

A Unilateral Multiphase Problem with Neumann Type Boundary Condition

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1 Introduction

In this paper we consider an open bounded subset Ω of \mathbb{R}^d , with smooth boundary $\Gamma = \partial\Omega$. Denoting by \mathbf{u} a vectorial function (u_1, \dots, u_N) and, given a normed vectorial space E , by \mathbf{E} the space E^N , $N \geq 2$, we consider the variational inequality

$$\left\{ \begin{array}{l} \mathbf{u}(t) \in \mathbb{K} \quad \text{for a.e. } t \in [0, T], \quad \mathbf{u}(0) = \mathbf{u}_0 : \\ \int_{\Omega} \partial_t \mathbf{u}(t) \cdot (\mathbf{v} - \mathbf{u}(t)) + \int_{\Omega} a(\mathbf{u}(t), \mathbf{v} - \mathbf{u}(t)) + \int_{\Omega} c \mathbf{u}(t) \cdot (\mathbf{v} - \mathbf{u}(t)) + \int_{\Gamma} b \mathbf{u}(t) \cdot (\mathbf{v} - \mathbf{u}(t)) \\ \geq \int_{\Omega} \mathbf{f}(t) \cdot (\mathbf{v} - \mathbf{u}(t)) + \int_{\Gamma} \mathbf{g}(t) \cdot (\mathbf{v} - \mathbf{u}(t)), \quad \forall \mathbf{v} \in \mathbb{K}, \quad \text{for a.e. } t \in [0, T], \end{array} \right. \quad (1)$$

where \mathbb{K} denotes the convex set associated with multiphase problems:

$$\mathbb{K} = \{ \mathbf{v} \in \mathbf{H}^1(\Omega) : \sum_{j=1}^N v_j \leq 1, \quad v_i \geq 0, \quad i = 1, \dots, N, \quad \text{in } \Omega \}, \quad (2)$$

$a(\mathbf{u}, \mathbf{v}) = a_{ij} u_{x_i} v_{x_j}$, with the summation convention for $i, j = 1, \dots, d$ and $\mathbf{u} \cdot \mathbf{v} = u_i v_i$ denotes the inner product.

Denoting $Q = \Omega \times (0, T)$ and $\Sigma = \Gamma \times (0, T)$, we assume time independent coefficients, for simplicity,

$$\left\{ \begin{array}{l} a_{ij} \in L^\infty(\Omega), \quad \exists \nu > 0 \quad \forall \xi \in \mathbb{R}^d \quad a_{ij} \xi_i \xi_j \geq \nu |\xi|^2, \\ c \in L^\infty(\Omega), \quad b \in L^\infty(\Gamma), \quad c \geq c_0 \geq 0, \quad b \geq b_0 \geq 0, \end{array} \right. \quad (3)$$

$$\mathbf{f} = (f_1, \dots, f_N) \in \mathbf{L}^2(Q), \quad \mathbf{g} = (g_1, \dots, g_N) \in H^1(0, T; \mathbf{L}^2(\Gamma)), \quad \mathbf{u}_0 \in \mathbb{K} \subset \mathbf{H}^1(\Omega). \quad (4)$$

In this work we extend some of the results presented in [7], where the operator considered was the Laplace operator, the reaction term \mathbf{f} might depend on \mathbf{u} and the boundary condition was homogeneous $\mathbf{g} = 0$. In what follows, we restrict the proofs to the new results which are not a simple adaptation of the results proved in [7]. We refer to this work for further references and motivation.

In Section 2 we approximate the variational inequality by penalization, proving the existence of a unique solution to the variational inequality (1). We deduce a kind of Lewy-Stampacchia inequalities and we estimate the order of convergence of the approximating solutions to the solution of the variational inequality.

Section 4 is dedicated to the stabilization of the evolutive solution and respective coincidence sets to the solution and coincidence sets of the stationary coercive variational inequality. The main tool of this section is the equivalence of the variational inequality with a system of nonlinear reaction-diffusion equations coupled through the characteristic functions of the coincidence sets, as recalled from [7] in Section 3, where we extend the continuous dependence result in the new framework.

2 Approximation of the variational solution

We prove the existence of solution of the variational inequality (1), by first considering a family of approximating systems of equations. Note that a different approach to the existence of the solution can be found in [3], [1] or [2], for instance.

