Complemented congruences on double Ockham algebras

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Abstract. For $n \in \mathbb{N}$ and $m \in \mathbb{N}_0$, an algebra $\mathcal{L} = (L, \wedge, \vee, f, g, 0, 1)$ of type $(2, 2, 1, 1, 0, 0)$ is said to be a double $\mathrm{K}_{n,m}$-algebra, if $\mathcal{L}$ is a double Ockham algebra that satisfies the identities $f^{2n+m} = f^m$, $g^{2n+m} = g^m$, $fg = g^{2zn}$ and $gf = f^{2zn}$, where z is the smallest natural number greater than or equal to $m/2n$. In [2], T. Blyth, A. Noor and J. Varlet study congruences on some double $\mathrm{K}_{1,1}$-algebras. They describe the complement (when it exists) of a principal congruence and, using this description, they also determine when the complement exists. In this paper we generalize this work for double $\mathrm{K}_{\mathrm{n,m}}$-algebras.

1. Preliminaries

The variety $\mathbf{O}$ of Ockham algebras is the class of all algebras $(L, \wedge, \vee, h, 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that $(L, \wedge, \vee, 0, 1)$ is a bounded distributive lattice and h is a dual endomorphism of this lattice, i.e., $h(0) = 1$, $h(1) = 0$, $h(x \wedge y) = h(x) \vee h(y)$ and $h(x \vee y) = h(x) \wedge h(y)$. These algebras were defined by J. Berman in [1]. We write (L, h) for an Ockham algebra $(L, \wedge, \vee, h, 0, 1)$ and we represent both the universe L and the lattice $(L, \wedge, \vee, 0, 1)$ by L. The subvariety of $\mathbf{O}$ characterized by the identity $h^{2n+m} = h^m$, $n \in \mathbb{N}$ and $m \in \mathbb{N}_0$, is denoted by $\mathbf{K}_{n,m}$ and the elements of this class are called $\mathrm{K}_{n,m}$-algebras. Further information about Ockham algebras and $\mathrm{K}_{n,m}$-algebras can be found in [1] and [3].

For each $\mathcal{L} = (L, h) \in \mathbf{O}$, and for all $n \in \mathbb{N}$ and $m \in \mathbb{N}_0$, the sets $h^m(L)$ and $L_{n,m} = \{x \in L : h^{2n+m}(x) = h^m(x)\}$ are subuniverses of $\mathcal{L}$. By $h^m(\mathcal{L})$ and $\mathcal{L}_{n,m}$ we denote the subalgebras $(h^m(L), h)$ and $(L_{n,m}, h)$ of $\mathcal{L}$, respectively. It is useful to notice that, if $\mathcal{L} \in \mathbf{K}_{n,m}$ then $h^m(\mathcal{L}) \in \mathbf{K}_{n,0}$.

Associated to Ockham algebras we have the notion of double Ockham algebras, introduced by M. Sequeira in [5]. A double Ockham algebra is an algebra $\mathcal{L} = (L, \wedge, \vee, f, g, 0, 1)$ of type $(2, 2, 1, 1, 0, 0)$ such that $(L, \wedge, \vee, f, 0, 1)$ and $(L, \wedge, \vee, g, 0, 1)$ are Ockham algebras. The variety of double Ockham algebras is represented by $\mathbf{O}_2$. We denote a double Ockham algebra $\mathcal{L} = (L, \wedge, \vee, f, g, 0, 1)$ by $\mathcal{L} = (L, f, g)$ and we represent by L, both, the universe L and the distributive lattice $(L, \wedge, \vee, 0, 1)$. For the Ockham algebras that are reduct of $\mathcal{L} = (L, f, g)$ we write (L, f) and (L, g).

Let $\mathcal{L} = (L, f, g) \in \mathbf{O}_2$. For each $h \in \{f, g\}$, and all $n \in \mathbb{N}$ and all $m \in \mathbb{N}_0$, we represent by $L^h_{m,n}$ the set $\{x \in L : h^{2n+m}(x) = h^m(x)\}$. We write $(L^f_{n,m}, f)$ and

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$(L^g_{n,m}, g)$ for the greatest subalgebras of (L, f) and (L, g), respectively, that belong to $\mathbf{K}_{n,m}$.

