



Sustainable Bridges

Assessment for Future Traffic Demands and Longer Lives

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Bayesian updating, a powerful tool for updating engineering models using results of testing and monitoring

Luis NEVES, Dawid WIŚNIEWSKI & Paulo CRUZ

Many of the European railway bridges are getting close to the end of their service life. At the same time the railway operators demand higher axle loads for freight trains and higher speeds for passenger trains. This requires new and better methods, models and tools that can be used in the assessment of existing bridges, which will let to more realistic evaluation of their load carrying capacity and also more accurate evaluation of their remaining service life. This paper presents a mathematical approach that allows to incorporate the results of testing or monitoring in the assessment of existing structures.

1. INTRODUCTION

Validation and updating (if necessary) of the assumptions made when deriving theoretical models of resistance and loading for bridge assessment, can be achieved by testing and monitoring. The term “testing” is usually associated with a point in time (discrete) observation of the bridge behaviour, whereas “monitoring” refers to the continuous or repetitive (long-term) observation of the bridge response by means of sensors that are usually permanently installed on the bridge.

The available testing and monitoring methods can provide information on different parameters regarding the structure itself and the loads acting on the structure. The provided information about the structure itself can be at different detail level including material level, element level, and structural level. On the other hand, the performed measurements can provide information about required parameters directly, for example measuring the concrete compressive strength on cores retrieved from the structure, or indirectly by using correlation with some other parameters (e.g. surface hardness). In the assessment of existing railway bridges the amount of information obtained indirectly can be much bigger due to availability of various non-destructive testing methods. The procedures and methodologies of using indirectly obtained information in the assessment should be different than those for directly obtained information due to different level of confidence to the obtained results. The Bayesian updating

approach, presented in this paper, allows the consideration of the difference in the level of confidence for the results of performed test or monitoring and, in its general form, is suitable for most of the possible structural engineering applications.

2. BACKGROUND

When destructive or non-destructive tests, monitoring, or load tests are performed, new information becomes available. This new information can be used in the safety or durability assessment of a structure, resulting in a reduction in uncertainty on the assessment or, in some cases, leading to significant changes in the values of the parameters used.

In current practice, this data is directly included in the assessment considering that it is perfectly accurate, disregarding previous information available. However, in most cases, the new information can not be assumed to be absolutely accurate, as uncertainty exists on both the results obtained due to the tests and on the relation between those and the safety of the structure.

For example, when the compressive strength of concrete is analysed in an existing structure using the Schmidt Hammer test, in most cases, for several tests performed, the results will differ significantly. This occurs for two main reasons. First, the properties of concrete change from element to element within the same structure, second, even if the test is repeated at the same point, the results will differ as a consequence of small changes in the execution of the test. In other words, the uncertainty in measured results arises from uncertainty in the material properties and from lack of accuracy of the testing methodology.

In a Bayesian updating framework, both sources of uncertainty are considered in a consistent manner, resulting in a more reliable indication of the real properties of materials and/or better resistance models leading finally to more accurate assessment of a structure (Box and Tiao, 1973).

3. UPDATING OF SINGLE RANDOM PROPERTY

Due to the cost of non-destructive or destructive tests, these are only performed after all available data was collected and analyzed. As a consequence, before performing such tests, the bridge engineer is already capable of estimating the most significant parameters necessary for the analysis of the structure. This estimative can have different levels of confidence, depending on the data available, experience with similar structures, and previous analysis of the structure. This initial knowledge will be denoted as *prior*, since it is acquired before any test is carried out.

Due to the nature of this information, it is usually associated with large uncertainty. The main objective of performing destructive or non-destructive tests is to reduce this uncertainty, and adjust the values obtained.

Defining the property under analysis (e.g., concrete compressive strength) by θ , it is possible (using existing codes or previous experience) to define a probability density function for this variable as:

$$\theta \sim f_{\theta}(\theta) \quad (1)$$

If no information exists on the property so-called a non-informative *prior* can be used. This is simply a function that is constant for all possible values of the property θ .