We introduce A and B , the differential and boundary operators

$$Av = \partial_t v - (a_{ij}v_{x_i})_{x_j} + cv, \quad \text{in } Q, \quad (5)$$

$$Bv = a_{ij}v_{x_i}n_j + bv, \quad \text{on } \Sigma, \quad (6)$$

where $\mathbf{n} = (n_1, \dots, n_d)$ is the unit outward normal vector to Γ .

We denote by $A\mathbf{v} = (Av_1, \dots, Av_N)$, $\partial_t \mathbf{v} = (\partial_t v_1, \dots, \partial_t v_N)$ and, for each $0 < \varepsilon < 1$, we define $\theta_\varepsilon : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\theta_\varepsilon(s) = \begin{cases} 0 & \text{if } s \geq 0 \\ s/\varepsilon & \text{if } -\varepsilon < s < 0 \\ -1 & \text{if } s \leq -\varepsilon. \end{cases} \quad (7)$$

Given $\mathbf{h} = (h_1, \dots, h_N)$, the penalization operator $\Theta_\varepsilon^{\mathbf{h}}$ is defined by

$$\begin{aligned} \Theta_\varepsilon^{\mathbf{h}} \mathbf{u} \cdot \mathbf{v} &= \sum_{i=1}^N \left[h_i^- \theta_\varepsilon(u_i) - \sum_{\substack{1 \leq i_1 < \dots < i_k \leq N \\ i \in \{i_1, \dots, i_k\}}} \frac{1}{k} (h_{i_1} + \dots + h_{i_k})^+ \theta_\varepsilon(1 - u_{i_1 \dots i_k}) \right] v_i \\ &= \sum_{i=1}^N h_i^- \theta_\varepsilon(u_i) v_i - \sum_{1 \leq i_1 < \dots < i_k \leq N} \frac{1}{k} h_{i_1 \dots i_k}^+ \theta_\varepsilon(1 - u_{i_1 \dots i_k}) v_{i_1 \dots i_k}, \end{aligned} \quad (8)$$

where $\sum_{\substack{1 \leq i_1 < \dots < i_k \leq N \\ i \in \{i_1, \dots, i_k\}}}$ denotes the summation over all the subsets $\{i_1, \dots, i_k\}$ of $\{1, \dots, N\}$ to which i belongs, in particular, k varies from 1 to N , and

$$\forall \mathbf{v} = (v_1, \dots, v_N) \quad \forall \{i_1, \dots, i_k\} \subseteq \{1, \dots, N\} \quad v_{i_1 \dots i_k} = v_{i_1} + \dots + v_{i_k}. \quad (9)$$

The approximating problems are given by the following weakly coupled parabolic system with Neumann boundary condition

$$\begin{cases} A\mathbf{u}_\varepsilon + \Theta_\varepsilon^{\mathbf{f}} \mathbf{u}_\varepsilon = \mathbf{f} & \text{in } Q, \\ B\mathbf{u}_\varepsilon + \Theta_\varepsilon^{\mathbf{g}} \mathbf{u}_\varepsilon = \mathbf{g} & \text{on } \Sigma, \\ \mathbf{u}_\varepsilon(0) = \mathbf{u}(0) & \text{in } \Omega, \end{cases}$$

or, equivalently,

$$\begin{aligned} \int_{\Omega} \partial_t \mathbf{u}_\varepsilon(t) \cdot \mathbf{v} + \int_{\Omega} a(\mathbf{u}_\varepsilon(t), \mathbf{v}) + \int_{\Omega} c \mathbf{u}_\varepsilon(t) \cdot \mathbf{v} + \int_{\Gamma} b \mathbf{u}_\varepsilon(t) \cdot \mathbf{v} \\ + \int_{\Omega} \Theta_\varepsilon^{\mathbf{f}} \mathbf{u}_\varepsilon(t) \cdot \mathbf{v} + \int_{\Gamma} \Theta_\varepsilon^{\mathbf{g}} \mathbf{u}_\varepsilon(t) \cdot \mathbf{v} = \int_{\Omega} \mathbf{f}(t) \cdot \mathbf{v} + \int_{\Gamma} \mathbf{g}(t) \cdot \mathbf{v}, \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega). \end{aligned} \quad (10)$$

Proposition 2.1. *Assuming (3), (4) and $\mathbf{u}_0 \in \mathbb{K}$, the problem (10) has a unique solution $\mathbf{u}_\varepsilon \in H^1(0, T; \mathbf{L}^2(\Omega)) \cap L^\infty(0, T; \mathbf{H}^1(\Omega))$.*

