Research Article

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Recognizing pro-R closures of regular languages

Abstract: Given a regular language L, we effectively construct a unary semigroup that recognizes the topological closure of L in the free unary semigroup relative to the variety of unary semigroups generated by the pseudovariety R of all finite $\mathcal{R}$-trivial semigroups. In particular, we obtain a new effective solution of the separation problem of regular languages by R-languages.

Keywords: Free profinite semigroup, unary semigroup, regular language, profinite closure, R-trivial semigroup, algebraic recognition, omega-term

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1 Introduction

There is a remarkable connection between the theories of finite semigroups and regular languages. At its basis is the well known and simple fact of the finiteness of the syntactic semigroups of such languages, which may be effectively computed as the transition semigroups of their minimal automata. This suggests a method for testing whether a regular language has a certain combinatorial property, namely by verifying whether its syntactic semigroup enjoys an associated algebraic property. A general framework for this kind of problems and a characterization of which properties may be handled in this way was given by Eilenberg [16]. On the semigroup side, the relevant algebraic properties define so-called pseudovarieties, which are nonempty classes of finite semigroups closed under taking homomorphic images, subsemigroups and finite

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direct products. In particular, Eilenberg's result prompted considerable interest in studying pseudovarieties.

For the above method to be successful for a suitable combinatorial property, one needs to be able to test membership of a given finite semigroup in the corresponding pseudovariety. Thus, a key question on a pseudovariety is to determine whether its membership problem admits an algorithmic solution, in which case the pseudovariety is said to be decidable. It turns out that several combinatorial constructions on classes of languages correspond to operations on pseudovarieties that are known not to preserve decidability in general [1, 14]. This fact has led to the search for stronger algorithmic properties that may be preserved by such operations. The notion of a tame pseudovariety, in its various flavors, has emerged from this approach [10], inspired by seminal work of Ash [13]. A quick introduction to this line of ideas and its applications may be found in [4].

Tameness is intimately connected with profinite topologies. Roughly speaking, tameness of a pseudovariety V means that there is a natural algebraic structure on profinite semigroups, with the same homomorphisms, enjoying special properties. Profinite semigroups are then naturally viewed as algebras of that kind and one may speak of the variety of such algebras generated by V. One of the key properties is the word problem in such relatively free algebras. The other key property has to do with the solution, modulo V, of finite systems of semigroup equations with clopen constraints: should the system admit a solution, does it also have a solution in the restricted algebraic language?

Even rather simple systems of equations as that reduced to the single equation $x = y$ lead to highly nontrivial problems on pseudovarieties of interest. Determining whether that equation with clopen constraints has a solution modulo V is equivalent to the following V-separation problem: given two regular languages, determine whether there is a language whose syntactic semigroup belongs to V which contains one of them and is disjoint from the other; in topological terms, this means that the closures in the free pro-V semigroup of the given languages are disjoint [3]. The algorithmic solution of this problem for various pseudovarieties turns out to have numerous applications (see, for instance, [18]).

Among pseudovarieties that have deserved a lot of attention, for their connections with formal language theory or for their inherent algebraic interest, is the pseudovariety R of all finite $\mathcal{R}$-trivial semigroups, that is, finite semigroups in which every principal right ideal admits only one element as a generator. Its word problem for the signature consisting of multiplication and the ω-power (which, in a finite semigroup, gives the idempotent power of the base) has a particularly nice solution [12] (see also [17]). Moreover, R has very strong tameness properties [7] with respect to this signature.

The main contribution of this paper is to show that the pro-R closure of a regular language in the free ω-semigroup relatively to the pseudovariety R is recognized by a homomorphism into a finite ω-semigroup. The proof is constructive: starting from a finite automaton recognizing the given regular language, we construct a finite recognizer for the pro-R closure of the language, in which the image of the language is effectively computable. As a consequence, we obtain a new algorithm to test whether the intersection of the pro-R closures of finitely many regular languages is empty or not. Indeed, this property is clearly decidable given finite recognizers for these closures. This problem is known to be equivalent to testing whether a subset of a finite semigroup is R-pointlike [3]. Therefore, our result provides an algorithmic solution for it, and also for testing whether such a subset is R-idempotent pointlike. In particular, we solve the R-separation problem for regular languages.

The paper is organized as follows. In Section 2, we introduce the necessary terminology and background. Section 3 serves to construct a first finite approximation to a semigroup modeling the ω-words in the closure of a regular language. A suitable (but unnatural) ω-power and a natural partial order on such a finite semigroup are considered respectively in Sections 4 and 5. Finally, in Section 6, it is shown that the previously constructed unary semigroup recognizes the topological closure of the given regular language, and some decidability applications are drawn.

2 Preliminaries

The reader is referred to [4, 5] for quick introductions to the topics of this paper. Nevertheless, we briefly recall the notions involved in our discussions.

Finite semigroups are viewed as discrete topological spaces. A *profinite semigroup* is an inverse limit of an inverse system of finite semigroups; equivalently, it is a (multiplicative) semigroup with a topology for which the multiplication is continuous and such that the topology is compact (Hausdorff) and zero-dimensional. Given an element s of a profinite semigroup and an integer k, the sequence $(s^{n!+k})_n$ converges to an element, denoted $s^{\omega+k}$. In particular, for $k = 0$, we get the element $s^\omega = s^{\omega+0}$, which is idempotent.

By a *pseudovariety* we mean a (nonempty) class V of finite semigroups that is closed under taking homomorphic images, subsemigroups and finite direct products. The pseudovariety of all finite semigroups is denoted S. A profinite semigroup S is said to be *pro-*V if distinct points may be separated by continuous homomorphisms into semigroups from V. For a finite set A, a pro-V semigroup S is said to be *free pro-*V *over* A if there is a mapping $\iota : A \to S$ whose image generates a dense subsemigroup of S and such that, for every function $\varphi : A \to T$ into a pro-V semigroup T, there is a unique continuous homomorphism $\hat{\varphi} : S \to T$ such that $\hat{\varphi} \circ \iota = \varphi$. Such a pro-$\mathsf{V}$ semigroup S always exists and it is clearly unique up to homeomorphic isomorphism. It is denoted $\overline{\Omega}_A\mathsf{V}$. The elements of $\overline{\Omega}_A\mathsf{V}$ are called *pseudowords over* V or simply *pseudowords* if $\mathsf{V} = \mathsf{S}$. The unique continuous homomorphism $\overline{\Omega}_A\mathsf{S} \to \overline{\Omega}_A\mathsf{V}$ induced by the generating mapping $A \to \overline{\Omega}_A\mathsf{V}$ is denoted p_{V}.

Consider the pseudovariety Sl of all finite semilattices, commutative semigroups in which all elements are idempotents. As is well known, we may view $\overline{\Omega}_A\mathsf{Sl}$ as the semigroup of nonempty subsets of A under the operation of union. The continuous homomorphism $p_{\mathsf{Sl}} : \overline{\Omega}_A\mathsf{S} \to \overline{\Omega}_A\mathsf{Sl}$ that sends each free generator $a \in A$ to $\{a\}$ is also denoted c and it is called the *content* function.

A key pseudovariety in our study is the class R of all finite semigroups in which Green's relation $\mathcal{R}$ is trivial, that is, if two elements generate the same right ideal then they are equal.

The *cumulative content* $\vec{c}(w)$ of a pseudoword $w \in \overline{\Omega}_A\mathsf{S}$ consists of all letters $a \in A$ for which there exists a factorization $w = uv$ with $p_{\mathsf{R}}(v)$ idempotent and $a \in c(v)$. The terminology comes from [11], where it was used in a more restrictive sense, and [12], where the definition is easily recognized to be equivalent to the one adopted here.

This paper deals specially with unary semigroups, that is semigroups with an additional unary operation, which will be usually denoted as an ω-power. Such unary semigroups will, therefore, often be called *ω-semigroups*. As we have observed above, profinite semigroups have a natural structure of ω-semigroups. In particular, we may consider the variety of ω-semigroups generated by the semigroups of a given pseudovariety V; it is denoted V^ω. The ω-semigroup in V^ω freely generated by a (finite) set A may be obtained as the ω-subsemigroup of $\overline{\Omega}_A\mathsf{V}$ generated by A; it is denoted $\Omega^\omega_A\mathsf{V}$. Elements of the free ω-semigroup are called (semigroup) *ω-terms*.

The generating mapping $\iota : A \to \overline{\Omega}_A\mathsf{V}$ extends uniquely to a homomorphism $A^+ \to \overline{\Omega}_A\mathsf{V}$ defined on the semigroup A^+ freely generated by A; it is also denoted ι. For a language $L \subseteq A^+$, we denote $\mathrm{cl}_{\omega,\mathsf{V}}(L)$ the topological closure of $\iota(L)$ in the subspace $\Omega^\omega_A\mathsf{V}$. A property of a pseudovariety V introduced in [10] that plays an important role is that of being *ω-full*. We take as the definition the equivalent formulation given in [9, Proposition 4.3]: a pseudovariety V is ω-full if and only if the equality $p_{\mathsf{V}}(\mathrm{cl}_{\omega,\mathsf{S}}(L)) = \mathrm{cl}_{\omega,\mathsf{V}}(L)$ holds for every regular language $L \subseteq A^+$.