As an example, a case where the results of a test are described by a variable x can be considered. Knowing the nature of the test, the probability of the observation being made can be calculated for each value of the parameter θ . For example, if cores from several concrete

elements using a concrete of class C30 are tested, it is possible to measure the probability of each result of the core compressive strength. If this work is repeated for several cores, the distribution for values of the unknown parameter θ can be found. It must be noted that this information can be found in the literature for the most common tests.

The posterior distribution of the unknown variable θ can be found through expression:

$$f'_\theta(\theta) = \frac{f_\theta(\theta) \cdot \prod_{i=1}^n f_{x|\theta}(x_i | \theta)}{\int_{-\infty}^{+\infty} f_\theta(\theta) \cdot \prod_{i=1}^n f_{x|\theta}(x_i | \theta)} \quad (2)$$

The denominator of the above expression serves only as a normalization parameter, so that the area below the function is 1. It is, therefore, not important at this stage. As a result, the expression (2) can be replaced by:

$$f'_\theta(\theta) \propto f_\theta(\theta) \cdot \prod_{i=1}^n f_{x|\theta}(x_i | \theta) \quad (3)$$

The first term of expression (3) refers to the prior distribution probability. The second term describes the likelihood of a certain value of x being output from the test, for a certain value of the parameter θ . For example, represents the probability of a concrete with compressive strength θ yielding a core resistance x , when tested. Direct use of this equation can be made for relatively simple cases. For more complex situations simulation can be employed.

An example of typical results obtained using this methodology is presented in Figure 1. The prior distribution represents knowledge before carrying out the test. Considering that it is based on limited information, it is associated with significant dispersion. The likelihood distribution is associated with a smaller dispersion, meaning that it refers to a test with high accuracy. The posterior distribution represented shows a dispersion between the two previous curves, meaning that execution of the test implies a significant reduction in the dispersion of the parameter (e.g., concrete compressive strength) under analysis. Moreover, since the test yielded a result different from the mean of the prior distribution, there is a small shift in the mean of the parameter.

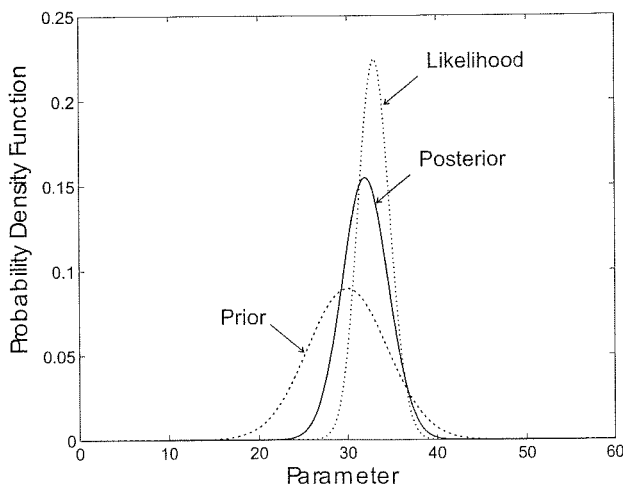


Figure 1. Comparison of Prior, likelihood and posterior distributions

5. Bridge performance and resistance for higher loads and speeds

Considering the most common case, in which a structural parameter with Gaussian distribution is investigated, the characteristic values of this parameter, using the results of tests, can be found using following equation:

$$x_k = m - t_{vd} \cdot s \sqrt{\left(1 + \frac{1}{n}\right)} \quad (4)$$

where n is the number of samples or tests, m is the mean value obtained from tests and prior knowledge, s is the standard deviation obtained from tests and prior knowledge and t_{vd} is the coefficient of the Student distribution dependent on the sample size, prior standard deviation, and probability of occurrence as given in Table 1.

Table 1. Parameter t_{vd} for normal distributed variables

Probability	Degrees of freedom, $\nu = n - 1$								
$F(-\beta)$	1	2	3	5	7	10	20	50	∞
0.10	3.08	1.89	1.64	1.48	1.42	1.37	1.33	1.30	1.28
0.05	6.31	2.92	2.35	2.02	1.89	1.81	1.72	1.68	1.64
0.01	31.8	6.97	4.54	3.37	3.00	2.76	2.53	2.40	2.33
0.005	63.7	9.93	5.84	4.03	2.50	3.17	2.84	2.68	2.58
0.001	318	22.33	10.21	5.89	4.78	4.14	3.55	3.26	3.09

If the standard deviation is known from past experience than $\nu = \infty$, and s should be replaced by the known standard deviation, σ .

