On the Drazin index of regular elements

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Abstract

It is known that the existence of the group inverse $a^{\#}$ of a ring element a is equivalent to the invertibility of $a^2a^- + 1 - aa^-$, independently of the choice of the von Neumann inverse a^- of a. In this paper, we relate the Drazin index of a with the Drazin index of $a^2a^- + 1 - aa^-$. We give an alternative characterization when considering matrices over an algebraically closed field. We close with some questions and remarks.

1 Introduction

Let R denote a ring with unity 1. We say $a \in R$ is regular provided $a \in aRa$. We shall also denote $a\{1\} = \{x \in R \,|\, axa = a\}$, whose elements are called *von Neumann inverses* of a. As usual, a^- is an element of $a\{1\}$. If some power of a is regular then a is said to be weak-regular. As an example, $2 \in \mathbb{Z}_8$ is not regular and still it is weak-regular.

In this paper, we will consider Drazin invertibility ([3]) on general associative rings with unity 1. An element a is said to be Drazin invertible provided there is a common solution to the equations

$$a^k xa = a^k,\ xax = x,\ ax = xa,$$

for some $k \geq 0$. It is well known the uniqueness of the solution, if it exists. As usual, it will be denoted by a^D. The smallest k for which the equations have a common solution is called the *Drazin index* of a, and denoted by $i(a)$. Two special cases deserve our attention: when $i(a) = 0$ means a is a unit, and when $i(a) \leq 1$ defines the so called *group invertible*

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elements. In the former case, the Drazin inverse will be denoted by $a^{\#}$. That is to say, group invertibility is a special case of Drazin invertibility. However, it can be proved that a has a Drazin inverse provided it has a power which is group invertible. Furthermore, the smallest k for which $\left(a^k\right)^{\#}$ exists equals the Drazin index $i(a)$ of a, and $a^D = a^{k-1}\left(a^k\right)^{\#} = \left(a^k\right)^{\#} a^{k-1}$.

We will make use of left and right ideals generated by a power of a. In fact, $i(a) = k$ if and only if k is the smallest for which $a^kR = a^{k+1}R$ and $Ra^k = Ra^{k+1}$, or equivalently, $a^k \in a^{k+1}R \cap Ra^{k+1}$. This implies, for any $n \geq k$, the relation $a^n \in a^{n+1}R \cap Ra^{n+1}$. As a special case, $a^{\#}$ exists if and only if $a \in a^2R \cap Ra^2$ if and only if $aR = a^2R, Ra = Ra^2$. The left [resp. right] index of a is the smallest value p [resp. q] for which $a^{p+1}R = a^pR$ [resp. $Ra^{q+1} = Ra^q$]. It was shown in [3] (cf. [7, page 11]) that if p and q are finite then $p = q = i(a)$.

R. Cline showed in [2] how to relate $(ab)^D$ with $(ba)^D$, namely $(ab)^D = a\left((ba)^D\right)^2 b$. This equality is known as Cline's formula. According to [7, page 16], the indices $i(ab)$ and $i(ba)$ differ at most by unity. That is to say, $|i(ab) - i(ba)| \leq 1$. When considering matrices over a field $\mathbb{F}$, this corresponds to $\psi_{AB}(\lambda) = \lambda^{0,\pm 1}\psi_{BA}(\lambda)$, where ψ_{AB} and ψ_{BA} denote, respectively, the minimal polynomial of AB and BA. If, in addition, $\mathbb{F}$ is algebraically closed, then every matrix is similar to a diagonal block matrix with Jordan blocks, known as the Jordan canonical (or normal) form. This gives, in particular, the core-nilpotent decomposition: given a matrix A over $\mathbb{F}$, there are (possibly absent) matrices U invertible and N nilpotent with nilpotency index, say, k, for which $A \approx \begin{bmatrix} U & 0 \\ 0 & N \end{bmatrix}$, where $\approx$ denotes matrix similarity. In this case, $A^D \approx \begin{bmatrix} U^{-1} & 0 \\ 0 & 0 \end{bmatrix}$. Note that Drazin invertibility is invariant to matrix similarity, and recall similar matrices have the same minimal polynomial. Therefore, this means $\psi_A = lcm(\psi_U, \psi_N)$. As U is invertible and N is nilpotent with nilpotency index k then $\psi_U(0) \neq 0$ and $\psi_N(\lambda) = \lambda^k$, and hence $\psi_A(\lambda) = \lambda^k\psi_U(\lambda)$. As a conclusion, the Drazin index of A equals the algebraic multiplicity (possibly zero) of 0 as a root of the minimal polynomial ψ_A of A. With no surprise, the multiplicity of the root 0 (if any) of the minimal polynomial of a matrix over a field is usually called the index of the matrix.