The main result of this paper (Theorem 6.1) is that $\mathrm{cl}_{\omega,\mathsf{R}}(L)$ is recognized by a homomorphism onto an effectively constructible finite ω-semigroup. In contrast, it should be noted that the analogous result does not hold for the pseudovariety G of all finite groups. Indeed, the variety of ω-semigroups G^ω satisfies the identities $x^\omega y = y = yx^\omega$, which forces the interpretation of the ω-power in finite members of G^ω to be the natural one. It follows that the members of G^ω are finite groups and the subsets of $\Omega_A^\omega \mathsf{G} = A^+ \cup \{1\}$ recognized by homomorphisms into members of G^ω are the closures of G-languages. Thus, members of G^ω cannot recognize the closures in $\Omega_A^\omega \mathsf{G}$ of arbitrary regular languages of A^+.

Similar considerations apply to the language in which the ω-power is replaced by the $(\omega-1)$-power, which is more suitable to capture group phenomena. It is not excluded though that there is some even richer language that will be sufficient to obtain a result similar to our main theorem for the pseudovariety of groups. A somewhat related phenomenon is that G is not tame for the language of $(\omega-1)$-semigroups for arbitrary finite systems of word equations [15]. The quest for a richer language capturing tameness is also open. We do not know if there is a connection between the two properties, namely recognition of closures of regular languages in $\Omega_A^\sigma \mathsf{V}$ by homomorphisms into finite σ-algebras and σ-tameness with respect to arbitrary finite systems of equations.

3 A semigroup modeled after R

We introduce in this section a finite semigroup which is meant to capture certain parameters of pseudowords over R. The precise connection is delayed until Section 5, where it plays an important role.

Let A be a finite alphabet. Consider the following pseudovarieties of bands:

$$\mathsf{LRB} = [\![xyx = xy,\ x^2 = x]\!] \qquad \text{(left regular bands)}$$
$$\mathsf{MNB} = [\![xyxzx = xyzx,\ x^2 = x]\!] \quad \text{(regular bands).}$$

Note that the solution of the word problem in the relatively free semigroup $\Omega_A\mathsf{LRB}$, that is the identity problem for LRB, is obtained by reducing each word w to the canonical form which retains from w only the leftmost occurrence of each letter.

In the following result, we consider a first approximation to the behavior of pseudowords over R. This is done taking a pair where the first component models the cumulative content, while the second registers the order of the first occurrences of letters.

Lemma 3.1. *Let* $\mathcal{L}_A$ *be the subset of the Cartesian product* $\mathcal{P}(A) \times (\Omega_A \mathsf{LRB})^1$ *consisting of all pairs* (B, u) *such that* $B \subseteq c(u)$*. For* (B, u) *and* (C, v) *in* $\mathcal{L}_A$*, let* $(B, u)(C, v) = (D, uv)$*, where*

$$D = \begin{cases} B & \text{if } c(v) \subseteq B \\ C & \text{otherwise.} \end{cases}$$

This defines an associative operation on $\mathcal{L}_A$ *which turns it into a band.*

Proof. We first check that the operation is associative. Consider three elements (B, u), (C, v), and (D, w) of $\mathcal{L}_A$. We verify that

$$(B, u)(C, v) \cdot (D, w) = (B, u) \cdot (C, v)(D, w). \tag{3.1}$$

If $c(vw) \subseteq B$, then $(B, u)(C, v) = (B, uv)$, and so $(B, u)(C, v) \cdot (D, w) = (B, uvw)$, while $(B, u) \cdot (C, v)(D, w) = (B, u)(E, vw) = (B, uvw)$, independently of the value of E. In the remaining cases, namely when $c(vw) \nsubseteq B$, one of the following must hold:

(*i*) $c(v) \subseteq B$;
(*ii*) $c(v) \nsubseteq B$ and $c(w) \nsubseteq C$;
(*iii*) $c(v) \nsubseteq B$ and $c(w) \subseteq C$;

In case (*i*), we have $c(w) \nsubseteq B$ and, since $C \subseteq c(v)$, we conclude that $c(w) \nsubseteq C$, which entails that both sides of (3.1) give (D, uvw). In case (*ii*), both sides of (3.1) also give (D, uvw) while in case (*iii*), they both give (C, uvw).

It is immediate that every element of $\mathcal{L}_A$ is idempotent, so that $\mathcal{L}_A$ is a band. □

One may ask how low $\mathcal{L}_A$ falls in the lattice of pseudovarieties of bands. It is not hard to show that, whenever $|A| \geq 2$, the pseudovariety generated by $\mathcal{L}_A$ is precisely MNB.

For a word $v \in A^*$ and a subset B of A, we let $\mathrm{i}_B(v)$ denote the leftmost letter of v that does not belong to B, if such a letter exists, or else the empty word.

For a finite alphabet A, let $A^1 = A \uplus \{1\}$. Given two elements (B, u) and (C, v) of $\mathcal{L}_A$, we define a function

$$\begin{aligned} \chi^B_{u,v} : A^1 &\to A^1 \times A^1 \\ a &\mapsto \begin{cases} (a, \mathrm{i}_B(v)) & \text{if } a \in c(u) \vee c(v) \subseteq B \\ (1, a) & \text{if } a \in A^1 \setminus c(u) \wedge c(v) \nsubseteq B. \end{cases} \end{aligned}$$

Given two functions $f : X \to X^m$ and $g : X \to X^n$ with respective components f_i $(i = 1, \dots, m)$ and g_j $(j = 1, \dots, n)$, let $(f, g) : X \times X \to X^{m+n}$ be defined by the formula

$$(f, g)(x, y) = (f_1(x), \dots, f_m(x), g_1(y), \dots, g_n(y)).$$

Further, let Id_X denote the identity function on the set X.

Lemma 3.2. *Let (B, u), (C, v), and (D, w) be elements of $\mathcal{L}_A$. Then, the following equality holds, where $(X, uv) = (B, u)(C, v)$:*

$$(\chi^B_{u,v}, \mathrm{Id}_{A^1}) \circ \chi^X_{uv,w} = (\mathrm{Id}_{A^1}, \chi^C_{v,w}) \circ \chi^B_{u,vw}. \tag{3.2}$$

Proof. Both sides of Equation (3.2) are functions $A^1 \to A^1 \times A^1 \times A^1$. We show that they coincide on each $a \in A^1$. Consider the following function values:

$$\begin{aligned} (x, a_3) &= \chi^X_{uv,w}(a) & (a_1, a_2) &= \chi^B_{u,v}(x) \\ (b_1, y) &= \chi^B_{u,vw}(a) & (b_2, b_3) &= \chi^C_{v,w}(y). \end{aligned}$$

We verify that $(a_1, a_2, a_3) = (b_1, b_2, b_3)$. This is a somewhat tedious case-by-case calculation which is summarized in the following table.

<table>
<tr><td rowspan="3"></td><td colspan="2">$c(v) \subseteq B$</td><td colspan="3">$c(v) \nsubseteq B$</td><td>$c(vw) \nsubseteq B$</td></tr>
<tr><td rowspan="2">$c(w) \subseteq B$</td><td>$c(w) \nsubseteq B$</td><td rowspan="2">$a \in c(u)$</td><td>$c(w) \subseteq C$</td><td colspan="2">$c(w) \nsubseteq C$</td></tr>
<tr><td>$a \in c(u)$</td><td>$a \notin c(u)$</td><td>$a \in c(v) \setminus c(u)$</td><td>$a \notin c(uv)$</td></tr>
<tr><td>X</td><td colspan="2">B</td><td colspan="3">C</td><td></td></tr>
<tr><td>(x, a_3)</td><td colspan="2">$(a, \mathrm{i}_B(w))$</td><td colspan="3">$(a, \mathrm{i}_C(w))$</td><td>$(1, a)$</td></tr>
<tr><td>(a_1, a_2)</td><td colspan="3">$(a, \mathrm{i}_B(v))$</td><td colspan="2">$(1, a)$</td><td>$(1, 1)$</td></tr>
<tr><td>(b_1, y)</td><td colspan="3">$(a, \mathrm{i}_B(vw))$</td><td colspan="3">$(1, a)$</td></tr>
<tr><td>(b_2, b_3)</td><td>$(1, 1)$</td><td>$(1, \mathrm{i}_B(w))$</td><td>$(\mathrm{i}_B(v), \mathrm{i}_C(w))$</td><td colspan="2">$(a, \mathrm{i}_C(w))$</td><td>$(1, a)$</td></tr>
<tr><td>(a_1, a_2, a_3)
(b_1, b_2, b_3)</td><td colspan="2">$(a, 1, \mathrm{i}_B(w))$</td><td>$(a, \mathrm{i}_B(v), \mathrm{i}_C(w))$</td><td colspan="2">$(1, a, \mathrm{i}_C(w))$</td><td>$(1, 1, a)$</td></tr>
</table>

The conditions in each column in the top part of the table define a partition of all relevant cases and, with only one exception, where the value of X remains undetermined, are sufficient to determine the values corresponding to the entries in the first column in the remainder of the table. Those values are obtained by simply applying the definition of the χ functions. In the last column, although, except for the value of X, the remaining values obtained do not depend on whether or not $c(v)$ is contained in B, it is useful to distinguish the two cases in the calculation. We leave it to the reader to check that all the values are correct. □

Consider a finite set Q. Let $\mathcal{B}(Q)$ be the monoid of all binary relations on Q.

