NORMAL FORMS AND LINEARIZATION OF RESONANT VECTOR FIELDS WITH MULTIPLE EIGENVALUES

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Abstract. We discuss the linearization and normal forms of resonant vector fields $X(x) = Ax + a(x)$, where A has one double or triple eigenvalue or a pair of double eigenvalues: we present a simple way of identifying the resonant monomials that have to appear in its normal form, and also effective conditions on the nonlinearity $a(x)$ for the resonant vector field to be linearizable.

1. Introduction

Normal forms for vector fields, or (autonomous) differential equations, are very important from the theoretical point of view, and also from the point of view of applications; in particular they are the main technique in bifurcation theory, involving families of differential equations depending on parameters [2]. The study of resonances becomes fundamental when considering families of vector fields, depending even on only one parameter.

Given a germ of a nonlinear vector field: $X(x) = Ax + a(x)$, with $a(x) = O(x^2)$, it follows from the classical results that if there are no resonance relations between the eigenvalues of A, the vector field is linearizable for any nonlinearity $a(x)$; otherwise, it is reducible to a resonant normal form: the nonlinear part contains resonant monomials only.

Remark 1. If the nonlinear terms contain no resonant monomials, this does not mean that the corresponding vector field is linearizable [3].

If the matrix A is diagonalizable, and the nonlinear terms contain only resonant monomials, or start with a resonant monomial, the corresponding vector field is not linearizable; however, this is not true if A is not diagonalizable : linearizability depends on the monomials that are actually present in the nonlinear part, it is not determined by the linear part, in contrast to the classical linearization results.

Our main objective here is, given a resonant matrix A with multiple eigenvalues, to present effective conditions on the nonlinearity $a(x)$ for

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the resonant vector field $X(x) = Ax + a(x)$ to be linearizable, and also a simple way of identifying the resonant monomials that have to appear in the normal form of a given resonant vector field, in particular those of smaller degree, when holomorphic or C^∞ linearization is impossible.

We restrict our considerations to the linearization problem in the formal category: in the holomorphic category, if the Brjuno condition is verified, the existence of a formal linearizing change of variables implies the existence of a holomorphic one [6]; in the smooth case, assuming hyperbolicity [7] or quasi-hyperbolicity [5], the existence of a formal linearizing change of variables implies the existence of a C^∞ one.

We will consider our vector fields in complex variables, but the results are also valid for real vector fields; however, in that case they are effective essentially only when the eigenvalues are also real.

2. Basic results and definitions

Let $\mathbb{K}$ be the field of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$, and denote by $\mathcal{F} = \mathbb{K}[[x_1, \dots, x_n]]$ the formal power series algebra over $\mathbb{K}$. A formal vector field X can be seen as a derivation on $\mathcal{F}$:

$$X(fg) = X(f)g + fX(g), \quad f, g \in \mathcal{F}$$

As usual, we identify the set $\mathrm{D}(\mathcal{F})$ of derivations on $\mathcal{F}$ with $\mathcal{F}^n$ by:

$$X = \sum_{i=1}^{n} X_i \frac{\partial}{\partial x_i}, \quad X_i \in \mathcal{F}, \qquad \text{and} \quad \frac{\partial}{\partial x_i} = e_i.$$

Let X be a vector field on a domain U in $\mathbb{C}^n$, a formal (holomorphic, smooth) map $X : U \longrightarrow \mathbb{C}^n$; it will always be supposed to have a singular point at the origin in $\mathbb{C}^n$:

$$X(x) = Ax + a(x), \quad a(x) = O(x^2).$$

and that the linear part A is in the Jordan canonical form:

$$A = \begin{bmatrix} \lambda_1 & 0 & \cdots & \cdots & \cdots & 0 \\ \varepsilon_1 & \lambda_2 & \ddots & & & \vdots \\ 0 & \varepsilon_2 & \lambda_3 \vdots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \varepsilon_{n-1} & \lambda_n \end{bmatrix}, \qquad \begin{array}{c} \varepsilon_i \in \{0, 1\} \\ \varepsilon_i = 1 \Longrightarrow \lambda_i = \lambda_{i+1} \end{array}$$

The formal (holomorphic, smooth) vector field X is said to be formally (biholomorphically, smoothly) linearizable, or conjugate to its linear part, if there exists a formal (holomorphic, smooth) change of coordinates $z = \psi(x)$, preserving the origin, such that in the new coordinates the nonlinear part is zero:

$$\frac{\partial \psi}{\partial x}(\xi(z)) X(\xi(z)) = Az, \qquad \xi = \psi^{-1}.$$

Formal linearization can be accomplished whenever the homological equation:

$$L_A h(x) = m(x), \qquad \text{where} \quad L_A h(x) = Ah(x) - \frac{\partial h}{\partial x}(x)Ax$$

can be solved for any monomial or homogeneous component that appears in the nonlinear part of X, or that appears subsequently after the changes of coordinates that kill the lower order terms of X.

Let $\lambda = (\lambda_1, \dots \lambda_n) \in \mathbb{C}^n$ be the eigenvalues of the linear part A of X; they are said to be *resonant* if, for some i, there exists $I = (i_1, \dots, i_n)$, with i_j nonnegative integers and $|I| = i_1 + \dots + i_n = k \geq 2$, such that:

$$I \cdot \lambda - \lambda_i = 0.$$

Then $|I| = k$ is the order of this resonance.

A monomial $x^I e_i = x_1^{i_1} \dots x_n^{i_n} e_i$ is said to be resonant if $I \cdot \lambda - \lambda_i = 0$.

If the eigenvalues λ of A are non resonant, the linear operator L_A is an isomorphism on $\mathcal{F}^n$, and formal linearization is always possible, independently of the actual nonlinearity (Poincaré Theorem [2]); otherwise we can only linearize those X whose nonlinearity is such that at every step the lower order terms are in the image of L_A.

When there are resonances, the Poincaré-Dulac theorem [2] allows the elimination of all nonresonant terms by a formal change of variables. This can be improved when the nilpotent part of A is not zero:

Belitskii Theorem [4]. *A formal vector field X is formally conjugate to a normal form $Ax + \Phi(x)$ consisting of its linear part Ax and a nonlinearity $\Phi(x)$ such that $L_A(\Phi(x)) = [A^T x, \Phi(x)] = 0$.*

If A is not semisimple, a monomial being resonant means that it belongs to the generalized eigenspace of the linear operator L_A corresponding to the zero eigenvalue, but that monomial can still be in the image of L_A. These resonant monomials can be dealt with as long as they do not subsequently generate monomials that do not belong to the image of the linear operator L_A.

Let $\mathcal{M}$ be the set of vectors in $\mathbb{Z}^n$ such that at most one coordinate is -1 and all the others are non negative, and consider a representation of the monomial $x^I e_i$ by a vector $P_i^I = I - e_i$ in $\mathcal{M}$.

We construct $\mathcal{G}$ as a subset of the set $\mathcal{R}$ of resonant monomials for which there exists another subset $\mathcal{U} \subset \mathcal{R}$ such that:

$$\mathcal{G} \subset L_A(\mathcal{U})_{\mathbb{R}} = \mathcal{G}_{\mathbb{R}}, \qquad \mathcal{G} + \mathcal{U} \subset \mathcal{G}$$

where $L_A(\mathcal{U})_{\mathbb{R}}$, respectively $\mathcal{G}_{\mathbb{R}}$, denotes the set of linear combinations with real coefficients of elements of $L_A(\mathcal{U})$, respectively $\mathcal{G}$.