Let $n, m \in \mathbb{N}$ and let q be the smallest natural number that is greater than or equal to $m/2n$; in what follows this element will be denoted by $\lceil m/2n \rceil$. The subvariety of $\mathbf{O}_2$ characterized by the identities $f^{2n+m} = f^m$, $g^{2n+m} = g^m$, $gf = f^{2qn}$, $fg = g^{2qn}$ is represented by $\mathbf{DK}_{n,m}$, [5], and the elements of this variety are called double $\mathrm{K}_{n,m}$-algebras.

Given $\mathcal{L} = (L, f, g) \in \mathbf{DK}_{n,m}$, we have that $f^m(L)$ is a subuniverse of $\mathcal{L}$. So, $(f^m(L), f, g)$ is a subalgebra of $\mathcal{L}$, that we denote by $f^m(\mathcal{L})$, and the Ockham algebras $(f^m(L), f)$ and $(f^m(L), g)$ are subalgebras of (L, f) and (L, g), respectively.

About double $\mathrm{K}_{n,m}$-algebras it is useful to remind that if $\mathcal{L} = (L, f, g) \in \mathbf{DK}_{n,m}$, then $f^{2n+k} = f^k$ and $g^{2n+k} = g^k$, for all $k \geq m$. We denote by $r(t)$ the remainder of the integer t on division by $2n$ and, for $1 \leq i, j \leq 2n+m-1$, let $z_{i,j} = m + r(j-i-m)$. Taking into account the relation between operations f and g it follows that:

Lemma 1.1. [5, Proposition 2] *Let $n, m \in \mathbb{N}$, $\mathcal{L} = (L, f, g) \in \mathbf{DK}_{n,m}$ and $q = \lceil m/2n \rceil$. Then*

i) $f^i g^i = g^{q2n}$, $g^i f^i = f^{q2n}$, $1 \leq i \leq 2n + m - 1$.
ii) $g^i f^j = f^{z_{i,j}}$, $f^j g^i = g^{z_{j,i}}$, $1 \leq i, j \leq 2n + m - 1$.
iii) $f^m(L) = g^m(L)$. □

We now present some notation related to congruences. Given an algebra $\mathcal{L}$ (element of $\mathbf{O}$ or element of $\mathbf{O_2}$) we denote by:

- $\mathrm{Con_{lat}}\,\mathcal{L}$ and $\mathrm{Con}\,\mathcal{L}$, the congruence lattice of the distributive lattice L (reduct of $\mathcal{L}$) and the algebra $\mathcal{L}$, respectively;
- $\theta_{\mathrm{lat}}(a, b)$ and $\theta(a, b)$ the least congruence of $\mathrm{Con_{lat}}\,\mathcal{L}$ and $\mathrm{Con}\,\mathcal{L}$, respectively, that identifies the elements a and b of L;
- $\mathbf{0}$ and $\mathbf{1}$ the identity and the universal congruence of $\mathcal{L}$, respectively;
- $\theta_{L'}$ a congruence defined on a subalgebra $\mathcal{L}'$ of $\mathcal{L}$ ($\mathbf{0}_{L'}$ and $\mathbf{1}_{L'}$ represent, respectively, the identity and the universal congruences of $\mathcal{L}'$).

For $\mathcal{L} = (L, f, g) \in \mathbf{DK}_{n,m}$ we represent by:

- $\mathrm{Con}_f\,\mathcal{L}$ and $\mathrm{Con}_g\,\mathcal{L}$, the congruence lattice of the algebra (L, f) and the algebra (L, g), respectively;
- $\theta_f(a, b)$ and $\theta_g(a, b)$ the least congruence of $\mathrm{Con}_f\,\mathcal{L}$ and $\mathrm{Con}_g\,\mathcal{L}$, respectively, that identifies the elements a and b of L;
- $\theta_{f,f^m(L)}(a, b)$, $\theta_{g,f^m(L)}(a, b)$ the least congruence of $\mathrm{Con}_f\,f^m(\mathcal{L})$ and $\mathrm{Con}_g\,f^m(\mathcal{L})$, respectively, that identifies the elements a and b of $f^m(L)$.