Given two functions $F, G \in \mathcal{B}(Q)^{A^1}$, we denote by $F \times G$ the function $A^1 \times A^1 \to \mathcal{B}(Q)$ defined by $(F \times G)(a, b) = F(a)G(b)$.

Definition 3.3. Let $R^\omega(Q, A)$ denote the set of all triples (F, B, u) such that $F \in \mathcal{B}(Q)^{A^1}$, $B \in \mathcal{P}(A)$, $u \in \Omega_A\mathsf{LRB}$, $F(a) = 1$ for all $a \in A^1 \setminus c(u)$, and $B \subseteq c(u)$. For two elements (F, B, u) and (G, C, v) of $R^\omega(Q, A)$, we define their product to be

$$(F, B, u)(G, C, v) = \big((F \times G) \circ \chi^B_{u,v}, D, uv\big),$$

where the product $(D, uv) = (B, u)(C, v)$ is computed in $\mathcal{L}_A$.

The triples in Definition 3.3 provide a refined model of pseudowords over R, where we add a first component to the two that were previously considered. The underlying idea is to capture the action on the states a finite automaton of the suffix of a pseudoword starting with the first occurrence of a given letter.

The following result is a first requirement for the above definition to be a good choice.

Proposition 3.4. *The set $R^\omega(Q, A)$ is a semigroup for the above multiplication.*

Proof. In view of Lemma 3.1, associativity is expressed by the formula

$$\Big(\big((F \times G) \circ \chi^B_{u,v}\big) \times H\Big) \circ \chi^X_{uv,w} = \Big(F \times \big((G \times H) \circ \chi^C_{v,w}\big)\Big) \circ \chi^B_{u,vw},$$

where $(X, uv) = (B, u)(C, v)$. Under the natural extension of the notation $F \times G$ to three factors, the above equality may be rewritten as

$$(F \times G \times H) \circ (\chi^B_{u,v}, \mathrm{Id}_{A^1}) \circ \chi^X_{uv,w} = (F \times G \times H) \circ (\mathrm{Id}_{A^1}, \chi^C_{v,w}) \circ \chi^B_{u,vw}.$$

The proposition now follows from Lemma 3.2. □

The following result amounts to a simple calculation in the semigroup $R^\omega(Q, A)$.

Lemma 3.5. *For an arbitrary element (F,B,u) of $R^\omega(Q,A)$, its natural ω-power is given by $(F,B,u)^\omega = (F_\omega,B,u)$, where*

$$F_\omega(a) = \begin{cases} 1 & \text{if } a \in A^1 \setminus c(u) \\ F(a)F(\mathrm{i}_B(u))^{\omega-1} & \text{if } a \in c(u). \end{cases}$$

Proof. One can easily show by induction on n that, for $n > 1$, we have $(F,B,u)^n = (F_n,B,u)$, where

$$F_n(a) = \begin{cases} 1 & \text{if } a \in A^1 \setminus c(u) \\ F(a)\big(F(\mathrm{i}_B(u))\big)^{n-1} & \text{if } a \in c(u). \end{cases} \qquad \square$$

In case the base of the ω-power is given as the product of two elements of $R^\omega(Q,A)$, the formula becomes somewhat more complicated. We only sketch the routine proof, leaving the details to the reader.

Lemma 3.6. *For arbitrary elements (F,B,u) and (G,C,v) of $R^\omega(Q,A)$, the natural ω-power of their product is given by $\big((F,B,u)(G,C,v)\big)^\omega = (H,D,uv)$, where $D = B$ if $c(v) \subseteq B$ while $D = C$ otherwise, and*

$$H(a) = \begin{cases} 1 & \text{if } a \in A^1 \setminus c(uv) \\ F(a)G(\mathrm{i}_B(v))\big(F(\mathrm{i}_D(u))G(\mathrm{i}_B(v))\big)^{\omega-1} & \\ & \text{if } a \in c(u) \wedge \big(c(v) \subseteq B \vee \mathrm{i}_D(u) \in c(u)\big) \\ F(a)G(\mathrm{i}_B(v))G(\mathrm{i}_C(v))^{\omega-1} & \\ & \text{if } a \in c(u) \wedge \mathrm{i}_C(uv) \notin c(u) \wedge c(v) \nsubseteq B \\ G(a)\big(F(\mathrm{i}_C(u))G(\mathrm{i}_B(v))\big)^{\omega-1} & \\ & \text{if } a \in c(v) \setminus c(u) \wedge \mathrm{i}_C(u) \in c(u) \\ G(a)G(\mathrm{i}_C(v))^{\omega-1} & \text{if } a \in c(v) \setminus c(u) \wedge \mathrm{i}_C(uv) \notin c(u). \end{cases}$$

Proof. Taking into account that $\mathrm{i}_C(u) \in c(u)$ if and only if $\mathrm{i}_C(uv) \in c(u)$, in which case $\mathrm{i}_C(u) = \mathrm{i}_C(uv)$, it is easy to check that the conditions defining each case in the expression for $H(a)$ given in the statement of the lemma are mutually exclusive and cover all possibilities. It requires then only a simple calculation using Lemma 3.5 to verify that the values of $H(a)$ are correctly given in each case. $\square$

4 An alternative ω-power

Consider next an A-labeled digraph $\mathcal{G} = (Q,A,\delta)$, with finite set of vertices Q, and labeling given by a function $\delta : A \to \mathcal{B}(Q)$, which is to be interpreted as meaning that there is an edge $p \xrightarrow{a} q$ if and only if $(p,q) \in \delta(a)$. The function δ determines a continuous homomorphism $(\overline{\Omega}_A \mathsf{S})^1 \to \mathcal{B}(Q)$, which is also denoted δ. We also write $q \in pw$ to indicate that $(p,q) \in \delta(w)$.

Given a subset B of A, we let $\varepsilon(B) = \bigcup \delta(B^*)$; in other words, a pair (p,q) of elements of Q belongs to $\varepsilon(B)$ if and only if there is some $w \in B^*$ such that $q \in pw$. For $u \in (\overline{\Omega}_A \mathsf{S})^1$, we also let $\varepsilon(u) = \varepsilon(c(u))$.

Definition 4.1. We associate with the finite A-labeled digraph $\mathcal{G} = (Q, A, \delta)$ an interpretation of the ω-power in $R^\omega(Q,A)$ as follows. For $(F,B,u) \in R^\omega(Q,A)$, let (F_ω, B, u) be the natural ω-power of (F,B,u) in the finite semigroup $R^\omega(Q,A)$. Then $(F,B,u)^{[\omega]}$ is defined to be the triple $(G, c(u), u)$, where $G(a) = F_\omega(a)\varepsilon(u)$ for each $a \in c(u)$ and $G(a) = 1$ for all $a \in A \setminus c(u)$. This defines a unary semigroup structure on $R^\omega(Q,A)$, which depends on the choice of labeling δ. We denote this unary semigroup $R^\omega(\mathcal{G})$.

A word of warning is perhaps needed at this point. In a unary semigroup, we most often use the notation $x \mapsto x^\omega$ to denote the unary operation and we also use it for the abstract operation. However, in a finite semigroup, the standard notation is to indicate x^ω as the idempotent power of x. Since, in the unary semigroup $R^\omega(\mathcal{G})$, we consider a different unary operation, the notation $x^{[\omega]}$ has been adopted. From hereon, we talk about ω-semigroups instead of unary semigroups.

For a triple x in $R^\omega(Q,A)$, let $\pi_i(x)$ denote its ith component. The following proposition shows that the ω-semigroup $R^\omega(\mathcal{G})$ has some nice properties.

Proposition 4.2. *For every finite A-labeled digraph $\mathcal{G} = (Q, A, \delta)$, the ω-semigroup $R^\omega(\mathcal{G})$ satisfies the following identities of ω-semigroups:*

$$(x^\omega)^\omega = (x^r)^\omega = x^\omega \quad (r \geq 2),$$
$$(xy)^\omega x = (xy)^\omega x^\omega = (xy)^\omega.$$

Proof. Let $(F', c(u), u) = x^{[\omega]}$. We first note that, from the definition of the multiplication it follows that $(F', c(u), u)(H, D, w) = (F', c(u), u)$ for every element (H, D, w) of $R^\omega(\mathcal{G})$ such that $c(w) \subseteq c(u)$. In particular, we obtain the identities $(xy)^{[\omega]}x = (xy)^{[\omega]}x^{[\omega]} = (xy)^{[\omega]}$ and that $x^{[\omega]}$ is idempotent. Hence, for $a \in c(u)$, $\pi_1\big((x^{[\omega]})^{[\omega]}\big)(a)$ is $F'(a)\varepsilon(u)^2$ while it is 1 at $a \in A \setminus c(u)$. Since the relation $\varepsilon(u)$ is idempotent, it follows that $(x^{[\omega]})^{[\omega]} = x^{[\omega]}$. Finally, that $(x^r)^{[\omega]} = x^{[\omega]}$ follows from the fact $(x^r)^\omega = x^\omega$ in every finite semigroup. □

We now consider the subset $\tilde{R}^\omega(\mathcal{G})$ of $R^\omega(Q,A)$ consisting of the triples (F,B,u) such that the following conditions hold for every $a \in c(u)$:

$$F(a) \subseteq \varepsilon(u); \tag{4.1}$$

$$F(a)\varepsilon(B) = F(a). \tag{4.2}$$

Note that Property (4.1) implies that the inclusion $F(a)\varepsilon(u) \subseteq \varepsilon(u)$ holds.

