\ln f) = \Delta_y(\ln f) = 0$.
The relation between the corresponding eigenvalues is given by

$$\lambda = \bar{\lambda} + \Delta_z(\ln f). \tag{4.102}$$

Proof: To solve the eigenvalue-eigenvector equations

$$\underset{3}{M}_b{}^c z^b = \lambda z^c \tag{4.103}$$

and

$$\underset{3}{\bar{M}}{}_b{}^c z^b = \bar{\lambda} z^c, \tag{4.104}$$

one must contract these equations with x_c, y_c and z_c. Thereby, the exposed results are obtained in the following way. The first condition in a) is a consequence of

$$\underset{3}{M}{}_b{}^c z^b x_c = 0, \tag{4.105}$$

and (4.38). Joining

$$\underset{3}{\bar{M}}{}_b{}^c z^b x_c = 0 \tag{4.106}$$

to those condition yields $\Delta_x(\ln f) = 0$ given in b). From

$$\underset{3}{M}{}_b{}^c z^b y_c = 0 \tag{4.107}$$

and (4.38) one calculates the second condition in a), which together with

$$\underset{3}{\bar{M}}{}_b{}^c z^b y_c = 0 \tag{4.108}$$

results in $\Delta_y(\ln f) = 0$. Inserting (4.38) in

$$\underset{3}{M}{}_b{}^c z^b z_c = \lambda, \tag{4.109}$$

one gets the eigenvalue presented in a). Considering

$$\underset{3}{\bar{M}}{}_b{}^c z^b z_c = \bar{\lambda} \tag{4.110}$$

leads to expression relating λ and $\bar{\lambda}$.

The converse is true as well. The conditions given in a) and b), respectively, imply (4.105-4.108). The expressions for the eigenvalues satisfy (4.109) and (4.110), respectively. Thus, z is an eigenvector for $\underset{3}{M}$ in case a), and z is an eigenvector for $\underset{3}{M}$ and $\underset{3}{\bar{M}}$ in case b).

$\square$

4.3.4 Concluding remarks

Analysing the results of the eigenvalue-eigenvector problem obtained for the tensor $\underset{1}{M}$, knowing that $\underset{1}{M}$ is related with $\underset{1}{\bar{M}}$ by

$$\underset{1}{M}_{bc} = \underset{1}{\bar{M}}_{bc} + \frac{1}{f}\left(x_c D_b f + x_b D_c f - \frac{1}{n_1^2} k_{bc} x^m D_m f\right)$$

given in (4.36), one can say the following.

The conditions that must be satisfied, so that the eigenvectors of k remain eigenvectors for $\underset{1}{M}$, depend on the conformal function f through a derivative along a spatial vector and on contractions of the tensor $\underset{1}{\bar{M}}$ with two spatial tetrad vectors[7]. In particular, two such conditions must be verified for x to be an eigenvector of $\underset{1}{M}$, whereas for y and z to be eigenvectors only one of the two conditions depends on the conformal function. Regarding the expressions for the eigenvalues of $\underset{1}{\bar{M}}$, the eigenvalue corresponding to x depends on the derivative of f along x and on $\underset{1}{\bar{M}}$ contracted totally with x. The eigenvalue corresponding to y (or z) depend on the contraction of $\underset{1}{\bar{M}}$ with the respective eigenvector y (or z), on the derivative of f along x and additionally on the eigenvalues of k: n_1^2 and n_2^2 (or n_1^2 and n_3^2).

Observing the conditions that must be satisfied in order to have x as eigenvector for $\underset{1}{M}$ and $\underset{1}{\bar{M}}$, f must be invariant along the other two eigenvectors of k, namely y and z. In this case, one obtains an expression showing the relation between the eigenvalues λ and $\bar{\lambda}$ corresponding to x. The eigenvalues differ absolutely in the quantity $\Delta_x(\ln f)$. For y (or z) to be eigenvectors of $\underset{1}{M}$ and $\underset{1}{\bar{M}}$, the function f must be invariant along the same eigenvector y (or z) and the quantity $\underset{1}{\bar{M}}_b{}^c y^b z_c$ must vanish, which due to the symmetry of $\underset{1}{\bar{M}}$ equals $\underset{1}{\bar{M}}_b{}^c z^b y_c$. The eigenvalue λ depends on $\bar{\lambda}$, on the derivative of f along x and on the eigenvalues of k: n_1^2 and n_2^2 (or n_1^2 and n_3^2). In this case, the absolute difference between the eigenvalues λ and $\bar{\lambda}$ is given by $\frac{n_2^2}{n_1^2}\Delta_x(\ln f)$, (or $\frac{n_3^2}{n_1^2}\Delta_x(\ln f)$).

Considering Theorem 15, Theorem 16 and Theorem 17, where the eigenvalue-eigenvector problem is analysed for the tensors $\underset{2}{M}$ and $\underset{2}{\bar{M}}$, one can observe a similar behaviour. The role that x plays for $\underset{1}{M}$ and $\underset{1}{\bar{M}}$ is now played by y. Interchanging x with y, $\underset{1}{M}$

[7]See Theorem 12, Theorem 13 and Theorem 14.

with $\underset{2}{M}$ and n_1^2 with n_2^2 in the preceding results leads to the conclusions concerning this case.

As for the tensors $\underset{3}{M}$ and $\underset{3}{\bar{M}}$, whose eigenvalue-eigenvector problem is dealt in Theorem 18, Theorem 19 and Theorem 20, the same presented remarks are valid by interchanging x with z, $\underset{1}{M}$ with $\underset{3}{M}$ and n_1^2 with n_3^2.

As seen in the previous chapter, the eigenvectors of k (respectively $\bar{k}$) are not, in general, eigenvectors for $\underset{\alpha}{M}$ (respectively $\underset{\alpha}{\bar{M}}$). Here, one can additionally say that the eigenvectors of k and $\bar{k}$, which are conformally related, do not remain eigenvectors for the tensors $\underset{\alpha}{M}$ and $\underset{\alpha}{\bar{M}}$ simultaneously and in order to remain, restrictions must be imposed on the conformal factor f and other restrictions involving the tensor $\underset{\alpha}{\bar{M}}$.

4.3.5 Summarizing the results

Using the orthonormal tetrad $e^a_\mu = (e^a_0, e^a_1, e^a_2, e^a_3) = (u^a, x^a, y^a, z^a)$ to reformulate the results presented in the last theorems, one can summarize them in the two following theorems:

Theorem 21 *a) The tetrad vector e_α is an eigenvector for $\underset{\alpha}{M}$ iff*
$\underset{\alpha}{\bar{M}}{}_b{}^c e^b_\alpha e_{\beta c} + \Delta_{e_\beta}(\ln f) = 0$ for each $\beta \neq \alpha$.
The corresponding eigenvalue is $\lambda = \underset{\alpha}{\bar{M}}{}_b{}^c e^b_\alpha e_{\alpha c} + \Delta_{e_\alpha}(\ln f)$.
b) The tetrad vector e_α is an eigenvector for $\underset{\alpha}{M}$ and $\underset{\alpha}{\bar{M}}$ iff f remains invariant along the two spatial tetrad vectors e_β, such that $\beta \neq \alpha$, i.e. $\Delta_{e_\beta}(\ln f) = 0$ whenever $\beta \neq \alpha$. The relation between the corresponding eigenvalues is given by
$\lambda = \bar{\lambda} + \Delta_{e_\alpha}(\ln f)$.

Proof: The eigenvalue-eigenvector problem $\underset{\alpha}{M}{}_b{}^c e^b_\alpha = \lambda e^c_\alpha$ is solved using the following algebraic conditions

$$\underset{\alpha}{M}{}_b{}^c e^b_\alpha e_{\alpha c} = \lambda, \tag{4.111}$$

$$\underset{\alpha}{M}{}_b{}^c e^b_\alpha e_{\beta c} = 0 \tag{4.112}$$

and

$$\underset{\alpha}{M}_b{}^c e^b_\alpha e_{\gamma c} = 0 \tag{4.113}$$

together with the relation (4.39), where $\gamma \neq \beta \neq \alpha$. From (4.112) and (4.113) one obtains the two conditions summarised in the expression given in a). The eigenvalue λ in a) is calculated from (4.111). Considering in addition to (4.111), (4.112) and (4.113) the conditions

$$\underset{\alpha}{\bar{M}}_b{}^c e^b_\alpha e_{\alpha c} = \bar{\lambda}, \tag{4.114}$$

$$\underset{\alpha}{\bar{M}}_b{}^c e^b_\alpha e_{\beta c} = 0 \tag{4.115}$$

and

$$\underset{\alpha}{\bar{M}}_b{}^c e^b_\alpha e_{\gamma c} = 0, \tag{4.116}$$

where $\gamma \neq \beta \neq \alpha$, one gets the results presented in b).

On the other hand, suppose that the conditions presented in a) and b), respectively, hold. Then, it follows that (4.112), (4.113) and (4.115), (4.116), respectively, are satisfied. The eigenvalues given in a) and b), respectively, satisfy (4.111) and (4.114), respectively. This proves that e_α is an eigenvector for $\underset{\alpha}{M}$ in case a) and that e_α is an eigenvector for $\underset{\alpha}{M}$ and $\underset{\alpha}{\bar{M}}$ in case b).

□

Theorem 22 *a) The tetrad vector e_β is an eigenvector for $\underset{\alpha}{M}$ iff*
$\underset{\alpha}{\bar{M}}_b{}^c e^b_\beta e_{\alpha c} + \Delta_{e_\beta}(\ln f) = 0$ *and* $\underset{\alpha}{\bar{M}}_b{}^c e^b_\beta e_{\gamma c} = 0$.
The corresponding eigenvalue is $\lambda = \underset{\alpha}{\bar{M}}_b{}^c e^b_\beta e_{\beta c} - \frac{n^2_\beta}{n^2_\alpha}\Delta_{e_\alpha}(\ln f)$.
b) The tetrad vector e_β is an eigenvector for $\underset{\alpha}{M}$ and $\underset{\alpha}{\bar{M}}$ iff $\Delta_{e_\beta}(\ln f) = 0$, for a fixed $\beta \neq \alpha$, i.e. f remains invariant along the direction of e_β, and $\underset{\alpha}{\bar{M}}_b{}^c e^b_\beta e_{\gamma c} = 0$.
The relation between the corresponding eigenvalues is given by $\lambda = \bar{\lambda} - \frac{n^2_\beta}{n^2_\alpha}\Delta_{e_\alpha}(\ln f)$.

Proof: Contracting the eigenvalue-eigenvector problems $\underset{\alpha}{M}{}_b{}^c e^b_\beta = \lambda e^c_\beta$ and $\underset{\alpha}{\bar{M}}{}_b{}^c e^b_\beta = \bar{\lambda} e^c_\beta$ with $e_{\alpha c}$, $e_{\beta c}$ and $e_{\gamma c}$, where $\gamma \neq \beta \neq \alpha$, the results are obtained as follows. Substituting (4.39) in

$$\underset{\alpha}{M}{}_b{}^c e^b_\beta e_{\alpha c} = 0 \tag{4.117}$$

and in

$$\underset{\alpha}{M}{}_b{}^c e^b_\beta e_{\gamma c} = 0 \tag{4.118}$$

leads to the first two respective conditions given in a). Joining (4.117) and the equation

$$\underset{\alpha}{\bar{M}}{}_b{}^c e^b_\beta e_{\alpha c} = 0 \tag{4.119}$$

implies $\Delta_{e_\beta}(\ln f) = 0$, the condition appearing in b). From $\underset{\alpha}{M}{}_b{}^c e^b_\beta e_{\gamma c} = 0$ and (4.39), one gets $\underset{\alpha}{\bar{M}}{}_b{}^c e^b_\beta e_{\gamma c} = 0$, the same condition in a) and b). The eigenvalue λ in a) is calculated from

$$\underset{\alpha}{M}{}_b{}^c e^b_\beta e_{\beta c} = \lambda \tag{4.120}$$

by using (4.39). Finally, introducing

$$\underset{\alpha}{\bar{M}}{}_b{}^c e^b_\beta e_{\beta c} = \bar{\lambda} \tag{4.121}$$

in those expression reveals that the eigenvalues λ and $\bar{\lambda}$ are related by
$\lambda = \bar{\lambda} - \frac{n^2_\beta}{n^2_\alpha}\Delta_{e_\alpha}(\ln f)$.
Also, if the conditions given in a) and b), respectively, hold, then (4.117-4.119) are satisfied, so that e_β is an eigenvector for $\underset{\alpha}{M}$ in case a), and e_β is an eigenvector for $\underset{\alpha}{M}$ and $\underset{\alpha}{\bar{M}}$ in case b), associated with the corresponding eigenvalues.

□

The following conclusions can be drawn. Solving the eigenvalue-eigenvector problem for $\underset{\alpha}{M}$, knowing that $\underset{\alpha}{M}$ is related with $\underset{\alpha}{\bar{M}}$ by

$$\underset{\alpha}{M}_{bc} = \underset{\alpha}{\bar{M}}_{bc} + \frac{1}{f}\left(e^\alpha_c D_b f + e^\alpha_b D_c f - \frac{1}{n^2_\alpha} k_{bc} e^{\alpha m} D_m f\right),$$

the two conditions, that must be imposed for e_α to be an eigenvector, depend on the derivative of the function f along the other spatial tetrad vector e_β and on the contraction of $\underset{\alpha}{M}$ with e_α and e_β. In this case the corresponding eigenvalue depends on the derivative of f along e_α and on $\underset{\alpha}{\bar{M}}$ contracted totally with e_α.

For e_α to be an eigenvector for $\underset{\alpha}{M}$ and $\underset{\alpha}{\bar{M}}$, f must be invariant along the other two eigenvectors of k: $\Delta_{e_\beta}(\ln f) = 0$. The eigenvalue λ depends on $\bar{\lambda}$ and on the derivative of f along e_α.

Solving the problem for e_β to be an eigenvector of $\underset{\alpha}{M}$ one concludes that the contraction of $\underset{\alpha}{\bar{M}}$ with the two vectors e_β and e_γ must vanish and additionally a condition depending on the derivative of f along e_β and on $\underset{\alpha}{\bar{M}}$ contracted with e_α and e_β must be satisfied. The eigenvalue corresponding to the eigenvector e_β depends on $\underset{\alpha}{\bar{M}}$ contracted totally with the same eigenvector, on the derivative of f along the eigenvector e_α and on the eigenvalues of k: n_α^2 and n_β^2.

In order to have e_β as eigenvector for both tensors $\underset{\alpha}{M}$ and $\underset{\alpha}{\bar{M}}$, f must be invariant along e_β and the contraction of $\underset{\alpha}{\bar{M}}$ with e_β and e_γ must vanish. In this case, one obtains an expression relating the eigenvalues λ, $\bar{\lambda}$ and n_β^2, all three obtained from different tensors - $\underset{\alpha}{M}$, $\underset{\alpha}{\bar{M}}$ and k - but corresponding to the same eigenvector e_β, and another eigenvalue of k: n_α^2. The absolute difference between λ and $\bar{\lambda}$ is given by $\frac{n_\beta^2}{n_\alpha^2}\Delta_{e_\alpha}(\ln f)$.

Based on this analysis one can state that in general the eigenvectors of k do not remain eigenvectors for $\underset{\alpha}{M}$ and $\underset{\alpha}{\bar{M}}$. Only if one imposes restrictions on the conformal function, the eigenvectors of k are also eigenvectors of $\underset{\alpha}{M}$ and $\underset{\alpha}{\bar{M}}$. Under those restrictions one obtains an expression relating the mentioned eigenvalues of $\underset{\alpha}{M}$, $\underset{\alpha}{\bar{M}}$ and k.

Chapter 5

Applications to spherically and axially symmetric space-times

In this chapter, the analysis developed in Chapter 3 is applied to a static spherically symmetric space-time, a non-static spherically symmetric space-time and to a particular case of an axially symmetric space-time. Moreover, considering the static spherically symmetric space-time, the results are obtained for two material metrics which are conformally related, one of them being flat, in order to have also a practical application of the analysis developed in Chapter 4. For the non-static spherically symmetric space-time, the attempt to specify all results also for both conformally related material metrics was not completely successful. The calculations revealed that it was almost impossible to write down the results, because the expressions are complicated and long due to the fact that in the non-static case additional terms involving derivatives with respect to the coordinate t appear. So, in this case, only the eigenvalues are specified for both material metrics. The other results are written as functions of n_1 and n_2 without specifying them.

The spherically symmetric space-times and the axially symmetric space-time with cylindrical symmetry are considered due to their significance on modelling neutron stars and due to the motivation presented in the literature to study elasticity for these stellar objects.

The elasticity difference tensor $S^a{}_{bc}$ and the constant volume shear tensor s_{ab} are calculated for the mentioned space-times and the eigenvalue-eigenvector problem associated with the elasticity difference tensor through its decomposition into the three tensors $\underset{\alpha}{M}$ is studied.

5.1 Static spherically symmetric space-time

Neutron stars can approximately be modelled by spherically symmetric metrics. The metric regarded here, for example, can be thought of as the interior metric of a non rotating star composed by an elastic material[1].

Consider a static spherically symmetric space-time, whose metric g is given by the following line-element

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \tag{5.1}$$

with coordinates $\omega^a = \{t, r, \theta, \phi\}$, where r represents the radial coordinate, ϕ, the axial coordinate and θ, the azimuthal coordinate. The space-time can be specified by defining the orthonormal tetrad $\{u, x, y, z\}$ with the following basis vectors and basis one-forms

$$\begin{aligned}
u^a &= \left[\tfrac{1}{e^{\nu(r)}}, 0, 0, 0\right] & \qquad u_a &= \left[-e^{\nu(r)}, 0, 0, 0\right] \\
x^a &= \left[0, \tfrac{1}{e^{\lambda(r)}}, 0, 0\right] & \qquad x_a &= \left[0, e^{\lambda(r)}, 0, 0\right] \\
y^a &= \left[0, 0, \tfrac{1}{r}, 0\right] & \qquad y_a &= [0, 0, r, 0] \\
z^a &= \left[0, 0, 0, \tfrac{1}{r\sin\theta}\right] & \qquad z_a &= [0, 0, 0, r\sin\theta],
\end{aligned}$$

so that $g_{ab} = -u_a u_b + x_a x_b + y_a y_b + z_a z_b$. The line-element corresponding to the projection tensor $h_{ab} = x_a x_b + y_a y_b + z_a z_b$ is given by

$$ds^2 = e^{2\lambda(r)}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{5.2}$$

[1]See for instance Magli and Kijowski (1992) [46].

In the spherically symmetric case, two of the eigenvalues of $k^a{}_b$ are equal. Let $n_2 = n_3$, so that $n_2^2 = n_3^2$ are the degenerate eigenvalues, implying that the pulled-back material metric takes the form $k_{ab} = n_1^2 x_a x_b + n_2^2 y_a y_b + n_2^2 z_a z_b$ with corresponding line-element

$$ds^2 = n_1^2 e^{2\lambda(r)} dr^2 + n_2^2 r^2 d\theta^2 + n_2^2 r^2 \sin^2\theta d\phi^2. \tag{5.3}$$

Let $\xi^A = \{\tilde{r}, \tilde{\theta}, \tilde{\phi}\}$ be the coordinate system in the material space X. Because of the assumption that the space-time is static and spherically symmetric, the material radius $\tilde{r}$ depends only on r, $\tilde{r}(r)$, and the material angles $\tilde{\theta}$ and $\tilde{\phi}$ can be chosen to be equal to the physical angles: $\tilde{\theta} = \theta$ and $\tilde{\phi} = \phi$. Thus, the configuration of the material is described by the material radius $\tilde{r}(r)$. And the relativistic deformation gradient has only $\dfrac{d\xi^1}{d\omega^1} = \dfrac{d\tilde{r}}{dr} = \tilde{r}'$, $\dfrac{d\xi^2}{d\omega^2} = 1$ and $\dfrac{d\xi^3}{d\omega^3} = 1$ as non-zero components. The derivative with respect to the coordinate r is here represented by a prime.

