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Incorporating a Four-Dimensional Filter Line Search Method into an Interior Point Framework

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Abstract

Here we incorporate a four-dimensional filter line search method into an infeasible primal-dual interior point framework for nonlinear programming. Each entry in the filter has four components measuring dual feasibility, complementarity, primal feasibility and optimality. Three measures arise directly from the first order optimality conditions of the problem and the fourth is the objective function, so that convergence to a stationary point that is a minimizer is guaranteed. The primary assessment of the method has been done with a well-known collection of small problems.

*Key words: Nonlinear optimization, interior point, filter method
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1 Introduction

In this paper we consider a nonlinear constrained optimization problem in the following form:

$$\begin{array}{ll} \min_{x \in \mathbb{R}^n} & F(x) \\ \text{s.t.} & h(x) \geq 0 \end{array} \qquad (1)$$

where $h_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, m$ and $F : \mathbb{R}^n \to \mathbb{R}$ are nonlinear and twice continuously differentiable functions. Interior point methods based on a logarithmic barrier function have been widely used for nonlinear programming [9, 11, 12]. To allow convergence from poor starting points, barrier and augmented Lagrangian merit functions may be used [7]. Some line search frameworks use penalty merit functions to enforce progress toward the solution. As an alternative to merit functions, Fletcher and Leyffer [4] proposed a filter method as a tool to guarantee global convergence in algorithms for nonlinear optimization. This technique incorporates the concept of non-dominance to build a filter that is able to accept trial points if they improve either the objective function or the constraints violation, instead of a combination of those two measures defined by a merit function. The filter replaces the use of merit functions, so avoiding the update of penalty parameters that are associated with the penalization of the constraints in a merit function. The filter technique has already been adapted to interior point methods. In [13, 14, 15], a filter line search strategy incorporated in a barrier type method is used. The two components of each entry in the filter are the barrier objective function and the constraints violation. In [10], a two-dimensional filter is used in a primal-dual interior point method context. The two entries, measuring quasi-centrality and optimality, combine the three criteria of the first order optimality conditions. A three-dimensional filter based line search strategy has already been tested in [2, 3]. The three components of the filter measure feasibility, centrality and optimality and are present in the first order KKT conditions of the barrier problem associated with the problem (1). The optimality measure relies on the norm of the gradient of the Lagrangian function. Convergence to stationary points may been proved, although convergence to a local minimizer is not guaranteed.

In this paper we propose a four-dimensional filter line search method to incorporate into a primal-dual interior point framework. The three criteria of the first order optimality conditions are used separately to define three measures, and the objective function, $F(x)$, is the other so that convergence to a stationary point that is a minimizer is guaranteed.

The paper is organized as follows. Section 2 presents the interior point paradigm and Section 3 introduces the novel filter line search method that relies on four components and presents the acceptance conditions used to accept a point in the filter. The experimental results and the conclusions make Section 4.

2 The primal-dual interior point paradigm

In this interior point paradigm, problem (1) is reformulated as an equality constrained problem by using nonnegative slack variables w, as follows:

$$\begin{array}{rl} \min_{x\in \mathbb{R}^n, w\in \mathbb{R}^m} & F(x) \\ \text{s.t.} & h(x) - w = 0 \\ & w \geq 0, \end{array} \tag{2}$$

and the first order or Karush-Kuhn-Tucker (KKT) optimality conditions for a minimum of (2) are written as

$$\begin{array}{rl} \nabla_x \mathcal{L}(x, w, y, v) & = 0 \\ y - v & = 0 \\ Wy & = 0 \\ h(x) - w & = 0 \\ w \geq 0,\ v & \geq 0 \end{array} \tag{3}$$

where $y, v \in \mathbb{R}^m$ are the vector of Lagrange multipliers, $W = diag(w_i)$ is a diagonal matrix, and $\nabla_x \mathcal{L}$ is the gradient with respect to x of the Lagrangian function defined by

$$\mathcal{L}(x,w,y,v) = F(x) - y^T(h(x)-w) - v^T w.$$

The system (3) is equivalent to the system

$$\begin{array}{rl} \nabla F(x) - A(x)^T y & = 0 \\ Wy & = 0 \\ h(x) - w & = 0 \\ w \geq 0,\ y & \geq 0 \end{array} \tag{4}$$