Remark: Let $\mathcal{L} = (L, f, g) \in \mathbf{DK}_{n,m}$. Given $\theta_f \in \mathrm{Con}_f\,\mathcal{L}$ and $\theta_g \in \mathrm{Con}_g\,\mathcal{L}$, since θ_f, $\theta_g \in \mathrm{Con}_{\mathrm{lat}}\,\mathcal{L}$, we represent by $\theta_f \vee \theta_g$ and $\theta_f \wedge \theta_g$, respectively, the join and the meet of θ_f and θ_g on $\mathrm{Con}_{\mathrm{lat}}\,\mathcal{L}$.

To study principal congruences of $\mathcal{L} = (L, f, g) \in \mathbf{O_2}$ it suffices to consider the congruence $\theta(a, b)$ for $a \leq b$ since, for any congruence θ of a lattice L' and any $x, y \in L'$, we have $(x, y) \in \theta$ if and only if $(x \wedge y, x \vee y) \in \theta$.

For any $\mathcal{L} \in \mathbf{O}$ (resp. $\mathbf{O}_2$), the lattice $\mathrm{Con}\,\mathcal{L}$ is distributive. Also, for any subalgebra $\mathcal{L}'$ of an algebra $\mathcal{L} \in \mathbf{O}$, each congruence defined on $\mathcal{L}'$ is the restriction of some congruence defined on $\mathcal{L}$. This means that the variety $\mathbf{O}$ satisfies the congruence extension property. Consequently we have the following:

Lemma 1.2. *If $\mathcal{L} \in \mathbf{O}$, $\mathcal{L}'$ is a subalgebra of $\mathcal{L}$ and $a, b \in L'$, then*

$$\theta(a, b)|_{L'} = \theta_{L'}(a, b). \qquad \square$$

The following result, that establishes that any principal congruence on $\mathcal{L} \in \mathbf{K}_{n,m}$ is the join of principal congruences on the distributive lattice L, is fundamental in the investigation of congruences defined on $\mathrm{K}_{n,m}$-algebras.

Lemma 1.3. [1, Corollary Theorem 1] *If $\mathcal{L} = (L, h) \in \mathbf{K}_{n,m}$ and $a, b \in L$ with $a \leq b$ then*

$$\theta(a, b) = \bigvee_{i=0}^{2n+m-1} \theta_{\mathrm{lat}}\big(h^i(a), h^i(b)\big). \qquad \square$$

For double $\mathrm{K}_{n,m}$-algebras it is also possible to establish a result similar to this one:

Lemma 1.4. [5] If $\mathcal{L} = (L, f, g) \in \mathbf{DK}_{n,m}$ and $a, b \in L$ with $a \leq b$, then

$$\theta(a, b) = \theta_{\mathrm{lat}}(a, b) \vee \bigvee_{i=1}^{2n+m-1} \theta_{\mathrm{lat}}\big(f^i(a), f^i(b)\big) \vee \bigvee_{j=1}^{2n+m-1} \theta_{\mathrm{lat}}\big(g^j(a), g^j(b)\big). \qquad \square$$

From Lemmas 1.3 and 1.4 is immediate that:

Lemma 1.5. *If $\mathcal{L} = (L, f, g) \in \mathbf{DK}_{n,m}$ and $a, b \in L$ are such that $a \leq b$, then*

$$\theta(a, b) = \theta_f(a, b) \vee \theta_g(a, b). \qquad \square$$

Definition 1.6. By a *p-ladder* in an ordered set E we shall mean a subset of E that consists of two p-chains $a_1 < ... < a_p$ and $b_1 < ... < b_p$ such that $a_i \leq b_i$ for $i = 1, ..., p$. We shall denote a p-ladder by $(a_i, b_i)_p$.