Besides that

$$-\varepsilon \leq u_i^\varepsilon, \quad i = 1, \dots, N, \quad \sum_{i=1}^N u_i^\varepsilon \leq 1 + \varepsilon, \quad (11)$$

and

$$f_i - \sum_{\substack{1 \leq i_1 < \dots < i_k \leq N \\ i \in \{i_1, \dots, i_k\}}} \frac{1}{k} (f_{i_1} + \dots + f_{i_k})^+ \leq Au_i^\varepsilon \leq f_i^+ \quad \text{a.e. in } Q, \quad (12)$$

$$g_i - \sum_{\substack{1 \leq i_1 < \dots < i_k \leq N \\ i \in \{i_1, \dots, i_k\}}} \frac{1}{k} (g_{i_1} + \dots + g_{i_k})^+ \leq Bu_i^\varepsilon \leq g_i^+ \quad \text{a.e. on } \Sigma. \quad (13)$$

Proof. Since the operator $\Theta_\varepsilon^{\mathbf{h}}$ is monotone, for any \mathbf{h} , the existence and uniqueness of solution of (10) is immediate by applying the theory of monotone operators ([3], [8]).

The proof of (11) can be adjusted following the steps of the proof in [7].

To prove (12), choosing an arbitrary $\varphi \in \mathcal{D}(Q)$ such that $\varphi \geq 0$ and $v_i = \varphi$, $v_j = 0$ for $j \neq i$, we prove that

$$f_i - \sum_{\substack{1 \leq i_1 < \dots < i_k \leq N \\ i \in \{i_1, \dots, i_k\}}} \frac{1}{k} (f_{i_1} + \dots + f_{i_k})^+ \leq Au_i^\varepsilon = f_i - \Theta_\varepsilon(\mathbf{u}_\varepsilon) \leq f_i + f_i^- = f_i^+ \quad \text{a.e. in } Q.$$

Choosing now $\varphi \in \mathcal{D}(\overline{Q})$ such that $\varphi \geq 0$ and $v_i = \varphi$, $v_j = 0$ for $j \neq i$, we obtain (13).

Theorem 2.2. *Assuming (3), (4) and $\mathbf{u}_0 \in \mathbb{K}$, the variational inequality (1) has a unique solution \mathbf{u} such that*

$$\mathbf{u} \in H^1(0, T; \mathbf{L}^2(\Omega)) \cap L^\infty(0, T; \mathbf{H}^1(\Omega)) \quad (14)$$

and

$$A\mathbf{u} \in \mathbf{L}^2(Q), \quad B\mathbf{u} \in \mathbf{L}^2(\Sigma). \quad (15)$$

Proof. If \mathbf{u}_ε is the solution of the problem (10), using $\partial_t \mathbf{u}_\varepsilon$ as test function, integrating between 0 and t and calling $Q_t = \Omega \times (0, t)$ and $\Sigma_t = \Gamma \times (0, t)$, we obtain

$$\begin{aligned} & \int_{Q_t} |\partial_t \mathbf{u}_\varepsilon|^2 + \frac{1}{2} \int_\Omega a(\mathbf{u}_\varepsilon(t), \mathbf{u}_\varepsilon(t)) + \frac{1}{2} \int_\Omega c|\mathbf{u}_\varepsilon(t)|^2 + \frac{1}{2} \int_\Gamma b|\mathbf{u}_\varepsilon(t)|^2 \\ &= \int_{Q_t} (\mathbf{f} - \Theta_\varepsilon^{\mathbf{f}} \mathbf{u}_\varepsilon) \partial_t \mathbf{u}_\varepsilon + \int_{\Sigma_t} (\mathbf{g} - \Theta_\varepsilon^{\mathbf{g}} \mathbf{u}_\varepsilon) \partial_t \mathbf{u}_\varepsilon + \frac{1}{2} \int_\Omega a(\mathbf{u}_0, \mathbf{u}_0) + \frac{1}{2} \int_\Omega c|\mathbf{u}_0|^2 + \frac{1}{2} \int_\Gamma b|\mathbf{u}_0|^2. \end{aligned}$$