where $A(x)$ is the Jacobian matrix of the constraint functions $h(x)$. If the second equation of conditions (4) is perturbed, we get the KKT perturbed system of equations

$$\begin{array}{rl} \nabla F(x) - A(x)^T y & = 0 \\ Wy - \mu e & = 0 \\ h(x) - w & = 0 \\ w \geq 0,\ y & \geq 0 \end{array} \tag{5}$$

where e is a vector of unit m elements and μ is a positive parameter called barrier parameter [9, 12]. This perturbed system is equivalent to the KKT conditions of the barrier problem associated with problem (2), in the sense that they have the same solution,

$$\begin{array}{rl} \min_{x \in \mathbb{R}^n, w \in \mathbb{R}^m} & \varphi_\mu(x,w) \\ \text{s.t.} & h(x) - w = 0 \end{array} \tag{6}$$

where $\varphi_\mu(x,w) \equiv F(x) - \mu \sum_{i=1}^{m} \log(w_i)$ is the logarithmic barrier function. Applying the Newton's method to solve (5), the following system, after symmetrization, arises

$$\begin{bmatrix} -H & 0 & A(x)^T \\ 0 & -W^{-1}Y & -I \\ A(x) & -I & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta w \\ \Delta y \end{bmatrix} = \begin{bmatrix} \nabla F(x) - A(x)^T y \\ -\mu W^{-1} e + y \\ w - h(x) \end{bmatrix} \tag{7}$$

where $Y = diag(y_i)$ is a diagonal matrix,

$$H = \nabla^2 F(x) - \sum_{i=1}^{m} y_i \nabla^2 h_i(x)$$

is the Hessian matrix of the Lagrangian function.

Since the second equation in (7) can be used to eliminate Δw without producing any off-diagonal fill-in in the remaining system, one obtains

$$\Delta w = WY^{-1}\left(\mu W^{-1} e - y - \Delta y\right), \tag{8}$$

and the resulting reduced KKT system

$$\begin{bmatrix} -H & A(x)^T \\ A(x) & WY^{-1} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} \nabla F(x) - A(c)^T y \\ w - h(x) + WY^{-1}(\mu W^{-1} e - y) \end{bmatrix} \tag{9}$$

to compute the search directions Δx, Δw, Δy. This interior point based method implements a line search procedure combined with a backtracking strategy to compute a step size α_k, at each iteration k, and define a new approximation by

$$\begin{aligned} x_{k+1} &= x_k + \alpha_k \Delta x_k \\ w_{k+1} &= w_k + \alpha_k \Delta w_k \\ y_{k+1} &= y_k + \alpha_k \Delta y_k \end{aligned}$$

where equal step sizes are used with primal and dual directions. The choice of the step size α_k is a very important issue in nonconvex optimization and in the interior point context aims:

1. to ensure the nonnegativity of the slack and dual variables;
2. to enforce progress towards feasibility, complementarity and optimality.

Here we propose a four-dimensional filter method combined with a backtracking strategy to define new approximations to the primal, slack and dual variables that give a sufficient reduction in one of the filter measures. The backtracking strategy defines a decreasing sequence of step sizes

$$\alpha_{k,l} \in (0, \alpha_k^{\max}], l = 0, 1, \ldots,$$

with $lim_l \alpha_{k,l} = 0$, until a set of acceptance conditions are satisfied. Here, the index l denotes the iteration counter for the inner loop. The parameter $\alpha_k^{\max}$ represents the longest step size that can be taken along the direction before violating the nonnegativity conditions $w_k \geq 0, y_k \geq 0$. If the initial approximations for the slack and dual variables satisfy $w_0 > 0, y_0 > 0$, the maximal step size $\alpha_k^{\max} \in (0, 1]$ is defined by

$$\alpha_k^{\max} = \min \left\{1, \varepsilon \min\{-w_k^i (\Delta w_k^i)^{-1}, -y_k^i (\Delta y_k^i)^{-1}\}\right\} \tag{10}$$

for all i such that $\Delta w_k^i < 0$ and $\Delta y_k^i < 0$, and $\varepsilon \in (0, 1)$ is a fixed parameter.