Let $T = \{0, 1, ..., n-1\}$ and, for $s \in \{1, ..., n\}$, let $T_s = \{J : J \subseteq T, |J| = s\}$. Let $\mathcal{L} = (L, h) \in \mathbf{K}_{n,m}$ and $a, b \in L$ be such that $a \leq b$. For $s \in \{1, ..., n\}$, let

$$\widetilde{a}_{h,s} = \bigwedge_{J \in T_s} \bigvee_{j \in J} h^{2j}(a), \qquad \widetilde{b}_{h,s} = \bigwedge_{J \in T_s} \bigvee_{j \in J} h^{2j}(b).$$

It is easy to prove that the set $\{\widetilde{a}_{h,s}, \widetilde{b}_{h,s} : s = 1, ..., n\}$ is an n-ladder consisting of elements that belong to the subalgebra $\mathcal{L}_{1,m}$. In the following theorem, which is an unpublished result of M. Sequeira, this n-ladder is used to establish that any principal congruence defined on a double $\mathrm{K}_{n,m}$-algebra $\mathcal{L} = (L, f, g)$ is the join of principal congruences generated by elements of $L_{1,m}$.

Theorem 1.7. *Let $\mathcal{L} = (L, h) \in \mathbf{K}_{n,m}$ and $a, b \in L$ be such that $a \leq b$. Then*

$$\theta(a, b) = \bigvee_{s=1}^{n} \theta(\widetilde{a}_{h,s}, \widetilde{b}_{h,s}). \qquad \square$$

Next Lemma follows immediately from Theorem 1.7 and Lemma 1.5 and describes each principal congruence defined on a double $\mathrm{K}_{n,m}$-algebra $\mathcal{L} = (L, f, g)$ by means of elements of $L^f_{1,m}$ and elements of $L^g_{1,m}$.

Lemma 1.8. *If $\mathcal{L} = (L, f, g) \in \mathbf{DK}_{n,m}$ and $a, b \in L$ with $a \leq b$, then*

$$\theta(a, b) = \bigvee_{s=1}^{n} \theta_f(\widetilde{a}_{f,s}, \widetilde{b}_{f,s}) \vee \bigvee_{t=1}^{n} \theta_g(\widetilde{a}_{g,t}, \widetilde{b}_{g,t}). \qquad \square$$

The purpose of this paper is to characterize the principal congruences $\theta(a, b)$ on double $\mathrm{K}_{n,m}$-algebras that are complemented. The study of these congruences is strongly related to the following theorem which establishes that, given $\mathcal{L} = (L, f) \in \mathbf{O}$, all congruences generated by elements of $L_{1,0}$ are complemented. This theorem is, also, an unpublished result of M. Sequeira [5].

Theorem 1.9. *If $\mathcal{L} = (L, h) \in \mathbf{O}$ and $a, b \in L_{1,0}$ with $a \leq b$, then $\theta(a, b)$ is complemented in* $\mathrm{Con}(\mathcal{L})$*, and*

$$\begin{aligned} \theta(a, b)' &= \theta\big(h(a) \vee b, 1\big) \vee \theta\big(h(a), h(a) \vee a\big) \vee \theta\big(b, b \vee h(b)\big) \\ &= \theta\big(0, a \wedge h(b)\big) \vee \theta\big(a \wedge h(a), a\big) \vee \theta\big(b \wedge h(b), h(b)\big). \qquad \square \end{aligned}$$

2. Congruences

Let $\mathcal{L} = (L, f, g) \in \mathbf{DK}_{n,m}$ and $a, b \in L$ be such that $a \leq b$. By Lemma 1.5, the congruence $\theta(a, b)$ is the join, on $\mathrm{Con}_{\mathrm{lat}}\, \mathcal{L}$, of a principal congruence on (L, f) and a principal congruence on (L, g). So, it is natural that the study of $\theta(a, b)$ uses various results obtained on [4]; where the author studies complemented congruences on $\mathrm{K}_{n,m}$-algebras. Moreover, similar results for double $\mathrm{K}_{n,m}$ algebras, involving the relation between the operations f and g, need to be established. We start this section establishing and proving that results.

Lemma 2.1. *Let $\mathcal{L} = (L, f, g) \in \mathbf{DK}_{n,m}$, $i \in \mathbb{N}$, $k \in \mathbb{N}$ be such that $k \geq m$ and $a, b \in L$ with $a \leq b$. Then, given $x, y \in L$*

$$(x, y) \in \theta_{\mathrm{lat}}\big(g^i(a), g^i(b)\big) \Rightarrow \big(f^k(x), f^k(y)\big) \in \theta_{\mathrm{lat}}\big(g^t(a), g^t(b)\big),$$

for some $t \in \{m, ..., 2n + m - 1\}$.