A ring R is said to be *Dedekind finite* if $xy = 1$ implies $yx = 1$. An important property of these rings is that, given $e^2 = e, f^2 = f \in R$, then, as in [4, Theorem 1], the equivalence of the following hold:

1. R is Dedekind finite;
2. $eR \subseteq fR$ and $e \sim f$ imply $eR = fR$;
3. $Re \subseteq Rf$ and $e \sim f$ imply $Re = Rf$;

where $e \sim f$ means $eR \cong fR$ as right R-modules, or equivalently, $Re \cong Rf$ as left R-modules.

As a consequence (cf. [4, Theorem 2]), if a^k is regular (that is, a is weak-regular) then the equality $a^kR = a^{k+1}R$ is equivalent to the existence of the Drazin inverse of a, with $i(a) \leq k$, provided R is Dedekind finite. In this case, the equality $a^kR = a^{k+1}R$ implies

$Ra^k \cong Ra^{k+1}$ as left R-modules by taking $\varphi(ya^k) = ya^{k+1}$ as the desired isomorphism. Since trivially $Ra^{k+1} \subseteq Ra^k$ then $Ra^{k+1} = Ra^k$, and therefore $i(a) \leq k$.

If R is *not* Dedekind finite, then such an outcome cannot be expected. Indeed, if $uv = 1 \neq vu$ then u^D does *not* exist and still $u^\ell R = R = u^{\ell+1}R$, for any natural ℓ.

2 Main results

The Puystjens-Hartwig Theorem ([10]) characterizes the group invertibility of a regular element in terms of units. We may rewrite it as the equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (4) in the proposition below. We add two more simpler equivalences.

Proposition 2.1. *Given a regular $a \in R$, the following conditions are equivalent:*

1. $i(a) \leq 1$;
2. $i(a^2a^- + 1 - aa^-) = 0$ *for one and hence all choices of* $a^- \in a\{1\}$;
3. $i(a + 1 - aa^-) = 0$ *for one and hence all choices of* $a^- \in a\{1\}$;
4. $i(a^-a^2 + 1 - a^-a) = 0$ *for one and hence all choices of* $a^- \in a\{1\}$;
5. $i(a + 1 - a^-a) = 0$ *for one and hence all choices of* $a^- \in a\{1\}$.

Proof. Note that $1 + aa^-(a-1)$ is a unit if and only if $1 + (a-1)aa^- = a^2a^- + 1 - aa^-$ is a unit, and so (2) $\Leftrightarrow$ (3). The equivalence (4) $\Leftrightarrow$ (5) is obtained similarly. $\square$

Recently [13], the existence of the group inverse of a regular element was characterized by means of another unit. We give a proof for the sake of completeness.

Proposition 2.2 (Schmoeger)**.** *Given a regular $a \in R$ then $i(a) \leq 1$ if and only if $i(1 - aa^- - a^-a) = 0$, for some $a^- \in a\{1\}$.*

Proof. Setting $u = 1 - aa^- - a^-a$ then obviously $ua = -a^-a^2$ and $au = -a^2a^-$, which lead to $a \in a^2R \cap Ra^2$.

Conversely, taking $a^- = a^\#$ one can show that $\left(1 - aa^\# - aa^\#\right)^2 = 1$. $\square$

Using the same reasoning of the previous result, we may state the following:

Proposition 2.3. *Let $a \in R$ be a regular element, and consider the following conditions:*

(A) $i(a) \leq 1$.

(B) $i(aa^- + 1 - a^-a) = 0$, *for some* $a^- \in a\{1\}$.

(C) $i(a^-a + 1 - aa^-) = 0$, *for some* $a^- \in a\{1\}$.

(D) R is Dedekind-finite.

Then

1. $(A) \Leftrightarrow ((B) \wedge (C))$.

2. $(D) \Rightarrow (((B) \vee (C)) \Rightarrow (A))$.

Proof. (1). (A) means $a^{\#}$ exists, and so (B) and (C) both hold by taking $a^- = a^{\#}$. Conversely if both $aa^- + 1 - a^-a$ and $a^=a + 1 - aa^=$ are units for some $a^=, a^- \in a\{1\}$, and since $a(aa^- + 1 - a^-a) = a^2a^-$ and $(a^=a + 1 - aa^=)a = a^=a^2$, then $a \in a^2R \cap Ra^2$, which in turn means $i(a) \leq 1$.