Since every three-dimensional spherically symmetric metric is conformally flat, the following two material metrics are considered in X, whose line-elements are given by

1. $ds^2 = d\tilde{r}^2 + \tilde{r}^2 d\tilde{\theta}^2 + \tilde{r}^2 sin^2\tilde{\theta} d\tilde{\phi}^2$

2. $ds^2 = f^2(\tilde{r})(d\tilde{r}^2 + \tilde{r}^2 d\tilde{\theta}^2 + \tilde{r}^2 sin^2\tilde{\theta} d\tilde{\phi}^2)$.

The material metric 1. is flat and the material metric 2. is conformally related with the flat one. Denoting the metric tensor 1. by $\bar{K}$ and the metric tensor 2. by K, then it becomes obvious that both metrics satisfy the relation $K = f^2(\tilde{r})\bar{K}$. Therefore, this is a particular case of the more general case treated in Chapter 4, where two conformally related material metrics are investigated and no particular restrictions about flatness are considered.

The results listed in the next paragraphs concerning these two material metrics are numbered 1. and 2., respectively.

Pulled-back material metric

1. Calculating the pull-back of the material metric 1. one obtains

$$
\begin{aligned}
k^a{}_b &= g^{ac}k_{cb} = g^{ac}(\xi^C_c \xi^B_b K_{CB}) \\
&= \tilde{r}'^2\, e^{-2\lambda}\, \delta^a{}_1\, \delta^1{}_b + \frac{\tilde{r}^2}{r^2}\, \delta^a{}_2\, \delta^2{}_b + \frac{\tilde{r}^2}{r^2}\, \delta^a{}_3\, \delta^3{}_b.
\end{aligned}
$$

The line-element corresponding to the pulled-back material metric k_{ab} is

$$
ds^2 = \tilde{r}'^2\, dr^2 + \tilde{r}^2\, d\theta^2 + \tilde{r}^2\, sin^2\theta\, d\phi^2. \tag{5.4}
$$

Comparing this expression with the line-element given by (5.3), one concludes that the eigenvalues of k are given by

$$
n_1^2 = \tilde{r}'^2\, e^{-2\lambda} = n_1^2(r) \tag{5.5}
$$

$$
n_2^2 = n_3^2 = \frac{\tilde{r}^2}{r^2} = n_2^2(r). \tag{5.6}
$$

The linear particle densities have the following form

$$
n_1 = n_1(r) = \sqrt{\tilde{r}'^2\, e^{-2\lambda}} = \tilde{r}'\, e^{-\lambda} \tag{5.7}
$$

$$
n_2 = n_2(r) = n_3(r) = \frac{\tilde{r}}{r}. \tag{5.8}
$$

2. Calculating the pull-back of the material metric 2. one obtains

$$
\begin{aligned}
k^a{}_b &= g^{ac}k_{cb} = g^{ac}(\xi^C_c \xi^B_b K_{CB}) \\
&= f^2(\tilde{r}) \left[\tilde{r}'^2\, e^{-2\lambda}\, \delta^a{}_1\, \delta^1{}_b + \frac{\tilde{r}^2}{r^2}\, \delta^a{}_2\, \delta^2{}_b + \frac{\tilde{r}^2}{r^2}\, \delta^a{}_3\, \delta^3{}_b \right].
\end{aligned}
$$

The line-element corresponding to the pulled-back material metric k_{ab} is

$$
ds^2 = f^2(\tilde{r})[\tilde{r}'^2\, dr^2 + \tilde{r}^2\, d\theta^2 + \tilde{r}^2\, sin^2\theta\, d\phi^2]. \tag{5.9}
$$

The eigenvalues of k are given by

$$
n_1^2 = f^2(\tilde{r})\tilde{r}'^2\, e^{-2\lambda} = n_1^2(r) \tag{5.10}
$$

$$
n_2^2 = n_3^2 = f^2(\tilde{r})\frac{\tilde{r}^2}{r^2} = n_2^2(r). \tag{5.11}
$$

The linear particle densities have the following form

$$n_1 = n_1(r) = f(\tilde{r})\sqrt{\tilde{r}'^2\, e^{-2\lambda}} = f(\tilde{r})\tilde{r}'\, e^{-\lambda} \tag{5.12}$$

$$n_2 = n_2(r) = n_3(r) = f(\tilde{r})\frac{\tilde{r}}{r}. \tag{5.13}$$

The linear particle densities are positive quantities, for this reason, the function $f(\tilde{r})$ must be positive.

Comparing the eigenvalues of the metric 1. with those of the metric 2., they differ in the factor $f^2(\tilde{r})$, confirming the result obtained in Section 4.2.1.

Constant volume shear tensor

The components of the constant volume shear tensor $s_{ab} = \frac{1}{2}(h_{ab} - n^{-\frac{2}{3}}k_{ab})$ are:

$$s_{rr} = \frac{1}{2}e^{2\lambda}\left(1 - n^{-\frac{2}{3}}n_1^2\right)$$
$$s_{\theta\theta} = \frac{1}{2}r^2\left(1 - n^{-\frac{2}{3}}n_2^2\right)$$
$$s_{\phi\phi} = \frac{1}{2}r^2 sin^2\theta\left(1 - n^{-\frac{2}{3}}n_2^2\right)$$

The components of the constant volume shear tensor vanish iff $n_1^2 = n_2^2$.

Substituting n_1^2 and n_2^2 explicitly in the last expressions one obtains the following results, considering the two material metrics

1.

$$s_{rr} = \frac{1}{2}\left(e^{2\lambda} - n^{-\frac{2}{3}}\tilde{r}'^2\right)$$
$$s_{\theta\theta} = \frac{1}{2}\left(r^2 - \tilde{r}^2 n^{-\frac{2}{3}}\right)$$
$$s_{\phi\phi} = \frac{1}{2}sin^2\theta\left(r^2 - \tilde{r}^2 n^{-\frac{2}{3}}\right)$$

The constant volume shear tensor is zero if $\tilde{r}$ is of the form $\tilde{r} = ce^{\pm\int \frac{1}{r}e^{\lambda}dr}$, $c > 0$. Since the linear particle densities are positive, it follows that the allowable form for $\tilde{r}$ in the unsheared state is $\tilde{r} = ce^{\int \frac{1}{r}e^{\lambda}dr}$, $c > 0$.

2.

$$\begin{aligned} s_{rr} &= \frac{1}{2}\left(e^{2\lambda} - n^{-\frac{2}{3}}\tilde{r}'^2 f^2\right) \\ s_{\theta\theta} &= \frac{1}{2}\left(r^2 - \tilde{r}^2 n^{-\frac{2}{3}} f^2\right) \\ s_{\phi\phi} &= \frac{1}{2} sin^2\theta \left(r^2 - \tilde{r}^2 n^{-\frac{2}{3}} f^2\right) \end{aligned}$$

The constant volume shear tensor is zero if $\tilde{r}$ is of the form $\tilde{r} = ce^{\pm\int \frac{1}{r}e^{\lambda}dr}$, $c > 0$. Also in this case, since the linear particle densities are positive, $\tilde{r}$ is restricted to the form $\tilde{r} = ce^{\int \frac{1}{r}e^{\lambda}dr}$, $c > 0$, for the unsheared state.

Analysing the last results, one can observe that the condition forcing the constant volume shear tensor to vanish is the same for the two considered material metrics. The case of the material being in an unsheared state, which means $s_{ab} = 0$, is independent of the conformal factor. One concludes that the conformal factor has no influence on the state of shear of the material.
Since $n^{-\frac{2}{3}} = (n_1^2 n_2^4)^{-\frac{1}{3}}$ and by introducing in this expression the eigenvalues (5.10) and (5.11), one can prove that the components of the constant volume shear tensor given in 2. equal the components of the constant volume shear tensor given in 1. .
These results are in accordance with the result stated in Section 4.2.3, where it is shown that two conformally related pulled-back material metrics belonging to the same space-time have the same constant volume shear tensor.

Elasticity difference tensor

The non-zero components of the elasticity difference tensor ${S^a}_{bc}$ are:

$$S^r{}_{rr} = \frac{n_1'}{n_1}$$
$$S^\theta{}_{\theta r} = \frac{n_2'}{n_2}$$
$$S^\phi{}_{\phi r} = \frac{n_2'}{n_2}$$
$$S^r{}_{\theta\theta} = re^{-2\lambda} - re^{-2\lambda}\frac{n_2^2}{n_1^2} - e^{-2\lambda}r^2\frac{n_2}{n_1^2}n_2'$$
$$S^r{}_{\phi\phi} = e^{-2\lambda}rsin^2\theta - e^{-2\lambda}rsin^2\theta\frac{n_2^2}{n_1^2} - e^{-2\lambda}r^2sin^2\theta\frac{n_2}{n_1^2}n_2'$$

Since $S^a{}_{bc} = S^a{}_{cb}$, there are only seven non-zero components for this tensor on the coordinate system.

These components can alternatively be written in the following form, after substituting the quantities n_1 and n_2 by their expressions for the two material metrics:

1.

$$S^r{}_{rr} = \frac{\tilde{r}''}{\tilde{r}'} - \lambda'$$
$$S^\theta{}_{\theta r} = \frac{\tilde{r}'}{\tilde{r}} - \frac{1}{r}$$
$$S^\phi{}_{\phi r} = \frac{\tilde{r}'}{\tilde{r}} - \frac{1}{r}$$
$$S^r{}_{\theta\theta} = re^{-2\lambda} - \frac{\tilde{r}}{\tilde{r}'}$$
$$S^r{}_{\phi\phi} = sin^2\theta\left(re^{-2\lambda} - \frac{\tilde{r}}{\tilde{r}'}\right)$$

One can verify that the components $S^\theta{}_{\theta r}$ and $S^\phi{}_{\phi r}$ are zero if the function $\tilde{r}$ is of the form $\tilde{r} = c_1 r$, where c_1 is a positive constant.
$S^r{}_{rr}$ is zero if $\tilde{r} = c_2 + c_3\int e^\lambda dr$, where $c_2 + c_3\int e^\lambda dr > 0$, and the components $S^r{}_{\theta\theta}$ and $S^r{}_{\phi\phi}$ are zero if $\tilde{r} = c_4 e^{\int \frac{e^{2\lambda}}{r}dr}$, $c_4 > 0$.

2.

$$
\begin{aligned}
S^r{}_{rr} &= \frac{\tilde{r}''}{\tilde{r}'} - \lambda' + \frac{1}{f}\frac{df}{d\tilde{r}}\tilde{r}' \\
S^\theta{}_{\theta r} &= \frac{\tilde{r}'}{\tilde{r}} - \frac{1}{r} + \frac{1}{f}\frac{df}{d\tilde{r}}\tilde{r}' \\
S^\phi{}_{\phi r} &= \frac{\tilde{r}'}{\tilde{r}} - \frac{1}{r} + \frac{1}{f}\frac{df}{d\tilde{r}}\tilde{r}' \\
S^r{}_{\theta\theta} &= re^{-2\lambda} - \frac{\tilde{r}}{\tilde{r}'} - \frac{\tilde{r}^2}{\tilde{r}'}\frac{1}{f}\frac{df}{d\tilde{r}} \\
S^r{}_{\phi\phi} &= sin^2\theta\left(re^{-2\lambda} - \frac{\tilde{r}}{\tilde{r}'} - \frac{\tilde{r}^2}{\tilde{r}'}\frac{1}{f}\frac{df}{d\tilde{r}}\right)
\end{aligned}
$$

The components $S^\theta{}_{\theta r}$ and $S^\phi{}_{\phi r}$ are zero if the function $\tilde{r}$ is of the form $\tilde{r} = \frac{cr}{f}$, where $c > 0$ is a constant.
$S^r{}_{rr}$ is zero if $\frac{\tilde{r}''}{\tilde{r}'} - \lambda' + \frac{1}{f}\frac{df}{d\tilde{r}}\tilde{r}' = 0$ and the components $S^r{}_{\theta\theta}$ and $S^r{}_{\phi\phi}$ are zero if $1 + \tilde{r}\frac{d\ln(f)}{dr} - e^{-2\lambda}r\frac{d\ln(\tilde{r})}{dr} = 0$.

Analysing these results one may conclude that it is most unlikely that the elasticity difference tensor vanishes totally, because very strict conditions must be imposed on $\tilde{r}$, r and λ and, concerning the non-flat material metric, additionally on the function f.

One can observe that the components of the elasticity difference tensor for the non-flat material metric 2. depend on the components of the elasticity difference tensor for the flat metric 1. and on additional terms. These additional terms making the difference between the elasticity difference tensors can be calculated according to the formula (4.25) given in Section 4.2.4:

$$
S^a{}_{bc} - \bar{S}^a{}_{bc} = \frac{1}{f}\left(h^a{}_c D_b f + h^a{}_b D_c f - \bar{k}^{-1am}\bar{k}_{bc} D_m f\right),
$$

where $\bar{S}^a{}_{bc}$ represents the elasticity difference tensor associated with the flat material metric 1..

The tetrad components of the elasticity difference tensor can be obtained using the formula (3.16) given in Section 3.3, yielding:

$$
\begin{aligned}
S^1{}_{11} &= e^{-\lambda}\frac{n_1'}{n_1} \\
S^2{}_{21} &= e^{-\lambda}\frac{n_2'}{n_2} \\
S^3{}_{31} &= e^{-\lambda}\frac{n_2'}{n_2} \\
S^1{}_{22} &= e^{-\lambda}\frac{1}{r} - e^{-\lambda}\frac{1}{r}\frac{n_2^2}{n_1^2} - e^{-\lambda}\frac{n_2}{n_1^2}n_2' \\
S^1{}_{33} &= e^{-\lambda}\frac{1}{r} - e^{-\lambda}\frac{1}{r}\frac{n_2^2}{n_1^2} - e^{-\lambda}\frac{n_2}{n_1^2}n_2'.
\end{aligned}
$$

Expressions for $\underset{1}{M}$, $\underset{2}{M}$ and $\underset{3}{M}$

The second order symmetric tensors $\underset{1}{M}$, $\underset{2}{M}$ and $\underset{3}{M}$ have the following non-zero components.

$$
\begin{aligned}
\underset{1}{M}_{rr} &= e^{\lambda}\frac{n_1'}{n_1} \\
\underset{1}{M}_{\theta\theta} &= e^{-\lambda}r - e^{-\lambda}r\frac{n_2^2}{n_1^2} - e^{-\lambda}r^2\frac{n_2}{n_1^2}n_2' \\
\underset{1}{M}_{\phi\phi} &= e^{-\lambda}rsin^2\theta - e^{-\lambda}rsin^2\theta\frac{n_2^2}{n_1^2} - e^{-\lambda}r^2sin^2\theta\frac{n_2}{n_1^2}n_2' \\
\underset{2}{M}_{r\theta} &= \underset{2}{M}_{\theta r} = r\frac{n_2'}{n_2} \\
\underset{3}{M}_{r\phi} &= \underset{3}{M}_{\phi r} = rsin\theta\frac{n_2'}{n_2}
\end{aligned}
$$

One can observe, that the components of $\underset{2}{M}$ are very similar to the components of $\underset{3}{M}$, they differ only in the factor $\sin\theta$.

Using the explicit expressions for n_1^2 and n_2^2, concerning the two material metrics, one obtains:

1. Flat material metric

$$\underset{1}{M}_{rr} = e^{\lambda}\left(\frac{\tilde{r}''}{\tilde{r}'} - \lambda'\right)$$
$$\underset{1}{M}_{\theta\theta} = e^{\lambda}\left(re^{-2\lambda} - \frac{\tilde{r}}{\tilde{r}'}\right)$$
$$\underset{1}{M}_{\phi\phi} = sin^2\theta e^{\lambda}\left(re^{-2\lambda} - \frac{\tilde{r}}{\tilde{r}'}\right)$$
$$\underset{2}{M}_{r\theta} = \underset{2}{M}_{\theta r} = \frac{\tilde{r}'}{\tilde{r}}r - 1$$
$$\underset{3}{M}_{r\phi} = \underset{3}{M}_{\phi r} = sin\theta\left(\frac{\tilde{r}'}{\tilde{r}}r - 1\right)$$

2. Non-flat material metric

$$\underset{1}{M}_{rr} = e^{\lambda}\left(\frac{\tilde{r}''}{\tilde{r}'} - \lambda' + \frac{1}{f}\frac{df}{d\tilde{r}}\tilde{r}'\right)$$
$$\underset{1}{M}_{\theta\theta} = e^{\lambda}\left(re^{-2\lambda} - \frac{\tilde{r}}{\tilde{r}'} - \frac{\tilde{r}^2}{f}\frac{df}{d\tilde{r}}\frac{1}{\tilde{r}'}\right)$$
$$\underset{1}{M}_{\phi\phi} = e^{\lambda}sin^2\theta\left(re^{-2\lambda} - \frac{\tilde{r}}{\tilde{r}'} - \frac{\tilde{r}^2}{f}\frac{df}{d\tilde{r}}\frac{1}{\tilde{r}'}\right)$$
$$\underset{2}{M}_{r\theta} = \underset{2}{M}_{\theta r} = r\left(\frac{\tilde{r}'}{\tilde{r}} - \frac{1}{r} + \frac{1}{f}\frac{df}{d\tilde{r}}\tilde{r}'\right)$$
$$\underset{3}{M}_{r\phi} = \underset{3}{M}_{\phi r} = r\sin\theta\left(\frac{\tilde{r}'}{\tilde{r}} - \frac{1}{r} + \frac{1}{f}\frac{df}{d\tilde{r}}\tilde{r}'\right)$$

Comparing the tensor components of $\underset{1}{M}$ for the flat material metric with those for the non-flat material metric, also here one can recognize that the components of $\underset{1}{M}$ in 2. depend on the components of $\underset{1}{M}$ for the flat material metric. This property is inherited from the elasticity difference tensor via its decomposition to the tensors $\underset{\alpha}{M}$, since it is a consequence of the existing relation between the elasticity difference tensors associated with the two metrics. The additional terms in the components of $\underset{1}{M}$ in 2., indicating the difference between the tensors $\underset{1}{M}$ for both cases, result according to the formula (4.36):

$$\underset{1}{M}_{bc} - \underset{1}{\bar{M}}_{bc} = \frac{1}{f}\left(x_c D_b f + x_b D_c f - \frac{1}{n_1^2}k_{bc}x^m D_m f\right),$$

deduced in Section 4.2.5.

The following relations between the components of S and the second order tensors can be established.

$$\underset{1}{M}_{rr} = e^{\lambda} S^{r}{}_{rr}$$

$$\underset{1}{M}_{\theta\theta} = e^{\lambda} S^{r}{}_{\theta\theta}$$

$$\underset{1}{M}_{\phi\phi} = e^{\lambda} S^{r}{}_{\phi\phi}$$

$$\underset{2}{M}_{r\theta} = r S^{\theta}{}_{\theta r}$$

$$\underset{3}{M}_{r\phi} = r sin\theta S^{\phi}{}_{\phi r}$$

Eigenvalue-eigenvector problem

The following paragraphs deal with the eigenvalue-eigenvector problems for $\underset{1}{M}$, $\underset{2}{M}$ and $\underset{3}{M}$, respectively.