Lemma 4.3. *The set $\tilde{R}^\omega(\mathcal{G})$ is a subsemigroup of $R^\omega(Q,A)$.*

Proof. We verify only that Property (4.2) is preserved by multiplication, leaving it to the reader to verify that the same is true for Property (4.1). Let (F,B,u) and (G,C,v) be arbitrary elements of $\tilde{R}^\omega(\mathcal{G})$ and consider the product $(H,D,uv) = (F,B,u)(G,C,v)$. We need to show that $H(a)\varepsilon(D) = H(a)$ for every $a \in c(uv)$.

In case $c(v) \subseteq B$, we have $D = B$, $\mathrm{i}_B(v) = 1$, and we may compute

$$H(a) = F(a)G(\mathrm{i}_B(v)) = F(a) = F(a)\varepsilon(B) = H(a)\varepsilon(D).$$

Assume next that $c(v) \nsubseteq B$, so that $D = C$ and $\mathrm{i}_B(v) \in c(v)$. In case, additionally, $a \in c(u)$, we obtain

$$H(a) = F(a)G(\mathrm{i}_B(v)) = F(a)G(\mathrm{i}_B(v))\varepsilon(C) = H(a)\varepsilon(D).$$

Finally, otherwise, that is when, additionally, $a \in c(uv) \setminus c(u)$, we get

$$H(a) = G(a) = G(a)\varepsilon(C) = H(a)\varepsilon(D). \qquad \square$$

Note that, for every $(F,B,u) \in \tilde{R}^\omega(\mathcal{G})$, its ω-power $(F,B,u)^{[\omega]}$ belongs to $\tilde{R}^\omega(\mathcal{G})$. Hence, $\tilde{R}^\omega(\mathcal{G})$ is in fact an ω-subsemigroup of $R^\omega(\mathcal{G})$. In particular, $\tilde{R}^\omega(\mathcal{G})$ satisfies all the identities of Proposition 4.2.

Lemma 4.4. *Let $\mathcal{G} = (Q,A,\delta)$ be a finite A-labeled digraph, B a subset of A, and $s,t \in \mathcal{B}(Q)$ be relations contained in $\varepsilon(B)$. Then, the following equality holds: $(st)^\omega s\,\varepsilon(B) = (st)^\omega \varepsilon(B)$.*

Proof. We have already observed that the definition of $\varepsilon(B)$ implies that $s\,\varepsilon(B)$ and $t\,\varepsilon(B)$ are both contained in $\varepsilon(B)$. Hence, the relation $(st)^\omega s\,\varepsilon(B)$ is certainly contained in $(st)^\omega\varepsilon(B)$. The reverse inclusion is obtained by noting that $(st)^\omega\varepsilon(B) = (st)^\omega s \cdot t(st)^{\omega-1}\varepsilon(B) \subseteq (st)^\omega s\,\varepsilon(B)$. $\square$

While the ω-semigroup $R^\omega(\mathcal{G})$ in general fails the identity $(xy)^\omega = x(yx)^\omega$, it turns out that the ω-subsemigroup $\tilde{R}^\omega(\mathcal{G})$ does satisfy it.

Proposition 4.5. *The ω-semigroup $\tilde{R}^\omega(\mathcal{G})$ satisfies the identity $(xy)^\omega = x(yx)^\omega$.*

Proof. Let (F, B, u) and (G, C, v) be arbitrary elements of $\tilde{R}^\omega(\mathcal{G})$ and consider the corresponding expressions

$$\begin{aligned}(\tilde{H}, c(uv), uv) &= \big((F, B, u)(G, C, v)\big)^{[\omega]}\\ (\tilde{I}, c(uv), vu) &= \big((G, C, v)(F, B, u)\big)^{[\omega]}\\ (J, c(uv), uv) &= (F, B, u)\big((G, C, v)(F, B, u)\big)^{[\omega]}.\end{aligned}$$

Then, taking into account Lemma 3.6, we may compute

$$\tilde{H}(a) = \begin{cases} 1 & \text{if } a \in A^1 \setminus c(uv)\\ F(a)G(\mathrm{i}_B(v))\big(F(\mathrm{i}_D(u))G(\mathrm{i}_B(v))\big)^{\omega-1}\varepsilon(uv) & \\ & \text{if } a \in c(u) \wedge \big(c(v) \subseteq B \vee \mathrm{i}_D(u) \in c(u)\big)\\ F(a)G(\mathrm{i}_B(v))G(\mathrm{i}_C(v))^{\omega-1}\varepsilon(uv) & \\ & \text{if } a \in c(u) \wedge \mathrm{i}_C(uv) \notin c(u) \wedge c(v) \nsubseteq B\\ G(a)\big(F(\mathrm{i}_C(u))G(\mathrm{i}_B(v))\big)^{\omega-1}\varepsilon(uv) & \\ & \text{if } a \in c(v) \setminus c(u) \wedge \mathrm{i}_C(u) \in c(u)\\ G(a)G(\mathrm{i}_C(v))^{\omega-1}\varepsilon(uv) & \text{if } a \in c(v) \setminus c(u) \wedge \mathrm{i}_C(uv) \notin c(u)\end{cases}$$

and, dually,

$$\tilde{I}(a) = \begin{cases} 1 & \text{if } a \in A^1 \setminus c(vu)\\ G(a)F(\mathrm{i}_C(u))\big(G(\mathrm{i}_E(v))F(\mathrm{i}_C(u))\big)^{\omega-1}\varepsilon(uv) & \\ & \text{if } a \in c(v) \wedge \big(c(u) \subseteq C \vee \mathrm{i}_E(v) \in c(v)\big)\\ G(a)F(\mathrm{i}_C(u))F(\mathrm{i}_B(u))^{\omega-1}\varepsilon(uv) & \\ & \text{if } a \in c(v) \wedge \mathrm{i}_B(vu) \notin c(v) \wedge c(u) \nsubseteq C\\ F(a)\big(G(\mathrm{i}_B(v))F(\mathrm{i}_C(u))\big)^{\omega-1}\varepsilon(uv) & \\ & \text{if } a \in c(u) \setminus c(v) \wedge \mathrm{i}_B(v) \in c(v)\\ F(a)F(\mathrm{i}_B(u))^{\omega-1}\varepsilon(uv) & \text{if } a \in c(u) \setminus c(v) \wedge \mathrm{i}_B(vu) \notin c(v)\end{cases}$$

from which it follows that

$$J(a) = \begin{cases} 1 & \text{if } a \in A^1 \setminus c(uv)\\ F(a) & \text{if } a \in c(uv) = B\\ F(a)G(\mathrm{i}_B(v))F(\mathrm{i}_C(u))\big(G(\mathrm{i}_E(v))F(\mathrm{i}_C(u))\big)^{\omega-1}\varepsilon(uv) & \\ & \text{if } a \in c(u) \wedge c(v) \nsubseteq B\\ F(a)F(\mathrm{i}_B(u))^{\omega}\varepsilon(uv) & \\ & \text{if } a \in c(u) \wedge \mathrm{i}_B(vu) \in c(u) \setminus c(v)\\ G(a)F(\mathrm{i}_C(u))\big(G(\mathrm{i}_E(v))F(\mathrm{i}_C(u))\big)^{\omega-1}\varepsilon(uv) & \\ & \text{if } a \in c(v) \setminus c(u) \wedge \big(c(u) \subseteq C \vee \mathrm{i}_E(v) \in c(v)\big)\end{cases}$$

It remains to show that $J(a) = \tilde{H}(a)$ for every $a \in A$. We test this equality following the separation in cases in the above description of J.

Case 1. In case $a \in A^1 \setminus c(uv)$, we get $J(a) = 1 = \tilde{H}(a)$.

Case 2. Suppose now that $a \in c(uv) = B$. Since $B \subseteq c(u)$, it follows that $c(v) \subseteq B = c(u)$, which yields $D = B$ and $\mathrm{i}_D(u) = \mathrm{i}_B(v) = 1$. Hence, the only possible alternative in the above description of $\tilde{H}$ is the second one. Moreover, it gives $\tilde{H}(a) = F(a)\varepsilon(uv)$. Since (F, B, u) belongs to $\tilde{R}^\omega(\mathcal{G})$ and $B = c(uv)$, we do have $\tilde{H}(a) = F(a)\varepsilon(uv) = F(a) = J(a)$.

Case 3. Suppose next that $a \in c(u)$ and $c(v) \nsubseteq B$. The latter assumption implies that $D = C$ and $\mathrm{i}_B(vu) = \mathrm{i}_B(v)$. There are now two possibilities, the first of which is to fall in the second alternative of the formula for computing $\tilde{H}$, with $\mathrm{i}_D(u) \in c(u)$, that is, $c(u) \nsubseteq D = C$, which entails $E = B$. In this case, we obtain $J(a) = F(a)G(\mathrm{i}_B(v))F(\mathrm{i}_C(u))\big(G(\mathrm{i}_B(v))F(\mathrm{i}_C(u))\big)^{\omega-1}\varepsilon(uv)$ while $\tilde{H}(a) = F(a)G(\mathrm{i}_B(v))\big(F(\mathrm{i}_C(u))G(\mathrm{i}_B(v))\big)^{\omega-1}\varepsilon(uv)$ so that the equality $J(a) = \tilde{H}(a)$ follows from Lemma 4.4. The other possibility is to fall in the third alternative of the formula for $\tilde{H}$, with $\mathrm{i}_C(uv) \notin c(u)$, which yields $c(u) \subseteq C$, so that $E = C$, and $\mathrm{i}_C(u) = 1$. Hence, we obtain directly $J(a) = F(a)G(\mathrm{i}_B(v))G(\mathrm{i}_C(v))^{\omega-1}\varepsilon(uv) = \tilde{H}(a)$.