3 Four-dimensional filter line search method

In order to define the components of each entry in the filter and the corresponding acceptance conditions, the following notation is used:

$$\begin{aligned} &u = (x, w, y), && \Delta = (\Delta x, \Delta w, \Delta y), \\ &u^1 = (x, w), && \Delta^1 = (\Delta x, \Delta w), \\ &u^2 = (w, y), && \Delta^2 = (\Delta w, \Delta y), \\ &u^3 = (x, y), && \Delta^3 = (\Delta x, \Delta y). \end{aligned}$$

The optimality conditions (4) define a set of natural measures to assess the algorithm progress. Some combinations of these measures may be used to define the components of each entry in the filter, see for example [10]. We use the three conditions separately. Further, to be able to guarantee convergence to stationary points that are minimizers, we introduce F as the fourth measure in the filter [5]. Table 1 lists the four components for the herein proposed filter.

Table 1: Components of the four-dimensional filter

measure	
primal feasibility	$\theta_{pf}(u^1) \equiv \|h(x)-w\|_2$
complementarity	$\theta_c(u^2) \equiv \|Wy\|_2$
dual feasibility	$\theta_{df}(u^3) \equiv \left\|\nabla F(x) - A(x)^T y\right\|_2$
optimality	$F(x)$

3.1 The acceptance conditions

In this algorithm, the trial point $u_k(\alpha_{k,l}) = u_k + \alpha_{k,l}\Delta_k$ is acceptable by the filter, if it leads to sufficient progress in one of the four measures compared to the current iterate,

$$\begin{array}{l} \theta_{pf}(u_k^1(\alpha_{k,l})) \leq (1-\gamma_1)\,\theta_{pf}(u_k^1) \ \text{ or } \ \theta_c(u_k^2(\alpha_{k,l})) \leq (1-\gamma_2)\,\theta_c(u_k^2) \\ \text{or } \ \theta_{df}(u_k^3(\alpha_{k,l})) \leq (1-\gamma_3)\,\theta_{df}(u_k^3) \ \text{ or } \ F(x_k(\alpha_{k,l})) \leq F(x_k) - \gamma_4\theta_{pf}(u_k^1) \end{array} \tag{11}$$

where $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in (0,1)$ are fixed constants.

However, to prevent convergence to a point that is nonoptimal, and whenever for the trial step size $\alpha_{k,l}$, the following switching conditions

$$\begin{array}{l} m_k(\alpha_{k,l}) < 0 \text{ and } \left[-m_k(\alpha_{k,l})\right]^{s_o}\left[\alpha_{k,l}\right]^{1-s_o} > \delta\left[\theta_{pf}(u_k^1)\right]^{s_1} \\ \text{and } \left[-m_k(\alpha_{k,l})\right]^{s_o}\left[\alpha_{k,l}\right]^{1-s_o} > \delta\left[\theta_c(u_k^2)\right]^{s_2} \text{ and } \left[-m_k(\alpha_{k,l})\right]^{s_o}\left[\alpha_{k,l}\right]^{1-s_o} > \delta\left[\theta_{df}(u_k^3)\right]^{s_3} \end{array} \tag{12}$$

hold, with fixed constants $\delta > 0$, $s_1, s_2, s_3 > 1$, $s_o \geq 1$, where

$$m_k(\alpha) = \alpha \nabla F(x_k)^T \Delta x_k,$$

then the trial point must satisfy the Armijo condition with respect to the optimality measure

$$F(x_k(\alpha_{k,l})) \leq F(x_k) + \eta_1\, m_k(\alpha_{k,l}), \tag{13}$$

instead of (11) to be acceptable. Here, $\eta_1 \in (0, 0.5)$ is a constant.

According to previous publications on filter methods (for example [13]), a trial step size $\alpha_{k,l}$ is called a F-step if (13) holds. Similarly, if a F-step is accepted as the final step size α_k in iteration k, then k is referred to as a F-type iteration.