(2). If R is Dedekind finite, and as in (1), (B) shows $a \in a^2R$ and therefore $a \in Ra^2$ (see [4]), or (C) implies $a \in Ra^2$ and therefore $a \in a^2R$. In either case, $a^{\#}$ exists. □

Condition (2) is the best possible, as if R is not Dedekind finite, it is possible to exist a regular $a \in R$ which has no group inverse, and still $aa^- + 1 - a^-a$ or $a^-a + 1 - aa^-$ are units for some $a^- \in a\{1\}$. Take $R = \mathcal{B}(\ell^2)$, and the usual orthonormal basis $(e_i)_{i=1}^{\infty}$ in ℓ^2. Define $a \in R$ as $a(e_i) = e_{i+1}$, which is regular and a^- defined as $a^-(e_i) = \begin{cases} e_{i-1} & \text{if } i \geq 2 \\ 0 & \text{otherwise} \end{cases}$ is a von Neumann inverse of a. Note $aa^- \neq 1 = a^-a$, $aa^- + 1 - a^-a$ is not a unit and $a^-a + 1 - aa^- = 2 - aa^-$ is invertible. In fact, $(2 - aa^-)^{-1}(e_i) = \begin{cases} \frac{1}{2}e_1 & \text{if } i = 1 \\ e_i & \text{otherwise} \end{cases}$.

In the next result, we extend Proposition 2.1.

Theorem 2.4. *Let $a \in R$ be a regular non-invertible element. The following conditions are equivalent:*

1. $i(a) = k + 1$.

2. $i(a^2a^- + 1 - aa^-) = k$, *for some* $a^- \in a\{1\}$.

3. $i(a^-a^2 + 1 - a^-a) = k$, *for some* $a^- \in a\{1\}$.

Proof. (1) $\Leftrightarrow$ (2). When $k = 0$ we get Proposition 2.1. So we may consider $k \geq 1$.

Firstly, note that $a^{k+1}a^- = (a^2a^-)^k$, for $k \geq 1$, and secondly $a^2a^- \in eRe$, where $e = aa^-$, from which $(a^2a^-)^D \in eRe$ with index k if and only if $i(a^2a^- + 1 - aa^-) = k$ (see [9]). Alternatively, $x + y$ with $xy = 0 = yx$ has Drazin index k if and only if x, y have Drazin inverses in which case $k = \max\{i(x), i(y)\}$.

If $i(a^2a^- + 1 - aa^-) = k$ then $i(a^2a^-) = k$. This means $(a^2a^-)^{k+1}R = (a^2a^-)^kR$ and $R(a^2a^-)^{k+1} = R(a^2a^-)^k$, which in turn gives $a^{k+2}R = a^{k+1}R$ and $Ra^{k+2} = Ra^{k+1}$. Hence, $i(a) \leq k + 1$. Now, if $i(a) = l \leq k$ then $a^{l+1}a^-R = a^{l+1}R = a^lR = a^la^-R$, from which $(a^2a^-)^lR = (a^2a^-)^{l-1}R$, and therefore $k = i(a^2a^-) \leq l - 1 < k$.

Conversely, if $i(a) = k+1$ then $a^{k+2}a^-R = a^{k+1}a^-R$ and $Ra^{k+2}a^- = Ra^{k+1}a^-$, which give $(a^2a^-)^{k+1}R = (a^2a^-)^kR$ and $R(a^2a^-)^{k+1} = R(a^2a^-)^k$. Therefore, $i\left(a^2a^-\right) \leq k$. Assuming $i\left(a^2a^-\right) = l < k$ then this would give $a^{l+2}R = (a^2a^-)^{l+1}R = (a^2a^-)^lR = a^{l+1}$ and therefore $i(a) \leq l + 1 < k + 1$. Hence, $i(a^2a^-) = k$, which in turn implies $i(a^2a^- + 1 - aa^-) = k$.

The equivalence (1) $\Leftrightarrow$ (3) is similar to (1) $\Leftrightarrow$ (2). □

We remark the index of the elements in the Theorem is *independent* of the choice of the von Neumann inverse of a. Therefore, we may state the following result:

Corollary 2.5. *Given a regular $a \in R$ and $a^- \in a\{1\}$, if $i(a^2a^- + 1 - aa^-) = k$ then $i(a^2a^= + 1 - aa^=) = k$ for any $a^= \in a\{1\}$.*

When $k = 0$, this gives the known fact that the invertibility of $a^2a^- + 1 - aa^-$ is independent of the choice of a^-, as in Proposition 2.1.