If no prior information exists on the problem, the standard deviation and the mean are given by the sample mean and standard deviation as:

$$m = \frac{\sum x_i}{n} \quad (5)$$

$$s = \sqrt{\frac{1}{n} \sum (x_i - m)^2} \quad (6)$$

When prior information exists, the updated mean and standard deviation must conjugate the two sources of information. For example, it can be considered that prior information indicates that the parameter has a normal distribution with unknown mean and standard deviation. Prior information gives an estimative on the values of the mean and standard deviation in terms of the expected values and uncertainty. In other words, prior information gives a belief on the distribution of the parameter. Since this belief is not certain, it is defined in a probabilistic form. Defining the expected value and the coefficient of variation of the prior mean as $m(\mu')$ and $V(\mu')$, the expected value and the coefficient of variation of the prior standard deviation are given by $m(\sigma')$ and $V(\sigma')$. This prior can be considered equivalent to n' associated with a parameter ν' as (Rücker et al., 2006):

$$n' = \left[\frac{m(\sigma')}{m(\mu')} \frac{1}{V(\mu')} \right]^2 \quad (7)$$

$$v' = \frac{1}{2} \frac{1}{[V(\sigma')]^2} \quad (8)$$

Continuing, the Equation (4) changes to:

$$x_k = m(\mu'') - t_{v''} \cdot m(\sigma'') \sqrt{1 + \frac{1}{n''}} \quad (9)$$

where $m(\mu'')$ is the updated expected mean value, $m(\sigma'')$ is the updated standard deviation, n'' is the number of samples, and $t_{v''}$ is the updated coefficient of the Student distribution as given in Table 1.

The updated mean, $m(\mu'')$, updated standard deviation, $m(\sigma'')$, updated number of samples n'' and updated v'' are:

$$n'' = n + n' \quad (10)$$

$$m(\mu'') = \frac{n \cdot m + n' \cdot m(\mu')}{n''} \quad (11)$$

$$v'' = \begin{cases} v + v' + 1 & \text{if } n' > 0 \\ v + v' & \text{if } n' = 0 \end{cases} \quad (12)$$

$$m(\sigma'') = \sqrt{\frac{[v' \cdot m(\sigma')^2 + n' \cdot m(\mu')^2] + [v \cdot s^2 + n \cdot m^2] - n'' \cdot m(\mu'')^2}{v''}} \quad (13)$$

4. UPDATING OF UNCERTAIN RELATIONS

In civil engineering, it is quite common that a measured parameter depends of a large number of uncertain basic variables. For example, the deflection at mid-span of a bridge can be measured with certain accuracy, although it depends on a large set of parameters including the load applied, the type of concrete, the geometry of the structure and the resistance of critical cross sections. The probability of an analyzed event (e.g. structure failure, initiation of corrosion, etc.) can be updated considering a measured property I as:

$$P(F|I) = \frac{P(F \cap I)}{P(I)} \quad (14)$$

where F represent the analysed event, and I the observed event.

The analysis of this type of problems can become extremely complex due to the possible existence of correlation between the different parameters. However, an analysis performed using Monte-Carlo simulation is relatively simple. For example, when considering a structural problem where the probability of failure of some structure is computed using simulation, this analysis will yield the probabilistic distribution of a set of results for different levels of loading. Some of these results can be measured in the real structure, although these measurements will always include some degree of error. Denote the measured quantity by θ , the new probability of failure can be defined as:

$$p_f = \frac{\sum I(g_i \leq 0) p(x_i | \theta)}{\sum p(x_i | \theta)} \quad (15)$$

where I is the identity function, and $p(x_i | \theta)$ is the probability of one of the results (e.g. displacement) associated with the sample x_i occurring.