The complement of $\mathcal{G}$ in $\mathcal{R}$ will be denoted by $\mathcal{B}$. The sets $\mathcal{G}, \mathcal{U}$ and $\mathcal{B}$ are not unique.

Remark 2. Abusing notation, we denote by the same symbol a set of monomials and the set of vectors in $\mathbb{Z}^n$ that represent them: in $L_A(\mathcal{U})$

the Lie derivative is applied to the monomials in $\mathcal{U}$, while in $\mathcal{G}+\mathcal{U}\subset\mathcal{G}$ the sum involves the vectors in $\mathbb{Z}^n$, and is considered only if the result is in $\mathcal{M}$. The context should make this clear.

To a nonlinearity $a(x)$ there corresponds a set:

$$\mathcal{A}=\{P_i^I=I-e_i,\ \text{such that}\ a_i^I\neq 0\}\subset\mathcal{M}\subset\mathbb{Z}^n$$

We extend $\mathcal{A}$ to a set $\mathcal{A}_{ext}$ so that:

- $\mathcal{A}\subset\mathcal{A}_{ext},\quad \mathcal{A}_{ext}+\mathcal{U}\subset\mathcal{A}_{ext}$
- $\mathcal{A}_{ext}$ is closed for the following permutations, whenever the resulting vector belongs to $\mathcal{M}-\mathcal{R}$ (corresponds to some nonresonant monomial):

$$P_i^I=I-e_i\in\mathcal{A}_{ext},\ \varepsilon_i=1\Longrightarrow P_{i+1}^I\in\mathcal{A}_{ext}$$

$$P_i^I=I-e_i\in\mathcal{A}_{ext},\ \varepsilon_k=1\Longrightarrow P_i^J\in\mathcal{A}_{ext},\ J=I-e_{k+1}+e_k$$

or, in terms of monomials:

$$x^Ie_i\in\mathcal{A}_{ext},\ \varepsilon_i=1,\ i<n \quad\Longrightarrow\quad x^Ie_{i+1}\in\mathcal{A}_{ext}$$

$$x^Ie_i\in\mathcal{A}_{ext},\ \varepsilon_k=1,\ i_{k+1}>0\Longrightarrow x_kx_{k+1}^{-1}x^Ie_i\in\mathcal{A}_{ext} \tag{1}$$

We define $\mathcal{C}$ as the set of all those linear combinations with non negative integers (not all zero) of vectors in $\mathcal{A}_{ext}$ that belong to $\mathcal{M}$.

Theorem 1 ([3])**.** *Let $X(x)=Ax+a(x)$ be a formal (holomorphic, C^∞) vector field on a neighbourhood U of the origin in $\mathbb{C}^n$; if the nonlinearity $a(x)$ is such that all resonant monomials in $\mathcal{C}$ are in $\mathcal{G}$ (the Brjuno condition is verified, the critical point is hyperbolic), there exists a formal (holomorphic, C^∞) change of coordinates $y=\psi(x)$ linearizing the vector field X.*

We also have information on the normal form of the vector field X when it is not formally linearizable:

Corollary 1 ([3])**.** *Let $X(x)=Ax+a(x)$ be a formal (holomorphic, C^∞) vector field on a neighbourhood U of the origin in $\mathbb{C}^n$; a resonant normal form for X can be obtained involving only the nonlinear resonant monomials corresponding to points in $\mathcal{C}\cap\mathcal{B}$.*

Remark 3. The resonant normal form can be further simplified in many cases [11, 12]. The changes of coordinates then do not necessarily correspond to monomials in the image of L_A.

3. Normal forms

Given a formal (holomorphic, C^∞) vector field $X(x)=Ax+a(x)$ on a neighbourhood U of the origin in $\mathbb{C}^n$, we can associate an oriented graph to the resonant monomials (relative to the eigenvalues of A) of a certain degree:

- the vertices are the resonant monomials;

- there is an arrow from $x^I e_i$ to $x^{I-e_{r+1}+e_r} e_i$ if $\varepsilon_r = 1$, $i_{r+1} > 0$.
- there is an arrow from $x^I e_i$ to $x^I e_{i+1}$ if $\varepsilon_i = 1$, $i < n$.
- there are no other arrows.

We will be interested in the non trivial (not reduced to a vertex) connected components (ignoring orientation). A straightforward computation leads to:

Lemma 1. *The following are equivalent:*

- *There is an arrow from monomial m_1 to monomial m_2.*
- *m_2 appears in the expression of $L_A(m_1)$ with a non zero coefficient.*
- *m_2 results from applying a permutation of the form (1) to m_1.*

Thus all monomials corresponding to trivial components of that oriented graph are outside the image of L_A.

It follows from lemma 1 that we can simplify the study of the connected components:

- if there exists a monomial m_1 which is the source of an unique arrow and that one leads to m_2, and there is no arrow from m_2, we eliminate m_2 and all arrows leading to it;
- the preceding process is applied to the reduced graph until no further simplification is possible.

In fact if at a given step there exists a monomial m_1 which is the source of an unique arrow and that one leads to m_2, then $L_A(m_1)$ is a linear combination of m_2 and eventually other monomials in the image of L_A already removed; it follows that m_2 is also in the image of L_A.

The set of monomials m_2 as above gives $\mathcal{G}$, whereas $\mathcal{U}$ is formed by those monomials together with the monomials m_1; the elements of $\mathcal{B}$ are exactly those remaining in the graph after all simplifications.

The objective of this section is the construction of $\mathcal{B}$ using this approach when there is only one Jordan block, of dimension 2 or 3, or two Jordan blocks, both of dimension 2. A different way of constructing $\mathcal{B}$ is used in section 4

3.1. **One Jordan block case.** Assume there is only one Jordan block of dimension m bigger than 1; we take $\varepsilon_1 = \cdots = \varepsilon_{m-1} = 1$ and $\varepsilon_m = \cdots = \varepsilon_n = 0$, $\bar{x} = (x_{m+1}, \ldots, x_n)$ and $\bar{I} = (i_{m+1}, \ldots, i_n)$. Note that the eigenvalues λ_1, λ_{m+1},..., λ_n are not necessarily distinct.