Case 4. Assume now that $a \in c(u)$ and $\mathrm{i}_B(vu) \in c(u) \setminus c(v)$. The second condition implies that $\mathrm{i}_B(v) = 1$, that is, $c(v) \subseteq B$, whence $D = B$, and $\mathrm{i}_B(vu) = \mathrm{i}_B(u)$. This means that we are in second alternative of the formula for $\tilde{H}$ and we obtain $\tilde{H}(a) = F(a)F(\mathrm{i}_B(u))^{\omega-1}\varepsilon(uv)$ while $\tilde{H}(a) = F(a)F(\mathrm{i}_B(u))^{\omega}\varepsilon(uv)$ and so the equality $\tilde{H}(a) = J(a)$ follows from Lemma 4.4.

Case 5a. Here, we consider the case where $a \in c(v) \setminus c(u)$ and $c(u) \subseteq C$. The latter condition means that $\mathrm{i}_C(u) = 1$ and implies that $E = C$. This falls in the third alternative of the formula for $\tilde{H}$ and we obtain $\tilde{H}(a) = G(a)G(\mathrm{i}_C(v))^{\omega-1}\varepsilon(uv) = J(a)$.

Case 5b. Assume finally that $a \in c(v) \setminus c(u)$, $c(u) \nsubseteq C$, and $\mathrm{i}_E(v) \in c(v)$. Since $c(u) \nsubseteq C$, we have $E = B$. Taking into account that $\mathrm{i}_E(v) \in c(v)$, we deduce that $D = C$. We fall in the fourth alternative of the formula for $\tilde{H}$, which gives the equality $\tilde{H}(a) = G(a)\big(F(\mathrm{i}_C(u))G(\mathrm{i}_B(v))\big)^{\omega-1}\varepsilon(uv)$ while the fifth alternative of the formula for J provides the equality $J(a) = G(a)F(\mathrm{i}_C(u))\big(G(\mathrm{i}_B(v))F(\mathrm{i}_C(u))\big)^{\omega-1}\varepsilon(uv)$. Applying Lemma 4.4, we conclude that $\tilde{H}(a) = J(a)$. □

Combining Proposition 4.5 with Proposition 4.2, we are led to the following key result.

Proposition 4.6. *The ω-semigroup $\tilde{R}^\omega(\mathcal{G})$ belongs to the variety R^ω.*

Proof. It remains to invoke the result from [12, Theorem 6.1] that the identities in Propositions 4.2 and 4.5 define the variety R^ω. ☐

We introduce a further restriction on the elements of $\tilde{R}^\omega(\mathcal{G})$, namely we consider the subset $S^\omega(\mathcal{G})$ consisting of the elements (F, B, u) of $\tilde{R}^\omega(\mathcal{G})$ such that

$$X \subseteq Y \subseteq A \implies F(\mathrm{i}_X(u)) \subseteq \varepsilon(Y)F(\mathrm{i}_Y(u)). \tag{4.3}$$

Proposition 4.7. *The set $S^\omega(\mathcal{G})$ is an ω-subsemigroup of $R^\omega(\mathcal{G})$.*

Proof. Consider two elements (F, B, u) and (G, C, v) of $R^\omega(Q, A)$ and their product $(H, D, uv) = (F, B, u)(G, C, v)$.

Suppose that (F, B, u) and (G, C, v) satisfy Property (4.3). We claim that so does their product (H, D, uv).

Consider subsets X and Y of A such that $X \subseteq Y$. Assume first that $c(v) \subseteq B$, so that $c(v) \subseteq c(u)$, $\mathrm{i}_Y(uv) = \mathrm{i}_Y(u)$, and

$$H(\mathrm{i}_X(uv)) = F(\mathrm{i}_X(uv))G(\mathrm{i}_B(v)) \text{ and } H(\mathrm{i}_Y(uv)) = F(\mathrm{i}_Y(uv))G(\mathrm{i}_B(v)).$$

If $\mathrm{i}_X(uv) \notin c(u)$, then $\mathrm{i}_Y(uv) \notin c(u)$ must also hold and $F(\mathrm{i}_X(uv)) = 1 = F(\mathrm{i}_Y(uv))$, whence $H(\mathrm{i}_X(uv)) = H(\mathrm{i}_Y(uv)) \subseteq \varepsilon(Y)H(\mathrm{i}_Y(uv))$. On the other hand, if $\mathrm{i}_X(uv) \in c(u)$, then we have $\mathrm{i}_X(uv) = \mathrm{i}_X(u)$. Hence, we may apply the assumption that (F, B, u) satisfies Property (4.3) to deduce that

$$H(\mathrm{i}_X(uv)) = F(\mathrm{i}_X(u))G(\mathrm{i}_B(v)) \subseteq \varepsilon(Y)F(\mathrm{i}_Y(u))G(\mathrm{i}_B(v)) = \varepsilon(Y)H(\mathrm{i}_Y(uv)).$$

Assume next that $c(v) \nsubseteq B$. In case $\mathrm{i}_X(uv) \notin c(u)$, then $\mathrm{i}_Y(uv) \notin c(u)$ also holds, and we obtain

$$H(\mathrm{i}_X(uv)) = G(\mathrm{i}_X(v)) \subseteq \varepsilon(Y)G(\mathrm{i}_Y(v)) = \varepsilon(Y)H(\mathrm{i}_Y(uv)).$$

We may, therefore, assume that $\mathrm{i}_X(uv) \in c(u)$. The additional assumption that $\mathrm{i}_Y(uv) \notin c(u)$ entails that $c(u) \subseteq Y$ and so $B \subseteq Y$, as $B \subseteq c(u)$. Since both (F, B, u) and (G, C, v) satisfy Property (4.3), we may deduce the following relations:

$$\begin{aligned} H(\mathrm{i}_X(uv)) &= F(\mathrm{i}_X(u))G(\mathrm{i}_B(v)) \subseteq \varepsilon(Y)F(\mathrm{i}_Y(u))G(\mathrm{i}_B(v)) = \varepsilon(Y)G(\mathrm{i}_B(v)) \\ &\subseteq \varepsilon(Y) \cdot \varepsilon(Y)G(\mathrm{i}_Y(v)) = \varepsilon(Y)H(\mathrm{i}_Y(uv)). \end{aligned}$$

It remains to consider the case where both $\mathrm{i}_X(uv)$ and $\mathrm{i}_Y(uv)$ belong to $c(u)$. Taking into account that (F, B, u) satisfies Property (4.3), we obtain:

$$H(\mathrm{i}_X(uv)) = F(\mathrm{i}_X(u))G(\mathrm{i}_B(v)) \subseteq \varepsilon(Y)F(\mathrm{i}_Y(u))G(\mathrm{i}_B(v)) = \varepsilon(Y)H(\mathrm{i}_Y(uv)).$$

To conclude the proof, we must show that the ω-power $(I, c(u), u) = (F, B, u)^{[\omega]}$ satisfies Property (4.3) if so does (F, B, u). Let $(F_\omega, B, u) = (F, B, u)^\omega$, so that the function I is given by the formula $I(a) = F_\omega(a)\varepsilon(u)$ if $a \in c(u)$ and $I(a) = 1$ otherwise. Let X and Y be such that $X \subseteq Y \subseteq A$. In case $c(u) \subseteq X$, we get

$$I(\mathrm{i}_X(u)) = 1 \subseteq \varepsilon(Y)1 = \varepsilon(Y)I(\mathrm{i}_Y(u)).$$

For the remainder of the proof, we assume that $c(u) \nsubseteq X$. From the assumption that (F, B, u) satisfies Property (4.3) and the previous step of the proof, we know that $(F_\omega, B, u) = (F, B, u)^\omega$, which is a finite power of (F, B, u), also satisfies Property (4.3). Hence, we obtain

$$I(\mathrm{i}_X(u)) = F_\omega(\mathrm{i}_X(u))\varepsilon(u) \subseteq \varepsilon(Y)F_\omega(\mathrm{i}_Y(u))\varepsilon(u).$$

In case $c(u) \nsubseteq Y$, the rightmost expression in the preceding inclusion is equal to $\varepsilon(Y)I(\mathrm{i}_Y(u))$. Otherwise, that expression reduces to $\varepsilon(Y)\varepsilon(u)$ and

$$\varepsilon(Y)\varepsilon(u) \subseteq \varepsilon(Y)\varepsilon(Y) = \varepsilon(Y) = \varepsilon(Y)I(\mathrm{i}_Y(u)),$$

which concludes the proof. □

5 A natural partial order and generators

Given two elements x and y of $R^\omega(Q, A)$, we write $x \leq y$ if $\pi_1(x) \subseteq \pi_1(y)$, $\pi_2(x) \subseteq \pi_2(y)$, and $\pi_3(x) = \pi_3(y)$. This defines a partial order on $R^\omega(Q, A)$.

Proposition 5.1. *The order $\leq$ is stable under multiplication on the left. The restriction of the order $\leq$ to $S^\omega(\mathcal{G})$ is stable under multiplication on the right.*

Proof. Let (F, B, u), (G, C, v), and (H, D, w) be elements of $R^\omega(Q, A)$.