Proof. Let $x, y \in L$. If $(x,y) \in \theta_{\rm lat}\big(g^i(a), g^i(b)\big)$, for some $i \in \mathbb{N}$, then $\big(f^k(x), f^k(y)\big) \in \theta_{\rm lat}\big(f^k(g^i(a)), f^k(g^i(b))\big)$. From Lemma 1.1 it follows that $f^k\big(g^i(a)\big) = g^t(a)$ and $f^k\big(g^i(b)\big) = g^t(b)$, with $t \in \{m, ..., 2n+m-1\}$. □

Lemma 2.2. *Let $\mathcal{L} = (L, f, g) \in \mathbf{DK}_{n,m}$ and $a, b \in L$ with $a \le b$. Then*

$$\theta_g(a,b)|_{f^m(L)} = \bigvee_{k=0}^{2n+m-1} \theta_{\rm lat}\big(g^k(a), g^k(b)\big)|_{f^m(L)}.$$

Proof. The result follows immediately from [4, Lemma 2.3] since $f^m(L) = g^m(L)$, $\theta_g(a,b) \in \mathrm{Con}_g\,\mathcal{L}$ and $(L,g) \in \mathbf{K}_{n,m}$. □

Lemma 2.3. *Let $\mathcal{L} = (L, f, g) \in \mathbf{DK}_{n,m}$ and $a, b \in L$ be such that $a \le b$. Then,*

$$\theta(a,b)|_{f^m(L)} = \theta_{\rm lat}(a,b)|_{f^m(L)} \vee \bigvee_{i=1}^{2n+m-1} \theta_{\rm lat}\big(f^i(a), f^i(b)\big)|_{f^m(L)} \vee \bigvee_{j=1}^{2n+m-1} \theta_{\rm lat}\big(g^j(a), g^j(b)\big)|_{f^m(L)}.$$

Proof. By Lemma 1.4 we have

$$\theta(a,b) = \theta_{\rm lat}(a,b) \ \vee \bigvee_{i=1}^{2n+m-1} \theta_{\rm lat}\big(f^i(a), f^i(b)\big) \ \vee \bigvee_{j=1}^{2n+m-1} \theta_{\rm lat}\big(g^j(a), g^j(b)\big)$$

and it is obvious that

$$\theta_{\rm lat}(a,b)|_{f^m(L)} \ \vee \bigvee_{i=1}^{2n+m-1} \theta_{\rm lat}\big(f^i(a), f^i(b)\big)|_{f^m(L)} \ \vee \bigvee_{j=1}^{2n+m-1} \theta_{\rm lat}\big(g^j(a), g^j(b)\big)|_{f^m(L)}$$
$$\le \Big[\theta_{\rm lat}(a,b) \ \vee \bigvee_{i=1}^{2n+m-1} \theta_{\rm lat}\big(f^i(a), f^i(b)\big) \ \vee \bigvee_{j=1}^{2n+m-1} \theta_{\rm lat}\big(g^j(a), g^j(b)\big)\Big]\Big|_{f^m(L)}.$$

Let x, y be elements of L such that $(x,y) \in \theta(a,b)|_{f^m(L)}$, i.e., such that

$$(x,y) \in \Big[\theta_{\rm lat}(a,b) \ \vee \bigvee_{i=1}^{2n+m-1} \theta_{\rm lat}\big(f^i(a), f^i(b)\big) \ \vee \bigvee_{j=1}^{2n+m-1} \theta_{\rm lat}\big(g^j(a), g^j(b)\big)\Big]\Big|_{f^m(L)}.$$

Then $x, y \in f^m(L)$ and there exist $s \in \mathbb{N}$ and $x_0 = x$, x_1, ..., $x_s = y \in L$ such that, for all $v \in \{0, ..., s-1\}$,

- $(x_v, x_{v+1}) \in \theta_{\rm lat}\big(f^{i_v}(a), f^{i_v}(b)\big)$, for some $i_v \in \{0, ..., 2n+m-1\}$

or

- $(x_v, x_{v+1}) \in \theta_{\rm lat}\big(g^{j_v}(a), g^{j_v}(b)\big)$, for some $j_v \in \{1, ..., 2n+m-1\}$.