Example 1. If $m = 2$, the trivial components correspond to $\bar{x}^{\bar{I}} e_j$, $j \geq 3$, and the non trivial connected components are of the following types:

$$x_2^k \bar{x}^{\bar{I}} e_i \longrightarrow x_1 x_2^{k-1} \bar{x}^{\bar{I}} e_i \longrightarrow \cdots \longrightarrow x_1^{k-1} x_2 \bar{x}^{\bar{I}} e_i \longrightarrow x_1^k \bar{x}^{\bar{I}} e_i$$

if $i \geq 3$, and:

$$\bar{x}^{\bar{I}} e_1 \longrightarrow \bar{x}^{\bar{I}} e_2$$

or (for $k>0$):

$$\begin{array}{ccccccccc}
x_2^k\bar{x}^{\bar{I}}e_1 & \longrightarrow & x_1x_2^{k-1}\bar{x}^{\bar{I}}e_1 & \longrightarrow & \cdots & \longrightarrow & x_1^{k-1}x_2\bar{x}^{\bar{I}}e_1 & \longrightarrow & x_1^k\bar{x}^{\bar{I}}e_1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
x_2^k\bar{x}^{\bar{I}}e_2 & \longrightarrow & x_1x_2^{k-1}\bar{x}^{\bar{I}}e_2 & \longrightarrow & \cdots & \longrightarrow & x_1^{k-1}x_2\bar{x}^{\bar{I}}e_2 & \longrightarrow & x_1^k\bar{x}^{\bar{I}}e_2
\end{array}$$

Theorem 2. *Let $X(x)=Ax+a(x)$ be a vector field on a neighbourhood U of the origin in $\mathbb{C}^n$. If there is only one Jordan block, of dimension $m=2$, a resonant normal form for X can be obtained from the resonant monomials:*

$$\{\bar{x}^{\bar{I}}e_1, \qquad \bar{x}^{\bar{I}}e_j \quad j\geq 3, \qquad x_2^k\bar{x}^{\bar{I}}e_j \quad j\geq 1,\ k>0\}$$

Proof. The monomials appearing in the normal form have to generate a complement of the image of L_A; as said before, all monomials in the image of L_A can be killed by a convenient change of coordinates.

For the case $m=2$ the trivial components correspond to monomials $\bar{x}^{\bar{I}}e_j$ ($j\geq 3$) in the last $n-2$ components, not involving any variable x_1 or x_2. The nontrivial components are those of example 1; after reduction (from right to left and from bottom to top) they become:

$$\begin{array}{cccccc}
x_2^k\bar{x}^{\bar{I}}e_i\ (i\geq 3), & x_2^k\bar{x}^{\bar{I}}e_1 & \longrightarrow & x_1x_2^{k-1}\bar{x}^{\bar{I}}e_1, & & \bar{x}^{\bar{I}}e_1 \\
& \downarrow & & & & \\
& x_2^k\bar{x}^{\bar{I}}e_2 & & & &
\end{array}$$

Clearly $\bar{x}^{\bar{I}}e_1$, $x_2^k\bar{x}^{\bar{I}}e_i$ and $x_2^k\bar{x}^{\bar{I}}e_1$ are not in the image of L_A, as there is no arrow leading to them; also $L_A(x_2^k\bar{x}^{\bar{I}}e_1)$ is a linear combination of $x_1x_2^{k-1}\bar{x}^{\bar{I}}e_1$ and $x_2^k\bar{x}^{\bar{I}}e_2$, therefore we can kill $x_1x_2^{k-1}\bar{x}^{\bar{I}}e_1$ by creating new terms $x_2^k\bar{x}^{\bar{I}}e_2$. Thus the proof is complete for this case. $\square$

Remark 4. The set $\mathcal{B}$ includes, besides $\bar{x}^{\bar{I}}e_1$, $\bar{x}^{\bar{I}}e_j$ ($j\geq 3$), and $x_2^k\bar{x}^{\bar{I}}e_j$, all monomials of the form $x_1x_2^{k-1}\bar{x}^{\bar{I}}e_1$, but these latter are not necessary for the normal form: $\mathcal{B}$ is the set of monomials that are not in the image of L_A, but each linear combination of these monomials that belongs to that image allows the exclusion from the normal form of one of the monomials appearing in it.

Theorem 3. *Let $X(x)=Ax+a(x)$ be a vector field on a neighbourhood U of the origin in $\mathbb{C}^n$. If there is only one Jordan block, of dimension $m=3$, a normal form for X can be obtained from the resonant monomials:*

$$\{\bar{x}^{\bar{I}}e_1, \qquad \bar{x}^{\bar{I}}e_j \quad j\geq 4, \qquad x_1^sx_3^{r-s}\bar{x}^{J}e_j \quad s=0,\dots,[r/2] \quad j\geq 1\}$$

Proof. For the case $m=3$ the trivial components correspond to monomials in the last $n-3$ components, not involving any variable x_1, x_2 or x_3; these same monomials give rise to the graph:

$$\bar{x}^{\bar{I}}e_1 \longrightarrow \bar{x}^{\bar{I}}e_2 \longrightarrow \bar{x}^{\bar{I}}e_3$$

in the first three components, which of course reduces to $\bar{x}^{\bar{I}} e_1$.

Given a resonant monomial $M^i(x_1, x_2, x_3)\bar{x}^J e_j$, with $j \geq 4$, and omitting $\bar{x}^J e_j$, the corresponding graph is as shown in figure 1.

$$
\begin{array}{ccccccccc}
x_3^i & & & & & & & & \\
\downarrow & & & & & & & & \\
x_2 x_3^{i-1} & \longrightarrow & x_1 x_3^{i-1} & & & & & & \\
\downarrow & & \downarrow & & & & & & \\
\vdots & \dots & \vdots & \dots & & & & & \\
\downarrow & & \downarrow & & \downarrow & & & & \\
x_2^{i-1} x_3 & \longrightarrow & x_i x_2^{i-2} x_3 & \longrightarrow & \dots & \longrightarrow & x_i^{i-1} x_3 & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
x_2^i & \longrightarrow & x_1 x_2^{i-1} & \longrightarrow & \dots & \longrightarrow & x_1^{i-1} x_2 & \longrightarrow & x_1^i
\end{array}
$$

FIGURE 1. Graph corresponding to $M^i(x_1, x_2, x_3)\bar{x}^J e_j$

After reduction (from right to left and from bottom to top) it becomes one of the graphs in figure 2 or figure 3.

We analyze the reduced graphs from top to bottom, from left to right, along diagonals:

- x_3^{2l} does not belong to the image of L_A;
- $x_2 x_3^{2l-1}$ belongs to that image;
- We can kill $x_2^2 x_3^{2l-2}$, as $L_A(x_2 x_3^{2l-1})$ is a linear combination of it with $x_1 x_3^{2l-1}$, at the expense of creating new terms in $x_1 x_3^{2l-1}$;
- All monomials in the next downward diagonal are in the image of L_A: $x_1 x_2 x_3^{2l-2} = L_A(x_1 x_3^{2l-1})$ and as $L_A(x_2 x_3^{2l-1})$ is a linear combination of $x_1 x_2 x_3^{2l-2}$ and $x_2^3 x_3^{2l-3}$, the latter is also in the image of L_A;
- All monomials in the next diagonal can be killed except for $x_1^2 x_3^{2l-2}$: we follow the upward diagonal, we first kill the terms in $x_2^4 x_3^{2l-4}$ using $x_2^3 x_3^{2l-3}$, creating new terms in $x_1 x_2^2 x_3^{2l-3}$, and these in turn can be killed using $x_1 x_2 x_3^{2l-2}$, creating new terms in $x_1^2 x_3^{2l-2}$;
- the next diagonals are alternately formed by monomials all in the image of L_A, which can be seen going downwards, or by monomials that can be killed creating new terms in the last monomial in the diagonal, going upwards, of the form $x_1^s x_3^{2l-s}$.

This shows that, from all resonant monomials $M^{2l}(x_1, x_2, x_3)\bar{x}^J e_j$, only those of the form $x_1^s x_3^{2l-s}\bar{x}^J e_j$, $s = 0, \dots, l$, are necessary for the resonant normal form.