Suppose that the inequality $(F, B, u) \leq (G, C, v)$ holds so that, in particular, we have $u = v$. Let

$$(I, X, wu) = (H, D, w)(F, B, u) \text{ and } (J, Y, wu) = (H, D, w)(G, C, u).$$

In case $c(u) \subseteq D$, we get $X = D = Y$ and

$$I(a) = H(a)F(\mathrm{i}_D(u)) \subseteq H(a)G(\mathrm{i}_D(u)) = J(a);$$

note that the conditions in the previous line also hold if $a \in c(w)$. We now assume that $c(u) \nsubseteq D$, which yields $X = B \subseteq C = Y$. It remains to consider the case where $a \notin c(w)$, in which we obtain $I(a) = F(a) \subseteq G(a) = J(a)$. This completes the proof of left stability.

For the proof of right stability within $S^\omega(\mathcal{G})$, we assume that the triples (F, B, u), (G, C, v), and (H, D, w) are elements of $S^\omega(\mathcal{G})$ such that the inequality $(F, B, u) \leq (G, C, v)$ holds, so that $u = v$. Consider the products

$$(I, X, uw) = (F, B, u)(H, D, w) \text{ and } (J, Y, uw) = (G, C, u)(H, D, w).$$

Suppose first that $c(w) \subseteq B$, whence also $c(w) \subseteq C$ holds. It follows that $X = B \subseteq C = Y$ and

$$I(a) = F(a)H(\mathrm{i}_B(w)) = F(a) \subseteq G(a) = G(a)H(\mathrm{i}_C(w)) = J(a).$$

From hereon, we suppose that $c(w) \nsubseteq B$. In case $c(w) \subseteq C$, we get $X = D \subseteq c(w) \subseteq C = Y$. In case $c(w) \nsubseteq C$, we obtain $c(w) \nsubseteq B$ and $X = D = Y$.

Next, we assume that $a \in c(u)$, so that

$$I(a) = F(a)H(\mathrm{i}_B(w)) \subseteq G(a)\varepsilon(C)H(\mathrm{i}_C(w)) = G(a)H(\mathrm{i}_C(w)) = J(a),$$

where the inclusion uses the inequality $F \subseteq G$ and the assumption that (H, D, w) satisfies Property (4.3), and the second equality comes from the hypothesis that (G, C, v) satisfies Property (4.2).

Finally, consider the case where $a \notin c(u)$. In case $c(w) \subseteq C$, since $c(w) \subseteq C \subseteq c(u)$, we get $I(a) = H(a) = 1$, while we also have $J(a) = G(a)H(\mathrm{i}_C(w)) = 1$. Otherwise, that is in the case where $c(w) \nsubseteq C$, we simply get $I(a) = H(a) = J(a)$. This concludes the proof of right stability. ☐

Let $\mathcal{T}_A^\omega$ denote the algebra of ω-terms over A, that is, the unary algebra freely generated by A, in which the unary operation is represented by the ω-power.

Next, we choose special elements in $R^\omega(Q, A)$.

Definition 5.2. Consider a finite A-labeled digraph $\mathcal{G} = (Q, A, \delta)$. For each letter $a \in A$, let the triple $\nu_{[\omega]}(a) = (F_a, \emptyset, a)$ be determined by

$$F_a(b) = \begin{cases} \delta(a) & \text{if } b = a, \\ 1 & \text{otherwise.} \end{cases}$$

Note that $(F_a, \emptyset, a)$ belongs to $S^\omega(\mathcal{G})$. We define two homomorphisms $\mathcal{T}_A^\omega \to S^\omega(\mathcal{G})$ of ω-semigroups by letting $\nu_\omega(a) = \nu_{[\omega]}(a) = (F_a, \emptyset, a)$ for each $a \in A$: for ν_ω, we consider the natural structure of ω-semigroup of $S^\omega(\mathcal{G})$ while, for $\nu_{[\omega]}$, we take its alternative ω-power defined in Section 4.

The unique homomorphism of ω-semigroups $\mathcal{T}_A^\omega \to \Omega_A^\omega\mathsf{S}$ mapping each generator $a \in A$ to itself is denoted η. In view of Proposition 4.6, we may consider the unique homomorphism of ω-semigroups $\rho_{\mathcal{G}} : \Omega^\omega_A\mathsf{R} \to S^\omega(\mathcal{G})$ that maps each generator $a \in A$ to the triple $(F_a, \emptyset, a)$.

The following result further explains our choice of multiplication in $\mathcal{L}_A$.

Lemma 5.3. *For each $\alpha \in \mathcal{T}^\omega_A$, the following properties hold:*
(i) $c(\pi_3(\nu_{[\omega]}(\alpha))) = c(\eta(\alpha))$;
(ii) $\pi_2(\nu_{[\omega]}(\alpha)) = \vec{c}(\eta(\alpha))$.

Proof. The proof is done by induction on the construction of the ω-term α. If α is a letter, then the result is obtained by direct inspection. Assuming that $\alpha = \beta\gamma$, the definitions and the induction hypothesis for both β and γ yield

$$\begin{aligned} c(\pi_3(\nu_{[\omega]}(\alpha))) &= c(\pi_3(\nu_{[\omega]}(\beta\gamma))) = c(\pi_3(\nu_{[\omega]}(\beta))) \cup c(\pi_3(\nu_{[\omega]}(\gamma))) \\ &= c(\eta(\beta)) \cup c(\eta(\gamma)) = c(\eta(\alpha)). \end{aligned}$$

Similarly, since $\vec{c}(\eta(\alpha))$ is equal to $\vec{c}(\eta(\beta)) = \pi_2(\nu_{[\omega]}(\beta))$ if $c(\eta(\gamma)) \subseteq \vec{c}(\eta(\beta))$, and to $\vec{c}(\eta(\gamma)) = \pi_2(\nu_{[\omega]}(\gamma))$ otherwise, we get $\vec{c}(\eta(\alpha)) = \pi_2(\nu_{[\omega]}(\alpha))$ by definition of the multiplication in $\mathcal{L}_A$.

Suppose next that the induction hypothesis holds for the ω-term α. By definition of the $[\omega]$-power and since $\nu_{[\omega]}(\alpha^\omega) = \nu_{[\omega]}(\alpha)^{[\omega]}$, we must have $\pi_2(\nu_{[\omega]}(\alpha^\omega)) = c(\pi_3(\nu_{[\omega]}(\alpha^\omega))) = c(\pi_3(\nu_{[\omega]}(\alpha)))$. As α satisfies (i) and $\vec{c}(\eta(\alpha^\omega)) = c(\eta(\alpha))$, we deduce that α^ω still satisfies both (i) and (ii). □

In particular, we obtain the following result.

Proposition 5.4. *An ω-term $\alpha \in \mathcal{T}^\omega_A$ is such that $p_{\mathsf{R}}(\eta(\alpha))$ is idempotent if and only if the equality $\pi_2(\nu_{[\omega]}(\alpha)) = c(\pi_3(\nu_{[\omega]}(\alpha)))$ holds.* □

Further properties of the order relation are established in the following lemma.

Lemma 5.5. *For a finite A-labeled digraph $\mathcal{G}$, the following conditions hold for all elements x and y of the ω-semigroup $S^\omega(\mathcal{G})$, every ω-term $\alpha \in \mathcal{T}^\omega_A$, and every letter $a \in A$:*
(i) $x^\omega \le x^{[\omega]}$;
(ii) $x \le y$ *implies* $x^{[\omega]} \le y^{[\omega]}$;
(iii) $\nu_\omega(\alpha) \le \nu_{[\omega]}(\alpha)$.

Proof. Properties (i) and (ii) follow immediately from the interpretation of the ω-power given by Definition 4.1. Property (iii) can then be deduced easily by induction on the construction of the ω-term α in terms of the application of the operations of multiplication and ω-power taking into account that the order $\le$ in $S^\omega(\mathcal{G})$ is stable under multiplication by Proposition 5.1. □

Following standard terminology, we say that $(S, \leq)$ is an *ordered ω-semigroup* if S is an ω-semigroup and $\leq$ is a partial order on S which is compatible with multiplication and ω-power.

Proposition 5.6. *The pair $(S^\omega(\mathcal{G}), \leq)$ is an ordered ω-semigroup.*

Proof. The order is compatible with multiplication by Proposition 5.1 and with ω-power by Lemma 5.5(ii). ☐

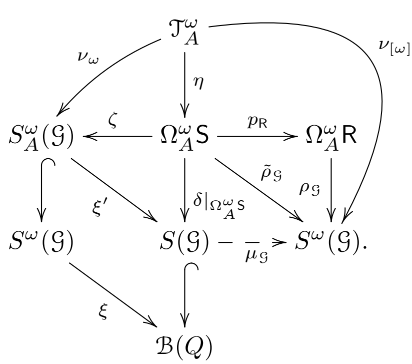
6 Recognition of R-closures

Given two ω-semigroups S and T, a *relational morphism* $S \to T$ is a binary relation $\mu \subseteq S \times T$ with domain S which is an ω-subsemigroup of $S \times T$. For a finite A-labeled digraph $\mathcal{G} = (Q, A, \delta)$, let $S(\mathcal{G})$ be the semigroup $\delta(A^+)$. The relational morphism of ω-semigroups $\mu_\mathcal{G} : S(\mathcal{G}) \to S^\omega(\mathcal{G})$ is the composite relation $\rho_\mathcal{G} \circ p_{\mathsf{R}} \circ (\delta|_{\Omega_A^\omega \mathsf{S}})^{-1}$.