Since, if $\tilde{\theta}^\varepsilon(v_i) = \int_0^{v_i} \theta^\varepsilon(s) ds$ denotes the primitive of (7), and $(\tilde{\boldsymbol{\theta}}^\varepsilon(\mathbf{v}))_i = \tilde{\theta}^\varepsilon(v_i)$, then $|\tilde{\boldsymbol{\theta}}^\varepsilon(\mathbf{v})| \leq |\mathbf{v}|$ and we have

$$\begin{aligned} \int_{\Sigma_t} (\mathbf{g} - \Theta_\varepsilon^{\mathbf{g}} \mathbf{u}_\varepsilon) \cdot \partial_t \mathbf{u}_\varepsilon &= \int_\Gamma \mathbf{g}(t) \cdot \mathbf{u}_\varepsilon(t) - \int_\Gamma \mathbf{g}(0) \cdot \mathbf{u}_0 - \int_{\Sigma_t} \partial_t \mathbf{g} \cdot \mathbf{u}_\varepsilon + \int_{\Sigma_t} \partial_t \mathbf{g}^- \cdot \tilde{\boldsymbol{\theta}}^\varepsilon(\mathbf{u}_\varepsilon) \\ &\quad - \int_\Gamma \mathbf{g}^-(t) \cdot \tilde{\boldsymbol{\theta}}^\varepsilon(\mathbf{u}_\varepsilon(t)) + \sum_{1 \leq i_1 < \dots < i_k \leq N} \int_{\Sigma_t} \frac{1}{k} \partial_t g_{i_1 \dots i_k}^+ \tilde{\theta}^\varepsilon(1 - u_{i_1 \dots i_k}^\varepsilon) \\ &\quad - \sum_{1 \leq i_1 < \dots < i_k \leq N} \int_\Gamma \frac{1}{k} g_{i_1 \dots i_k}^+(t) \tilde{\theta}^\varepsilon(1 - u_{i_1 \dots i_k}^\varepsilon(t)), \end{aligned}$$

then the sequence

$$\mathbf{u}_\varepsilon \text{ is bounded in } H^1(0, T; \mathbf{L}^2(\Omega)) \cap L^\infty(0, T; \mathbf{H}^1(\Omega)) \quad (16)$$

and, consequently, there exists \mathbf{u} such that

$$\begin{aligned} \mathbf{u}_\varepsilon &\xrightarrow[\varepsilon]{} \mathbf{u} && \text{in } \mathbf{L}^2(Q) \text{ strong,} \\ \mathbf{u}_\varepsilon &\xrightarrow[\varepsilon]{} \mathbf{u} && \text{in } L^\infty(0, T; \mathbf{H}^1(\Omega)) \text{ weak-*,} \\ \partial_t \mathbf{u}_\varepsilon &\xrightarrow[\varepsilon]{} \partial_t \mathbf{u} && \text{in } \mathbf{L}^2(Q) \text{ weak,} \\ A\mathbf{u}_\varepsilon &\xrightarrow[\varepsilon]{} A\mathbf{u} && \text{in } \mathbf{L}^2(Q) \text{ weak,} \quad B\mathbf{u}_\varepsilon \xrightarrow[\varepsilon]{} B\mathbf{u} \quad \text{in } \mathbf{L}^2(\Sigma) \text{ weak.} \end{aligned}$$

Using the monotonicity of $\Theta_\varepsilon^{\mathbf{f}}$ and $\Theta_\varepsilon^{\mathbf{g}}$, choosing $\mathbf{v} \in L^2(0, T; \mathbb{K})$ as test function in (10) and integrating in time, we have

$$\begin{aligned} \int_Q \partial_t \mathbf{u}_\varepsilon \cdot (\mathbf{v} - \mathbf{u}_\varepsilon) + \int_Q a(\mathbf{u}_\varepsilon, \mathbf{v} - \mathbf{u}_\varepsilon) + \int_Q c \mathbf{u}_\varepsilon \cdot (\mathbf{v} - \mathbf{u}_\varepsilon) + \int_\Sigma b \mathbf{u}_\varepsilon \cdot (\mathbf{v} - \mathbf{u}_\varepsilon) \\ \geq \int_Q \mathbf{f} \cdot (\mathbf{v} - \mathbf{u}_\varepsilon) + \int_\Sigma \mathbf{g} \cdot (\mathbf{v} - \mathbf{u}_\varepsilon). \end{aligned}$$