The same process applied to the other reduced graph leads to a similar conclusion: first we see that the graph can be further reduced,

$$
\begin{array}{ccccccccccc}
x_3^{2l} & & & & & & & & & & \\
\downarrow & & & & & & & & & & \\
x_2x_3^{2l-1} & \longrightarrow & x_1x_3^{2l-1} & & & & & & & & \\
\downarrow & & \downarrow & & & & & & & & \\
x_2^2x_3^{2l-2} & \longrightarrow & x_1x_2x_3^{2l-2} & \longrightarrow & x_1^2x_3^{2l-2} & & & & & & \\
\downarrow & & \downarrow & & \downarrow & & & & & & \\
x_2^3x_3^{2l-3} & \longrightarrow & x_1x_2^2x_3^{2l-3} & \longrightarrow & x_1^2x_2x_3^{2l-3} & \longrightarrow & x_1^3x_3^{2l-3} & & & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \\
x_2^4x_3^{2l-4} & \longrightarrow & x_1x_2^3x_3^{2l-4} & \longrightarrow & \dots & \longrightarrow & \dots & & & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \\
\vdots & \dots & \vdots & \dots & \vdots & \dots & \vdots & \dots & & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
x_2^lx_3^l & \longrightarrow & x_1x_2^{l-1}x_3^l & \longrightarrow & \dots & \longrightarrow & \dots & \longrightarrow & \dots & \longrightarrow & x_1^lx_3^l \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\vdots & \dots & \vdots & \dots & \vdots & \dots & \vdots & \dots & & & \\
\downarrow & & \downarrow & & \downarrow & & & & & & \\
x_2^{2l-2}x_3^2 & \longrightarrow & x_1x_2^{2l-3}x_3^2 & \longrightarrow & x_1^2x_2^{2l-4}x_3^2 & & & & & & \\
\downarrow & & \downarrow & & & & & & & & \\
x_2^{2l-1} & \longrightarrow & x_1x_2^{2l-2}x_3 & & & & & & & & \\
\downarrow & & & & & & & & & & \\
x_2^{2l} & & & & & & & & & &
\end{array}
$$

FIGURE 2. Reduced graph for $i = 2l$

as $x_1^lx_2x_3^l$ can be eliminated and then successively all terms in the downward diagonal until x_2^{2l+1}, then reasoning as above we conclude that from all resonant monomials $M^{2l+1}(x_1, x_2, x_3)\bar{x}^Je_j$, only those of the form $x_1^sx_3^{2l+1-s}\bar{x}^Je_j$, $s = 0, \dots, l$, are necessary for the resonant normal form. This proves our result for all components e_j, $j = 4, \dots, n$..

We consider now resonant monomials in the first three components. The corresponding graph can be thought of as three copies of the first graph considered above, one for each component e_i, connected by arrows that lead from one monomial in the first (second) component to the same monomial in the second (third) component.

The analysis of the part of the graph corresponding the third component is absolutely similar to what we have done before, as there are no new arrows leading from any of the vertexes nor any of the incoming arrows from the second component allows the conclusion that any more

$$
\begin{tikzcd}
x_3^{2l+1} \arrow[d] & & & & \\
x_2x_3^{2l} \arrow[r] \arrow[d] & x_1x_3^{2l} \arrow[d] & & & \\
x_2^2x_3^{2l-1} \arrow[r] \arrow[d] & x_1x_2x_3^{2l-1} \arrow[r] \arrow[d] & x_1^2x_3^{2l-1} \arrow[d] & & \\
\vdots \arrow[r, phantom, "\cdots"] \arrow[d] & \vdots \arrow[r, phantom, "\cdots"] \arrow[d] & \vdots \arrow[r, phantom, "\cdots"] \arrow[d] & {} \arrow[d] & \\
x_2^lx_3^{l+1} \arrow[r] \arrow[d] & x_1x_2^{l-1}x_3^{l+1} \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & x_1^lx_3^{l+1} \arrow[d] \\
x_2^{l+1}x_3^l \arrow[r] \arrow[d] & x_1x_2^lx_3^l \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & x_1^lx_2x_3^l \\
\vdots \arrow[r, phantom, "\cdots"] \arrow[d] & \vdots \arrow[r, phantom, "\cdots"] \arrow[d] & \vdots \arrow[r, phantom, "\cdots"] \arrow[d] & {} & \\
x_2^{2l-1}x_3^2 \arrow[r] \arrow[d] & x_1x_2^{2l-2}x_3^2 \arrow[r] \arrow[d] & x_1^2x_2^{2l-3}x_3^2 & & \\
x_2^{2l}x_3 \arrow[r] \arrow[d] & x_1x_2^{2l-1}x_3 & & & \\
x_2^{2l+1} & & & &
\end{tikzcd}
$$

FIGURE 3. Reduced graph for $i = 2l + 1$

monomials in the third component are in the image of L_A: it is true that there is an unique arrow leading from $x_1^i e_2$, and that arrow goes to $x_1^i e_3$, but this monomial could already be killed as we also have the same situation involving $x_1^{i-1}x_2e_3$.

Thus we conclude that in the third component the monomials of the normal form can again be chosen to be $x_1^s x_3^{i-s}\bar{x}^J e_3$, $s = 0, \ldots, [i/2]$.

We can reduce the graph so that the part corresponding to the third component is just as in figure 2 or figure 3, and eliminate from the part corresponding to the second component all arrows that would lead to the eliminated vertexes (in the third component).

The reduced graph at the next step, to be more precise, the reduced part corresponding to the second component, is similar to the one obtained for the third component but contains one more diagonal: the monomials in the diagonal from x_2^{2l} to $x_1^l x_3^l$, respectively from x_2^{2l+1} to $x_1^{l+1}x_3^l$, have an extra arrow leading to the same monomial in the third component (figure 4), and therefore we cannot show that the monomials in the next diagonal are in the image of L_A.

$$
\begin{array}{ccccccccccc}
x_3^{2l+1} & & & & & & & & & & \\
\downarrow & & & & & & & & & & \\
x_2x_3^{2l} & \longrightarrow & x_1x_3^{2l} & & & & & & & & \\
\downarrow & & \downarrow & & & & & & & & \\
x_2^2x_3^{2l-1} & \longrightarrow & x_1x_2x_3^{2l-1} & \longrightarrow & x_1^2x_3^{2l-1} & & & & & & \\
\downarrow & & \downarrow & & \downarrow & & & & & & \\
\vdots & \dots & \vdots & \dots & \vdots & \dots & & & & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \\
x_2^lx_3^{l+1} & \longrightarrow & x_1x_2^{l-1}x_3^{l+1} & \longrightarrow & \dots & \longrightarrow & \dots & \longrightarrow & x_1^lx_3^{l+1} & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
x_2^{l+1}x_3^l & \longrightarrow & x_1x_2^lx_3^l & \longrightarrow & \dots & \longrightarrow & \dots & \longrightarrow & x_1^lx_2x_3^l & \longrightarrow & x_1^{l+1}x_3^l \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
\vdots & \dots & \vdots & \dots & \vdots & \dots & \vdots & \dots & & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \\
x_2^{2l-1}x_3^2 & \longrightarrow & x_1x_2^{2l-2}x_3^2 & \longrightarrow & x_1^2x_2^{2l-3}x_3^2 & \longrightarrow & x_1^3x_2^{2l-4}x_3^2 & & & & \\
\downarrow & & \downarrow & & \downarrow & & & & & & \\
x_2^{2l}x_3 & \longrightarrow & x_1x_2^{2l-1}x_3 & \longrightarrow & x_1^2x_2^{2l-2}x_3 & & & & & & \\
\downarrow & & \downarrow & & & & & & & & \\
x_2^{2l+1} & \longrightarrow & x_1x_2^{2l} & & & & & & & &
\end{array}
$$