3.2 The four-dimensional filter

The filter is a set that contains combinations of the four measures θ_{pf}, θ_c, θ_{df} and F that are prohibited for a successful trial point and is initialized to

$$\overline{F}_0 \subseteq \left\{(\theta_{pf}, \theta_c, \theta_{df}, F) \in \mathrm{IR}^4 : \theta_{pf} \geq \theta_{pf}^{\max}, \theta_c \geq \theta_c^{\max}, \theta_{df} \geq \theta_{df}^{\max}, F \geq F^{\max}\right\}, \tag{14}$$

for the nonnegative constants $\theta_{pf}^{\max}$, $\theta_c^{\max}$, $\theta_{df}^{\max}$ and $F^{\max}$. The filter is updated according to

$$\begin{array}{rl} \overline{F}_{k+1} & = \overline{F}_k \cup \left\{(\theta_{pf}, \theta_c, \theta_{pf}, F) \in \mathrm{IR}^4 : \theta_{pf} \geq (1-\gamma_1)\,\theta_{pf}(u_k^1) \text{ and } \theta_c \geq (1-\gamma_2)\,\theta_c(u_k^2) \right. \\ & \left. \text{and } \theta_{df} \geq (1-\gamma_3)\,\theta_{df}(u_k^3) \text{ and } F \geq F(x_k) - \gamma_4\theta_{pf}(u_k^1)\right\}, \end{array} \tag{15}$$

whenever the accepted step size satisfies (11). However, when for the accepted step size the conditions (12) and (13) hold, the filter remains unchanged.

Finally, when the backtracking line search cannot find a trial step size $\alpha_{k,l}$ that satisfies the above criteria, we define a minimum desired step size $\alpha_k^{\min}$, using linear models of the involved functions,

$$\alpha_k^{\min} = \xi \begin{cases} \min\{\gamma_1, \pi_1, \pi_2, \pi_3, \pi_4\}, & \text{if } m_k(\alpha_{k,l}) < 0 \\ & \text{and } (\theta_{pf}(u_k^1) \leq \theta_{pf}^{\min} \text{ or } \theta_c(u_k^2) \leq \theta_c^{\min} \text{ or } \theta_{df}(u_k^3) \leq \theta_{df}^{\min}) \\ \min\{\gamma_1, \pi_1\}, & \text{if } m_k(\alpha_{k,l}) < 0 \\ & \text{and } (\theta_{pf}(u_k^1) > \theta_{pf}^{\min} \text{ and } \theta_c(u_k^2) > \theta_c^{\min} \text{ and } \theta_{df}(u_k^3) > \theta_{df}^{\min}) \\ \gamma_1, & \text{otherwise} \end{cases} \tag{16}$$

where

$$\pi_1 = \frac{\gamma_4 \theta_{pf}(u_k^1)}{-m_k(\alpha_{k,l})}, \quad \pi_2 = \frac{\delta\left[\theta_{pf}(u_k^1)\right]^{s_1}}{\left[-m_k(\alpha_{k,l})\right]^{s_o}}, \quad \pi_3 = \frac{\delta\left[\theta_c(u_k^2)\right]^{s_2}}{\left[-m_k(\alpha_{k,l})\right]^{s_o}}, \quad \pi_4 = \frac{\delta\left[\theta_{df}(u_k^3)\right]^{s_3}}{\left[-m_k(\alpha_{k,l})\right]^{s_o}}$$

for positive constants $\theta_{pf}^{\min}, \theta_c^{\min}, \theta_{df}^{\min}$ and a safety factor $\xi \in (0,1]$.

Like in [15] and whenever the backtracking line search finds a trial step size $\alpha_{k,l} < \alpha_k^{\min}$, the algorithm reverts to a restoration phase. Here, the algorithm tries to find a new iterate u_{k+1} that is acceptable to the current filter, *i.e.*, (11) holds, by reducing either the primal feasibility measure or the complementarity within an iterative process.