Using the fact that

$$\liminf_{\varepsilon \rightarrow 0} \int_Q (\partial_t \mathbf{u}_\varepsilon \cdot \mathbf{u}_\varepsilon + a(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon)) + \int_\Sigma b \mathbf{u}_\varepsilon \cdot \mathbf{u}_\varepsilon \geq \int_Q (\partial_t \mathbf{u} \cdot \mathbf{u} + a(\mathbf{u}, \mathbf{u})) + \int_\Sigma b \mathbf{u} \cdot \mathbf{u},$$

we obtain

$$\begin{aligned} \int_Q \partial_t \mathbf{u} \cdot (\mathbf{v} - \mathbf{u}) + \int_Q a(\mathbf{u}, \mathbf{v} - \mathbf{u}) + \int_Q c \mathbf{u} \cdot (\mathbf{v} - \mathbf{u}) + \int_\Sigma b \mathbf{u} \cdot (\mathbf{v} - \mathbf{u}) \\ \geq \int_Q \mathbf{f} \cdot (\mathbf{v} - \mathbf{u}) + \int_\Sigma \mathbf{g} \cdot (\mathbf{v} - \mathbf{u}), \quad \forall \mathbf{v} \in L^2(0, T; \mathbb{K}), \end{aligned}$$

and this inequality is equivalent to (1). The uniqueness of solution is immediate.

To obtain (15) it is enough to pass to the limit in ε in (12) and (13).

Theorem 2.3. *Assume (3) and (4). Let \mathbf{u}^ε and \mathbf{u} be, respectively, the unique solution of the approximating problem (10) and of the variational inequality (1). Then there exists a positive constant c , independent of ε , such that*

$$\|\mathbf{u}_\varepsilon - \mathbf{u}\|_{L^\infty(0, T; \mathbf{L}^2(\Omega)) \cap L^2(0, T; \mathbf{H}^1(\Omega))} \leq c\sqrt{\varepsilon}. \quad (17)$$

Proof. To prove this result choose $\mathbf{v}_\varepsilon = ((u_1^\varepsilon - \frac{\varepsilon}{N})^+, \dots, (u_N^\varepsilon - \frac{\varepsilon}{N})^+)$ as test function in (1) and \mathbf{u} as test function in (10) and integrate in time their difference (see [7], for details).

3 The reaction-diffusion system using the coincidence sets

Let \mathbf{u} be the solution of (1) and set

$$w_i = 1 - \sum_{j \neq i} u_j, \quad i = 1, \dots, N. \quad (18)$$

As in the homogeneous case, each component u_i satisfies a double obstacle problem

$$0 \leq u_i(x, t) \leq w_i(x, t) \quad \text{for a.e. } (x, t) \in Q, \quad i = 1, \dots, N. \quad (19)$$

For an arbitrary nonnegative and bounded function $\varphi = \varphi(x, t)$ defined for $(x, t) \in Q$, such that

$$\mathbb{K}_0^\varphi = \{v \in L^2(0, T; H^1(\Omega)) : 0 \leq v \leq \varphi \text{ in } Q\} \neq \emptyset, \quad (20)$$

and for given $f \in L^2(Q)$ and $g \in L^2(\Sigma)$, we consider the parabolic double obstacle scalar problem

$$u \in \mathbb{K}_0^\varphi : \quad \int_Q \partial_t u(v - u) + \int_Q a(u, v - u) + \int_Q cu(v - u) + \int_\Sigma bu(v - u) \\ \geq \int_Q f(v - u) + \int_\Sigma g(v - u) \quad \forall v \in \mathbb{K}_0^\varphi, \quad (21)$$

subject to a given compatible initial condition

$$u(0) = u_0 \quad \text{in } \Omega, \quad (22)$$

satisfying a , b and c the assumption (3).

By adapting the general theory of the obstacle problem to the parabolic double obstacle problem as in [7] where $\mathbf{g} = 0$, we may conclude that if $f^-, (A\varphi - f)^- \in L^2(\Omega)$ and $g^-, (B\varphi - g)^- \in L^2(\Gamma)$ then

$$f \wedge A\varphi \leq Au \leq f \vee 0, \quad \text{a.e. in } Q, \quad (23)$$

$$g \wedge B\varphi \leq Bu \leq g \vee 0, \quad \text{a.e. on } \Sigma \quad (24)$$

and, in addition,

$$Au = f + f^- \chi_{\{u=0\}} - (A\varphi - f)^- \chi_{\{u=\varphi\}} \quad \text{a.e. in } Q. \quad (25)$$

Applying the above result to the N coupled obstacle problems we have, exactly as in [7] and taking Proposition 2.1 into account.