Note that the composite mapping $\tilde{\rho}_\mathcal{G} = \rho_\mathcal{G} \circ p_{\mathsf{R}}$ is a homomorphism of ω-semigroups. On the other hand, the restriction of $\tilde{\rho}_\mathcal{G}$ to A extends to a continuous homomorphism $\overline{\Omega}_A \mathsf{S} \to S^\omega(\mathcal{G})$. Its restriction to $\Omega_A^\omega \mathsf{S}$ is denoted ζ. For an ω-term $\alpha \in \mathcal{T}_A^\omega$ on the alphabet A representing the ω-word $w \in \Omega_A^\omega \mathsf{S}$, note that $\nu_\omega(\alpha) = \zeta(w)$ and $\nu_{[\omega]}(\alpha) = \tilde{\rho}_\mathcal{G}(w)$.

Denote by $S_A^\omega(\mathcal{G})$ the subsemigroup of $S^\omega(\mathcal{G})$ generated by $\tilde{\rho}_\mathcal{G}(A)$. Note that $S_A^\omega(\mathcal{G})$ consists of elements of $S^\omega(\mathcal{G})$ of the form $(F, \emptyset, u)$. There is another mapping that plays a role in our construction. It is the mapping $\xi : S_A^\omega(\mathcal{G}) \to \mathcal{B}(Q)$ which sends the triple (F, B, u) to the binary relation $F(\mathrm{i}_\emptyset(u))$. It follows from the definition of the multiplication in $R^\omega(Q, A)$ that the restriction $\xi' = \xi|_{S_A^\omega(\mathcal{G})}$ is a homomorphism of semigroups, taking its values in $S(\mathcal{G})$.

The relevant mappings are depicted in the following diagram:



Note that the diagram commutes. In view of Lemma 5.5(*iii*), the inequality $\zeta(w) \leq \tilde{\rho}_{\mathcal{G}}(w)$ holds for every $w \in \Omega_A^\omega \mathsf{S}$.

Assuming that the language $L \subseteq A^+$ is recognized by some automaton obtained from the A-labeled digraph $\mathcal{G}$ by adding an appropriate choice of sets I and T, respectively of initial and terminal vertices, the language L is also recognized by the transition homomorphism $\delta|_{A^+} : A^+ \to \mathcal{B}(Q)$, namely

$$L = (\delta|_{A^+})^{-1}\{\theta \in \mathcal{B}(Q) : \theta \cap (I \times T) \neq \emptyset\}.$$

It follows that the homomorphism $\delta|_{\Omega_A^\omega \mathsf{S}} : \Omega_A^\omega \mathsf{S} \to \mathcal{B}(Q)$ recognizes $\mathrm{cl}_{\omega,\mathsf{S}}(L)$ as

$$\mathrm{cl}_{\omega,\mathsf{S}}(L) = (\delta|_{\Omega_A^\omega \mathsf{S}})^{-1}\{\theta \in \mathcal{B}(Q) : \theta \cap (I \times T) \neq \emptyset\}.$$

whence so does ζ.

Theorem 6.1. *Let $\mathcal{A} = (Q, A, \delta, I, T)$ be a finite automaton and consider the underlying A-labeled digraph $\mathcal{G} = (Q, A, \delta)$ and the set*

$$P = \{x \in S^\omega(\mathcal{G}) : \xi(x) \cap (I \times T) \neq \emptyset\}.$$

Then the equality $\rho_{\mathcal{G}}^{-1}(P) = \mathrm{cl}_{\omega,\mathsf{R}}(L)$ holds for the language L recognized by $\mathcal{A}$.

Proof. Let w be an arbitrary element of $\mathrm{cl}_{\omega,\mathsf{R}}(L)$. Since R is ω-full [9, Theorem 7.4], there is some $v \in \mathrm{cl}_{\omega,\mathsf{S}}(L)$ such that $p_{\mathsf{R}}(v) = w$. Let $\alpha \in \mathcal{T}_A^\omega$ be an ω-term such that $\eta(\alpha) = v$ and let v_n be the word that is obtained from α by replacing each subterm of the form u^ω by $u^{n!}$. Then $\lim v_n = v$. Since the closure of L in $\overline{\Omega}_A \mathsf{S}$ is an open set [2, Theorem 3.6.1], it follows that $v_n \in L$ for all sufficiently large n. For each n, $\rho_{\mathcal{G}}(v_n)$ is an element of $S_A^\omega(\mathcal{G})$. Since $S^\omega(\mathcal{G})$ is finite, there is some subsequence $(v_{n_k})_k$ such that $s = \rho_{\mathcal{G}}(v_{n_k}) \in P$ is independent of k. Note that the sequence $(\rho_{\mathcal{G}}(v_n))_n$ is eventually constant with value $\nu_\omega(\alpha)$. In particular, that value must be s. On the other hand, $\rho_{\mathcal{G}}(w) = \tilde{\rho}_{\mathcal{G}}(v) = \nu_{[\omega]}(\alpha)$. In view of Lemma 5.5(*iii*), it follows that $\rho_{\mathcal{G}}(w) \in P$ since P is upward closed with respect to the order $\leq$. We have thus established the inclusion $\mathrm{cl}_{\omega,\mathsf{R}}(L) \subseteq \rho_{\mathcal{G}}^{-1}(P)$.

For the reverse inclusion, let w be an arbitrary element of $\tilde{\rho}_{\mathcal{G}}^{-1}(P)$. We claim that there is $v \in \mathrm{cl}_{\omega,\mathsf{S}}(L)$ such that $p_{\mathsf{R}}(v) = p_{\mathsf{R}}(w)$, which shows that $\rho_{\mathcal{G}}^{-1}(P) \subseteq \mathrm{cl}_{\omega,\mathsf{R}}(L)$. Since ζ recognizes $\mathrm{cl}_{\omega,\mathsf{S}}(L)$, it suffices to show that there is $v \in \Omega_A^\omega \mathsf{S}$ such that $\zeta(v) \in P$ and $p_{\mathsf{R}}(v) = p_{\mathsf{R}}(w)$. More generally, suppose that $\alpha \in \mathcal{T}_A^\omega$ is such that the pair of states (p, q) belongs to $\xi(\nu_{[\omega]}(\alpha))$. We claim that there is some $\beta \in \mathcal{T}_A^\omega$ such that $p_{\mathsf{R}}(\eta(\alpha)) = p_{\mathsf{R}}(\eta(\beta))$ and $(p, q) \in \xi(\nu_\omega(\beta))$. We prove the claim by induction on the construction of the ω-term α in terms of the operations of multiplication and ω-power.

If $\alpha = \alpha_1\alpha_2$, then there exists $r \in Q$ such that $(p, r) \in \xi(\nu_{[\omega]}(\alpha_1))$ and $(r, q) \in \xi(\nu_{[\omega]}(\alpha_2))$. Assuming the claim holds for both α_i, there is $\beta_i \in \mathcal{T}_A^\omega$ such that $p_{\mathsf{R}}(\eta(\alpha_i)) = p_{\mathsf{R}}(\eta(\beta_i))$ $(i = 1, 2)$, $(p, r) \in \xi(\nu_\omega(\beta_1))$, and $(r, q) \in \xi(\nu_\omega(\beta_2))$. Then the ω-term $\beta = \beta_1\beta_2$ has the required properties.

Suppose next that $\alpha = \alpha_0^\omega$, where the claim holds for the ω-term α_0. By hypothesis, the pair of states (p, q) belongs to the relation $\xi(\nu_{[\omega]}(\alpha)) = \xi(\nu_{[\omega]}(\alpha_0)^{[\omega]})$. Let $n \geq 1$ be such that $S^\omega(\mathcal{G})$ satisfies the identity $x^\omega = x^n$ for the natural ω-power. In view of the definition of the $[\omega]$-power in $S^\omega(\mathcal{G})$, it follows that there is some state $r \in Q$ such that $(p, r) \in \xi(\nu_{[\omega]}(\alpha_0^n))$, where $\nu_{[\omega]}(\alpha_0)^\omega = \nu_{[\omega]}(\alpha_0)^n = \nu_{[\omega]}(\alpha_0^n)$, and there is a path in $\mathcal{G}$ from r to q labeled by some word u of $c(\eta(\alpha_0))^*$. Since $\nu_{[\omega]}(\alpha_0^n)$ is idempotent, by the pigeonhole principle there is some state $r' \in Q$ such that each of the pairs (p, r'), (r', r'), and (r', r) belongs to $\xi(\nu_{[\omega]}(\alpha_0^n))$. By the case of the product, already handled in the preceding paragraph, since α_0 is assumed to satisfy the claim, so does α_0^n. Hence, there are ω-terms $\beta_i \in \mathcal{T}_A^\omega$ such that $p_{\mathsf{R}}(\eta(\beta_i)) = p_{\mathsf{R}}(\eta(\alpha_0^n))$ $(i = 1, 2, 3)$, $(p, r') \in \xi(\nu_\omega(\beta_1))$, $(r', r') \in \xi(\nu_\omega(\beta_2))$, and $(r', r) \in \xi(\nu_\omega(\beta_3))$. Since elements of $\overline{\Omega}_A\mathsf{S}$ with the same image under p_{R} have the same content, we obtain the equalities

$$p_{\mathsf{R}}(\eta(\beta_1\beta_2^\omega\beta_3 u)) = p_{\mathsf{R}}(\eta(\beta_1\beta_2^\omega\beta_3)) = p_{\mathsf{R}}(\eta(\alpha_0^\omega)) = p_{\mathsf{R}}(\eta(\alpha)).$$

We have thus shown that the ω-term $\beta = \beta_1\beta_2^\omega\beta_3 u$ has all the required properties, thereby concluding the induction step and the proof of the theorem. ☐

Note that we may use the same labeled digraph to recognize several languages.