Solving the eigenvalue-eigenvector problem for $\underset{1}{M}$ one obtains the results exposed in Table 5.1.

Table 5.1: Eigenvectors and eigenvalues for $\underset{1}{M}$

Eigenvectors	Eigenvalues
x	$\mu_1 = \frac{e^{-\lambda}}{n_1} n_1'$
y	$\mu_2 = \frac{e^{-\lambda}}{r} - \frac{e^{-\lambda}}{r} \frac{n_2^2}{n_1^2} - e^{-\lambda} \frac{n_2}{n_1^2} n_2'$
z	$\mu_3 = \frac{e^{-\lambda}}{r} - \frac{e^{-\lambda}}{r} \frac{n_2^2}{n_1^2} - e^{-\lambda} \frac{n_2}{n_1^2} n_2'$

Substituting n_1 and n_2 by their explicit expressions yields the results exposed in Table 5.2 for the flat metric 1. and in Table 5.3 for the non-flat metric 2..

Table 5.2: Eigenvectors and eigenvalues for $\underset{1}{M}$ considering the material metric 1.

Eigenvectors	Eigenvalues
x	$\mu_1 = e^{-\lambda}\left(\frac{\tilde{r}''}{\tilde{r}'} - \lambda'\right)$
y	$\mu_2 = \frac{e^{\lambda}}{r^2}\left(re^{-2\lambda} - \frac{\tilde{r}}{\tilde{r}'}\right)$
z	$\mu_3 = \frac{e^{\lambda}}{r^2}\left(re^{-2\lambda} - \frac{\tilde{r}}{\tilde{r}'}\right)$

Table 5.3: Eigenvectors and eigenvalues for $\underset{1}{M}$ considering the material metric 2.

Eigenvectors	Eigenvalues
x	$\mu_1 = e^{-\lambda}\left(\frac{\tilde{r}''}{\tilde{r}'} - \lambda' + \frac{1}{f}\frac{df}{d\tilde{r}}\tilde{r}'\right)$
y	$\mu_2 = \frac{e^{\lambda}}{r^2}\left(re^{-2\lambda} - \frac{\tilde{r}}{\tilde{r}'} - \frac{1}{f}\frac{df}{d\tilde{r}}\frac{\tilde{r}^2}{\tilde{r}'}\right)$
z	$\mu_3 = \frac{e^{\lambda}}{r^2}\left(re^{-2\lambda} - \frac{\tilde{r}}{\tilde{r}'} - \frac{1}{f}\frac{df}{d\tilde{r}}\frac{\tilde{r}^2}{\tilde{r}'}\right)$

The eigendirections x, y and z of k are directly eigenvectors for $\underset{1}{M}$. One notices that y and z are eigenvectors associated with the same eigenvalue. The results imply that the canonical form for $\underset{1}{M}$ is $\underset{1}{M}_{bc} = \mu_1 x_b x_c + \mu_2(y_b y_c + z_b z_c)$, where μ_1 and μ_2 $(= \mu_3)$ are the eigenvalues with which x and y (or z), respectively, are associated. The algebraic multiplicity of μ_2 coincides with the geometric multiplicity and is equal to 2. The eigenvalues depend on n_i, $i = 1, 2$ and on the variation of n_i due to the variation of r. Moreover, n_1 appears explicitly in all eigenvalues.

Table 5.4 contains the eigenvalues and associated eigenvectors for the tensor $\underset{2}{M}$.

Table 5.4: Eigenvectors and eigenvalues for $\underset{2}{M}$

Eigenvectors	Eigenvalues
$x+y$	$\mu_4 = e^{-\lambda}\frac{n_2'}{n_2}$
$x-y$	$\mu_5 = -e^{-\lambda}\frac{n_2'}{n_2}$
z	$\mu_6 = 0$

Replacing n_2 by its explicit expression leads to the results presented in Table 5.5 for the flat metric 1. and in Table 5.6 for the metric 2..

Table 5.5: Eigenvectors and eigenvalues for $\underset{2}{M}$ considering the material metric 1.

Eigenvectors	Eigenvalues
$x+y$	$\mu_4 = e^{-\lambda}(\frac{\tilde{r}'}{\tilde{r}} - \frac{1}{r})$
$x-y$	$\mu_5 = -e^{-\lambda}(\frac{\tilde{r}'}{\tilde{r}} - \frac{1}{r})$
z	$\mu_6 = 0$

Table 5.6: Eigenvectors and eigenvalues for $\underset{2}{M}$ considering the material metric 2.

Eigenvectors	Eigenvalues
$x+y$	$\mu_4 = e^{-\lambda}\left(\frac{\tilde{r}'}{\tilde{r}} - \frac{1}{r} + \frac{1}{f}\frac{df}{d\tilde{r}}\tilde{r}'\right)$
$x-y$	$\mu_5 = -e^{-\lambda}\left(\frac{\tilde{r}'}{\tilde{r}} - \frac{1}{r} + \frac{1}{f}\frac{df}{d\tilde{r}}\tilde{r}'\right)$
z	$\mu_6 = 0$

One can observe that all eigenvalues of $\underset{2}{M}$ are distinct. Since z is an eigenvector associated with a zero eigenvalue, the canonical form for $\underset{2}{M}$ can be expressed as $\underset{2}{M}_{bc} = 2\mu_4(x_b y_c + y_b x_c)$, where $\mu_4 = e^{-\lambda}\left(\frac{\tilde{r}'}{\tilde{r}} - \frac{1}{r}\right)$ for case 1. and $\mu_4 = e^{-\lambda}\left(\frac{\tilde{r}'}{\tilde{r}} - \frac{1}{r} + \frac{1}{f}\frac{df}{d\tilde{r}}\tilde{r}'\right)$ for case 2..

Solving the particular eigenvalue-eigenvector problems $\underset{2}{M}_b{}^c x^b = \mu x^c$, $\underset{2}{M}_b{}^c y^b = \mu y^c$ and $\underset{2}{M}_b{}^c z^b = \mu z^c$ leads to the following results.

i) x is an eigenvector for $\underset{2}{M}$ associated with the eigenvalue 0 iff $e^{-\lambda}\frac{n_2'}{n_2} = 0$.

ii) y is an eigenvector for $\underset{2}{M}$ associated with the eigenvalue 0 iff $e^{-\lambda}\frac{n_2'}{n_2} = 0$.

iii) z is an eigenvector for $\underset{2}{M}$ associated with the eigenvalue 0.

The eigenvector z of k is automatically an eigenvector for $\underset{2}{M}$, however, for x and y to be eigenvectors for $\underset{2}{M}$, n_2 must be a constant scalar field with respect to r: $n_2' = 0$. For case 1. this equation implies that $\tilde{r}$ must be of the form $\tilde{r} = cr$, with c a positive constant. If $\tilde{r} = cr$, then x, y and z are associated with the same eigenvalue 0. In this case, $\underset{2}{M}$ would vanish.
Considering the non-flat material metric, $n_2' = 0$ is satisfied iff

$$f'\tilde{r}r + f\tilde{r}'r - f\tilde{r} = 0.$$

Table 5.7 contains the eigenvalues and associated eigenvectors for the tensor $\underset{3}{M}$.

Table 5.7: Eigenvectors and eigenvalues for $\underset{3}{M}$

Eigenvectors	Eigenvalues
$x + z$	$\mu_7 = e^{-\lambda}\frac{n_2'}{n_2}$
$x - z$	$\mu_8 = -e^{-\lambda}\frac{n_2'}{n_2}$
y	$\mu_9 = 0$

Writing the results specifically for the metric 1. and the metric 2., one obtains the results given in Table 5.8 and Table 5.9, respectively.

Table 5.8: Eigenvectors and eigenvalues for $\underset{3}{M}$ considering the material metric 1.

Eigenvectors	Eigenvalues
$x+z$	$\mu_7 = e^{-\lambda}(\frac{\tilde{r}'}{\tilde{r}} - \frac{1}{r})$
$x-z$	$\mu_8 = -e^{-\lambda}(\frac{\tilde{r}'}{\tilde{r}} - \frac{1}{r})$
y	$\mu_9 = 0$

Table 5.9: Eigenvectors and eigenvalues for $\underset{3}{M}$ considering the material metric 2.

Eigenvectors	Eigenvalues
$x+z$	$\mu_7 = e^{-\lambda}\left(\frac{\tilde{r}'}{\tilde{r}} - \frac{1}{r} + \frac{1}{f}\frac{df}{d\tilde{r}}\tilde{r}'\right)$
$x-z$	$\mu_8 = -e^{-\lambda}\left(\frac{\tilde{r}'}{\tilde{r}} - \frac{1}{r} + \frac{1}{f}\frac{df}{d\tilde{r}}\tilde{r}'\right)$
y	$\mu_9 = 0$

The tensor $\underset{3}{M}$ has distinct eigenvalues, the eigenvalue corresponding to y is zero. The canonical form of $\underset{3}{M}$ can be written as $\underset{3}{M}_{bc} = 2\mu_7(x_b z_c + z_b x_c)$, where $\mu_7 = e^{-\lambda}(\frac{\tilde{r}'}{\tilde{r}} - \frac{1}{r})$ for case 1. and $\mu_7 = e^{-\lambda}\left(\frac{\tilde{r}'}{\tilde{r}} - \frac{1}{r} + \frac{1}{f}\frac{df}{d\tilde{r}}\tilde{r}'\right)$ for case 2..

Considering the eigenvalue-eigenvector problems $\underset{3}{M}_b{}^c x^b = \mu x^c$, $\underset{3}{M}_b{}^c y^b = \mu y^c$ and $\underset{3}{M}_b{}^c z^b = \mu z^c$ one gets:

i) x is an eigenvector for $\underset{3}{M}$ associated with the eigenvalue 0 iff $e^{-\lambda}\frac{n_2'}{n_2} = 0$.

ii) y is an eigenvector for $\underset{3}{M}$ associated with the eigenvalue 0.

iii) z is an eigenvector for $\underset{3}{M}$ associated with the eigenvalue 0 iff $e^{-\lambda}\frac{n_2'}{n_2} = 0$.

For $\underset{3}{M}$, y is an eigenvector associated with the eigenvalue 0 and x and z are eigenvectors associated with the same zero eigenvalue if and only if $n_2' = 0$. Considering the flat material metric 1. and substituting n_2 by its expression given in (5.8) one deduces that y and z are eigenvectors for $\underset{3}{M}$ iff $\tilde{r} = cr$, where $c > 0$. Considering the non-flat material metric 2. and substituting n_2 by (5.13), y and z are eigenvectors for $\underset{3}{M}$ iff $f'\tilde{r}r + f\tilde{r}'r - f\tilde{r} = 0$. In this case $\underset{3}{M}$ reduces to $\underset{3}{M}_{bc} = 0$.

Ricci rotation coefficients

The expressions for the Ricci rotation coefficients are

$$
\begin{aligned}
\gamma_{010} &= e^{-\lambda}\nu' \\
\gamma_{122} &= \frac{e^{-\lambda}}{r} \\
\gamma_{133} &= \frac{e^{-\lambda}}{r} \\
\gamma_{233} &= \frac{\cos\theta}{r\sin\theta}.
\end{aligned}
$$

Kinematical quantities

Calculating Θ, $\dot{u}_a$, σ_{ab}, and ω_{ab}, contained in[2]

$$u_{a;b} = -\dot{u}_a u_b + u_{a;c}h^c{}_b = -\dot{u}_a u_b + \frac{1}{3}\Theta h_{ab} + \sigma_{ab} + \omega_{ab},$$

one obtains

$$
\begin{aligned}
&\Theta = 0 \\
&\dot{u}_a = (0, \nu', 0, 0) \\
&\sigma_{ab} = 0 \\
&\omega_{ab} = 0.
\end{aligned}
$$

5.2 Non-static spherically symmetric space-time

Consider a non-static spherically symmetric space-time and write the line-element of the corresponding space-time metric g in the form

$$ds^2 = -e^{2\nu(t,r)}dt^2 + e^{2\lambda(t,r)}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2, \tag{5.14}$$

[2]See page 14 in Section 2.1.4.

where the space-time coordinates are given by the set $\omega^a = \{t, r, \theta, \phi\}$. This space-time metric differs from the static spherically symmetric one in the coordinate functions $e^{2\nu}$ and $e^{2\lambda}$, which depend here additionally on the coordinate t.

Let $\xi^A = \{\tilde{r}, \tilde{\theta}, \tilde{\phi}\}$ be the coordinate system in the material space X. Because of the assumption that the space-time is non static and spherically symmetric, the material radius $\tilde{r}$ depends on t and r. The material angles $\tilde{\theta}$ and $\tilde{\phi}$ can be chosen to be equal to the physical angles: $\tilde{\theta} = \theta$ and $\tilde{\phi} = \phi$. Thus, the configuration of the material is described by the material radius $\tilde{r}(t, r)$. The relativistic deformation gradient $\xi^A_a = \dfrac{\partial \xi^A}{\partial \omega^a}$ has $\dfrac{\partial \xi^1}{\partial \omega^0} = \dot{\tilde{r}}$, $\dfrac{\partial \xi^1}{\partial \omega^1} = \tilde{r}'$, $\dfrac{\partial \xi^2}{\partial \omega^2} = 1$ and $\dfrac{\partial \xi^3}{\partial \omega^3} = 1$ as the only non-zero components.

In X, consider two forms of the material metric:

1. $ds^2 = d\tilde{r}^2 + \tilde{r}^2 d\tilde{\theta}^2 + \tilde{r}^2 sin^2\tilde{\theta} d\tilde{\phi}^2$

2. $ds^2 = f^2(\tilde{r})(d\tilde{r}^2 + \tilde{r}^2 d\tilde{\theta}^2 + \tilde{r}^2 sin^2\tilde{\theta} d\tilde{\phi}^2)$.

Again, here, the metric 2. is conformally related with the flat metric 1., this being again a particular case of that one considered in Chapter 4.

At this point one can see the reasons for the difference between the non-static configuration considered in this section and the static configuration treated in the previous section. As already mentioned, the functions $e^{2\nu}$ and $e^{2\lambda}$ of the non-static space-time metric depend on both coordinates t and r. Here, the material radius $\tilde{r}$ and the conformal factor f^2 depend in addition to r also on the coordinate t. Moreover, the relativistic deformation gradient has one more non-zero component, namely $\dot{\tilde{r}}$, the time derivative of the material radius.

In the subsequent paragraphs, the results corresponding to the flat material metric 1. and to the non-flat material metric 2. are listed subordinately to the items 1. and 2., respectively.

Pulled-back material metrics

1. Pulling back the material metric 1. gives

$$
\begin{aligned}
k^a{}_b &= g^{ac} k_{cb} = g^{ac}(\xi^C_c \xi^B_b K_{CB}) \\
&= -\dot{\tilde{r}}^2\, e^{-2\nu}\, \delta^a{}_0\, \delta^0{}_b - \dot{\tilde{r}}\, \tilde{r}'\, e^{-2\nu}\, \delta^a{}_0\, \delta^1{}_b + \tilde{r}'\, \dot{\tilde{r}}\, e^{-2\lambda}\, \delta^a{}_1\, \delta^0{}_b \\
&\quad + \tilde{r}'^2\, e^{-2\lambda}\, \delta^a{}_1\, \delta^1{}_b + \frac{\tilde{r}^2}{r^2}\, \delta^a{}_2\, \delta^2{}_b + \frac{\tilde{r}^2}{r^2}\, \delta^a{}_3\, \delta^3{}_b.
\end{aligned}
$$

The line-element corresponding to the pulled-back material metric k_{ab} is

$$
\begin{aligned}
ds^2 &= -\dot{\tilde{r}}'^2\, dt^2 + \dot{\tilde{r}}\, \tilde{r}'\, dtdr + \tilde{r}'\, \dot{\tilde{r}}\, drdt \\
&\quad + \tilde{r}'^2\, dr^2 + \tilde{r}^2\, d\theta^2 + \tilde{r}^2\, sin^2\theta\, d\phi^2.
\end{aligned} \tag{5.15}
$$

Calculating the eigenvalues of k one obtains:

$$
n_1^2 = \tilde{r}'^2\, e^{-2\lambda} - \dot{\tilde{r}}^2\, e^{-2\nu} = n_1^2(t, r) \tag{5.16}
$$

$$
n_2^2 = n_3^2 = \frac{\tilde{r}^2}{r^2} = n_2^2(t, r). \tag{5.17}
$$

The linear particle densities have the following form

$$
n_1 = n_1(t, r) = \sqrt{\tilde{r}'^2\, e^{-2\lambda} - \dot{\tilde{r}}^2\, e^{-2\nu}} \tag{5.18}
$$

$$
n_2 = n_2(t, r) = n_3(t, r) = \frac{\tilde{r}}{r}. \tag{5.19}
$$

2. Pulling back the material metric 2. implies

$$
\begin{aligned}
k^a{}_b &= g^{ac} k_{cb} = g^{ac}(\xi^C_c \xi^B_b K_{CB}) \\
&= f^2(\tilde{r})[-\dot{\tilde{r}}^2\, e^{-2\nu}\, \delta^a{}_0\, \delta^0{}_b - \dot{\tilde{r}}\, \tilde{r}'\, e^{-2\nu}\, \delta^a{}_0\, \delta^1{}_b + \tilde{r}'\, \dot{\tilde{r}}\, e^{-2\lambda}\, \delta^a{}_1\, \delta^0{}_b \\
&\quad + \tilde{r}'^2\, e^{-2\lambda}\, \delta^a{}_1\, \delta^1{}_b + \frac{\tilde{r}^2}{r^2}\, \delta^a{}_2\, \delta^2{}_b + \frac{\tilde{r}^2}{r^2}\, \delta^a{}_3\, \delta^3{}_b].
\end{aligned}
$$

The line-element corresponding to the pulled-back material metric k_{ab} is

$$
\begin{aligned}
ds^2 &= f^2(\tilde{r})[-\dot{\tilde{r}}'^2\, dt^2 + \dot{\tilde{r}}\, \tilde{r}'\, dtdr + \tilde{r}'\, \dot{\tilde{r}}\, drdt \\
&\quad + \tilde{r}'^2\, dr^2 + \tilde{r}^2\, d\theta^2 + \tilde{r}^2\, sin^2\theta\, d\phi^2].
\end{aligned} \tag{5.20}
$$

Calculating the eigenvalues of k one obtains

$$n_1^2 = f^2(\tilde{r})[\tilde{r}'^2\, e^{-2\lambda} - \dot{\tilde{r}}^2\, e^{-2\nu}] = n_1^2(t,r) \tag{5.21}$$

$$n_2^2 = n_3^2 = f^2(\tilde{r})\frac{\tilde{r}^2}{r^2} = n_2^2(t,r). \tag{5.22}$$

The linear particle densities have the following form

$$n_1 = n_1(t,r) = f(r)\sqrt{\tilde{r}'^2\, e^{-2\lambda} - \dot{\tilde{r}}^2\, e^{-2\nu}} \tag{5.23}$$

$$n_2 = n_2(t,r) = n_3(t,r) = f(r)\frac{\tilde{r}}{r}. \tag{5.24}$$

Multiplying the eigenvalues of the metric 1. by the conformal factor $f^2(\tilde{r})$ establishes the expressions of the eigenvalues of the metric 2.. This is a consequence of the result proved in Section 4.2.1.