Theorem 3.1. *Assuming (3) and (4), the solution \mathbf{u} of the variational inequality (1) satisfies the nonlinear parabolic system*

$$Au_i = f_i + f_i^- \chi_{\{u_i=0\}} - \sum_{\substack{1 \leq i_1 < \dots < i_k \leq N \\ i \in \{i_1, \dots, i_k\}}} \frac{1}{k} (f_{i_1} + \dots + f_{i_k})^+ \chi_{i_1 \dots i_k} \quad \text{a.e. in } Q, \quad (26)$$

where $\chi_{i_1 \dots i_k} = \chi_{I_{i_1 \dots i_k}}$, for $k = 1, \dots, N$, denotes each of the $2^N - 1$ characteristic functions of each

$$I_{i_1 \dots i_k} = \{(x, t) \in Q : u_{i_1 \dots i_k}(x, t) = 1, u_{i_j}(x, t) > 0 \text{ for all } j = 1, \dots, k\}. \quad (27)$$

Besides that,

$$f_i - \sum_{\substack{1 \leq i_1 < \dots < i_k \leq N \\ i \in \{i_1, \dots, i_k\}}} \frac{1}{k} (f_{i_1} + \dots + f_{i_k})^+ \leq Au_i \leq f_i^+ \quad \text{a.e. in } Q, \quad (28)$$

$$g_i - \sum_{\substack{1 \leq i_1 < \dots < i_k \leq N \\ i \in \{i_1, \dots, i_k\}}} \frac{1}{k} (g_{i_1} + \dots + g_{i_k})^+ \leq Bu_i \leq g_i^+ \quad \text{a.e. on } \Sigma. \quad (29)$$

Remark 3.2. *An interesting open question is to prove that, also for the Neumann condition of the solution \mathbf{u} , there exist some coefficients $\gamma^i, \gamma_{i_1 \dots i_k}$, involving the boundary data \mathbf{g} such that, similarly to (26),*

$$Bu_i = g_i + \gamma^i \chi_{\{u_i=0\}} - \sum_{\substack{1 \leq i_1 < \dots < i_k \leq N \\ i \in \{i_1, \dots, i_k\}}} \gamma_{i_1 \dots i_k} \hat{\chi}_{i_1 \dots i_k} \quad \text{a.e. in } \Sigma,$$

where

$$\hat{\chi}_{i_1 \dots i_k} = \{(x, t) \in \Sigma : u_{i_1}(x, t) = \dots = u_{i_k}(x, t)\}.$$

Notice that if all $g_i = 0$ on Σ , from (29) we obtain also $\gamma_i = 0$ and all $\gamma_{i_1 \dots i_k} = 0$. In particular, if the coefficients a_{ij} and b are Lipschitz continuous, $\mathbf{f} \in \mathbf{L}^p(Q)$ and $\mathbf{u}_0 \in \mathbb{K} \cap \mathbf{W}^{2-2/p, p}(\Omega)$, $p \geq \frac{2d+4}{d+4}$, the parabolic theory [4] gives the regularity $\mathbf{u} \in \mathbf{W}_p^{2,1}(Q) = L^p(0, T; \mathbf{W}^{2,p}(\Omega)) \cap W^{1,p}(0, T; \mathbf{L}^p(\Omega))$.

Corollary 3.3. *Let \mathbf{u} be the solution of the variational inequality (1), under the assumptions (3) and (4). Then, denoting by $|\mathcal{O}|$ the $(d+1)$ -Lebesgue measure of $\mathcal{O} \subseteq Q$, we have*

$$\left| \left\{ \sum_{j=1}^k f_{i_j} < 0 \right\} \cap \left\{ \sum_{j=1}^k u_{i_j} = 1, u_{i_j} > 0, j = 1, \dots, k \right\} \right| = 0 \quad (30)$$

for each partial coincidence subset $I_{i_1 \dots i_k}$, as well as

$$|\{f_i > 0\} \cap \{u_i = 0\}| = 0, \quad i = 1, \dots, N. \quad (31)$$

Suppose we have a sequence of data \mathbf{f}^ν , \mathbf{g}^ν and \mathbf{u}_0^ν satisfying (4) for each parameter ν , $0 < \nu < 1$, and such that

$$\mathbf{f}^\nu \xrightarrow[\nu]{} \mathbf{f} \text{ in } \mathbf{L}^2(Q), \quad \mathbf{g}^\nu \xrightarrow[\nu]{} \mathbf{g} \text{ in } \mathbf{L}^2(\Sigma), \quad \text{and} \quad \mathbf{u}_0^\nu \xrightarrow[\nu]{} \mathbf{u}_0 \text{ in } \mathbf{L}^2(\Omega). \quad (32)$$