Lemma 2.6. *Given a regular $t \in R$ and a natural k,*

$$\left(t + 1 - tt^-\right)^k = 1 + \sum_{i=1}^{k} \left(t^i - t^i t^-\right).$$

Proof. The proof is done by induction. The result holds trivially for $k = 1$.

Note that $(t + 1 - tt^-)^{k+1} = (t + 1 - tt^-)(t + 1 - tt^-)^k$ which equals, by the induction step,

$$\left(t + 1 - tt^-\right)\left(1 + \sum_{i=1}^{k} \left(t^i - t^i t^-\right)\right).$$

Hence, $\left(t + 1 - tt^-\right)^{k+1} = t + \sum_{i=2}^{k+1} \left(t^i - t^i t^-\right) + 1 + \sum_{i=1}^{k} \left(t^i - t^i t^-\right) - tt^- - \sum_{i=1}^{k} \left(t^i - t^i t^-\right) = 1 + t - tt^- + \sum_{i=2}^{k+1} \left(t^i - t^i t^-\right) = 1 + \sum_{i=1}^{k+1} \left(t^i - t^i t^-\right).$ □

Lemma 2.7. *Given a regular nilpotent $n \in R$ with $n^{k+1} = 0 \neq n^k$,*

$$\left(n + 1 - nn^-\right)^{k+1} = \left(n + 1 - nn^-\right)^k$$

Proof. By the previous Lemma, $\left(n + 1 - nn^-\right)^{k+1} = 1 + \sum_{i=1}^{k+1} \left(n^i - n^i n^-\right)$. Since $n^{k+1} = 0$, we have, $\left(n + 1 - nn^-\right)^{k+1} = 1 + \sum_{i=1}^{k} \left(n^i - n^i n^-\right) = \left(n + 1 - nn^-\right)^k$. □

Theorem 2.8. *Given a regular nilpotent $0 \neq n \in R$ then $n^{k+1} = 0 \neq n^k$ if and only if $i(n + 1 - nn^-) = k$, for some n^-.*

Proof. From Lemma 2.7, $i(n + 1 - nn^-) \leq k$. Note that since the nilpotency index of n is $k + 1$ then also $i(n) = k + 1$.

We may write $n + 1 - nn^-$ as $\begin{bmatrix} 1 & n \end{bmatrix} \begin{bmatrix} 1 - nn^- \\ 1 \end{bmatrix}$. Using [2], $\begin{bmatrix} 1 & n \end{bmatrix} \begin{bmatrix} 1 - nn^- \\ 1 \end{bmatrix}$ has a Drazin inverse if and only if $M = \begin{bmatrix} 1 - nn^- \\ 1 \end{bmatrix} \begin{bmatrix} 1 & n \end{bmatrix} = \begin{bmatrix} 1 - nn^- & 0 \\ 1 & n \end{bmatrix}$ has a Drazin inverse, and $|i(n + 1 - nn^-) - i(M)| \leq 1$. From [7, Theorem 1], and since $i(1 - nn^-) = 1$ then $i(n) \leq i(M) \leq i(n) + 1$, that is to say, $k + 1 \leq i(M) \leq k + 2$. Recall $i(n + 1 - nn^-) \leq k$.

Now $i(M) = k + 1$ implies the possible values for $i(n + 1 - nn^-)$ are $k, k + 1, k + 2$. If $i(M) = k + 2$ then the possible values for $i(n + 1 - nn^-)$ are $k + 1, k + 2, k + 3$. We are left with $i(n + 1 - nn^-) = k$.