Corollary 6.2. *Let $L_1, \ldots, L_n$ be regular languages over the same finite alphabet A and suppose that, for each $i \in \{1, \ldots, n\}$, a suitable choice of initial and terminal states in the A-labeled digraph $\mathcal{G} = (Q, A, \delta)$ yields an automaton recognizing L_i. Then $S^\omega(\mathcal{G})$ recognizes every Boolean combination of the sets $\mathrm{cl}_{\omega,\mathsf{R}}(L_i)$.*

Proof. It suffices to apply Theorem 6.1 and note that inverse functions behave well with respect to Boolean operations. ☐

For instance, one may take the same labeled digraph $\mathcal{G}$ to recognize several given regular languages over the same finite alphabet A, such as the disjoint union of their minimal automata where, naturally, the choice of initial and terminal states depends on the language. Corollary 6.2 then provides an algorithm to compute the intersection of the pro-R closures of the given languages in $\Omega_A^\omega\mathsf{R}$. Since R is completely tame for the signature ω, we thus obtain an algorithm to test whether the intersection of the pro-R closures of the given languages in $\overline{\Omega}_A\mathsf{R}$ is empty. The case of a pair of regular languages gives an algorithmic solution of the problem of separation by R-languages.

Recall that a subset P of a finite semigroup S is said to be V-*pointlike* if, for every relational morphism $\mu : S \to T$ into a semigroup from V, there is some $t \in T$ such that $P \times \{t\} \subseteq \mu$. Equivalently, one may consider an arbitrary onto homomorphism $\varphi : A^+ \to S$, where A is a finite alphabet, and require that the closures of the regular languages $\varphi^{-1}(s)$ $(s \in S)$ in $\overline{\Omega}_A\mathsf{V}$ have some point in common. Again, since R is completely tame for the signature ω, the preceding property for the pseudovariety R is equivalent to the sets $\mathrm{cl}_{\omega,\mathsf{R}}(\varphi^{-1}(s))$ $(s \in S)$ having some point in common. In view of Corollary 6.2, we obtain an algorithm establishing the following result.

Corollary 6.3. *It is decidable whether a given subset of a finite semigroup is* R-*pointlike.* □

The previous corollary is not new. It was first proved in [6]. The proof may also be derived from much more general results from [7] and yet another approach to compute R-pointlike sets was obtained in [8].

We next consider a further algorithmic property associated with a pseudovariety V, which is important in the computation of Mal'cev products with V. A subset P of a finite semigroup S is said to be a V-*idempotent-pointlike subset* if, for every relational morphism $\mu : S \to T$ into a semigroup from V, there is an idempotent $e \in T$ such that $P \times \{e\} \subseteq \mu$.

Corollary 6.4. *It is decidable whether a given subset of a finite semigroup is* R-*idempotent pointlike.*

Proof. Choose a finite alphabet A and an onto homomorphism $\varphi : A^+ \to S$, where S is a given finite semigroup. Let P be a finite subset of S. By tameness [7], P is R-idempotent pointlike if and only if there is, for each $s \in P$, some ω-word $w_s \in \mathrm{cl}_{\omega,\mathsf{S}}(\varphi^{-1}(s))$ such that $p_{\mathsf{R}}(w_s)$ is the same idempotent independent of s. Since $p_{\mathsf{R}}(\mathrm{cl}_{\omega,\mathsf{S}}(\varphi^{-1}(s))) = \mathrm{cl}_{\omega,\mathsf{R}}(\varphi^{-1}(s))$ by fullness, we conclude that P is R-idempotent pointlike if and only if the intersection $I = \bigcap_{s\in P} \mathrm{cl}_{\omega,\mathsf{R}}(\varphi^{-1}(s))$ contains some idempotent.

By Corollary 6.2, there is a finite A-labeled digraph $\mathcal{G}$ such that $S^\omega(\mathcal{G})$ recognizes I. In view of Proposition 5.4, I contains an idempotent if and only if the image of I under $\rho_{\mathcal{G}}$ contains some triple (F, B, u) such that $B = c(u)$, a condition that may be effectively tested. □

Again, Corollary 6.4 follows from the general tameness results of [7] but the algorithms that may be derived from tameness are merely theoretical, depending on enumerating in parallel all favorable and unfavorable cases, until our instance of the problem is produced [10]. The algorithm described in the proof of Corollary 6.4 is much more effective.

Proposition 6.5. *For a given finite labeled digraph $\mathcal{G} = (Q, A, \delta)$, let $m = |Q|$ and $n = |A|$. Then, the cardinality of $S^\omega(\mathcal{G})$ is bounded above by $2^{(m^2+1)n} \cdot 3 \cdot n!$.*

Proof. Recall that the elements of $S^\omega(\mathcal{G})$ are triples (F, B, u) where F is a function $A^1 \to \mathcal{B}(Q)$, $B \subseteq A$, and $u \in \overline{\Omega}_A\mathsf{LRB}$. The cardinality $|\overline{\Omega}_A\mathsf{LRB}|$ is the number of different words in n letters without repeated letters. It is well known to be equal to $n! \sum_{r=0}^{n} \frac{1}{r!} = \lfloor e \cdot n! \rfloor$, whence it is bounded above by $3 \cdot n!$. For an element (F, B, u) of $S^\omega(\mathcal{G})$, the function $F : A^1 \to \mathcal{B}(Q)$ is such that $F(1) = 1$. There are $2^{m^2 n}$ such functions. □

Note that in general, the cardinality of the transition semigroup $S(\mathcal{G})$ of $\mathcal{G}$ also grows exponentially with m and n. We do not know whether $|S^\omega(\mathcal{G})|$ may grow exponentially with $|S(\mathcal{G})|$. A finer analysis taking into account Properties (4.1)–(4.3) may lead to better estimates than those provided by Proposition 6.5. Even better estimates may perhaps hold for the ω-subsemigroup $\operatorname{Im} \tilde{\rho}_{\mathcal{G}}$.

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References

[1] D. Albert, R. Baldinger, and J. Rhodes, *The identity problem for finite semigroups (the undecidability of)*, J. Symbolic Logic **57** (1992), 179–192.

[2] J. Almeida, *Finite semigroups and universal algebra*, World Scientific, Singapore, 1995, English translation.

[3] ———, *Some algorithmic problems for pseudovarieties*, Publ. Math. Debrecen **54 Suppl.** (1999), 531–552.

[4] ———, *Profinite semigroups and applications*, Structural theory of automata, semigroups and universal algebra (New York) (V. B. Kudryavtsev and I. G. Rosenberg, eds.), Springer, 2005, pp. 1–45.

[5] J. Almeida and A. Costa, *Handbook of AutoMathA*, vol. I. Theoretical Foundations, ch. 17. Profinite topologies, pp. 615–652, European Math. Soc. Publ. House, 2021.

[6] J. Almeida, J. C. Costa, and M. Zeitoun, *Tameness of pseudovariety joins involving R*, Monatsh. Math. **146** (2005), 89–111.

[7] ______, *Complete reducibility of systems of equations with respect to R*, Portugal. Math. **64** (2007), 445–508.

[8] ______, *Pointlike sets with respect to R and J*, J. Pure Appl. Algebra **212** (2008), 486–499.

[9] ______, *Closures of regular languages for profinite topologies*, Semigroup Forum **80** (2014), 20–40.

[10] J. Almeida and B. Steinberg, *On the decidability of iterated semidirect products and applications to complexity*, Proc. London Math. Soc. **80** (2000), 50–74.

[11] J. Almeida and P. Weil, *Free profinite $\mathcal{R}$-trivial monoids*, Int. J. Algebra Comput. **7** (1997), 625–671.

[12] J. Almeida and M. Zeitoun, *An automata-theoretic approach to the word problem for ω-terms over R*, Theor. Comp. Sci. **370** (2007), 131–169.

[13] C. J. Ash, *Inevitable graphs: a proof of the type II conjecture and some related decision procedures*, Int. J. Algebra Comput. **1** (1991), 127–146.

[14] K. Auinger and B. Steinberg, *On the extension problem for partial permutations*, Proc. Amer. Math. Soc. **131** (2003), 2693–2703.

[15] T. Coulbois and A. Khélif, *Equations in free groups are not finitely approximable*, Proc. Amer. Math. Soc. **127** (1999), 963–965.

[16] S. Eilenberg, *Automata, languages and machines*, vol. B, Academic Press, New York, 1976.

[17] M. Kufleitner and J. Ph. Wächter, *The word problem for omega-terms over the Trotter–Weil hierarchy*, Theory Comput. Syst. **62** (2018), no. 3, 682–738.

[18] Th. Place and M. Zeitoun, *Separating regular languages with first-order logic*, Logical Methods in Computer Science **12** (2016), 31 pp.