FIGURE 4. Reduced graph for the second component, $i = 2l$: all vertexes above the bottom diagonal have an arrow to the corresponding vertex in the third component

It is still true that, from the same type of reasoning as before, we can conclude that all monomials can be killed except $x_1^s x_3^{i-s}\bar{x}^J e_2$, $s = 0, \dots, [i/2]$:

- Killing $x_2x_3^{2l-1}\bar{x}^J e_2$ using the fact that $L_A(x_3^{2l}\bar{x}^J e_2)$ is a linear combination of $x_2x_3^{2l-1}\bar{x}^J e_2$ and $x_3^{2l}\bar{x}^J e_3$ leads to new terms only in the third component, and those were already accounted for;
- We can kill $x_2^2x_3^{2l-2}\bar{x}^J e_2$, as $L_A(x_2x_3^{2l-1}\bar{x}^J e_2)$ is a linear combination of it with $x_1x_3^{2l-1}\bar{x}^J e_2$ and $x_2^2x_3^{2l-2}\bar{x}^J e_3$, at the cost of creating new terms in $x_1x_3^{2l-1}\bar{x}^J e_2$ and in the third component;
- All subsequent monomials are killed by the same process: creating new terms in $x_1^s x_3^{i-s}\bar{x}^J e_2$, $s = 0, \dots, [i/2]$, and in the third component, which as we have seen before can all be killed, maybe leading to more terms in $x_1^s x_3^{i-s}\bar{x}^J e_3$, $s = 0, \dots, [i/2]$.

Finally, when we consider the reduced part of the graph corresponding to the first component, yet another diagonal must be included, by an argument in every way similar to the one used before. Also reasoning as for the second component, all monomials in the first component can be killed by creating new terms in $x_1^s x_3^{i-s}\bar{x}^J e_1$, $s = 0,\dots,[i/2]$ and also in $x_1^s x_3^{i-s}\bar{x}^J e_2$, $s = 0,\dots,[i/2]$. □

Remark 5. Normal forms are of course not unique: we could have chosen a normal form based on the resonant monomials of the form $x_2^{2j}x_3^{r-2j}\bar{x}^J$, $j = 0,\dots,[r/2]$, in all components.

3.2. **Two Jordan blocks case.** Assume there are exactly two Jordan blocks of dimension m_1 and m_2 bigger than 1; we take $\varepsilon_1 = \cdots = \varepsilon_{m_1-1} = 1$ and $\varepsilon_{m_1+1} = \cdots = \varepsilon_{m_1+m_2-1} = 1$, $\varepsilon_{m_1} = \varepsilon_{m_1+m_2} = \dots = \varepsilon_n = 0$, $\bar{x} = (x_{m_1+m_2+1},\dots,x_n)$ and $\bar{I} = (i_{m_1+m_2+1},\dots,i_n)$.

Theorem 4. *Let $X(x) = Ax + a(x)$ be a vector field on a neighbourhood U of the origin in $\mathbb{C}^n$. If there are exactly two Jordan blocks, of dimension $m_1 = m_2 = 2$, a normal form for X can be obtained from the set of resonant monomials of the form:*

- $\bar{x}^J e_1$, $\bar{x}^J e_3$ *and* $\bar{x}^J e_j$ *for* $j = 5,\dots,n$*;*
- $x_2^i\bar{x}^J e_j$ *or* $x_4^i\bar{x}^J e_j$, $j = 1,\dots,n$
- $x_1^s x_2^{k-s} x_4^l \bar{x}^J e_j$, $s = 0,1,\dots,\min(k,l)$, $j = 1,\dots,n$.

Proof. As before, resonant monomials not involving (x_1,x_2,x_3,x_4), of the form $\bar{x}^J e_j$ for $j = 1,\dots,n$, give rise to trivial connected components in all components but the first four, where we have:

$$\bar{x}^{\bar{I}}e_1 \longrightarrow \bar{x}^{\bar{I}}e_2, \qquad \bar{x}^{\bar{I}}e_3 \longrightarrow \bar{x}^{\bar{I}}e_4$$

The monomials $x_2^i\bar{x}^J e_j$ or $x_4^i\bar{x}^J e_j$, $j = 1,\dots,n$, correspond to connected components containing $M^i(x_1,x_2)\bar{x}^J e_j$ or $M^i(x_3,x_4)\bar{x}^J e_j$ respectively; these components are in every way similar to those considered in theorem 2.

There remains to consider the case of monomials $M^i(x_1,\dots,x_4)\bar{x}^J e_j$; they give rise, for $j \ge 5$, to the graphs in figure 5, assuming $k \le l$ and reducing by going from right to left and from the bottom up.

Analyzing this graph as we have done in the last proof, we see that we can choose $x_1^s x_2^{k-s} x_4^l \bar{x}^J e_j$, $s = 0,1,\dots,k$, in all components $j \ge 5$, to appear in the normal form; if $k > l$ the conclusion is similar, now involving $x_1^s x_2^{k-s} x_4^l \bar{x}^J e_j$, $s = 0,1,\dots,l$.

For the two first components, and similarly for the third and fourth, we can think of the respective components as two copies of the graph above, one for each component, with arrows from an element in the first to the same element in the second; as before, the reduced graph for the second (or fourth) component is analogous to those obtained above for all components $j \ge 5$, but the reduced graph for the first (or third) contains an extra 'diagonal' (figure 6).

We have already seen that type of structure in the previous proof: the end result is that the second component behaves exactly as those

$$
\begin{array}{ccccccc}
x_2^k x_4^l \bar{x}^J e_j & \longrightarrow & x_2^k x_3 x_4^{l-1} \bar{x}^J e_j & \longrightarrow & \dots & \longrightarrow & x_2^k x_3^l \bar{x}^J e_j \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
x_1 x_2^{k-1} x_4^l \bar{x}^J e_j & \longrightarrow & x_1 x_2^{k-1} x_3 x_4^{l-1} \bar{x}^J e_j & \longrightarrow & \dots & \longrightarrow & x_1 x_2^{k-1} x_3^l \bar{x}^J e_j \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\dots & \longrightarrow & \dots & \longrightarrow & \dots & \longrightarrow & \dots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
x_1^k x_4^l \bar{x}^J e_j & \longrightarrow & x_1^k x_3 x_4^{l-1} \bar{x}^J e_j & \longrightarrow & \dots & \longrightarrow & x_1^k x_3^l \bar{x}^J e_j
\end{array}
$$

$$
\begin{array}{ccccccc}
x_2^k x_4^l \bar{x}^J e_j & \longrightarrow & x_2^k x_3 x_4^{l-1} \bar{x}^J e_j & \longrightarrow & \dots & \longrightarrow & x_2^k x_3^k x_4^{l-k} \bar{x}^J e_j \\
\downarrow & & \downarrow & & \downarrow & & \\
x_1 x_2^{k-1} x_4^l \bar{x}^J e_j & \longrightarrow & x_1 x_2^{k-1} x_3 x_4^{l-1} \bar{x}^J e_j & \longrightarrow & \dots & & \\
\downarrow & & \downarrow & & & & \\
\dots & \longrightarrow & \dots & & & & \\
\downarrow & & & & & & \\
x_1^k x_4^l \bar{x}^J e_j & & & & & &
\end{array}
$$