3.3 Restoration phase

The task of the restoration phase is to compute a new iterate acceptable to the filter by decreasing either the primal feasibility or the complementarity, whenever the backtracking line search procedure cannot make sufficient progress and the step size becomes too small. Thus, the restoration algorithm works with the new functions

$$\theta_{pf}^2(u^1) = \frac{1}{2}\|h(x) - w\|_2^2 \text{ and } \theta_c^2(u^2) = \frac{1}{2}\|Wy\|_2^2$$

and the steps Δ^1 and Δ^2 that are descent directions for $\theta_{pf}^2(u^1)$ and $\theta_c^2(u^2)$, respectively.

Using a backtracking strategy, the algorithm selects, at each iteration k, a step size $\alpha_k \in (0, \alpha_k^{\max}]$ to define a new trial point $u_k(\alpha_k) = u_k + \alpha_k \Delta_k$ that satisfies either

$$\theta_{pf}^2(u_k^1(\alpha_k)) \leq \theta_{pf}^2(u_k^1) + \alpha_k \eta_2 \nabla \theta_{pf}^2(u_k^1)^T \Delta_k^1$$

or

$$\theta_c^2(u_k^2(\alpha_k)) \leq \theta_c^2(u_k^2) + \alpha_k \eta_3 \nabla \theta_c^2(u_k^2)^T \Delta_k^2$$

for constants η_2 and η_3 in the set $(0, 0.5)$.

3.4 Setting the barrier parameter

To guarantee a positive decreasing sequence of μ values, the barrier parameter is updated by a formula that couples the theoretical requirement defined on the first order KKT conditions (5) with a simple heuristic. Thus, μ is updated by

$$\mu_{k+1} = \max\left\{\epsilon, \min\left\{\kappa_\mu \mu_k, \delta_\mu \frac{w_{k+1}^T y_{k+1}}{m}\right\}\right\} \tag{17}$$

where the constants $\kappa_\mu, \delta_\mu \in (0,1)$ and the tolerance ϵ is used to prevent μ from becoming too small so avoiding numerical difficulties at the end of the iterative process.

3.5 Termination criteria

The termination criteria consider dual and primal feasibility and complementarity measures

$$\max\left\{\frac{\|\nabla F(x) - A(x)^T y\|_\infty}{s}, \|h(x) - w\|_\infty, \frac{\|Wy\|_\infty}{s}\right\} \leq \epsilon_{tol}, \tag{18}$$

where

$$s = \max\left\{1, 0.01\frac{\|y\|_1}{m}\right\}$$

and $\epsilon_{tol} > 0$ is the error tolerance.

4 Experimental results and conclusions

To test this interior point framework with the herein proposed four-dimensional filter line search technique we selected 109 constrained problems from the Hock and Schittkowski (HS) collection [8]. This preliminary selection aims to consider small and simple to code problems. The tests were done in double precision arithmetic with a Pentium 4. The algorithm is coded in the C programming language and includes an interface to AMPL to read the problems that are coded in the AMPL modeling language [6].

Our algorithm is a quasi-Newton based method in the sense that a symmetric positive definite quasi-Newton BFGS approximation, B_k, is used to approximate the Hessian of the Lagrangian H, at each iteration k. In the first iteration, we may set $B_0 = I$ or $B_0 =$ positive definite modification of $\nabla^2 F(x_0)$, depending on the characteristics of the problem to be solved.

4.1 Initial approximations

The algorithm implements two alternatives to initialize the primal and the dual variables. One uses the usual published initial values, x_0, as mentioned in [8], and sets all the dual variables to one. The other uses the published x_0 to define the initial dual variables, y_0, and new primal variables, $\tilde{x}_0$, by solving the simplified reduced system:

$$\begin{bmatrix} -(B_0 + I) & A^T(x_0) \\ A(x_0) & I \end{bmatrix} \begin{bmatrix} \tilde{x}_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} \nabla F(x_0) \\ 0 \end{bmatrix}.$$

Further, if $\|y_0\|_\infty > 10^3$ then y_0 is component by component set to one. However, if $\|\tilde{x}_0\|_\infty > 10^3\|x_0\|_\infty$ then $\tilde{x}_0 = x_0$.