Now, calculating the spatial eigenvectors x, y and z of k associated with the eigenvalues n_1^2 and n_2^2 (which has algebraic multiplicity 2) and determining the matter velocity field u from $u^a\xi_a^A = 0$, $u^a u_a = -1$ and $u^0 > 0$, allow to define the orthonormal tetrad $\{u, x, y, z\}$:

$$u^a = \left[e^{-\nu}\,\gamma, -e^{-\nu}\,\frac{\dot{\tilde{r}}}{\tilde{r}'}\,\gamma, 0, 0\right] \qquad u_a = \left[-e^{\nu}\,\gamma, -e^{2\lambda-\nu}\,\frac{\dot{\tilde{r}}}{\tilde{r}'}\gamma, 0, 0\right]$$

$$x^a = \left[-e^{\lambda-2\nu}\,\frac{\dot{\tilde{r}}}{\tilde{r}'}\,\gamma, e^{-\lambda}\,\gamma, 0, 0\right] \qquad x_a = \left[e^{\lambda}\,\frac{\dot{\tilde{r}}}{\tilde{r}'}\,\gamma, e^{\lambda}\,\gamma, 0, 0\right]$$

$$y^a = \left[0, 0, \frac{1}{r}, 0\right] \qquad y_a = [0, 0, r, 0]$$

$$z^a = \left[0, 0, 0, \frac{1}{r\sin\theta}\right] \qquad z_a = [0, 0, 0, r\sin\theta]$$

where $\gamma = \sqrt{\dfrac{e^{2\nu}\tilde{r}'^2}{e^{2\nu}\tilde{r}'^2 - e^{2\lambda}\dot{\tilde{r}}^2}}$.

Note that $-u^a u_a = x^a x_a = y^a y_a = z^a z_a = 1$ and all other inner products are zero. Using this orthonormal tetrad, the space-time metric can be written as $g_{ab} = -u_a u_b + x_a x_b + y_a y_b + z_a z_b$. The line-element corresponding to the projection tensor $h_{ab} =$

$x_a x_b + y_a y_b + z_a z_b$ is given by

$$ds^2 = e^{2\lambda(t,r)} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{5.25}$$

And the pulled-back material metric takes the form $k_{ab} = n_1^2 x_a x_b + n_2^2 y_a y_b + n_2^2 z_a z_b$.

Constant volume shear tensor

The only non-zero components of the constant volume shear tensor $s_{ab} = \frac{1}{2}(h_{ab} - n^{-\frac{2}{3}} k_{ab})$ are

$$s_{tt} = \frac{1}{2}\frac{e^{2\lambda+2\nu}\dot{\tilde{r}}^2}{e^{2\nu}\tilde{r}'^2 - e^{2\lambda}\dot{\tilde{r}}^2}(1 - n^{-\frac{2}{3}} n_1^2)$$
$$s_{tr} = s_{rt} = \frac{1}{2}\frac{e^{2\lambda+2\nu}\dot{\tilde{r}}\tilde{r}'}{e^{2\nu}\tilde{r}'^2 - e^{2\lambda}\dot{\tilde{r}}^2}(1 - n^{-\frac{2}{3}} n_1^2)$$
$$s_{rr} = \frac{1}{2}\frac{e^{2\lambda+2\nu}\tilde{r}'^2}{e^{2\nu}\tilde{r}'^2 - e^{2\lambda}\dot{\tilde{r}}^2}(1 - n^{-\frac{2}{3}} n_1^2)$$
$$s_{\theta\theta} = \frac{1}{2} r^2 (1 - n^{-\frac{2}{3}} n_2^2)$$
$$s_{\phi\phi} = \frac{1}{2} r^2 sin^2\theta (1 - n^{-\frac{2}{3}} n_2^2)$$

Excluding the cases $\dot{\tilde{r}} = 0$, $\tilde{r}' = 0$, $\sin\theta = 0$, $r = 0$ and $f = 0$, the components of the constant volume shear tensor vanish iff $n_1^2 = n_2^2$, or, writing this condition explicitly, iff

1. $\tilde{r}'^2 e^{-2\lambda} - \dot{\tilde{r}}^2 e^{-2\nu} - \frac{\tilde{r}^2}{r^2} = 0$

2. $\tilde{r}'^2 e^{-2\lambda} - \dot{\tilde{r}}^2 e^{-2\nu} - \frac{\tilde{r}^2}{r^2} = 0.$

The condition obtained for the non-flat material metric coincides with the condition for the flat material metric, since the factor f^2 cancels on both sides of the equation $n_1^2 = n_2^2$, when substituting n_1^2 and n_2^2 by the eigenvalues (5.21) and (5.22), respectively. This is consistent with the result obtained in Section 4.2.3, where it has been shown that the constant volume shear tensor corresponding to the non-flat metric equals the constant volume shear tensor defined for the flat metric.

Elasticity difference tensor

The non-zero components of the elasticity difference tensor $S^a{}_{bc}$ are:

$$S^t{}_{tt} = \frac{\dot{\tilde{r}}^3 e^{2\lambda}}{n_1} \frac{e^{2\lambda}\dot{\tilde{r}}\dot{n}_1 - e^{2\nu}\tilde{r}' n_1'}{(e^{2\nu}\tilde{r}'^2 - e^{2\lambda}\dot{\tilde{r}}^2)^2}$$

$$S^t{}_{rt} = \frac{\dot{\tilde{r}}^2 \tilde{r}' e^{2\lambda}}{n_1} \frac{e^{2\lambda}\dot{\tilde{r}}\dot{n}_1 - e^{2\nu}\tilde{r}' n_1'}{(e^{2\nu}\tilde{r}'^2 - e^{2\lambda}\dot{\tilde{r}}^2)^2}$$

$$S^r{}_{tt} = -\frac{\dot{\tilde{r}}^2 \tilde{r}' e^{2\nu}}{n_1} \frac{e^{2\lambda}\dot{\tilde{r}}\dot{n}_1 - e^{2\nu}\tilde{r}' n_1'}{(e^{2\nu}\tilde{r}'^2 - e^{2\lambda}\dot{\tilde{r}}^2)^2}$$

$$S^r{}_{rt} = -\frac{\dot{\tilde{r}} \tilde{r}'^2 e^{2\nu}}{n_1} \frac{e^{2\lambda}\dot{\tilde{r}}\dot{n}_1 - e^{2\nu}\tilde{r}' n_1'}{(e^{2\nu}\tilde{r}'^2 - e^{2\lambda}\dot{\tilde{r}}^2)^2}$$

$$S^t{}_{rr} = \frac{\dot{\tilde{r}} \tilde{r}'^2 e^{2\lambda}}{n_1} \frac{e^{2\lambda}\dot{\tilde{r}}\dot{n}_1 - e^{2\nu}\tilde{r}' n_1'}{(e^{2\nu}\tilde{r}'^2 - e^{2\lambda}\dot{\tilde{r}}^2)^2}$$

$$S^r{}_{rr} = -\frac{\tilde{r}'^3 e^{2\nu}}{n_1} \frac{e^{2\lambda}\dot{\tilde{r}}\dot{n}_1 - e^{2\nu}\tilde{r}' n_1'}{(e^{2\nu}\tilde{r}'^2 - e^{2\lambda}\dot{\tilde{r}}^2)^2}$$

$$S^\theta{}_{\theta t} = -\frac{\dot{\tilde{r}}}{n_2} \frac{e^{2\lambda}\dot{\tilde{r}}\dot{n}_2 - e^{2\nu}\tilde{r}' n_2'}{(e^{2\nu}\tilde{r}'^2 - e^{2\lambda}\dot{\tilde{r}}^2)}$$

$$S^\phi{}_{\phi t} = -\frac{\dot{\tilde{r}}}{n_2} \frac{e^{2\lambda}\dot{\tilde{r}}\dot{n}_2 - e^{2\nu}\tilde{r}' n_2'}{(e^{2\nu}\tilde{r}'^2 - e^{2\lambda}\dot{\tilde{r}}^2)}$$

$$S^\theta{}_{\theta r} = -\frac{\tilde{r}'}{n_2} \frac{e^{2\lambda}\dot{\tilde{r}}\dot{n}_2 - e^{2\nu}\tilde{r}' n_2'}{(e^{2\nu}\tilde{r}'^2 - e^{2\lambda}\dot{\tilde{r}}^2)}$$

$$S^\phi{}_{\phi r} = -\frac{\tilde{r}'}{n_2} \frac{e^{2\lambda}\dot{\tilde{r}}\dot{n}_2 - e^{2\nu}\tilde{r}' n_2'}{(e^{2\nu}\tilde{r}'^2 - e^{2\lambda}\dot{\tilde{r}}^2)}$$

$$S^t{}_{\theta\theta} = -\frac{r\dot{\tilde{r}}}{n_1^2 e^{2\nu}} \frac{r n_2(\dot{n}_2\dot{\tilde{r}}e^{2\lambda} - n_2'\tilde{r}'e^{2\nu}) + \tilde{r}'e^{2\nu}(n_1^2 - n_2^2)}{e^{2\nu}\tilde{r}'^2 - e^{2\lambda}\dot{\tilde{r}}^2}$$

$$S^r{}_{\theta\theta} = \frac{r\tilde{r}'}{n_1^2 e^{2\lambda}} \frac{r n_2(\dot{n}_2\dot{\tilde{r}}e^{2\lambda} - n_2'\tilde{r}'e^{2\nu}) + \tilde{r}'e^{2\nu}(n_1^2 - n_2^2)}{e^{2\nu}\tilde{r}'^2 - e^{2\lambda}\dot{\tilde{r}}^2}$$

$$S^t{}_{\phi\phi} = -\frac{r\dot{\tilde{r}}\sin^2\theta}{n_1^2 e^{2\nu}} \frac{r n_2(\dot{n}_2\dot{\tilde{r}}e^{2\lambda} - n_2'\tilde{r}'e^{2\nu}) + \tilde{r}'e^{2\nu}(n_1^2 - n_2^2)}{e^{2\nu}\tilde{r}'^2 - e^{2\lambda}\dot{\tilde{r}}^2}$$

$$S^r{}_{\phi\phi} = \frac{r\tilde{r}'\sin^2\theta}{n_1^2 e^{2\lambda}} \frac{r n_2(\dot{n}_2\dot{\tilde{r}}e^{2\lambda} - n_2'\tilde{r}'e^{2\nu}) + \tilde{r}'e^{2\nu}(n_1^2 - n_2^2)}{e^{2\nu}\tilde{r}'^2 - e^{2\lambda}\dot{\tilde{r}}^2}$$

Since $S^a{}_{bc} = S^a{}_{cb}$, there are twenty non-zero components for this tensor on the coordinate system chosen above.

The components $S^t{}_{tt}$, $S^t{}_{rt}$, $S^r{}_{tt}$, $S^r{}_{rt}$, $S^t{}_{rr}$, $S^r{}_{rr}$ are zero if $e^{2\lambda}\dot{\tilde{r}}\dot{n}_1 - e^{2\nu}\tilde{r}'n_1' = 0$. $S^\theta{}_{\theta t}$, $S^\phi{}_{\phi t}$, $S^\theta{}_{\theta r}$ and $S^\phi{}_{\phi r}$ vanish if $e^{2\lambda}\dot{\tilde{r}}\dot{n}_2 - e^{2\nu}\tilde{r}'n_2' = 0$. $S^t{}_{\theta\theta}$, $S^r{}_{\theta\theta}$, $S^t{}_{\phi\phi}$ and $S^r{}_{\phi\phi}$ are zero if

$rn_2(\dot{n}_2\dot{\tilde{r}}e^{2\lambda} - n_2'\tilde{r}'e^{2\nu}) + \tilde{r}'e^{2\nu}(n_1^2 - n_2^2) = 0.$

For the non-static case, the components of the elasticity difference tensor and of the tensors $\underset{\alpha}{M}$ concerning the two material metrics are not listed here, since the expressions become quite long. The tensor components are expressed in terms of the eigenvalues n_1^2 and n_2^2 without specifying them for the two material metrics.

Expressions for $\underset{1}{M}$, $\underset{2}{M}$ and $\underset{3}{M}$

The second order symmetric tensors $\underset{1}{M}$, $\underset{2}{M}$ and $\underset{3}{M}$ have the following non-zero components.

$$\underset{1}{M}_{tt} = \frac{e^{\nu+\lambda}\dot{\tilde{r}}^2}{n_1}\,\frac{e^{2\nu}\tilde{r}'n_1' - e^{2\lambda}\dot{\tilde{r}}\dot{n}_1}{e^{2\nu}\tilde{r}'^2 - e^{2\lambda}\dot{\tilde{r}}^2}\sqrt{\frac{1}{e^{2\nu}\tilde{r}'^2 - e^{2\lambda}\dot{\tilde{r}}^2}}$$
$$\underset{1}{M}_{rt} = \frac{e^{\nu+\lambda}\dot{\tilde{r}}\tilde{r}'}{n_1}\,\frac{e^{2\nu}\tilde{r}'n_1' - e^{2\lambda}\dot{\tilde{r}}\dot{n}_1}{e^{2\nu}\tilde{r}'^2 - e^{2\lambda}\dot{\tilde{r}}^2}\sqrt{\frac{1}{e^{2\nu}\tilde{r}'^2 - e^{2\lambda}\dot{\tilde{r}}^2}}$$
$$\underset{1}{M}_{rr} = \frac{e^{\nu+\lambda}\tilde{r}'^2}{n_1}\,\frac{e^{2\nu}\tilde{r}'n_1' - e^{2\lambda}\dot{\tilde{r}}\dot{n}_1}{e^{2\nu}\tilde{r}'^2 - e^{2\lambda}\dot{\tilde{r}}^2}\sqrt{\frac{1}{e^{2\nu}\tilde{r}'^2 - e^{2\lambda}\dot{\tilde{r}}^2}}$$
$$\underset{1}{M}_{\theta\theta} = -\frac{r[rn_2(e^{2\nu}\tilde{r}'n_2' - e^{2\lambda}\dot{\tilde{r}}\dot{n}_2) + \tilde{r}'e^{2\nu}(n_2^2 - n_1^2)]}{e^{\nu+\lambda}n_1^2}\sqrt{\frac{1}{e^{2\nu}\tilde{r}'^2 - e^{2\lambda}\dot{\tilde{r}}^2}}$$
$$\underset{1}{M}_{\phi\phi} = -\frac{r\sin^2\theta[rn_2(e^{2\nu}\tilde{r}'n_2' - e^{2\lambda}\dot{\tilde{r}}\dot{n}_2) + \tilde{r}'e^{2\nu}(n_2^2 - n_1^2)]}{e^{\nu+\lambda}n_1^2}\sqrt{\frac{1}{e^{2\nu}\tilde{r}'^2 - e^{2\lambda}\dot{\tilde{r}}^2}}$$
$$\underset{2}{M}_{t\theta} = \frac{r\dot{\tilde{r}}(\tilde{r}'n_2'e^{2\nu} - \dot{\tilde{r}}\dot{n}_2e^{2\lambda})}{n_2(e^{2\nu}\tilde{r}'^2 - e^{2\lambda}\dot{\tilde{r}}^2)}$$
$$\underset{2}{M}_{r\theta} = \frac{r\tilde{r}'(\tilde{r}'n_2'e^{2\nu} - \dot{\tilde{r}}\dot{n}_2e^{2\lambda})}{n_2(e^{2\nu}\tilde{r}'^2 - e^{2\lambda}\dot{\tilde{r}}^2)}$$

$$\underset{3}{M}_{t\phi} = \frac{r\dot{\tilde{r}}\sin\theta\,(\tilde{r}'n_2'e^{2\nu} - \dot{\tilde{r}}\dot{n}_2e^{2\lambda})}{n_2(e^{2\nu}\tilde{r}'^2 - e^{2\lambda}\dot{\tilde{r}}^2)}$$
$$\underset{3}{M}_{r\phi} = \frac{r\tilde{r}'\sin\theta\,(\tilde{r}'n_2'e^{2\nu} - \dot{\tilde{r}}\dot{n}_2e^{2\lambda})}{n_2(e^{2\nu}\tilde{r}'^2 - e^{2\lambda}\dot{\tilde{r}}^2)}$$

The structure of the tensor $\underset{1}{M}$ is similar to the structure of the tensor k in the sense that both tensors consist of six non zero components having the same coordinate indices.

Eigenvalue-eigenvector problem

The subsequent paragraphs are concerned with the eigenvector-eigenvalue problems for $\underset{1}{M}$, $\underset{2}{M}$ and $\underset{3}{M}$, respectively.

Solving the eigenvalue-eigenvector problem for $\underset{1}{M}$ one obtains the results exposed in Table 5.10.

Table 5.10: Eigenvectors and eigenvalues for $\underset{1}{M}$

Eigenvectors	Eigenvalues
x	$\mu_1 = \frac{e^{2\nu}\tilde{r}' n_1' - e^{2\lambda}\dot{\tilde{r}}\dot{n}_1}{e^{\lambda+\nu} n_1} \sqrt{\frac{1}{e^{2\nu}\tilde{r}'^2 - e^{2\lambda}\dot{\tilde{r}}^2}}$
y	$\mu_2 = \frac{r n_2 (e^{2\lambda}\dot{\tilde{r}}\dot{n}_2 - e^{2\nu}\tilde{r}' n_2') + \tilde{r}' e^{2\nu}(n_1^2 - n_2^2)}{e^{\lambda+\nu} r n_1^2} \sqrt{\frac{1}{e^{2\nu}\tilde{r}'^2 - e^{2\lambda}\dot{\tilde{r}}^2}}$
z	$\mu_3 = \frac{r n_2 (e^{2\lambda}\dot{\tilde{r}}\dot{n}_2 - e^{2\nu}\tilde{r}' n_2') + \tilde{r}' e^{2\nu}(n_1^2 - n_2^2)}{e^{\lambda+\nu} r n_1^2} \sqrt{\frac{1}{e^{2\nu}\tilde{r}'^2 - e^{2\lambda}\dot{\tilde{r}}^2}}$

The eigendirections x, y and z of k are directly eigenvectors for $\underset{1}{M}$. One notices that y and z are eigenvectors associated with the same eigenvalue. The results indicate that the canonical form for $\underset{1}{M}$ can be written as $\underset{1}{M}_{bc} = \mu_1 x_b x_c + \mu_2 (y_b y_c + z_b z_c)$, where μ_1 and μ_2 $(= \mu_3)$ are the eigenvalues with which x and y (or z), respectively, are associated. The algebraic multiplicity of μ_2 coincides with the geometric multiplicity and is equal to 2. The eigenvalues depend on n_i, $i = 1, 2$, and on the variation of n_i due to the variation of r and of t. Moreover, n_1 appears explicitly in all eigenvalues.

For $\underset{2}{M}$, the eigenvalues and associated eigenvectors are listed in Table 5.11.