FIGURE 5. Graph and reduced graph, $k \leq l$ and $j \geq 5$

$$
\begin{array}{ccccccccc}
x_2^k x_4^l & \longrightarrow & x_2^k x_3 x_4^{l-1} & \longrightarrow & \dots & \longrightarrow & x_2^k x_3^k x_4^{l-k} & \longrightarrow & x_2^k x_3^{k+1} x_4^{l-l-1} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
x_1 x_2^{k-1} x_4^l & \longrightarrow & x_1 x_2^{k-1} x_3 x_4^{l-1} & \longrightarrow & \dots & \longrightarrow & x_1 x_2^{k-1} x_3^k x_4^{l-4} & & \\
\downarrow & & \downarrow & & \downarrow & & & & \\
\dots & \longrightarrow & \dots & \longrightarrow & \dots & & & & \\
\downarrow & & \downarrow & & & & & & \\
x_1^k x_4^l & \longrightarrow & x_1^k x_3 x_4^{l-1} & & & & & &
\end{array}
$$

FIGURE 6. Reduced graph, $k \leq l$ $(\bar{x}^J e_1, \bar{x}^J e_3$ omitted$)$

for $j \geq 5$, thus we can choose $x_1^s x_2^{k-s} x_4^l \bar{x}^J e_2$, $s = 0, 1, \dots, \min(k, l)$, for the normal form, and in the first component we see going along diagonals, that all other terms can be killed by creating new terms in $x_1^s x_2^{k-s} x_4^l \bar{x}^J e_1$, $s = 0, 1, \dots, \min(k, l)$, and eventually terms in the second component. The result is similar for the third and fourth components and this finishes the proof. $\square$

4. Linearization

In all cases considered below, we take $\mathcal{G}$ as a subset of the resonant monomials that belong to the image of L_A, and for which we can construct a vector $\mu \in \mathbb{R}^n$ such that:

- $\mathcal{G}$ is exactly the subset of resonant monomials for which the inner product of the corresponding vectors with μ is bigger than $c \geq 0$.
- $\mathcal{U}$ is a subset of resonant monomials for which the inner product with μ (abusing notation, see remark 2) is not smaller than c.

Remark 6. We will identify a vector (λ or μ) with the linear map on $\mathcal{M} \subset \mathbb{Z}^n$ given by the inner product with that vector. Abusing notation as before, the value of this linear map on a monomial $x^I e_i$ is its value at the vector $I - e_i \in \mathcal{M}$; when considering λ, for instance, this leads to the value $\lambda \cdot I - \lambda_i$.

It will be necessary to show that $\mathcal{G} \subset [A, \mathcal{U}]_{\mathbb{R}} = \mathcal{G}_{\mathbb{R}}$, but $\mathcal{G} + \mathcal{U} \subset \mathcal{G}$ will follow immediately:

$$\mu \cdot \mathcal{G} > c, \ \mu \cdot \mathcal{U} \geq c \Longrightarrow \mu \cdot (\mathcal{G} + \mathcal{U}) > c \Longrightarrow \mathcal{G} + \mathcal{U} \subset \mathcal{G}$$

Proposition 1. *The vector field $X(x) = Ax + a(x)$ is formally linearizable if the nonlinearity $a(x)$ is such that all resonant monomials in $\mathcal{C}$ are in $\mathcal{G}$, for μ and c as follows:*

- *If there is only one Jordan block*
  - *of dimension $m = 2$: $\mu = e_1, \quad c = 0$*
  - *of dimension $m = 3$: $\mu = e_1 - e_3, \quad c = 1$*
- *If there are exactly two Jordan blocks, of dimension $m_1 = m_2 = 2$: $\mu = e_1 - e_2 + e_3 - e_4, \ c = 1$.*

Remark 7. The above proposition does not assume knowledge of the eigenvalues; in concrete cases its statement can sometimes be improved, as shown in subsection 5.1

Proof. We consider $m = 2$, with $\mu = e_1$ and $c = 0$; then it follows that:

$$\mathcal{G} = \{x^{I+2e_1}e_1, x^{I+e_1}e_2, \dots, x^{I+e_1}e_n, \text{ resonant}\}$$

and $\mathcal{G} \subset \text{Im}\,(L_A)$ from the analysis of the graphs we have done in theorem 2. We take:

$$\mathcal{U} = \{x^{I+e_1+e_2}e_1, x^{I+e_2}e_2, \dots, x^{I+e_2}e_n\}$$

It is clear that $\mu \cdot \mathcal{U} \geq 0$ (the inner product involves the vectors that represent the monomials in $\mathcal{U}$: $\mu \cdot P_i^I = \mu \cdot I - \mu_i$).

Going back to the graph considered in the proof of theorem 2, and considering only monomials in $\mathcal{U}$, the connected components are:

$$x_2^k \bar{x}^{\bar{I}} e_i \longrightarrow x_1 x_2^{k-1} \bar{x}^{\bar{I}} e_i \longrightarrow \cdots \longrightarrow x_1^{k-1} x_2 \bar{x}^{\bar{I}} e_i \longrightarrow x_1^k \bar{x}^{\bar{I}} e_i$$

if $i \geq 3$, and those of the type of figure 7.

FIGURE 7. $L_A(\mathcal{U}) = \mathcal{G}$

Since any arrow ends in an element of $\mathcal{G}$, we conclude that $L_A(\mathcal{U}) = \mathcal{G}$, and the proof is finished for this case.

We consider $m = 3$ next, with $\mu = e_1 - e_3$ and $c = 1$; then:

$$\mathcal{G} = \{x_1^{i_1+3}x_2^{i_2}(x_1x_3)^k\bar{x}^{\bar{I}}e_1, x_1^{i_1+2}x_2^{i_2}(x_1x_3)^k\bar{x}^{\bar{I}}e_2,\\ x_1^{i_1+1}x_2^{i_2}(x_1x_3)^k\bar{x}^{\bar{I}}e_3, \ldots, x_1^{i_1+1}x_2^{i_2}(x_1x_3)^k\bar{x}^{\bar{I}}e_n, \text{ resonant}\}$$

and $\mathcal{G} \subset \mathrm{Im}\,(L_A)$ from the analysis of the graphs we have done in theorem 3: these are part of the monomials eliminated in the process of reducing the connected components. We take:

$$\mathcal{U} = \{x_1^{i_1+2}x_2^{i_2+1}(x_1x_3)^k\bar{x}^{\bar{I}}e_1, x_1^{i_1+1}x_2^{i_2+1}(x_1x_3)^k\bar{x}^{\bar{I}}e_2,\\ x_1^{i_1}x_2^{i_2+1}(x_1x_3)^k\bar{x}^{\bar{I}}e_3, \ldots, x_1^{i_1}x_2^{i_2+1}(x_1x_3)^k\bar{x}^{\bar{I}}e_n, \text{ resonant}\}$$

It is clear that $\mu \cdot \mathcal{U} \geq 1$, and it is easy to conclude that the monomials in $L_A(\mathcal{U})$ are exactly those in $\mathcal{G}$: applying L_A to a monomial $x^I e_i \in \mathcal{U}$ gives new monomials (see lemma 1) of the form $x^I e_{i+1}$ (if $i = 1$ or $i = 2$), or $x^{I-e_2+e_1}e_i$ (changing one x_2 into x_1), or $x^{I-e_3+e_2}e_i$ (changing one x_3 into x_2); in all three cases the resulting monomials are in $\mathcal{G}$.