The nonnegativity of the initial slack variables are ensured by computing $w_0 = \max\{|h(x_0)|, \epsilon_w\}$, for the previously defined x_0, and a fixed positive constant ϵ_w.

4.2 Setting user defined parameters

The chosen values for some of the constants are similar to the ones proposed in [15]: $\theta_{pf}^{\max} = 10^4 \max\left\{1, \theta_{pf}(u_0^1)\right\}$, $\theta_{pf}^{\min} = 10^{-4} \max\left\{1, \theta_{pf}(u_0^1)\right\}$, $\theta_{c}^{\max} = 10^4 \max\left\{1, \theta_{c}(u_0^2)\right\}$, $\theta_{c}^{\min} = 10^{-4} \max\left\{1, \theta_{c}(u_0^2)\right\}$, $\theta_{df}^{\max} = 10^4 \max\left\{1, \theta_{df}(u_0^3)\right\}$, $\theta_{df}^{\min} = 10^{-4} \max\left\{1, \theta_{df}(u_0^3)\right\}$, $F^{\max} = 10^4 \max\{1, F(x_0)\}$, $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = 10^{-5}$, $\delta = 1$, $s_1 = s_2 = s_3 = 1.1$, $s_o = 2.3$, $\eta_1 = \eta_2 = \eta_3 = 10^{-4}$, $\xi = 0.05$.

The other parameters are set as follows: $\varepsilon = 0.95$, $\delta_\mu = \kappa_\mu = 0.1$, $\epsilon = 10^{-9}$, $\epsilon_w = 0.01$ and $\epsilon_{tol} = 10^{-6}$.

4.3 Comparative results

Table 2 summarizes the results obtained with the herein proposed four-dimensional filter line search interior point method. The table reports the number of iterations required to obtain a solution according to the termination criteria in (18), Nit, and the objective function value, $F(x^*)$. Except in 11 problems, the number of function evaluations was $Nit+1$. Results inside parentheses were obtained with the parameter ϵ_{tol} set to $= 10^{-4}$. In all problems, our algorithm converges to the solution within a reasonable number of iterations.

For a comparative purpose we compare our results with the IPOPT, a filter line search barrier based method [13, 14, 15]. The results obtained by IPOPT are reported in the file "Ipopt-table.pdf" under
`http://www.research.ibm.com/people/a/andreasw/papers/Ipopt-table.pdf`.

We noticed differences, some are rather small, in the objective function value in 32 problems. They are listed in Table 3. For the remaining problems used in this study, the herein proposed filter line search interior point method converges to the solutions reported in "Ipopt-table.pdf". The table reveals that we were able to get better solutions in eight problems. They are *emphasized* in the table. We may then conclude that the four-dimensional filter line search interior point based method is effective in reaching the solution of small nonlinear constrained optimization problems. In the future, different combinations of the criteria involved in the first order optimality conditions (4) will be analyzed, tested and compared with the present proposal.

References

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Table 2: Number of iterations and function values of our study