Table 5.11: Eigenvectors and eigenvalues for $\underset{2}{M}$

Eigenvectors	Eigenvalues
$x+y$	$\mu_4 = -\frac{e^{2\lambda}\dot{\tilde{r}}\dot{n}_2 - e^{2\nu}\tilde{r}'n_2'}{e^{\lambda+\nu}n_2}\sqrt{\frac{1}{e^{2\nu}\tilde{r}'^2 - e^{2\lambda}\dot{\tilde{r}}^2}}$
$x-y$	$\mu_5 = \frac{e^{2\lambda}\dot{\tilde{r}}\dot{n}_2 - e^{2\nu}\tilde{r}'n_2'}{e^{\lambda+\nu}n_2}\sqrt{\frac{1}{e^{2\nu}\tilde{r}'^2 - e^{2\lambda}\dot{\tilde{r}}^2}}$
z	$\mu_6 = 0$

In this case, only z is simultaneously an eigenvector of k and $\underset{2}{M}$, but now the corresponding eigenvalue is zero. The tensor $\underset{2}{M}$ has three distinct eigenvalues, two of them differ only in a sign. The canonical form for $\underset{2}{M}$ can be expressed as $\underset{2}{M}_{bc} = 2\mu_4(x_b y_c + y_b x_c)$.

Solving the particular eigenvalue-eigenvector problems $\underset{2}{M}_b{}^c x^b = \mu x^c$, $\underset{2}{M}_b{}^c y^b = \mu y^c$ and $\underset{2}{M}_b{}^c z^b = \mu z^c$ leads to the following results.

i) x is an eigenvector for $\underset{2}{M}$ associated with the eigenvalue 0 iff $e^{2\lambda}\dot{\tilde{r}}\dot{n}_2 - e^{2\nu}\tilde{r}'n_2' = 0$.

ii) y is an eigenvector for $\underset{2}{M}$ associated with the eigenvalue 0 iff $e^{2\lambda}\dot{\tilde{r}}\dot{n}_2 - e^{2\nu}\tilde{r}'n_2' = 0$.

iii) z is an eigenvector for $\underset{2}{M}$ associated with the eigenvalue 0.

The condition for z to be an eigenvector for $\underset{2}{M}$ is automatically satisfied, however, for x and y to be eigenvectors for $\underset{2}{M}$, the condition $e^{2\lambda}\dot{\tilde{r}}\dot{n}_2 - e^{2\nu}\tilde{r}'n_2' = 0$ must hold. The eigenvectors x, y and z are then associated with the eigenvalue 0. In this case, $\underset{2}{M}$ vanishes.

Table 5.12 contains the eigenvalues and associated eigenvectors for the tensor $\underset{3}{M}$.

Table 5.12: Eigenvectors and eigenvalues for $\underset{3}{M}$

Eigenvectors	Eigenvalues
$x+z$	$\mu_7 = -\frac{e^{2\lambda}\dot{\tilde{r}}\dot{n}_2 - e^{2\nu}\tilde{r}'n_2'}{e^{\lambda+\nu}n_2}\sqrt{\frac{1}{e^{2\nu}\tilde{r}'^2 - e^{2\lambda}\dot{\tilde{r}}^2}}$
$x-z$	$\mu_8 = \frac{e^{2\lambda}\dot{\tilde{r}}\dot{n}_2 - e^{2\nu}\tilde{r}'n_2'}{e^{\lambda+\nu}n_2}\sqrt{\frac{1}{e^{2\nu}\tilde{r}'^2 - e^{2\lambda}\dot{\tilde{r}}^2}}$
y	$\mu_9 = 0$

Here, only the eigenvector y of k remains as eigenvector for $\underset{3}{M}$, now it is associated with the eigenvalue 0. The other two eigenvalues differ in a sign. One concludes that the canonical form for $\underset{3}{M}$ can be written as $\underset{3}{M}_{bc} = 2\mu_7(x_b z_c + z_b x_c)$.

Considering the eigenvalue-eigenvector problems $\underset{3}{M}_b{}^c x^b = \mu x^c$, $\underset{3}{M}_b{}^c y^b = \mu y^c$ and $\underset{3}{M}_b{}^c z^b = \mu z^c$, it follows:

i) x is an eigenvector for $\underset{3}{M}$ associated with the eigenvalue 0 iff $e^{2\lambda}\dot{\tilde{r}}\dot{n}_2 - e^{2\nu}\tilde{r}'n_2' = 0$.

ii) y is an eigenvector for $\underset{3}{M}$ associated with the eigenvalue 0.

iii) z is an eigenvector for $\underset{3}{M}$ associated with the eigenvalue 0 iff $e^{2\lambda}\dot{\tilde{r}}\dot{n}_2 - e^{2\nu}\tilde{r}'n_2' = 0$.

For $\underset{3}{M}$, y is directly an eigenvector associated with the eigenvalue 0 and x and z are eigenvectors associated with the same zero eigenvalue if and only if $e^{2\lambda}\dot{\tilde{r}}\dot{n}_2 - e^{2\nu}\tilde{r}'n_2' = 0$. In this case $\underset{3}{M}$ reduces to $\underset{3}{M}_{bc} = 0$.

Ricci rotation coefficients

The expressions for the Ricci rotation coefficients are

$$
\begin{aligned}
\gamma_{010} &= \frac{e^{\nu+\lambda}(\tilde{r}'^2\dot{\tilde{r}}\dot{\nu} + \tilde{r}'\dot{\tilde{r}}^2\lambda' - 2\tilde{r}'^2\dot{\tilde{r}}\dot{\lambda} - 2\tilde{r}'\dot{\tilde{r}}^2\nu' + 2\tilde{r}'\dot{\tilde{r}}\dot{\tilde{r}}' - \tilde{r}'^2\ddot{\tilde{r}} - \dot{\tilde{r}}^2\tilde{r}'')}{(e^{2\nu}\tilde{r}'^2 - e^{2\lambda}\dot{\tilde{r}}^2)^{3/2}} \\
&\quad + \frac{e^{4\lambda}\dot{\tilde{r}}^3\dot{\lambda} + e^{4\nu}\tilde{r}'^3\nu'}{e^{\nu+\lambda}(e^{2\nu}\tilde{r}'^2 - e^{2\lambda}\dot{\tilde{r}}^2)^{3/2}} \\
\gamma_{011} &= \frac{e^{2\lambda}(\dot{\tilde{r}}^3\nu' - \tilde{r}'\dot{\tilde{r}}^2\dot{\nu} + \tilde{r}'\dot{\tilde{r}}\ddot{\tilde{r}} - \dot{\tilde{r}}^2\dot{\tilde{r}}') + e^{2\nu}(\tilde{r}'^3\dot{\lambda} - \tilde{r}'^2\dot{\tilde{r}}\lambda' - \tilde{r}'^2\dot{\tilde{r}}' + \tilde{r}'\dot{\tilde{r}}\tilde{r}'')}{(e^{2\nu}\tilde{r}'^2 - e^{2\lambda}\dot{\tilde{r}}^2)^{3/2}} \\
\gamma_{022} &= -\frac{\dot{\tilde{r}}}{r}\sqrt{\frac{1}{e^{2\nu}\tilde{r}'^2 - e^{2\lambda}\dot{\tilde{r}}^2}} \\
\gamma_{033} &= -\frac{\dot{\tilde{r}}}{r}\sqrt{\frac{1}{e^{2\nu}\tilde{r}'^2 - e^{2\lambda}\dot{\tilde{r}}^2}} \\
\gamma_{122} &= \frac{e^{\nu}\tilde{r}'}{e^{\lambda}r}\sqrt{\frac{1}{e^{2\nu}\tilde{r}'^2 - e^{2\lambda}\dot{\tilde{r}}^2}} \\
\gamma_{133} &= \frac{e^{\nu}\tilde{r}'}{e^{\lambda}r}\sqrt{\frac{1}{e^{2\nu}\tilde{r}'^2 - e^{2\lambda}\dot{\tilde{r}}^2}} \\
\gamma_{233} &= \frac{\cos\theta}{r\sin\theta}.
\end{aligned}
$$

For the non-static case there are three more rotation coefficients (γ_{011}, γ_{022} and γ_{033}) than for the static case.

Kinematical quantities

Calculating the expressions for the expansion Θ, acceleration $\dot{u}_a$, shear σ_{ab} and vorticity ω_{ab}, one realizes that it is quite impossible to write them down due to their complexity, but in this case one can state the following:

i) The expansion is non zero.

ii) The acceleration has $\dot{u}_t$ and $\dot{u}_r$ as non zero components.

iii) For the shear tensor field the following components are non-zero: σ_{tt}, $\sigma_{tr} = \sigma_{rt}$, σ_{rr}, $\sigma_{\theta\theta}$ and $\sigma_{\phi\phi}$.

iv) The vorticity tensor ω_{ab} vanishes.

5.3 Axially symmetric non-rotating space-time

To begin with, consider an elastic, axially symmetric, uniformly rotating body in interaction with its gravitational field. The exterior and the interior of the body may be described by the following metric[3]

$$ds^2 = -e^{2\nu}dt^2 + e^{2\mu}dr^2 + e^{2\mu}dz^2 + e^{2\psi}(d\phi - \omega dt)^2, \tag{5.26}$$

where ν, ψ, ω, μ are functions of r and z.

Assume that the material metric is flat. Introducing in X cylindrical coordinates $\xi^A = \{R, \zeta, \Phi\}$, then the material metric takes the form

$$ds^2 = dR^2 + d\zeta^2 + R^2 d\Phi^2, \tag{5.27}$$

where the parameters R, ζ depend on r and z. Φ is given by $\Phi(t, r, z, \phi) = \phi - \Omega(r, z)t$. The space-time metric for the limiting case of an axially symmetric non-rotating elastic system can be written as

$$ds^2 = -e^{2\nu}dt^2 + e^{2\mu}dr^2 + e^{2\mu}dz^2 + e^{2\psi}d\phi^2. \tag{5.28}$$

This metric is obtained from (5.26), when $\omega = 0$ and $\Omega = 0$.

Imposing $R = R(r)$, $\zeta = z$ and $g_{ab} = g_{ab}(r)$, one obtains a further reduction to cylindrical symmetry. This reduction is mentioned in Magli (1993) [44].

The space-time metric used in this section is given by (5.28), where the functions ν, μ, ψ depend on r only. The space-time coordinates are taken as $\omega^a = \{t, r, z, \phi\}$.

Suppose that the orthonormal tetrad $\{u, x, y, z\}$ is defined by the following basis vectors and one-forms

$$\begin{aligned}
u^a &= \left[\tfrac{1}{e^{\nu(r)}}, 0, 0, 0\right] & \qquad u_a &= \left[-e^{\nu(r)}, 0, 0, 0\right] \\
x^a &= \left[0, \tfrac{1}{e^{\mu(r)}}, 0, 0\right] & \qquad x_a &= \left[0, e^{\mu(r)}, 0, 0\right] \\
y^a &= \left[0, 0, \tfrac{1}{e^{\mu(r)}}, 0\right] & \qquad y_a &= \left[0, 0, e^{\mu(r)}, 0\right] \\
z^a &= \left[0, 0, 0, \tfrac{1}{e^{\psi(r)}}\right] & \qquad z_a &= \left[0, 0, 0, e^{\psi(r)}\right].
\end{aligned}$$

[3] See Magli (1993) [44].

Using this tetrad, the space-time metric can be written as $g_{ab} = -u_a u_b + x_a x_b + y_a y_b + z_a z_b$.

In X, the material metric K_{AB} is given by the line-element

$$ds^2 = dR^2 + d\zeta^2 + R^2 d\phi^2, \tag{5.29}$$

where $\zeta = z$ and $R = R(r)$.

The relativistic deformation gradient $\xi^A_a = \dfrac{\partial \xi^A}{\partial \omega^a}$ has the following non-zero components $\dfrac{d\xi^1}{d\omega^1} = \dfrac{dR}{dr} = R'$, $\dfrac{d\xi^2}{d\omega^2} = 1$ and $\dfrac{d\xi^3}{d\omega^3} = 1$.

Calculating the pull-back of the material metric one obtains

$$\begin{aligned} k^a{}_b &= g^{ac} k_{cb} = g^{ac} \left(\xi^C_c \xi^B_b K_{CB} \right) \\ &= R'^2 e^{-2\mu} \delta^a{}_1 \delta^1{}_b + e^{-2\mu} \delta^a{}_2 \delta^2{}_b + R^2 e^{-2\psi} \delta^a{}_3 \delta^3{}_b. \end{aligned} \tag{5.30}$$

The line-element corresponding to the pulled-back material metric k_{ab} can be expressed as

$$ds^2 = R'^2 dr^2 + dz^2 + R^2 d\phi^2. \tag{5.31}$$

Writing the pulled-back material metric in terms of the orthonormal tetrad

$$k_{ab} = n_1^2 x_a x_b + n_2^2 y_a y_b + n_3^2 z_a z_b,$$

it follows that its line-element is given by

$$ds^2 = n_1^2 e^{2\mu} dr^2 + n_2^2 e^{2\mu} dz^2 + n_3^2 e^{2\psi} d\phi^2. \tag{5.32}$$

Comparing this line-element with the line-element (5.31) enables to determine the eigenvalues of k:

$$n_1^2 = R'^2 e^{-2\mu} = n_1^2(r) \tag{5.33}$$

$$n_2^2 = e^{-2\mu} = n_2^2(r) \tag{5.34}$$

$$n_3^2 = R^2 e^{-2\psi} = n_3^2(r). \tag{5.35}$$

The linear particle densities are defined by

$$n_1 = n_1(r) = e^{-\mu}R'$$
$$n_2 = n_2(r) = e^{-\mu}$$
$$n_3 = n_3(r) = Re^{-\psi}.$$

Constant volume shear tensor

The non-zero components of the constant volume shear tensor $s_{ab} = \frac{1}{2}(h_{ab} - n^{-\frac{2}{3}}k_{ab})$ are

$$s_{rr} = \frac{1}{2}e^{2\mu}\left(1 - n^{-\frac{2}{3}}n_1^2\right)$$
$$s_{zz} = \frac{1}{2}e^{2\mu}\left(1 - n^{-\frac{2}{3}}n_2^2\right)$$
$$s_{\phi\phi} = \frac{1}{2}e^{2\psi}\left(1 - n^{-\frac{2}{3}}n_3^2\right)$$

Equating the components s_{rr}, s_{zz} and $s_{\phi\phi}$ to zero and substituting n_1^2, n_2^2 and n_3^2 by its values (5.33)-(5.35), one derives that the constant volume shear tensor vanishes iff the condition $R(r) = r = e^{\psi-\mu}$ is satisfied.

Elasticity difference tensor

The non-zero components of the elasticity difference tensor are listed below:

$$S^r{}_{rr} = \frac{n_1'}{n_1}$$
$$S^z{}_{zr} = \frac{n_2'}{n_2}$$
$$S^\phi{}_{\phi r} = \frac{n_3'}{n_3}$$
$$S^r{}_{zz} = \mu' - \frac{n_2^2}{n_1^2}\mu' - \frac{n_2}{n_1^2}n_2'$$
$$S^r{}_{\phi\phi} = e^{-2\psi-2\mu}\left(\psi' - \frac{n_3^2}{n_1^2}\psi' - \frac{n_3}{n_1^2}n_3'\right).$$

One can see that only seven components of the elasticity difference tensor are non-zero. Replacing the eigenvalues, which appear in the components of the elasticity difference tensor, by their explicit expressions (5.33), (5.34) and (5.35), implies

$$S^{r}{}_{rr} = \frac{R''}{R'} - \mu'$$
$$S^{z}{}_{zr} = -\mu'$$
$$S^{\phi}{}_{\phi r} = \frac{R'}{R} - \psi'$$
$$S^{r}{}_{zz} = \mu'$$
$$S^{r}{}_{\phi\phi} = -\frac{R}{R'} + e^{2\psi - 2\mu}\psi'.$$

Calculating the conditions for these components to vanish, leads to the following results:

(i) $S^{r}{}_{rr}$ is zero whenever $R(r) = c_1 + c_2 \int e^{\mu(r)} dr$;

(ii) $S^{z}{}_{zr}$ is zero whenever $\mu(r) = c$, where c is a constant;

(iii) $S^{\phi}{}_{\phi r}$ is zero whenever $R(r) = c_3 e^{\psi(r)}$;

(iv) $S^{r}{}_{zz}$ is zero whenever $\mu(r) = c_4$, where c_4 is a constant;

(v) $S^{r}{}_{\phi\phi}$ is zero whenever $R(r) = c_5 e^{\int \frac{e^{-2\psi+2\mu}}{\psi'} dr}$.

The tetrad components of the elasticity difference tensor are:

$$S^{1}{}_{11} = e^{-\mu}\frac{n_1'}{n_1}$$
$$S^{2}{}_{21} = e^{-\mu}\frac{n_2'}{n_2}$$
$$S^{3}{}_{31} = e^{-\mu}\frac{n_3'}{n_3}$$
$$S^{1}{}_{22} = e^{-\mu}\mu' - e^{-\mu}\frac{n_2^2}{n_1^2}\mu' - e^{-\mu}\frac{n_2}{n_1^2}n_2'$$
$$S^{1}{}_{33} = e^{-\mu}\psi' - e^{-\mu}\frac{n_3^2}{n_1^2}\psi' - e^{-\mu}\frac{n_3}{n_1^2}n_3'.$$

Expressions for $\underset{1}{M}$, $\underset{2}{M}$ and $\underset{3}{M}$

The second-order tensors $\underset{1}{M}$, $\underset{2}{M}$ and $\underset{3}{M}$ have the following non-zero components:

$$\begin{aligned}
\underset{1}{M}_{rr} &= e^{\mu}\frac{n_1'}{n_1} \\
\underset{1}{M}_{zz} &= e^{\mu}\left(\mu' - \frac{n_2^2}{n_1^2\mu'} - \frac{n_2}{n_1^2}n_2'\right) \\
\underset{1}{M}_{\phi\phi} &= e^{2\psi-\mu}\left(\psi' - \frac{n_3^2}{n_1^2}\psi' - \frac{n_3}{n_1^2}n_3'\right) \\
\underset{2}{M}_{rz} &= \underset{2}{M}_{zr} = e^{\mu}\frac{n_2'}{n_2} \\
\underset{3}{M}_{r\phi} &= \underset{3}{M}_{\phi r} = e^{\psi}\frac{n_3'}{n_3}.
\end{aligned}$$

Inserting in these expressions the eigenvalues given in (5.33), (5.34) and (5.35) leads to:

$$\begin{aligned}
\underset{1}{M}_{rr} &= e^{\mu}\left(\frac{R''}{R'} - \mu'\right) \\
\underset{1}{M}_{zz} &= \mu' e^{\mu} \\
\underset{1}{M}_{\phi\phi} &= e^{\mu}\left(-\frac{R}{R'} + e^{2\psi-2\mu}\psi'\right) \\
\underset{2}{M}_{rz} &= \underset{2}{M}_{zr} = -\mu' e^{\mu} \\
\underset{3}{M}_{r\phi} &= \underset{3}{M}_{\phi r} = e^{\psi}\left(\frac{R'}{R} - \psi'\right)
\end{aligned}$$

The next equations show the relations between the components of ${S^a}_{bc}$ and the second order tensors.

$$\begin{aligned}
\underset{1}{M}_{rr} &= e^{\mu}{S^r}_{rr} \\
\underset{1}{M}_{zz} &= e^{\mu}{S^r}_{zz} \\
\underset{1}{M}_{\phi\phi} &= e^{\mu}{S^r}_{\phi\phi} \\
\underset{2}{M}_{rz} &= e^{\mu}{S^z}_{zr} \\
\underset{3}{M}_{r\phi} &= e^{\psi}{S^\phi}_{\phi r}
\end{aligned}$$

Recalling the decomposition of the elasticity difference tensor[4], these results show that the components of the elasticity difference tensor are linear combinations of the components of only one eigenvector of k in each case. Three of them, ${S^r}_{rr}$, ${S^r}_{zz}$ and ${S^r}_{\phi\phi}$, are linear combinations of the component of the vector x. The other two (four if we also count the symmetric components), ${S^z}_{zr}$ and ${S^\phi}_{\phi r}$, are linear combinations of the component of the vector y and z, respectively.