Finally, we consider $m_1 = m_2 = 2$; we take $\mu = e_1 - e_2 + e_3 - e_4$ and $c = 1$. We have seen before that, in this case, a resonant monomial $x_1^{i_1}x_2^{i_2}x_3^{i_3}x_4^{i_4}\bar{x}^{\bar{I}}e_i$ is in the image of L_A if:

$$\begin{aligned} &i_1 + i_3 > \min(k = i_1 + i_2, l = i_3 + i_4), && i \neq 1, 3\\ &i_1 + i_3 > \min(k = i_1 + i_2, l = i_3 + i_4) + 1, && i = 1, 3 \end{aligned}$$

On the other hand, since $\min(k, l) \leq (k + l)/2$ it follows that then $[i_1 - i_2 + i_3 - i_4 > 0] \Longrightarrow [i_1 + i_3 > \min(k = i_1 + i_2, l = i_3 + i_4)]$.

Thus if μ is bigger than $c = 1$ for a vector representing a resonant monomial, then that monomial is in the image of L_A.

We can take: $\mathcal{U} = \{x^I e_i \in \mathcal{R},\ \mu(P_i^I) = \mu \cdot I - \mu_i \geq c = 1\}$ and it is easy to see that all monomials in $L_A(\mathcal{U})$ belong to $\mathcal{G}$: if the monomial $x^J e_j$ appears in $L_A(x^I e_i)$, then $\mu(P_j^J) > \mu(P_i^I)$; therefore $L_A(\mathcal{U})_{\mathbb{R}} \subset \mathcal{G}_{\mathbb{R}}$. Returning to the graphs considered in the proof of theorem 4, we see that those corresponding to $\mathcal{U}$ are obtained from those corresponding to $\mathcal{G}$ by joining an extra 'diagonal' on the left, and the monomials in each diagonal are the image of linear combinations of those in the previous (to the left) diagonal; thus $\mathcal{G} \subset L_A(\mathcal{U})_{\mathbb{R}}$ and $\mathcal{G}_{\mathbb{R}} \subset L_A(\mathcal{U})_{\mathbb{R}}$. □

Example 2. If there is only one Jordan block, of dimension $m = 2$, then X is linearizable if:

$$a(x) = (x_1^2\varphi_1(x), x_1\varphi_2(x), \ldots, x_1\varphi_n(x))$$

This follows from proposition 1 with $\mu = e_1$.

Similarly, if there is only one Jordan block, of dimension $m = 3$, then X is linearizable if:

$$a(x) = (x_1^3\varphi_1(\xi), x_1^2\varphi_2(\xi), x_1\varphi_3(\xi), \ldots, x_1\varphi_n(\xi)) \qquad \xi = (x_1, x_2, x_1x_3, \bar{x})$$

In particular, if we consider a vector field X in $\mathbb{R}^3$ with nilpotent linear part, then X is linearizable if it has the form:

$$X(x, y, z) = (0, x, y) + (x^3\varphi_1(x, y, xz), x^2\varphi_2(x, y, xz), x\varphi_3(x, y, xz))$$

5. Applications: vector fields in $\mathbb{R}^4$

Here we will be concerned only with vector fields whose linear part (in the Jordan canonical form) is not diagonal, with two Jordan blocks of dimension two or one block of dimension three; more specifically, we consider an example of each type.

In these cases, assuming resonance, there are no small denominators problems; therefore for holomorphic, or real analytic, vector fields, formal linearization implies holomorphic, respectively real analytic, linearization. The situation is not as simple for smooth vector fields, when hyperbolicity or quasi-hyperbolicity are not guaranteed.

5.1. $\lambda = (1, 1, 1, 3)$. The resonant normal form (theorem 3) is:

$$\dot{x} = x, \quad \dot{y} = x + y, \quad \dot{z} = y + z, \quad \dot{w} = 3w + \alpha xz^2 + \beta z^3$$

Since all resonant monomials are in the fourth component, we see from the proof of proposition 1 that its statement can be improved: we can take $\mu = e_1 - e_3$ and $c = 0$. It is even better to consider the general results of theorem 1 and its corollary, by defining:

$$\mathcal{G} = \{xy^2e_4, x^2ze_4, x^2ye_4, x^3e_4, xyze_4, y^3e_4, yz^2e_4\}$$
$$\mathcal{U} = \{xy^2e_4, x^2ze_4, x^2ye_4, x^3e_4, xyze_4, y^3e_4, y^2ze_4, xz^2e_4, z^3e_4\}$$

and of course, then $\mathcal{B} = \{y^2ze_4, xz^2e_4, z^3e_4\}$. Note that the sum of any vector corresponding to a monomial in $\mathcal{G}$ with any other vector corresponding to a monomial in $\mathcal{U}$ does not belong to $\mathcal{M}$ and therefore $\mathcal{G} + \mathcal{U} \subset \mathcal{G}$ is verified in an empty way.

As there are no resonant monomials of degree bigger than 3, the normal form of X should be determined by its 3-jet j^3X:

- we can disregard all non resonant monomials of degree 3, and also those resonant ones for which μ is non negative (xy^2e_4, x^2ze_4, x^2ye_4, x^3e_4, $xyze_4$, y^3e_4) as these can all be killed.
- yz^2e_4, for which $\mu = -2$, belongs to $\mathcal{G}$ and so can also be killed.
- the presence of y^2ze_4 means that xz^2e_4, should be present in the normal form; of course the presence of z^3e_4, or xz^2e_4, in j^3X implies its presence in the normal form as well.

Here we consider a generic choice of coefficients: $k(2y^2ze_4 + xz^2e_4)$ can be killed since $L_A(-yz^2e_4) = 2y^2ze_4 + xz^2e_4$, but we do not treat these cases where there is a special numeric relation between the coefficients.