More information regarding upgrading uncertain relations can be found in (Faber, 2000).

5. EXAMPLE OF APPLICATION

In the following example the results of tests on the mechanical properties of materials are used to improve the models of the material properties of concrete used in the assessment. This is made using the analytical expressions described in section 3.

For the purpose of this example, it has been considered that to assess the compressive strength of concrete a set of 3 tests were performed, which yields to the following results: 30.1 MPa, 25.4 MPa, and 32.5 MPa. The mean and standard deviation of the observed values are:

$$m = \frac{\sum X_i}{n} = 29.33 \text{ MPa} \quad (16)$$

$$s = \sqrt{\frac{(X_i - m)^2}{n-1}} = 3.61 \text{ MPa} \quad (17)$$

Considering no prior information exists, the characteristic value of the compressive strength is given by the Equation 9 where t_{vd} is taken from Table 1. In the present case, for a percentile of 5%, t_{vd} is equal to 2.92 for $\nu = n - 1 = 2$. Consequently, Equation 9 becomes:

$$x_k = m - t_{vd} \cdot s \cdot \sqrt{\left(1 + \frac{1}{n}\right)} = 29.33 - 2.92 \cdot 3.61 \cdot \sqrt{\left(1 + \frac{1}{3}\right)} = 17.2 \text{ MPa} \quad (18)$$

Another approach to this problem is based on the use of Bayesian probabilities. It can be considered that the resistance of concrete follows a normal distribution, with unknown mean and standard deviation. Since the mean and standard deviation are unknown, they can be treated as random variables.

Although no prior information exists on the class of concrete used, the standard deviation of the concrete strength is generally quite constant, and it can be assumed close to 3 MPa. This is to say that the standard deviation of the resistance is a random variable with mean 3 MPa. Since we believe this value is quite consistent, it can be assumed that it is associated with a low coefficient of variation (e.g., 0.1 = 10%, see Table 2). For simplicity, it can be also assumed that the standard deviation follows a normal distribution.

On the other hand, no prior information exists on the mean, as any class of concrete could have been used. For this reason it can be assumed that the mean of the resistance is also defined by a normal variable with mean equal to 30 MPa, and a very high coefficient of variation (e.g., 10 = 1000%, see Table 2). Such a large coefficient of variation is close to saying that any value is reasonable, considering no information exists.

Table 2. Prior information on the compressive strength of concrete

Compressive strength of concrete			
Prior mean		Prior standard deviation	
Mean [MPa]	Coef. of var.	Mean [MPa]	Coef. of var.
$m(\mu')$	$V(\mu')$	$m(\sigma')$	$V(\sigma')$
30	10	3	0.1

The distributions of the mean and standard deviation of the concrete compressive resistance based on prior knowledge only are shown in Figure 1.

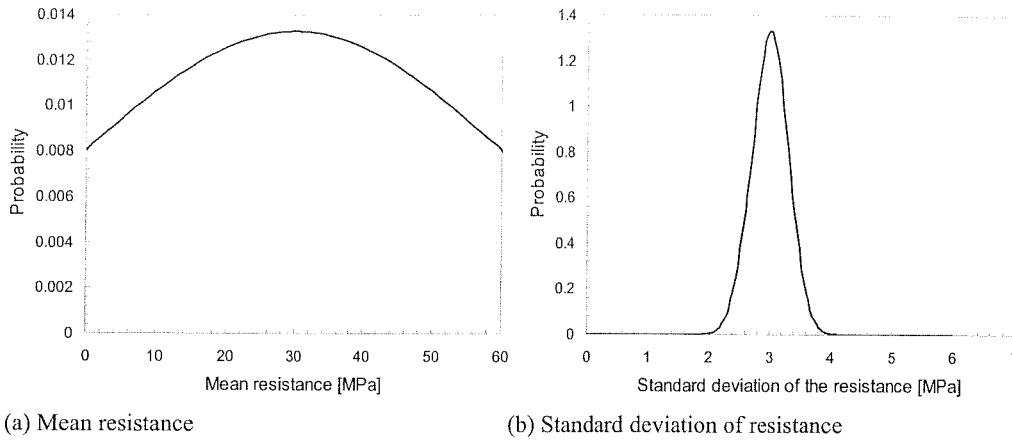


Figure 2. Mean and standard deviation of compressive resistance of concrete

Now, the prior knowledge can be combined with the results of tests. Using the mean value and the standard deviation of the observed values defined in Equations 16 and 17, Equations 7 and 8 can be rewritten as follows:

$$n' = \left[\frac{m(\sigma')}{m(\mu')} \frac{1}{V(\mu')} \right] = \left[\frac{3}{30} \frac{1}{10} \right]^2 \sim 0 \quad (19)$$

$$\nu' = \frac{1}{2} \frac{1}{[V(\sigma')]^2} = \frac{1}{2} \frac{1}{[0.1]^2} = 50 \quad (20)$$

In other words, the prior knowledge is equivalent to zero tests (since no information on the mean exists, but to 50 degrees of freedom, since the standard deviation is relatively well known).

The prior information can be now combined with the results of tests. The equivalent number of tests is the sum of the number of tests associated with prior knowledge and real tests:

$$n'' = n + n' = 3 + 0 = 3 \quad (21)$$

The updated expected value of the mean is the weighted average of the means of prior knowledge and tests:

$$m(\mu'') = \frac{n \cdot m + n' \cdot m(\mu')}{n''} = \frac{3 \cdot 29.33 + 0 \cdot 30}{3} = 29.33 \quad (22)$$

In other words, since there was no prior information on the mean value, the updated mean value is a result of tests alone.

The updated degrees of freedom are given by:

$$\nu'' = \begin{cases} \nu + \nu' + 1 & \text{if } n' > 0 \\ \nu + \nu' & \text{if } n' = 0 \end{cases} = 2 + 50 = 52 \quad (23)$$

5. Bridge performance and resistance for higher loads and speeds

The updated expected value of the standard deviation is given by:

$$m(\sigma'') = \sqrt{\frac{[v' \cdot m(\sigma')^2 + n' \cdot m(\mu')^2] + [v \cdot s^2 + n \cdot m^2] - n'' \cdot m(\mu'')^2}{v''}} \quad (24)$$

$$= \sqrt{\frac{[50 \cdot 3^2 + 0 \cdot 30^2] + [2 \cdot 3.61^2 + 3 \cdot 29.33^2] - 3 \cdot 29.33^2}{52}} = 3.026$$

As can be seen, the prior information on the standard deviation as a significant effect on the updated distribution. The results obtained are summarized in Table 3.

Table 3. Expected values of mean and standard deviation resulting from tests, prior information and Bayesian updating

	Compressive strength of concrete			
	Expected mean	Expected st. dev.	Number of samples	Degrees of freedom
	$m(\mu')$	$m(\sigma')$	n	v
Tests	29.33	3.61	3	2
Prior	30	3	0	50
Posterior	29.33	3.026	3	52

As a result, the new characteristic value can be computed as:

$$x_k = m'' - t_{v''} \cdot s'' \sqrt{\left(1 + \frac{1}{n''}\right)} = 29.33 - 1.68 \cdot 3.026 \sqrt{\left(1 + \frac{1}{3}\right)} = 23.48 \text{ MPa} \quad (25)$$

6. CONCLUSIONS

In this paper, the use of Bayesian updating in the assessment of existing structures is described. Bayesian updating is a consistent tool to combine different sources of uncertain information, as are results of non-destructive tests or engineering judgement. As the results of the example show, even very limited and initially considered useless information, as is the case of the diffuse available information on the expected standard deviation of the compressive strength of concrete, can be extremely useful in updating the information on this property. Although the mean value is unchanged, a significant reduction in the standard deviation occurs, leading to a significantly lower characteristic value.

As signalized in this paper, Bayesian updating is a powerful tool which can be used in the assessment of existing bridges, both to upgrade the theoretical or empirical engineering models, when some data obtained due to test or monitoring are available, but also to combine experts judgement with results of tests. As described most problems are much more complex than that presented in the paper, and use of simulation is often necessary.

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