Conversely, suppose $i(n + 1 - nn^-) = k$ and $i(n) = \ell$, or equivalently, $n^\ell = 0 \neq n^{\ell-1}$. We want to show $\ell = k+1$. If $\ell \leq k$ then $i(n+1-nn^-) \leq \ell - 1 < k$ from Lemma 2.7. Therefore $\ell > k$. Now suppose $\ell > k + 1$. Setting $M = \begin{bmatrix} 1 - nn^- \\ 1 \end{bmatrix} \begin{bmatrix} 1 & n \end{bmatrix} = \begin{bmatrix} 1 - nn^- & 0 \\ 1 & n \end{bmatrix}$ then $i(M) \in \{k - 1, k, k + 1\}$ and $\ell = i(n) \leq i(M) \leq i(n) + 1 = \ell + 1$. These inequalities do not hold for the possible values $k - 1, k, k + 1$ of $i(M)$. Therefore, and since n is nilpotent, $\ell = i(n) = k + 1$. □

Corollary 2.9. *Given a regular nilpotent $0 \neq n \in R$, $i(n) = k + 1$ if and only if $i(n + 1 - nn^-) = k$, for some n^-.*

Corollary 2.10. *Given a regular nilpotent $0 \neq n \in R$ and $n^- \in n\{1\}$ such that $i(n + 1 - nn^-) = k$ then $i(n + 1 - nn^=) = k$, for all $n^= \in n\{1\}$.*

Theorem 2.11. *Let A be a singular square matrix over an algebraically closed field. Then $i(A) = k + 1$ if and only if $i(A + 1 - AA^-) = k$ for some A^-.*

Proof. The case $k = 0$ follows from Proposition 2.1. So we may consider $k \geq 1$. For every matrix A there is C invertible and N nilpotent for which $A \approx \begin{bmatrix} C & 0 \\ 0 & N \end{bmatrix}$, where $\approx$ denotes matrix similarity. Recall this form is know as the core-nilpotent decomposition. Without loss of generalization, we may consider A to be in its core-nilpotent decomposition. Note that $i(N) = i(A) \geq 2$, and therefore $N \neq 0$. Setting $A^- = \begin{bmatrix} C^{-1} & 0 \\ 0 & N^- \end{bmatrix}$ and $U = A + I - AA^-$, then $U = \begin{bmatrix} C & 0 \\ 0 & N + I - NN^- \end{bmatrix}$. Now $i(A) = k + 1 \Leftrightarrow i(N) = k + 1 \Leftrightarrow i(N + I - NN^-) = k \Leftrightarrow i(U) = k$, which proves the theorem. □

3 Concluding remarks

We close this paper with some remarks and questions:

1. Cline's formula provides an alternative proof of the main results of [13], as $|i(ab) - i(ba)| \leq 1$. This implies if ab is a unit then $i(ba) \leq 1$, or equivalently, $(ba)^\#$ exists. Also if $((ab)^n)^\#$ exists then $i(ab) \leq n$, which implies $i(ba) \leq n + 1$, and therefore the existence of $\left((ba)^{n+1}\right)^\#$.

2. In this paper, we considered Drazin invertibility of regular elements. Still we must stress that a Drazin invertible element might not be regular. In this paper, we clearly addressed to the case where the element in regular.

3. When considering Drazin invertibility of a ring element, a usefull reasoning is by considering powers. The elements of the form $t+1-tt^-$ have powers with a special structure, as in Lemma 2.6:
Given a regular $t \in R$ and a natural k,
$$\left(t+1-tt^-\right)^{k+1} = t\left(t+1-tt^-\right)^k + 1 - tt^-.$$
The proof is done by induction. Simple calculations show the result holds for $k=1$.
Note that $(t+1-tt^-)^{k+1} = (t+1-tt^-)^k (t+1-tt^-)$ which equals, by the induction step,
$$\left(t\left(t+1-tt^-\right)^{k-1} + 1 - tt^-\right)\left(t+1-tt^-\right).$$
We obtain $t(t+1-tt^-)^k + 1 - tt^-$.

4. The invertibility of $a^2a^- + 1 - aa^-, a^-a^2 + 1 - a^-a, a + 1 - aa^-, a + 1 - a^-a$ is *independent* of the choice of a^-. What can be said when considering the units in Proposition 2.2 and in Proposition 2.3?

5. We have shown that $i(a) = k+1$ if and only $i(a^2a^- + 1 - aa^-) = k$, for $k \geq 1$. We have also proved $i(A) = k+1$ if and only $i(A + 1 - AA^-) = k$, for $k \geq 1$, if A is a square matrix over an algebraically closed field. Is the result also valid for, say, regular rings?

6. A positive answer for the previous item would provide the equivalence between $i(a^2a^- + 1 - aa^-) = k$ and $i(a + 1 - aa^-) = k$, and in this case it is independent of the choice of a^-.

7. The previous question is part of a more vast and structural one: does $i(1-xy) = k$ imply $i(1-yx) = k$? When $k = 0$ it is a well known result.

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