Eigenvalue-eigenvector problem

The following tables, Table 5.13 - Table 5.18, contain the eigenvalues and eigenvectors for the tensors $\underset{1}{M}$, $\underset{2}{M}$ and $\underset{3}{M}$. Their eigenvectors are then compared with the eigenvectors of the pulled-back material metric.

Solving the eigenvalue-eigenvector problem for the tensor $\underset{1}{M}$ one obtains the results appearing in Table 5.13.

Table 5.13: Eigenvectors and eigenvalues for $\underset{1}{M}$

Eigenvectors	Eigenvalues
x	$\lambda_1 = e^{-\mu}\frac{n_1'}{n_1}$
y	$\lambda_2 = e^{-\mu}\left(\mu' - \frac{n_2^2}{n_1^2}\mu' - \frac{n_2}{n_1^2}n_2'\right)$
z	$\lambda_3 = e^{-\mu}\left(\psi' - \frac{n_3^2}{n_1^2}\mu' - \frac{n_3}{n_1^2}n_3'\right)$

Substituting n_1^2, n_2^2 and n_3^2 by their explicit expressions leads to the eigenvalues exposed in Table 5.14.

[4]See (3.24) in Chapter 3.

Table 5.14: Eigenvectors and eigenvalues for $\underset{1}{M}$

Eigenvectors	Eigenvalues
x	$\lambda_1 = e^{-\mu}\left(\frac{R''}{R'} - \mu'\right)$
y	$\lambda_2 = e^{-\mu}\mu'$
z	$\lambda_3 = e^{-\mu}\left(-\frac{R}{R'}e^{2\mu-2\psi} + \psi'\right)$

One can observe that the eigendirections x, y and z of k are also eigenvectors for the tensor $\underset{1}{M}$. The eigenvectors are now associated with different eigenvalues, but still depending partially on the eigenvalues of k. The canonical form for $\underset{1}{M}$ can be written as $\underset{1}{M}_{bc} = \lambda_1 x_b x_c + \lambda_2 y_b y_c + \lambda_3 z_b z_c$.

Table 5.15 contains the eigenvalues and eigenvectors for the tensor $\underset{2}{M}$.

Table 5.15: Eigenvectors and eigenvalues for $\underset{2}{M}$

Eigenvectors	Eigenvalues
$x + y$	$\lambda_4 = e^{-\mu}\frac{n_2'}{n_2}$
$x - y$	$\lambda_5 = -e^{-\mu}\frac{n_2'}{n_2}$
z	$\lambda_6 = 0$

Writing n_2 explicitly, one obtains the results given in Table 5.16.

Table 5.16: Eigenvectors and eigenvalues for $\underset{2}{M}$

Eigenvectors	Eigenvalues
$x + y$	$\lambda_4 = -\mu' e^{-\mu}$
$x - y$	$\lambda_5 = \mu' e^{-\mu}$
z	$\lambda_6 = 0$

$\underset{2}{M}$ inherits only one eigenvector z from k, which is associated with the eigenvalue 0.

The other two eigenvectors of $\underset{2}{M}$ are linear combinations of x and y, namely $x+y$ and $x-y$. The corresponding eigenvalues are symmetric and depend on n_2^2, more accurately on n_2 and on the derivative n_2'. The canonical form for $\underset{2}{M}$ can be written as $\underset{2}{M}_{bc} = 2\lambda_4(x_b y_c + y_b x_c)$.

Solving the particular eigenvalue-eigenvector problems $\underset{2}{M}_b{}^c x^b = \lambda x^c$, $\underset{2}{M}_b{}^c y^b = \lambda y^c$ and $\underset{2}{M}_b{}^c z^b = \lambda z^c$ leads to the following results.

i) x is an eigenvector for $\underset{2}{M}$ associated with the eigenvalue 0 iff $e^{-\mu}\frac{n_2'}{n_2} = 0$.

ii) y is an eigenvector for $\underset{2}{M}$ associated with the eigenvalue 0 iff $e^{-\mu}\frac{n_2'}{n_2} = 0$.

iii) z is an eigenvector for $\underset{2}{M}$ associated with the eigenvalue 0.

The eigenvector z is directly an eigenvector for $\underset{2}{M}$. The other two eigenvectors, x and y, of k are eigenvectors for $\underset{2}{M}$ iff n_2 is constant with respect to r: $n_2' = 0$. This condition implies $\mu(r) = c$, in which case $\underset{2}{M} = 0$.
The eigenvalues and eigenvectors for the tensor $\underset{3}{M}$ are presented in Table 5.17.

Table 5.17: Eigenvectors and eigenvalues for $\underset{3}{M}$

Eigenvectors	Eigenvalues
$x+z$	$\lambda_7 = e^{-\mu}\frac{n_3'}{n_3}$
$x-z$	$\lambda_8 = -e^{-\mu}\frac{n_3'}{n_3}$
y	$\lambda_9 = 0$

Substituting n_3 by $Re^{-\psi}$, one obtains the results given in Table 5.18.

Table 5.18: Eigenvectors and eigenvalues for $\underset{3}{M}$

Eigenvectors	Eigenvalues
$x+z$	$\lambda_7 = e^{-\mu}\left(\frac{R'}{R} - \psi'\right)$
$x-z$	$\lambda_8 = -e^{-\mu}\left(\frac{R'}{R} - \psi'\right)$
y	$\lambda_9 = 0$

$\underset{3}{M}$ and k have the eigenvector y in common, the corresponding eigenvalue being equal to zero. The other two eigenvectors of $\underset{3}{M}$ are linear combinations of x and z, namely $x+z$ and $x-z$. These two eigenvectors are associated with symmetric eigenvalues, which depend on n_3^2 through its square root and the derivative n_3'. The canonical form for $\underset{3}{M}$ can be written as $\underset{3}{M}_{bc} = 2\lambda_7(x_b z_c + z_b x_c)$.

Solving the particular eigenvalue-eigenvector problems $\underset{3}{M}_b{}^c x^b = \lambda x^c$, $\underset{3}{M}_b{}^c y^b = \lambda y^c$ and $\underset{3}{M}_b{}^c z^b = \lambda z^c$ leads to the following results.

i) x is an eigenvector for $\underset{3}{M}$ associated with the eigenvalue 0 iff $e^{-\mu}\frac{n_3'}{n_3} = 0$.

ii) y is an eigenvector for $\underset{3}{M}$ associated with the eigenvalue 0.

iii) z is an eigenvector for $\underset{3}{M}$ associated with the eigenvalue 0 iff $e^{-\mu}\frac{n_3'}{n_3} = 0$.

$\underset{3}{M}$ inherits the eigenvector y from k, which is associated with the eigenvalue 0. For x and z to be eigenvectors for $\underset{3}{M}$ one must require that n_3 is constant with respect to r: $n_3' = 0$. Solving this equation yields $R(r) = ce^{\psi}$. In this case, one has $\underset{3}{M} = 0$.

Ricci rotation coefficients

Calculating the Ricci coefficients one obtains:

$$\gamma_{010} = \frac{\nu'}{e^{\mu}}$$
$$\gamma_{122} = \frac{\mu'}{e^{\mu}}$$
$$\gamma_{133} = \frac{\psi'}{e^{\mu}}.$$

Kinematical quantities

The kinematical quantities entering in the decomposition

$$u_{a;b} = -\dot{u}_a u_b + u_{a;c}h^c{}_b = -\dot{u}_a u_b + \frac{1}{3}\Theta h_{ab} + \sigma_{ab} + \omega_{ab} \tag{5.36}$$

are given by the following expressions:

$$\Theta = 0$$
$$\dot{u}_a = (0, \nu', 0, 0)$$
$$\sigma_{ab} = 0$$
$$\omega_{ab} = 0.$$

5.4 Concluding remarks

Static spherically symmetric space-time

For the static spherically symmetric space-time the results of the eigenvalue-eigenvector problem show that the eigendirections x, y and z of k remain eigenvectors for the tensor $\underset{1}{M}$ only. The eigenvalues associated with the three vectors are altered, when compared with the eigenvalues of k. In fact, the eigenvalues of $\underset{1}{M}$ are functions of the eigenvalues of k: the eigenvalue corresponding to x depends on n_1 and on n_1'; the eigenvalues corresponding to y and z are equal and they depend on both eigenvalues of k: n_1^2 and n_2^2 and on n_2'.

Considering $\underset{2}{M}$ and $\underset{3}{M}$, not all eigendirections of k remain eigendirections for those tensors. For $\underset{2}{M}$, only z continues to be an eigenvector, and for $\underset{3}{M}$, only y. Both are associated with the eigenvalue zero. Furthermore, for $\underset{2}{M}$, the eigendirections x and y of k are changed to be $x+y$ and $x-y$. For $\underset{3}{M}$, x and z are changed to $x+z$ and $x-z$. The vectors $x+y$ and $x+z$ are associated with the same eigenvalue, as well as $x-y$ and $x-z$ are associated with the same eigenvalue, both eigenvalues differing only in sign. Thus, the set of eigenvalues of $\underset{2}{M}$ coincides with the set of eigenvalues of $\underset{3}{M}$. The conditions for the vectors x, y and z to remain eigenvectors for $\underset{2}{M}$ and $\underset{3}{M}$ is that n_2 is constant with respect to the coordinate r, in which case $\underset{2}{M}$ and $\underset{3}{M}$ are reduced to a zero tensor.

The property that the eigenvectors y and z of k are associated with the same eigenvalue, namely n_2^2, goes over to the elasticity difference tensor, due to its formal definition, and consequently to the tensors $\underset{\alpha}{M}$. There, this property seems to be reflected by the new property that $\underset{2}{M}$ and $\underset{3}{M}$, the coefficients of y and z in the decomposition of S, have the same eigenvalues. One can see that the role that y (z) plays for $\underset{2}{M}$ is the same that z (y) plays for $\underset{3}{M}$. Interchanging y and z in Table 5.4 (Table 5.5 and Table 5.6) leads to the results given in Table 5.7 (Table 5.8 and Table 5.9) for $\underset{3}{M}$.

Non-static spherically symmetric space-time

Here, the behaviour of the eigenvectors and eigenvalues is similar to the static case. x, y and z are eigenvectors for $\underset{1}{M}$. $\underset{2}{M}$ has $x+y$, $x-y$ and z as eigenvectors, z being again associated with the eigenvalue 0. $\underset{3}{M}$ has $x+z$, $x-z$ and y as eigenvectors, y being associated with the eigenvalue 0. The eigenvalues corresponding to $x+y$ and $x+z$ are equal, as eigenvectors of $\underset{2}{M}$ and $\underset{3}{M}$, respectively, i.e. $\mu_4 = \mu_7$, as well as the eigenvalues corresponding to $x-y$ and $x-z$ are equal, $\mu_5 = \mu_8$. These eigenvalues are symmetric: $\mu_4 = \mu_7 = -\mu_5 = -\mu_8$. As in the previous case, all eigenvalues of $\underset{2}{M}$ are equal to those of $\underset{3}{M}$. The vectors y and z are identical to the vectors y and z in the static case, only x is here different. Other differences that can be observed in this case are that the expressions of the eigenvalues depend here on the coordinate t and on derivatives of the functions ν, λ and $\tilde{r}$ with respect to this coordinate.

One can conclude that the transition from the static to the non-static case does not change the structure of the eigenvectors. This means that the vectors which are eigenvectors for $\underset{1}{M}$, $\underset{2}{M}$ and $\underset{3}{M}$ in the static case are also eigenvectors for the three tensors in the non-static case, the only difference being that the expressions of some eigenvectors and eigenvalues are altered.

Axially symmetric space-time

For the axially symmetric space-time here considered one can conclude that the three eigenvectors x, y and z of k remain eigenvectors for the tensor $\underset{1}{M}$. The corresponding eigenvalues are now changed, but they still depend on the eigenvalues of k. The eigenvalue associated with the eigenvector x is a function of n_1 and n_1'. The eigenvalue associated with y depends on the eigenvalues n_1^2 and n_2^2 of k and the eigenvalue associated with z depends on the eigenvalues n_1^2 and n_3^2 of k. One can observe that all eigenvalues depend on n_1.

Considering $\underset{2}{M}$, z remains as eigenvector, now associated with a zero eigenvalue. The other two eigenvectors are $x + y$ and $x - y$, associated with symmetric eigenvalues. These new eigenvalues depend on n_2 and n_2'.

The tensor $\underset{3}{M}$ inherits only y as eigenvector from k. The other two eigenvectors are linear combinations of x and z: $x + z$ and $x - z$. Also in this case they are associated with symmetric eigenvalues depending on n_3 and n_3'.

One can observe that the vector x plays the same role for the tensors $\underset{2}{M}$ and $\underset{3}{M}$. Interchanging in Table 5.15 n_2 with n_3, y with z and $\underset{2}{M}$ with $\underset{3}{M}$, one obtains the results of Table 5.17.

Chapter 6

Generalizing results established for a spherically symmetric space-time with flat material metric

6.1 Introduction

Magli (1993) [45] applied the relativistic elasticity theory to a non-static spherically symmetric space-time and obtained a form for the Einstein field equations, which can be useful in the analysis of the relativistic interior dynamics of a spherically symmetric, non-rotating star composed of an elastic material. He expressed the energy-momentum tensor in terms of the material fields, see (2.53), in order to describe the interaction of the elastic material with the gravitational field by means of the Einstein field equations. In all this work, the material metric was assumed to be flat.

The results exposed in this chapter present a generalization of the results given in Magli (1993) [45] in the sense that here, the Einstein field equations are obtained for a non flat material metric K, conformally related with the flat material metric $\bar{K}$ used by Magli. Two cases are analysed. In the first case, two different space-time metrics still belonging to the same class of non-static spherically symmetric space-time metrics are considered: g, the space-time metric associated with the non-flat material metric K, and the space-time metric $\bar{g}$ associated with the flat material metric $\bar{K}$, which is the configuration established in Magli (1993) [45]. The second case consists in supposing that the space-time metric is the same both for the flat and the non-flat material metric, i.e. $g = \bar{g}$.

6.2 First case: $\mathbf{g \neq \bar{g}}$

Consider a non static spherically symmetric space-time M equipped with the coordinate system $\omega^a = \{t, r, \theta, \phi\}$. Let the space-time metric g_{ab} be given by the line-element

$$ds^2 = -a(t,r)dt^2 + b(t,r)dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2. \tag{6.1}$$

Consider a non-flat material space X with material space coordinates $\xi^A = \{y, \tilde{\theta}, \tilde{\phi}\}$, where $y = y(t,r)$, $\tilde{\theta} = \theta$ and $\tilde{\phi} = \phi$. Defining the non-flat material metric K_{AB} by the line-element

$$ds^2 = f^2(y)(dy^2 + y^2 d\tilde{\theta}^2 + \tilde{r}^2 sin^2\tilde{\theta} d\tilde{\phi}^2), \tag{6.2}$$

then K and the flat material metric $\bar{K}$ used in Magli (1993) [45] are conformally related, since

$$K_{AB} = f^2(y)\bar{K}_{AB}, \tag{6.3}$$

where the line-element of $\bar{K}$ is given by

$$ds^2 = dy^2 + y^2 d\tilde{\theta}^2 + \tilde{r}^2 sin^2\tilde{\theta} d\tilde{\phi}^2. \tag{6.4}$$

Let the line-element corresponding to the space-time metric $\bar{g}$ used by Magli[1] be written as

$$ds^2 = -\bar{a}(t,r)dt^2 + \bar{b}(t,r)dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2. \tag{6.5}$$

[1] Here, the space-time metric coefficients are chosen to be $\bar{a}$ and $\bar{b}$ instead of a and b; (c.f. Magli (1993) [45]).

In order to distinguish the quantities corresponding to the flat case considered by Magli, where the flat material metric $\bar{K}$ (6.4) and the space-time metric $\bar{g}$ (6.5) were used, from the quantities corresponding to the case of a non-flat material metric K (6.2) with space-time metric g (6.1), bars are placed over those functions and tensors referring to Magli's work, that is, to the flat case. Furthermore, the following notation is established: a dot indicates a derivative with respect to t and a prime, a derivative with respect to r.

Calculating the pull-back of the material metric K one obtains

$$\begin{aligned} k^a{}_b &= g^{ac}k_{cb} = g^{ac}K_{CB}\xi^C_c\xi^B_b = f^2(y)g^{ac}\bar{K}_{CB}\xi^C_c\xi^B_b = f^2(y)g^{ac}\bar{k}_{cb} \\ &= f^2(y)g^{ac}\left[\dot{y}^2\delta^0{}_c\delta^0{}_b + \dot{y}y'(\delta^0{}_c\delta^1{}_b + \delta^0{}_b\delta^1{}_c) + y'^2\delta^1{}_b\delta^1{}_c + y^2\delta^2{}_b\delta^2{}_c + y^2 sin^2\theta\delta^3{}_b\delta^3{}_c\right], \end{aligned}$$

where the relativistic deformation gradient ξ^A_a is of the form

$$\xi^A_a = \frac{\partial\xi^A}{\partial\omega^a} = \begin{pmatrix} \dot{y} & y' & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{6.6}$$

One can write the pulled-back material metric $k^a{}_b$ as

$$k^a{}_b = \begin{pmatrix} -f^2(y)(\dot{y}^2/a) & -f^2(y)(\dot{y}y'/a) & 0 & 0 \\ f^2(y)(\dot{y}y'/b) & f^2(y)(y'^2/b) & 0 & 0 \\ 0 & 0 & f^2(y)(y^2/r^2) & 0 \\ 0 & 0 & 0 & f^2(y)(y^2/r^2) \end{pmatrix}. \tag{6.7}$$

The velocity field of matter u defined by the conditions[2] $u^a\xi^A_a = 0$, $u^a u_a = -1$ and $u^0 > 0$ can be expressed as

$$u^a = \frac{\gamma}{\sqrt{a}}\left(1, -\frac{\dot{y}}{y'}, 0, 0\right), \tag{6.8}$$

where

$$\gamma = \left(1 - \frac{b}{a}\left(\frac{\dot{y}}{y'}\right)^2\right)^{-\frac{1}{2}}. \tag{6.9}$$

[2]See Section 2.1.4 in Chapter 2.

The projection tensor is defined by

$$h^a{}_b = \delta^a{}_b + u^a u_b = \begin{pmatrix} 1-\gamma^2 & -\gamma^2(b\dot{y}/ay') & 0 & 0 \\ \gamma^2(\dot{y}/y') & 1+\gamma^2(b/a)(\dot{y}/y')^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{6.10}$$

The strain operator $\tilde{g}^a{}_b = g^{ac}k_{cb} - u^a u_b$ (2.16), which can be used to measure the state of strain of the material by (2.15), has, due to the relation $\tilde{g}^a{}_b u^b = u^a$, one eigenvalue which is equal to one.