	$\lambda = -1$ *	$\lambda = 1$	$\lambda = 3$	$\lambda = 5$
$\mu = -3$ **		z^2e_1		
$\mu = -2$	z^2e_4	$z^2e_2,\ yze_1$	zwe_1	
$\mu = -1$	yze_4	$z^2e_3,\ xze_1$ $yze_2,\ y^2e_1$ zwe_4	$ywe_1,\ zwe_2$	w^2e_1
$\mu = 0$	$xze_4,\ y^2e_4$	$xye_1,\ xze_2$ $y^2e_2,\ yze_3$ ywe_4	$xwe_1,\ ywe_2$ $zwe_3,\ w^2e_4$	w^2e_2
$\mu = 1$	xye_4	$x^2e_1,\ xye_2$ $y^2e_3,\ xze_3$ xwe_4	$ywe_3,\ xwe_2$	w^2e_3
$\mu = 2$	x^2e_4	$x^2e_2,\ xye_3$	xwe_3	
$\mu = 3$		x^2e_3		

$^* : \lambda(x^I e_i) = \lambda \cdot I - \lambda_i, \qquad {}^{**} : \mu(x^I e_i) = \mu \cdot I - \mu_i$

TABLE 1. Quadratic monomials, $\lambda = (1, 1, 1, 3)$ and μ chosen as $\mu = e_1 - e_3$

All the same, this explains why y^2ze_4 is in $\mathcal{B}$ but it is not necessary for the normal form (as in remark 4), and also why the linearizable vector fields have codimension 2 in the space of all vector fields with this linear part: that is the codimension of the image of L_A, or the dimension of

$$\ker L_{A^T} = \{z^3e_4,\ y^2ze_4 - 2xz^2e_4\}_{\mathbb{R}}$$

when dealing with the Belitskii normal form:

$$\dot{x} = x, \quad \dot{y} = x + y, \quad \dot{z} = y + z, \quad \dot{w} = 3w + a(y^2z - 2xz^2) + bz^3$$

To study the influence of the quadratic monomials in the normal form, the only linear combinations we have to consider are sums of two vectors corresponding to them: these (can) correspond to monomials of degree 3, but monomials corresponding to linear combinations with bigger coefficients have bigger degree and cannot be resonant.

If for all monomials $\lambda > 1$, or $\mu \geq 0$ (table 1), there are no resonant terms in the normal form: either $\lambda > 0$ for the sum, therefore the corresponding monomials are not resonant, or $\lambda = 0$ with $\mu \geq 0$, and the corresponding monomials are resonant but can still be killed.

If quadratic terms for which $\mu < 0$ with $\lambda = 1$ and $\lambda = -1$ are present, then in general the normal form is not just the linear part, but we can in many cases identify it, taking in account the value of μ for the sum of the monomials, as in:

Example 3. Let $X = (x, x+y, y+z, 3w) + a(x, y, z, w)$ with:

$$a(x, y, z, w) = (ayz + bz^2, cz^2, dyz, exz) + \dots$$

where $\dots$ denotes terms of order at least 4. For generic values of the coefficients, its normal form can be written as:

$$\dot{x} = x, \quad \dot{y} = x + y, \quad \dot{z} = y + z, \quad \dot{w} = 3w + \beta z^3$$

In fact, as $\mu = 0$ for yze_3 and xze_4, $\mu = -2$ for z^2e_2 and yze_1, and $\mu = -3$ for z^2e_1, the normal form can only include terms with $\mu = -3$, i.e. $\alpha = 0$: the monomial z^3e_4 corresponds to $(-1, 0, 2, 0) + (1, 0, 1, -1) = (0, 0, 3, -1)$, and therefore z^3e_4 has to be present in the normal form.

The case $\lambda = (1, 1, 1, k)$, with $k > 3$ is similar, but the identification of the normal form is increasingly labour consuming.

Remark 8. We saw that all monomials in a diagonal, where the linear map μ is constant, can be killed by creating new terms on one monomial in that same diagonal; thus if we know the value of the linear map μ we can identify terms that do not appear in the normal form.

5.2. $\lambda = (1, 1, -1, -1)$**.** The resonant normal form (theorem 4) gives a nonlinearity of the form:

$$(y\varphi_1(xw, yw), y\varphi_2(xw, yw), w\varphi_3(xw, yw), w\varphi_4(xw, yw))$$

and thus, writing only the lower order terms of the vector field:

$$\begin{aligned}
\dot{x} &= x && + a_{11}xyw + a_{12}y^2w + a_{13}x^2yw^2 + a_{14}xy^2w^2 + a_{15}y^3w^2 + \dots \\
\dot{y} &= x + y && + a_{21}xyw + a_{22}y^2w + a_{23}x^2yw^2 + a_{24}xy^2w^2 + a_{25}y^3w^2 + \dots \\
\dot{z} &= -z && + a_{31}xw^2 + a_{32}yw^2 + a_{33}x^2w^3 + a_{34}xyw^3 + a_{35}y^2w^3 + \dots \\
\dot{w} &= z - w && + a_{41}xw^2 + a_{42}yw^2 + a_{43}x^2w^3 + a_{44}xyw^3 + a_{45}y^2w^3 + \dots
\end{aligned}$$

where $\dots$ stand for terms of order at least 7.

The resonant normal form is not polynomial, and we cannot identify it by studying a finite jet of the vector field X under consideration. We will consider vector fields with only linear and quadratic terms, as an example of the type of information we can get about the lower order terms in the normal form.

We remark that, for the resonant monomials in this case, $\mu > 0$ is equivalent to $\mu > 1$. The vector field X will be linearizable if for all quadratic terms we have $\lambda = 1, 3$ for all of them, or $\lambda = -1, -3$, or yet $\mu = 1, 3$; these are the simplest cases, but there are many other possibilities, for instance: we can take all terms for which $\lambda = 3$ and $\mu \geq -1$, then those with $\lambda = \pm 1$ and $\mu \geq 1$, and $\lambda = -3$ and $\mu = 3$.

If X is not linearizable, it is important to recognize when there will be no third order terms in the normal form; according to Sell theorem [9] (or Samovol theorem, [1]), then X is C^2 conjugate to its linear part.

	$\lambda=-3$	$\lambda=-1$	$\lambda=1$	$\lambda=3$
$\mu=-3$	w^2e_1	w^2e_3, ywe_1	ywe_3, y^2e_1	y^2e_3
$\mu=-1$	w^2e_2, zwe_1	ywe_2, w^2e_4 xwe_1, yze_1 zwe_3	y^2e_2, ywe_4 xye_1, yze_3 xwe_3	y^2e_4, xye_3
$\mu=1$	z^2e_1, zwe_2	z^2e_3 , xze_1 zwe_4, yze_2 xwe_2	x^2e_1, xze_3 xwe_4, yze_4 xye_2	x^2e_3, xye_4
$\mu=3$	z^2e_2	z^2e_4, xze_2	x^2e_2, xze_4	x^2e_4

TABLE 2. Quadratic monomials, $\lambda=(1,1,-1,-1)$

We can take, for instance, the monomials for which $\lambda=\pm3$ and $\mu=-3$, together with those for which $\lambda=\pm1$ and $\mu\geq1$: no sum of points for which $\lambda=\pm3$ corresponds to a monomial, and for all other sums we have $\lambda\neq0$ or $\lambda=0$ with $\mu>1$.

Remark 9. This can be extended to any vector field for which the 2-jet is as above, if $\mu>0$ for the 3-order resonant terms.

For these quadratic vector fields, as $\mu\geq0$ for all linear combinations leading to resonant monomials, the normal form is simpler (remark 8):

$$\begin{aligned}
\dot{x} &= x \\
\dot{y} &= x+y+\alpha_1x^2yw^2+\alpha_2x^3yw^3+\dots &&= x+y+yx^2w^2\psi_1(xw) \\
\dot{z} &= -z \\
\dot{w} &= z-w+\beta_1x^2w^3+\beta_2x^3w^4+\dots &&= z-w+x^2w^3\psi_2(xw)
\end{aligned}$$

where now ... stand for terms of order at least 9.

This analysis can in principle be extended to higher order terms, and to vector fields having a certain k-jet, but if that is certainly feasible in a given example, and this is the important fact, it does not seem worthwhile to try to study all possible cases.

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