Prob	Nit	$F(x^*)$	Prob	Nit	$F(x^*)$	Prob	Nit	$F(x^*)$
hs001	36	1.77130e-14	hs038	54	2.24116e-15	hs077	(39)	2.41505e-01
hs002	10	5.04261e-02	hs039	13	-1.00000e+00	hs078	12	-2.91970e+00
hs003	4	9.91001e-07	hs040	9	-2.50000e-01	hs079	10	7.87768e-02
hs004	6	2.66667e+00	hs041	15	1.92593e+00	hs080	11	5.39498e-02
hs005	8	-1.91322e+00	hs042	9	1.38579e+01	hs081	14	5.39498e-02
hs006	10	1.01337e-15	hs043	18	-4.40000e+01	hs083	16	-3.06655e+04
hs007	9	-1.73205e+00	hs044	24	-1.50000e+01	hs084	(21)	-5.28034e+06
hs008	10	-1.00000e+00	hs045	10	1.00000e+00	hs086	12	-3.23487e+01
hs009	15	-5.00000e-01	hs046	(15)	1.50567e-08	hs087	43	8.82760e+03
hs010	11	-1.00000e+00	hs047	24	1.98162e-09	hs088	35	1.36266e+00
hs011	10	-8.49846e+00	hs048	9	5.75184e-15	hs089	45	1.36266e+00
hs012	17	-3.00000e+01	hs049	(18)	4.98460e-07	hs090	39	1.36266e+00
hs014	9	1.39346e+00	hs050	9	6.20897e-15	hs091	45	1.36266e+00
hs015	12	3.06500e+02	hs051	9	2.88300e-15	hs092	40	1.36266e+00
hs016	11	2.50000e-01	hs052	11	5.32665e+00	hs093	19	1.35076e+02
hs017	10	1.00000e+00	hs053	12	4.09302e+00	hs095	14	1.56195e-02
hs018	12	5.00000e+00	hs054	13	1.92857e-01	hs096	13	1.56199e-02
hs019	35	-6.96181e+03	hs055	14	6.66667e+00	hs097	16	3.13581e+00
hs020	11	3.81987e+01	hs056	(9)	-8.88237e-10	hs098	17	3.13581e+00
hs021	9	-1.00000e+02	hs057	9	3.06476e-02	hs099	28	-8.31080e+08
hs022	8	1.00000e+00	hs059	12	-6.74950e+00	hs100	37	6.80630e+02
hs023	11	2.00000e+00	hs060	13	3.25682e-02	hs101	45	1.80976e+03
hs024	11	-1.00000e+00	hs061	16	-1.43646e+02	hs102	36	9.11880e+02
hs025	29	1.27017e-16	hs062	26	-2.62725e+04	hs103	36	5.43668e+02
hs026	(21)	1.83530e-07	hs063	13	9.61715e+02	hs104	15	3.95116e+00
hs027	21	4.00000e-02	hs064	51	6.29984e+03	hs105	48	1.13630e+03
hs028	7	1.38756e-13	hs065	9	9.53529e-01	hs106	60	7.04925e+03
hs029	14	-2.26274e+01	hs066	10	5.18164e-01	hs107	83	5.05501e+03
hs030	8	1.00000e+00	hs067	23	-1.16203e+03	hs108	25	-8.66025e-01
hs031	11	6.00000e+00	hs070	30	8.92318e-03	hs110	7	-4.57785e+01
hs032	14	1.00000e+00	hs071	13	1.70140e+01	hs112	60	-4.77611e+01
hs033	12	-4.58579e+00	hs072	20	7.27679e+02	hs113	17	2.43062e+01
hs034	12	-8.34032e-01	hs073	11	2.98944e+01	hs114	24	-1.76880e+03
hs035	8	1.11111e-01	hs074	18	5.12650e+03	hs116	89	9.75875e+01
hs036	14	-3.30000e+03	hs075	18	5.17441e+03	hs117	21	3.23487e+01
hs037	12	-3.45600e+03	hs076	8	-4.68182e+00	hs118	17	6.64820e+02
						hs119	23	2.44900e+02

Table 3: Function values obtained by IPOPT

Prob	$F(x^*)$	Prob	$F(x^*)$	Prob	$F(x^*)$
hs001	5.82781e-16	hs048	7.88861e-31	hs090	1.36265e+00
hs002	*4.94123e+00*	hs049	1.06000e-11	hs091	1.36265e+00
hs003	-7.49410e-09	hs050	0.00000e+00	hs092	1.36265e+00
hs016	2.31447e+01	hs051	4.93038e-32	hs095	1.56177e-02
hs025	*1.03460e-15*	hs054	-9.08075e-01	hs096	1.56177e-02
hs026	1.29138e-16	hs055	*6.77451e+00*	hs097	*4.07124e+00*
hs028	3.08149e-31	hs056	-3.45600e+00	hs098	*4.07124e+00*
hs038	3.34111e-19	hs059	*-7.80279e+00*	hs105	1.04461e+03
hs044	*-1.30000e+01*	hs070	7.49846e-03	hs108	*-6.74981e-01*
hs046	8.55335e-16	hs088	1.36265e+00	hs110	-9.96e+39
hs047	6.57516e-14	hs089	1.36265e+00		

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