The other eigenvalues are

$$\begin{aligned} s &= n_2^2 = n_3^2 = \; f^2(y)\,\frac{y^2}{r^2} \\ \eta &= n_1^2 = \; f^2(y)\,\frac{y'^2}{\gamma^2\, b}, \end{aligned} \tag{6.11}$$

so that these eigenvalues can be related with the eigenvalues $\bar{s}$ and $\bar{\eta}$ of the strain operator $\tilde{\bar{g}}^a{}_b = \bar{g}^{ac}\bar{k}_{cb} - u^a u_b$ considered in Magli (1993) [45] by

$$\begin{aligned} s &= n_2^2 = n_3^2 = \; f^2(y)\,\bar{s} \\ \eta &= n_1^2 = f^2(y)\,\frac{\bar{\gamma}^2}{\gamma^2}\,\frac{\bar{b}}{b}\,\bar{\eta} \end{aligned} \tag{6.12}$$

Note that s is the eigenvalue of algebraic multiplicity 2, which equals the eigenvalues n_2^2 and n_3^2 of k.

The three invariants of $\tilde{g}$, chosen by Magli[3], have the following expressions

$$\begin{aligned} I_1 &= \frac{1}{2}\left(\mathrm{Tr}\tilde{g} - 4\right) \\ I_2 &= \frac{1}{4}\left[\mathrm{Tr}\tilde{g}^2 - \left(\mathrm{Tr}\tilde{g}\right)^2\right] + 3 \\ I_3 &= \frac{1}{2}\left(\det\tilde{g} - 1\right), \end{aligned} \tag{6.13}$$

[3] See (2.17) in Chapter 2.

where

$$\begin{aligned}
&\mathrm{Tr}\,\tilde{g} = \eta\, +\, 2\,s\, +\, 1 \\
&\mathrm{Tr}\,\tilde{g}^2 = \eta^2\, +\, 2\,s^2\, +\, 1 \\
&\det\tilde{g} = \eta\,s^2.
\end{aligned} \tag{6.14}$$

Substituting (6.12) in (6.14) implies

$$\begin{aligned}
&\mathrm{Tr}\,\tilde{g} = f^2\,\mathrm{Tr}\,\bar{\tilde{g}}\, + f^2\left(\frac{\bar{\gamma}^2}{\gamma^2}\,\frac{\bar{b}}{b}\,\bar{\eta} - \bar{\eta} - 1\right) + 1 \\
&\mathrm{Tr}\,\tilde{g}^2 = f^4\,\mathrm{Tr}\,\bar{\tilde{g}}^2 + f^4\left(\frac{\bar{\gamma}^4}{\gamma^4}\,\frac{\bar{b}^2}{b^2}\,\bar{\eta}^2 - \bar{\eta}^2 - 1\right) +\, 1 \\
&\det\tilde{g} = f^6\,\frac{\bar{\gamma}^2}{\gamma^2}\,\frac{\bar{b}}{b}\,\det\bar{\tilde{g}},
\end{aligned} \tag{6.15}$$

showing the dependence of the traces $\mathrm{Tr}\tilde{g}$ and $\mathrm{Tr}\tilde{g}^2$ on the traces $\mathrm{Tr}\bar{\tilde{g}}$ and $\mathrm{Tr}\bar{\tilde{g}}^2$, respectively, and the dependence between the determinants of $\tilde{g}$ and $\bar{\tilde{g}}$.
Inserting (6.15) into (6.13) yields the following relation between the invariants I_1, I_2, I_3 and the invariants $\bar{I}_1$, $\bar{I}_2$, $\bar{I}_3$ of $\bar{\tilde{g}}$ corresponding to the flat material metric

$$\begin{aligned}
I_1 &= f^2\bar{I}_1 + \frac{3}{2}\left(f^2 - 1\right) + \frac{1}{2}f^2\bar{\eta}\left(\frac{\bar{\gamma}^2}{\gamma^2}\frac{\bar{b}}{b} - 1\right) \\
I_2 &= f^4\bar{I}_2 - 3f^4 + \frac{1}{4}f^4\left[\left(\frac{\bar{\gamma}^2}{\gamma^2}\frac{\bar{b}}{b}\right)^2\bar{\eta}^2 - \bar{\eta}^2 - 1\right] + \frac{1}{4} \\
&\quad - f^2\bar{I}_1\left[f^2\left(\frac{\bar{\gamma}^2}{\gamma^2}\frac{\bar{b}}{b}\bar{\eta} - \bar{\eta} - 1\right) + 1\right] - 2f^2\left[f^2\left(\frac{\bar{\gamma}^2}{\gamma^2}\frac{\bar{b}}{b}\bar{\eta} - \bar{\eta} - 1\right) + 1\right] \\
&\quad - \frac{1}{4}\left[f^2\left(\frac{\bar{\gamma}^2}{\gamma^2}\frac{\bar{b}}{b}\bar{\eta} - \bar{\eta} - 1\right) + 1\right]^2 + 3 \\
I_3 &= f^6\,\frac{\bar{\gamma}^2}{\gamma^2}\,\frac{\bar{b}}{b}\,\bar{I}_3 + \frac{1}{2}\,f^6\,\frac{\bar{\gamma}^2}{\gamma^2}\,\frac{\bar{b}}{b} - \frac{1}{2}.
\end{aligned} \tag{6.16}$$

Following the appendix of Magli (1993) [45], to deduce the expression for the energy-momentum tensor, one concludes that considering the metric g and the non-flat material metric K, the expression for the energy-momentum tensor is the same as (2.53).
Calculating the components of the energy-momentum tensor

$$\mathcal{T}^a_{\;b} = \epsilon\,\delta^a_{\;b} - \frac{\partial\epsilon}{\partial I_3}\det\tilde{g}\,h^a_{\;b} + \left(\mathrm{Tr}\tilde{g}\,\frac{\partial\epsilon}{\partial I_2} - \frac{\partial\epsilon}{\partial I_1}\right)k^a_{\;b} - \frac{\partial\epsilon}{\partial I_2}\,k^a_{\;c}\,k^c_{\;b}, \tag{6.17}$$

where the energy density is here represented by ϵ, agreeing with the notation used in Magli (1993) [45], yields

$$\begin{aligned}
\mathcal{T}^{0}_{\;0} &= \epsilon + \frac{\dot{y}^2}{a} \sum \\
\mathcal{T}^{1}_{\;0} &= -\frac{\dot{y}y'}{b} \sum \\
\mathcal{T}^{1}_{\;1} &= \epsilon - \frac{y'^2}{b} \sum \\
\mathcal{T}^{2}_{\;2} &= \epsilon - \frac{y^2}{r^2} \left[\sum + \left(\frac{\partial \epsilon}{\partial I_2} - f^2(y)\, \frac{y^2}{r^2}\, \frac{\partial \epsilon}{\partial I_3} \right) \left(f^4(y)\, \frac{y^2}{r^2} - f^4(y)\, \frac{y'^2}{\gamma^2 b} \right) \right],
\end{aligned} \tag{6.18}$$

where

$$\sum = f^2(y) \left[\frac{\partial \epsilon}{\partial I_1} - \frac{\partial \epsilon}{\partial I_2} \left(1 + 2\, f^2(y)\, \frac{y^2}{r^2} \right) + \frac{\partial \epsilon}{\partial I_3}\, f^4(y)\, \frac{y^4}{r^4} \right]. \tag{6.19}$$

The other two non vanishing components of the energy-momentum tensor are related by $\mathcal{T}^{0}_{\;1} = -\frac{b}{a}\mathcal{T}^{1}_{\;0}$ and $\mathcal{T}^{3}_{\;3} = \mathcal{T}^{2}_{\;2}$.

The energy density is defined in Magli (1993) [45] by

$$\epsilon = \rho\, v, \tag{6.20}$$

where

$$\rho = \rho_0 \sqrt{\det \tilde{g}} = \rho_0\, s \sqrt{\eta} \tag{6.21}$$

represents the actual density calculated in the rest frame, ρ_0 stands for the density of the relaxed material and

$$v = v(I_1(s,\eta), I_2(s,\eta), I_3(s,\eta)) = v(s,\eta) \tag{6.22}$$

represents the constitutive equation.

The quantities ρ and ϵ are related with $\bar{\rho}$ and $\bar{\epsilon}$ by

$$\rho = \bar{\rho}\sqrt{\frac{\det\tilde{g}}{\det\tilde{\bar{g}}}} = \bar{\rho} f^3 \sqrt{\frac{\bar{\gamma}^2 \bar{b}}{\gamma^2 b}}$$
$$\epsilon = \bar{\epsilon}\, f^3\, \frac{v}{\bar{v}}\sqrt{\frac{\bar{\gamma}^2 \bar{b}}{\gamma^2 b}}. \tag{6.23}$$

Using (6.13) and (6.14), one can prove the following relations:

$$\frac{\partial \epsilon}{\partial \eta} = \frac{1}{2\, f^2} \sum, \tag{6.24}$$

$$\frac{\partial \epsilon}{\partial s} = \frac{1}{f^2} \sum + \left(f^2 \frac{\partial \epsilon}{\partial I_2} - f^4 \frac{y^2}{r^2} \frac{\partial \epsilon}{\partial I_3} \right) \left(\frac{y^2}{r^2} - \frac{y'^2}{\gamma^2 b} \right). \tag{6.25}$$

Alternatively, one can express the components of the energy-momentum tensor in terms of the eigenvalues s and η by substituting the last results in (6.18) and using (6.20) and (6.21):

$$\begin{aligned}
\mathcal{T}^0_{\ 0} &= \rho \left[v + \left(v + 2\eta \frac{\partial v}{\partial \eta} \right) \gamma^2 \omega^2 \right], \\
\mathcal{T}^1_{\ 0} &= -\rho \sqrt{\frac{a}{b}} \left(v + 2\eta \frac{\partial v}{\partial \eta} \right) \gamma^2 \omega, \\
\mathcal{T}^1_{\ 1} &= -\rho \left[\gamma^2 \left(v + 2\eta \frac{\partial v}{\partial \eta} \right) - v \right], \\
\mathcal{T}^2_{\ 2} &= -\rho s \frac{\partial v}{\partial s},
\end{aligned} \tag{6.26}$$

where

$$\omega = \sqrt{1 - \frac{1}{\gamma^2}} = \sqrt{\frac{b}{a}} \frac{\dot{y}}{y'}. \tag{6.27}$$

The Einstein field equations $G^a{}_b = 8\pi \mathcal{T}^a_{\ b}$ can be written as

$G^0{}_0 = 8\pi \mathcal{T}^0_{\ 0}$:

$$-\frac{b'}{rb^2} - \frac{1}{r^2}\left(1 - \frac{1}{b}\right) = \rho \left[v + \left(v + 2\eta \frac{\partial v}{\partial \eta} \right) \gamma^2 \omega^2 \right] 8\pi, \tag{6.28}$$

$G^1{}_0 = 8\pi \mathcal{T}^1_{\ 0}$:

$$\frac{\dot{b}}{rb^2} = -\rho \sqrt{\frac{a}{b}} \left(v + 2\eta \frac{\partial v}{\partial \eta} \right) \gamma^2 \omega \, 8\,\pi, \tag{6.29}$$

$G^1{}_1 = 8\pi \mathcal{T}^1_1$:

$$\frac{a'}{rab} - \frac{1}{r^2}\left(1 - \frac{1}{b}\right) = -\rho\left[\gamma^2\left(v + 2\eta\frac{\partial v}{\partial \eta}\right) - v\right] 8\pi, \tag{6.30}$$

$G^2{}_2 = 8\pi \mathcal{T}^2_2$:

$$\begin{aligned}&\frac{1}{2b}\left[\frac{a''}{a} - \frac{a'^2}{2a^2} + \frac{1}{r}\left(\frac{a'}{a} - \frac{b'}{b}\right) - \frac{a'b'}{2ab}\right] - \frac{1}{2a}\left(\frac{\ddot{b}}{b} - \frac{\dot{b}^2}{2b^2} - \frac{\dot{a}\dot{b}}{2ab}\right) = \\ &- \rho s\frac{\partial v}{\partial s}\, 8\,\pi.\end{aligned} \tag{6.31}$$

It is possible to relate the expression for the energy-momentum tensor $\mathcal{T}$ (6.17) with the energy-momentum tensor $\bar{\mathcal{T}}$ corresponding to the case where the flat material metric is used. For that purpose, $\dfrac{\partial \epsilon}{\partial I_i}$ are written in terms of $\dfrac{\partial \bar{\epsilon}}{\partial \bar{I}_i}$, for $i = 1, 2, 3$ with the help of (6.16), the result being

$$\begin{aligned}\frac{\partial \epsilon}{\partial I_1} &= \frac{\partial \epsilon}{\partial \bar{\epsilon}}\left\{\frac{\partial \bar{\epsilon}}{\partial \bar{I}_1}\frac{1}{f^2} + \frac{\partial \bar{\epsilon}}{\partial \bar{I}_2}\frac{1}{f^4}\left[f^2\left(\frac{\bar{\gamma}^2\bar{b}}{\gamma^2 b}\bar{\eta} - \bar{\eta} - 1\right) + 1\right]\right\} \\ \frac{\partial \epsilon}{\partial I_2} &= \frac{\partial \epsilon}{\partial \bar{\epsilon}}\frac{\partial \bar{\epsilon}}{\partial \bar{I}_2}\frac{1}{f^4} \\ \frac{\partial \epsilon}{\partial I_3} &= \frac{\partial \epsilon}{\partial \bar{\epsilon}}\frac{\partial \bar{\epsilon}}{\partial \bar{I}_3}\frac{1}{f^6}\frac{\gamma^2 b}{\bar{\gamma}^2\bar{b}}.\end{aligned} \tag{6.32}$$

Introducing the results (6.32) and (6.15) in (6.17) and using (6.23) leads to

$$\mathcal{T}^a_b = f^3\frac{v}{\bar{v}}\sqrt{\frac{\bar{\gamma}^2\bar{b}}{\gamma^2 b}}\,\bar{\mathcal{T}}^a_b + \frac{\partial \bar{\epsilon}}{\partial \bar{I}_3}\,\mathrm{det}\bar{\tilde{g}}\, f^3\frac{v}{\bar{v}}\sqrt{\frac{\bar{\gamma}^2\bar{b}}{\gamma^2 b}}\,(1 - f^2)\,\bar{k}^a{}_b \tag{6.33}$$

6.3 Second case: $\mathbf{g} = \bar{\mathbf{g}}$

As for the analysis of the case $\bar{g}_{ab} = g_{ab}$, the quantities corresponding to the non-flat material metric K are related in a directer and simpler way with the quantities corresponding to the flat material metric $\bar{K}$.

The pull-back of the non-flat material metric K is calculated as follows

$$k^a{}_b = g^{ac}k_{cb} = f^2(y)g^{ac}\bar{K}_{CB}\xi^C_c\xi^B_b = f^2(y)g^{ac}[\dot{y}^2\delta^0{}_c\delta^0{}_b + \dot{y}y'(\delta^0{}_c\delta^1{}_b + \delta^0{}_b\delta^1{}_c)$$
$$+ y'^2\delta^1{}_b\delta^1{}_c + y^2\delta^2{}_b\delta^2{}_c + y^2 sin^2\theta\delta^3{}_b\delta^3{}_c].$$

One can verify that k can simply be related with the flat pulled-back material metric $\bar{k}$:

$$k^a{}_b = f^2(y)\bar{k}^a{}_b. \tag{6.34}$$

Thus, in this case, both the flat and non-flat material metric and its pull-backs are conformally related.

The eigenvalues s and η of k satisfy

$$\begin{aligned} s &= f^2\frac{y^2}{r^2} = f^2\bar{s} \\ \eta &= f^2\frac{y'^2}{\gamma^2 b} = f^2\bar{\eta}, \end{aligned} \tag{6.35}$$

where γ is given by (6.9).

The expressions (6.35) imply the following equivalences

$$\begin{aligned} \mathrm{Tr}\tilde{g} &= \eta + 2s + 1 = f^2\,\mathrm{Tr}\bar{\tilde{g}} + 1 - f^2 \\ \mathrm{Tr}\tilde{g}^2 &= \eta^2 + 2s^2 + 1 = f^4\,\mathrm{Tr}\bar{\tilde{g}}^2 + 1 - f^4 \\ \mathrm{det}\tilde{g} &= \eta\, s^2 = f^6\,\mathrm{det}\bar{\tilde{g}}. \end{aligned} \tag{6.36}$$

Using the last relations, one can show that the invariants I_1, I_2, I_3 associated with the operator $\tilde{g}$ depend on the invariants $\bar{I}_1$, $\bar{I}_2$, $\bar{I}_3$ associated with $\bar{\tilde{g}}$ as follows:

$$\begin{aligned} I_1 &= \frac{1}{2}\left(\mathrm{Tr}\tilde{g} - 4\right) = f^2\bar{I}_1 + \frac{3}{2}\left(f^2 - 1\right) \\ I_2 &= \frac{1}{4}\left[\mathrm{Tr}\tilde{g}^2 - (\mathrm{Tr}\tilde{g})^2\right] + 3 = f^4\bar{I}_2 + (f^4 - f^2)\bar{I}_1 + \frac{3}{2}\left(f^4 - f^2\right) - 3f^4 + 3 \\ I_3 &= \frac{1}{2}\left(\mathrm{det}\tilde{g} - 1\right) = f^6\bar{I}_3 + \frac{1}{2}\left(f^6 - 1\right). \end{aligned} \tag{6.37}$$

Taking into account the definitions (6.20) and (6.21), the energy density ϵ can be related with $\bar{\epsilon}$ by

$$\epsilon = f^3 \frac{\upsilon}{\bar{\upsilon}} \bar{\epsilon}. \tag{6.38}$$

Considering this equation for ϵ and observing (6.37), one can find the following expressions for $\frac{\partial \epsilon}{\partial I_i}$, $i = 1, 2, 3$, which appear in the definition of the energy-momentum tensor (6.17)

$$\begin{aligned} \frac{\partial \epsilon}{\partial I_1} &= \frac{1}{f^2} \frac{\partial \bar{\epsilon}}{\partial \bar{I}_1} - \frac{\partial \bar{\epsilon}}{\partial \bar{I}_2} \left(\frac{1}{f^2} - \frac{1}{f^4} \right) \\ \frac{\partial \epsilon}{\partial I_2} &= \frac{1}{f^4} \frac{\partial \bar{\epsilon}}{\partial \bar{I}_2} \\ \frac{\partial \epsilon}{\partial I_3} &= \frac{1}{f^6} \frac{\partial \bar{\epsilon}}{\partial \bar{I}_3}. \end{aligned} \tag{6.39}$$

Now, substituting (6.38), (6.39), (6.36) and (6.34) in the expression

$$\mathcal{T}^a_b = \epsilon\, \delta^a{}_b - \frac{\partial \epsilon}{\partial I_3} \det \tilde{g}\, h^a{}_b + \left(\mathrm{Tr} \tilde{g} \frac{\partial \epsilon}{\partial I_2} - \frac{\partial \epsilon}{\partial I_1} \right) k^a{}_b - \frac{\partial \epsilon}{\partial I_2} k^a{}_c\, k^c{}_b \tag{6.40}$$

leads to

$$\mathcal{T}^a_b = f^3 \frac{\upsilon}{\bar{\upsilon}} \bar{\mathcal{T}}^a_b, \tag{6.41}$$

where $\bar{\mathcal{T}}$ denotes the energy-momentum tensor associated with the flat metric $\bar{k}$. From the fact that $g = \bar{g}$, the energy-momentum tensor $\mathcal{T}$ associated with the metric k coincides with $\bar{\mathcal{T}}$: $\mathcal{T} = \bar{\mathcal{T}}$. Therefore, the following condition for the constitutive equations $\bar{\upsilon}$ and υ and the function f must be satisfied: $\upsilon = \frac{1}{f^3} \bar{\upsilon}$. This implies $\epsilon = \bar{\epsilon}$.