Then it is easy to show that the corresponding variational solutions to (1) satisfy the continuous dependence result

$$\mathbf{u}^\nu \xrightarrow[\nu]{} \mathbf{u} \quad \text{in } C^0([0, T]; \mathbf{L}^2(\Omega)) \cap L^2(0, T; \mathbf{H}^1(\Omega))$$

and, arguing as in Theorem 4.4 of [7] we have an interesting criteria of local stability of the corresponding coincidence sets in terms of their characteristic functions

$$\chi_{i_1 \dots i_k}^\nu = \chi_{\{u_{i_1 \dots i_k}^\nu = 1, u_{i_j}^\nu > 0 \ \forall j=1, \dots, k\}}, \quad 1 \leq i_1 < \dots < i_k \leq N, \quad \text{for } k = 1, \dots, N.$$

Theorem 3.4. *Assume (32) and that in some subset of positive measure $\mathcal{O} \subseteq Q$, the following stability conditions holds for the data of the limit problem*

$$\sum_{j=1}^k f_{i_j} \neq 0, \quad \text{a.e. in } \mathcal{O}, \quad 1 \leq i_1 < \dots < i_k \leq N, \quad k = 1, \dots, N. \quad (33)$$

Then the corresponding characteristic functions are such that ($1 < p < \infty$)

$$\begin{aligned} \chi_{\{u_i^\nu=0\}} &\xrightarrow[\nu]{} \chi_{\{u_i=0\}} && \text{in } L^p(\mathcal{O}), \quad \forall i = 1, \dots, N, \\ \chi_{i_1 \dots i_k}^\nu &\xrightarrow[\nu]{} \chi_{i_1 \dots i_k} && \text{in } L^p(\mathcal{O}), \quad \forall i_1, \dots, i_k. \end{aligned}$$

4 The stationary limit problem and stabilization as $t \rightarrow \infty$

Consider now the following stationary problem

$$\left\{ \begin{array}{l} \mathbf{u}^\infty \in \mathbb{K} \\ \int_{\Omega} a(\mathbf{u}^\infty, \mathbf{v} - \mathbf{u}^\infty) + \int_{\Omega} c \mathbf{u}^\infty \cdot (\mathbf{v} - \mathbf{u}^\infty) + \int_{\Gamma} b \mathbf{u}^\infty \cdot (\mathbf{v} - \mathbf{u}^\infty) \\ \geq \int_{\Omega} \mathbf{f}_\infty \cdot (\mathbf{v} - \mathbf{u}^\infty) + \int_{\Gamma} \mathbf{g}_\infty \cdot (\mathbf{v} - \mathbf{u}^\infty), \quad \forall \mathbf{v} \in \mathbb{K}, \end{array} \right. \quad (34)$$

where a , b and c are defined in (3) and we assume

$$\mathbf{f}^\infty \in \mathbf{L}^2(\Omega), \quad \mathbf{g}^\infty \in \mathbf{L}^2(\Gamma), \quad c_0 + b_0 > 0. \quad (35)$$

The existence and uniqueness of solution of the problem (34) is an immediate consequence of the theory of variational inequalities (see [3] or [5], for instance). Similarly to the Theorem 3.1, we have the following characterization of the solution.

Denote by $\chi_{i_1 \dots i_k}^\infty$ the characteristic function of

$$I_{i_1 \dots i_k}^\infty = \{x \in \Omega : u_{i_1 \dots i_k}^\infty(x) = 1, u_{i_j}^\infty(x) > 0, \forall j = 1, \dots, k\}.$$

Theorem 4.1. *The solution \mathbf{u}^∞ of (34) satisfies a nonlinear elliptic system a.e. in Ω with similar structure to (26) involving the characteristic functions $\chi_{\{u_i^\infty=0\}}$ and $\chi_{i_1 \dots i_k}^\infty$, $1 \leq i_1 < \dots < i_k \leq N$, $k = 1, \dots, N$, and the corresponding stationary Lewy-Stampacchia inequalities (28) and (29), respectively, a.e. in Ω and a.e. on Γ .*

Denote by \mathbf{u} the solution of the variational inequality (1) and by \mathbf{u}^∞ the solution of (34).