6.4 Concluding remarks

When considering a non-flat material metric instead of a flat material metric, the analysis of the results show that the Einstein field equations $G^a{}_b = 8\pi \mathcal{T}^a_b$ (6.28-6.31) itself have the same structure as the Einstein field equations derived by Magli for the flat case. The difference lies in the content of the equations. On the one hand, considering the Einstein tensor G, the difference is caused by the functions a and b appearing in the space-time metric used to construct G, since here, g differs from $\bar{g}$ through the coordinate functions: $a \neq \bar{a}$, $b \neq \bar{b}$. On the other hand, considering the energy-momentum tensor $\mathcal{T}$, its difference is due to the quantities building up $\mathcal{T}$ which depend on k or g or on both. Thus, in addition to the deviating functions a and b, here also the function f^2, since $k = f^2\bar{k}$, contributes to the discrepancy between $\mathcal{T}$ and $\bar{\mathcal{T}}$.

In addition to k and h, the quantities building up $\mathcal{T}$ are: the energy density ϵ, its derivatives with respect to the invariants $\dfrac{\partial \epsilon}{\partial I_i}$, $\mathrm{Tr}\tilde{g}$ and $\det\tilde{g}$. Investigating the construction of these objects in more detail, one can verify that the functions f^2, a and b, providing the discrepancy, enter into the objects through the eigenvalues of k. The eigenvalues are present in all mentioned quantities except in h.

In order to generalize the Einstein field equations, expressions showing the relation between quantities, such as the eigenvalues s and η, the invariants I_i, the energy density ϵ, the density ρ, the derivatives $\dfrac{\partial \epsilon}{\partial I_i}$ and the energy-momentum tensor $\mathcal{T}$, corresponding to the non-flat case and quantities corresponding to the flat case have been obtained. The functions f^2, a and b, which enables k and g to be distinct from $\bar{k}$ and $\bar{g}$, make the transition from the case where a flat material metric is used to the here considered generalized case.

As expected, for $f = 1$, $a = \bar{a}$ and $b = \bar{b}$, the expressions presented in this chapter are reduced to the expressions given in Magli (1993) [45].

Exploring the case $g = \bar{g}$, one can draw the following conclusion. For two conformally related material metrics $K = f^2\bar{K}$ belonging to the same space-time, the constitutive equations v associated with k and $\bar{v}$ associated with $\bar{k}$ must verify $v = \dfrac{1}{f^3}\bar{v}$.

Chapter 7

Conclusions

To end this thesis, the conclusions are presented within the context of the contributions of this work which suggest the prospect of future work.

7.1 Contributions

This thesis contributes to the advances in general relativistic elasticity in the following aspects.

- The leitmotif of this thesis is the elasticity difference tensor defined by Karlovini and Samuelsson (2003) [35]. The mathematical study of the elasticity difference tensor presented in this thesis underlines the importance and the interest of this object in the framework of general relativistic elasticity. In particular, it has been shown in Section 3.2 of Chapter 3, that the elasticity difference tensor arises from projecting the difference of two connections, one associated with the space-time metric $g_{ab} = -u_a u_b + h_{ab}$ and the other associated with the metric $\tilde{g}_{ab} = -u_a u_b + k_{ab}$, where k is the pulled-back material metric. Additionally, it has been demonstrated[1] that the elasticity difference tensor can be used to write the difference between the projected Riemann tensors (and Ricci tensors) associated with the two metrics entirely in terms of it.

[1]See (3.11) in Chapter 3.

Referring to its construction presented in Chapter 3, the elasticity difference describes the difference between two connections: the projected space-time connection and the projected connection originated by the material metric. One possible interpretation of this result is that the elasticity difference tensor reflects the difference between the curvatures of the actual space-time and the "relaxed" space-time. It gives a measure of deviation not on the level of the metrics, as for instance the strain tensor does, but on a higher level of curvatures.

- The mathematical study provides other new results for the elasticity difference tensor. In Chapter 3, a tetrad expression has been obtained for the elasticity difference tensor. This tetrad allows the elasticity difference tensor to be written in terms of the eigenvalues of the pulled-back material metric and in terms of the Ricci rotation coefficients. Also its traces have been calculated; these are invariants of that tensor.

- Furthermore, a new way of studying third order tensors, which are symmetric in the two covariant indices, has been presented through the analysis carried out for the elasticity difference tensor in Section 3.4 and Section 3.5. The elasticity difference tensor has been decomposed along the eigenvectors of the pulled-back material metric k into three second order tensors $\underset{\alpha}{M}$, which are the coefficients of the three eigenvectors in the decomposition. It has been studied under which conditions the eigenvectors of k are also eigenvectors of $\underset{\alpha}{M}$ and expressions for the corresponding eigenvalues have been found. This process can be viewed as an attempt to approach a classification for the elasticity difference tensor via the tensors $\underset{\alpha}{M}$. The derived results show that, in general, the eigendirections of k are not directly eigendirections of $\underset{\alpha}{M}$, unless conditions involving the Ricci rotation coefficients and the eigenvalues of k are satisfied.

- Another research field has been considered in this thesis: conformal transformations in general relativistic elasticity. In Chapter 4, first results in this topic have been obtained for the simplest case of having two conformally related material metrics in the same space-time. Among the results, relations have been established for the eigenvalues, the elasticity difference tensor and the tensors $\underset{\alpha}{M}$ associated with both material metrics. The constant volume shear tensors have been found to coincide for both material metrics. Also the eigenvalue-eigenvector problem for the tensors $\underset{\alpha}{M}$ has been reconsidered in connection with the conformally related material metrics. This work inspires the study of general relativistic elasticity in the context of conformally related metrics.

- The knowledge on physically significant space-times has been enriched. Indeed, results given in Chapter 2, Chapter 3 and Chapter 4 have been applied to: a static and a non-static spherically symmetric space-time, and to a particular case of an axially symmetric space-time. In particular, elements of the theory of general relativistic elasticity have been calculated and analysed and the elasticity difference tensor with the associated eigenvalue-eigenvector problem have been studied for these space-times. Moreover, in the static spherically symmetric case, two conformally related material metrics have been considered and some consequences have been explored.

- Existing results for a non-static spherically symmetric space-time with flat material metric have been generalized in Chapter 6. The generalization proceeds from working with a non-flat material metric, conformally related with a flat metric. The Einstein field equations have also been obtained in this case.

7.2 Future work

From the preceding topics and from the ideas and methods presented in this thesis, new problems arise. These problems are partly interesting study objectives for future work. Here the most straightforward ones are mentioned.

- Concerning the elasticity difference tensor, it would be interesting to study the relation between the metrics g, h, k and the elasticity difference tensor in more detail. As an example, the elasticity difference tensor vanishes if $k_{ab} = h_{ab}$, i.e. $k_{ab} = h_{ab} \quad \Rightarrow \quad S^a{}_{bc} = 0$, since in this case $D_a k_{bc} = D_a h_{bc} = 0$. However, it is not straightforward that $S^a{}_{bc} = 0$ implies $k_{ab} = h_{ab}$. It seems to be interesting to investigate this problem more carefully by studying in which cases or for which classes of metrics is valid:
  a) $k_{ab} = h_{ab} \quad \Longleftrightarrow \quad S^a{}_{bc} = 0$;
  b) $k_{ab} \neq h_{ab}$ and $S^a{}_{bc} = 0$.

- The construction of the elasticity difference tensor presented in Section 3.2 has been performed by specifying the alternative connection $\tilde{\nabla}$ as associated with the metric $\tilde{g}_{ab} = -u_a u_b + k_{ab}$. It is attractive to explore whether there are more possible choices of $\tilde{g}$ which via the projected connection lead to the elasticity difference tensor.

- The eigenvalue-eigenvector problem can be extended, and one can try to direct it more towards the elasticity difference tensor in order to achieve a better characterization or even a classification method for the elasticity difference tensor and thus for third order tensors which are symmetric in the two covariant indices (e.g. difference tensor). Up to now, it seems that a classification method for these tensors is neither established nor known.

- Another research field which contains many problems and can be addressed in the future is that of investigating the case of having conformally related metrics (space-time metrics and/or material metrics). A first step in this matter has been achieved in Chapter 4. Consequences for relativistic elastic objects can be studied and relations between these objects associated with both metrics can be determined. In this context it is interesting to study elasticity for warped space-times which are conformally related with locally decomposable space-times.

- Regarding the applications of relativistic elasticity, further problems can be investigated. Existing results for axially symmetric space-times can be generalized by considering non-flat material metrics and more general space-time metrics. The Einstein field equations for elastic spherically and axially symmetric space-times can be explored. Moreover, the dominant energy conditions and the equations of state can be taken into account in the study of relativistic elasticity.

Bibliography

[1] R. Beig and B. Schmidt. Relativistic elasticity. *Class. Quantum Grav.*, 20:889–904, 2003.

[2] R. Beig and B. Schmidt. Static, self-gravitating elastic bodies. *Proc. R. Soc. A*, 459:109–115, 2003.

[3] R. Beig and B. Schmidt. Relativistic elastostatics: I. Bodies in rigid rotation. *Class. Quantum Grav.*, 22:2249–2268, 2005.

[4] R. Beig and M. Wernig-Pichler. On the Motion of a Compact Elastic Body. *Commun. Math.Phys.*, 271:455–465, 2007.

[5] F. Belinfante. On the current and the density of the electric charge, the energy, the linear momentum and the angular momentum of arbitrary fields. *Physica*, 7:449–474, 1940.

[6] J. Bennoun. Sur les représentations hydrodynamique et thermodynamique des milieux élastiques en relativite générale. *C. R. Acad. Sci.*, 259:3705–3708, 1964.

[7] J. Bennoun. Étude des milieux continus élastiques et thermodynamiques en relativité générale. *Ann. Inst. H. Poincar A*, 3:41–110, 1965.

[8] M. Born. The determination of the number of oscillating particles in vapours, solutions, luminous gases. *Physik. Zeit.*, 12:569–575, 1911.

[9] A. Bressan. Onde ordinarie di discontinuità nei mezzi elastici con deformazioni finite in relatività generale. *Riv. Mat. Univ. Parma*, 4:23–40, 1963.

[10] Irene Brito, E.G.L.R. Vaz, and J. Carot. *General spherically symmetric elastic stars in relativity.* to be submitted.

[11] S. Calogero and J. Heinzle. Dynamics of Bianchi type I elastic spacetimes. *Class. Quantum Grav.*, 24:5173–5199, 2007.

[12] J. Carot and J. Costa. On the geometry of warped spacetimes. *Class. Quantum Grav.*, 10:461–482, 1993.

[13] J. Carot and L. Mas. Conformal transformations and viscous fluids in general relativity. *J. Math. Phys.*, 27:2336–2339, 1986.

[14] J. Carot and B. Tupper. Conformally reducible 2+2 spacetimes. *Class. Quantum Grav.*, 19:4141–4166, 2002.

[15] S. Carroll. *Spacetime and Geometry.* Addison Wesley, 2004.

[16] B. Carter. Elastic perturbation theory in general relativity and a variation principle for a rotating solid star. *Commun. Math. Phys.*, 30:261–280, 1973.

[17] B. Carter. Speed of sound in a high-pressure general relativistic solid. *Phys. Rev. D*, 7:1590–1593, 1973.

[18] B. Carter. Rheometric structure theory, convective differentiation and continuum electrodynamics. *Proc. R. Proc. Soc. Lond. A*, 372:169–200, 1980.

[19] B. Carter and H. Quintana. Foundations of general relativistic high-pressure elasticity theory. *Proc. R. Soc. Lond. A*, 331:57–83, 1972.

[20] B. Carter and H. Quintana. Stationary elastic rotational deformation of a relativistic neutron star model. *Astrophys. J.*, 202:511–522, 1975.

[21] C. Cattaneo. Elasticité Relativiste. *Symp. Math.*, 12:337–352, 1973.

[22] C. Cattaneo. *Teoria macroscopica dei continui relativistici.* Quad. Unione Mat. Ital., 1980.

[23] C. Cattaneo and A. Gerardi. Su un problema di equilibrio elastico in relativitá generale. *Rend. Mat.*, 8:187–200, 1974.

[24] D. Christodoulou. *The action principle and partial differential equations*. Princeton University Press, 2000.

[25] J. Ehlers. Contributions to the Relativistic Mechanics of Continuous Media. *Gen. Rel. Grav.*, 25:1225–1266, 1993.

[26] G. Ellis and H. Elst. Cosmological Models. *Cargèse Lectures*, 1998.

[27] G. Ellis and M. McCallum. A Class of Homogeneous Cosmological Models. *Commun. Math. Phys.*, 12:108–141, 1969.

[28] H. Elst and G. Ellis. Causal propagation of geometrical fields in relativistic cosmology. *Phys. Rev. D*, 59:024013, 1998.

[29] E. Glass and J. Winicour. Elastic general relativistic systems. *J. Math. Phys.*, 13:1934–1940, 1972.

[30] P. Haensel. Solid interiors of neutron stars and gravitational radiation. In *Astrophysical Sources of Gravitational Radiation*, 1995.

[31] G. Hall. Conformal Symmetries in General Relativity. In *Proceedings of the 3rd Hungarian Relativity Workshop*, 1989.

[32] G. Hall and J. Steele. Conformal vector fields in general relativity. *J. Math. Phys.*, 32:1847–1853, 1991.

[33] G. Herglotz. Über die Mechanik des deformierbaren Körpers vom Standpunkte der Relativitätstheorie. *Ann. Phys.*, 341:493–533, 1911.

[34] W. Hernandez. Elasticity Theory in General Relativity. *Phys. Rev. D*, 1:1013–1018, 1970.

[35] M. Karlovini and L. Samuelsson. Elastic stars in general relativity: I. Foundations and equilibrium models. *Class. Quantum Grav.*, 20:3613–3648, 2003.

[36] M. Karlovini and L. Samuelsson. Elastic stars in general relativity: III. Stiff ultrarigid exact solutions. *Class. Quantum Grav.*, 21:4531–4548, 2004.

[37] M. Karlovini, L. Samuelsson, and M. Zarroug. Elastic stars in general relativity: II.Radial perturbations. *Class. Quantum Grav.*, 21:1559–1581, 2004.

[38] J. Kijowski and G. Magli. Variational formulations of relativistic Elasticity and Thermoelasticity. In *Variational and Extremum principles in Macroscopic Systems*, pages 97–114. Elsevier, 1962.

[39] J. Kijowski and G. Magli. Relativistic elastomechanics as a lagrangian field theory. *J. Geom. Phys.*, 9:207–223, 1992.

[40] J. Kijowski and G. Magli. Relativistic elastomechanics is a gauge-type theory. *Preprint CPT-Luminy, 32/94, Marseille*, 1994.

[41] J. Kijowski and G. Magli. Unconstrained variational principle and canonical structure for relativistic elasticity. *Rep. Math. Phys.*, 39:99–112, 1997.

[42] J. Kijowski and G. Magli. Unconstrained Hamiltonian formulation of general relativity with thermo-elastic sources. *Class. Quantum Grav.*, 15:3891–3916, 1998.

[43] J. Kijowski, A. Smólski, and A. Górnicka. Hamiltonian theory of self-gravitating perfect fluid and a method of effective deparametrization of Einstein's theory of gravitation. *Phys. Rev. D*, 41:1875–1884, 1990.

[44] G. Magli. Axially Symmetric, Uniformly Rotating Neutron Stars in General Relativity: a Non-perturbative Approach. *Gen. Rel. Grav.*, 25:1277–1293, 1993.

[45] G. Magli. The Dynamical Structure of the Einstein Equations for a Non-rotating Star. *Gen. Rel. Grav.*, 25:441–460, 1993.

[46] G. Magli and J. Kijowski. A Generalization of the Relativistic Equilibrium Equations for a Non-rotating Star. *Gen. Rel. Grav.*, 24:139–158, 1992.

[47] G. Maugin. Magnetized deformable media in general relativity. *Ann. Inst. H. Poincar A*, 15:275–302, 1971.

[48] G. Maugin. Infinitesimal discontinuities in initially stressed relativistic elastic solids. *Commun. Math. Phys.*, 53:233–256, 1977.

[49] G. Maugin. On the covariant equations of the relativistic electrodynamics of continua III. Elastic solids. *J. Math. Phys.*, 19:1212–1219, 1978.

[50] P. McDermott, H. Van Horn, and C. Hansen. Nonradial Oscillations of Neutron Stars. *Astrophys. J.*, 325:725, 1988.

[51] C. Misner, K. Thorne, and J. Wheeler. *Gravitation.* W. H. Freeman and Company, 1970.

[52] F. Noether. Zur Kinematik des starren Körpers in der Relativtheorie. *Ann. Phys.*, 336:919–944, 1910.

[53] G. Nordström. The relativity mechanism of deformable compounds. *Physik. Zeit.*, 12:854–857, 1911.

[54] A. Papapetrou. Vibrations élastiques excitées par une onde gravitationnelle. *Ann. Inst. Henri Poincaré A*, 3:63–78, 1972.

[55] J. Park. Spherically symmetric static solutions of the Einstein equations with elastic matter source. *Gen. Rel. Grav.*, 32:235–252, 2000.

[56] D. Pines. Inside Neutron Stars. In *Proceedings of the twelfth international conference on low temperature physics*, 1971.

[57] C. Rayner. Elasticity in General Relativity. *Proc. R. Soc. Lond. A*, 272:44–53, 1963.

[58] N. Rosen. Flat-space Metric in General Relativity Theory. *Annals of Physics*, 22:1–11, 1963.

[59] L. Rosenfeld. Sur le tenseur d'impulsion-energie. *Acad. Roy. Belg.*, 18:1–30, 1940.

[60] S. Roy and P. Singh. Einstein's interior field equations in elastic media. *J. Phys. A: Math. Nucl. Gen.*, 6:1862–1866, 1973.

[61] S. Shapiro and S. Teukoplsky. *Black Holes, White Dwarfs and Neutron stars*. New York: Wiley, 1983.

[62] S. Souriau. *Géométrie et Relativité*. Paris: Hermann, 1965.

[63] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, and E. Herlt. *Exact Solutions of Einstein's Field Equations*. Cambridge University Press, second edition, 2003.

[64] J. Synge. A theory of elasticity in general relativity. *Math. Zeitschr.*, 72:82–87, 1959.

[65] A. Tahvildar-Zadeh. Relativistic and nonrelativistic elastodynamics with small shear strains. *Ann. Inst. Henri Poincaré*, 69:275–307, 1998.

[66] E.G.L.R. Vaz and Irene Brito. Analysing the elasticity difference tensor of general relativity. *Gen. Rel. Grav.*, 2008. DOI - 10.1007/s10714-008-0615-7; http://www.springerlink.com/content/7g17004285303803 (online article).

[67] R. Wald. *General Relativity*. University of Chicago Press, 1984.

[68] J. Weber. Detection and Generation of Gravitational Waves. *Phys. Rev.*, 117:306–313, 1960.

[69] J. Weber. *General relativity and gravitational waves*. Interscience, New York, 1961.

[70] J. Weber. Evidence for Discovery of Gravitational Radiation. *Phys. Rev. Lett.*, 22:1320–1324, 1969.

[71] B. Witt. The quantization of geometry. In *Gravitation: an introduction to current research*. Wiley, 1962.