Theorem 4.2. *Assume (35) and suppose the assumptions (3) and (4) hold for all $T < \infty$. Assume also that $\mathbf{f}(t) \xrightarrow[t \rightarrow +\infty]{} \mathbf{f}^\infty$ in $\mathbf{L}^2(\Omega)$ and $\mathbf{g}(t) \xrightarrow[t \rightarrow +\infty]{} \mathbf{g}^\infty$ in $\mathbf{L}^2(\Gamma)$.*

Then

$$\mathbf{u}(t) \xrightarrow[t \rightarrow +\infty]{} \mathbf{u}^\infty \quad \text{in } \mathbf{L}^2(\Omega). \quad (36)$$

Proof. Using \mathbf{u}^∞ as test function in (1) and \mathbf{u} in (34), we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\mathbf{u}(t) - \mathbf{u}^\infty|^2 + \alpha \int_{\Omega} |\nabla(\mathbf{u}(t) - \mathbf{u}^\infty)|^2 + \int_{\Omega} c |\mathbf{u}(t) - \mathbf{u}^\infty|^2 + \int_{\Gamma} b |\mathbf{u}(t) - \mathbf{u}^\infty|^2 \\ \leq \int_{\Omega} (\mathbf{f}(t) - \mathbf{f}^\infty) \cdot (\mathbf{u}(t) - \mathbf{u}^\infty) + \int_{\Gamma} (\mathbf{g}(t) - \mathbf{g}^\infty) \cdot (\mathbf{u}(t) - \mathbf{u}^\infty) \end{aligned}$$

and setting $y(t) = \|\mathbf{u}(t) - \mathbf{u}^\infty\|_{\mathbf{L}^2(\Omega)}^2$, we may easily show that there exists positive constants α and C , independent of t , such that

$$y'(t) + \alpha y(t) \leq \phi(t),$$

where $\phi(t) = C(\|\mathbf{f}(t) - \mathbf{f}^\infty\|_{\mathbf{L}^2(\Omega)}^2 + \|\mathbf{g}(t) - \mathbf{g}^\infty\|_{\mathbf{L}^2(\Gamma)}^2)$, and the conclusion (36) follows by standard arguments.

Once proved the stabilization (36) in time, of the solution $\mathbf{u}(t)$ of the variational inequality (1) to the solution \mathbf{u}^∞ of the stationary variational inequality (34), the proof of the asymptotic stabilization of the coincidence sets (under suitable assumptions) may be obtained as in Theorem 3.4 (see [7]).

Theorem 4.3. *Under the assumptions of Theorem 4.2, assume also, for all $k = 1, \dots, N$, and all $1 \leq i_1 < \dots < i_k \leq N$, that*

$$\sum_{j=1}^k f_{i_j}^\infty \neq 0 \quad \text{a.e. in } \omega \subseteq \Omega.$$

Then, for all $1 < p < \infty$, and all indexes as above, we have

$$\chi_{\{u_i(t)=0\}} \xrightarrow{t \rightarrow +\infty} \chi_{\{u_i^\infty=0\}} \quad \text{and} \quad \chi_{i_1 \dots i_k}(t) \xrightarrow{t \rightarrow +\infty} \chi_{i_1 \dots i_k}^\infty \quad \text{in } L^p(\omega),$$

where $\chi_{i_1 \dots i_k}(t)$ denotes the characteristic function of $I_{i_1 \dots i_k}$ defined in (27) at time t .

Remark 4.4. *In Theorem 4.2 it is clear that if, for instance, $\phi(t) = O(e^{-\lambda t})$, $\lambda > 0$, then we also obtain $\|\mathbf{u}(t) - \mathbf{u}^\infty\|_{\mathbf{L}^2(\Omega)} = O(e^{-\mu t})$, for some $\mu > 0$, as $t \rightarrow \infty$. As in [6], it would be interesting to obtain also sufficient conditions for the stronger stabilization of the characteristic functions.*

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