In what follows we consider $q = \lceil m/2n \rceil$. Thus, if $(x_v, x_{v+1}) \in \theta_{\rm lat}\big(f^{i_v}(a), f^{i_v}(b)\big)$ we have by [4, Lemma 2.2] that $\big(f^{q2n}(x_v), f^{q2n}(x_{v+1})\big) \in \theta_{\rm lat}\big(f^{t_v}(a), f^{t_v}(b)\big)$, for some $t_v \in \{m, ..., 2n+m-1\}$. Since $f^{q2n}(x_v), f^{q2n}(x_{v+1})$ are elements of $f^m(L)$, then $\big(f^{q2n}(x_v), f^{q2n}(x_{v+1})\big) \in \theta_{\rm lat}\big(f^{t_v}(a), f^{t_v}(b)\big)|_{f^m(L)}$.

If $(x_v, x_{v+1}) \in \theta_{\text{lat}}\big(g^{j_v}(a), g^{j_v}(b)\big)$ it is also possible to conclude, in this case using Lemma 2.1, that $\big(f^{q2n}(x_v), f^{q2n}(x_{v+1})\big) \in \theta_{\text{lat}}\big(g^{s_v}(a), g^{s_v}(b)\big)|_{f^m(L)}$, for some $s_v \in \{m, ..., 2n+m-1\}$.

Consequently

$$\big(f^{q2n}(x), f^{q2n}(y)\big) \in \theta_{\text{lat}}(a,b)|_{f^m(L)} \vee \bigvee_{i=1}^{2n+m-1} \theta_{\text{lat}}\big(f^i(a), f^i(b)\big)|_{f^m(L)} \vee \bigvee_{j=1}^{2n+m-1} \theta_{\text{lat}}\big(g^j(a), g^j(b)\big)|_{f^m(L)}$$

where $f^{q2n}(x) = x$ and $f^{q2n}(y) = y$ since $x, y \in f^m(L)$. Thus we have

$$\theta(a,b)|_{f^m(L)} \leq \theta_{\text{lat}}(a,b)|_{f^m(L)} \vee \bigvee_{i=1}^{2n+m-1} \theta_{\text{lat}}\big(f^i(a), f^i(b)\big)|_{f^m(L)} \vee \bigvee_{j=1}^{2n+m-1} \theta_{\text{lat}}\big(g^j(a), g^j(b)\big)|_{f^m(L)}. \qquad \square$$

This lemma is used to prove the following result:

Lemma 2.4. *Let $\mathcal{L} = (L, f, g) \in \mathbf{DK}_{n,m}$ and $a, b \in L$ be such that $a \leq b$. Then,*

$$\theta(a,b)|_{f^m(L)} = \bigvee_{s=1}^{n} \theta_f(\widetilde{a}_{f,s}, \widetilde{b}_{f,s})|_{f^m(L)} \vee \bigvee_{t=1}^{n} \theta_g(\widetilde{a}_{g,t}, \widetilde{b}_{g,t})|_{f^m(L)}.$$

Proof. By Lemma 2.3 we have

$$\theta(a,b)|_{f^m(L)} = \theta_{\text{lat}}(a,b)|_{f^m(L)} \vee \bigvee_{i=1}^{2n+m-1} \theta_{\text{lat}}\big(f^i(a), f^i(b)\big)|_{f^m(L)} \vee \bigvee_{j=1}^{2n+m-1} \theta_{\text{lat}}\big(g^j(a), g^j(b)\big)|_{f^m(L)}.$$

From [4, Lemma 2.3] and Lemma 2.2 it follows that

$$\theta(a,b)|_{f^m(L)} = \theta_f(a,b)|_{f^m(L)} \vee \theta_g(a,b)|_{f^m(L)}$$

and, by Theorem 1.7

$$\theta(a,b)|_{f^m(L)} = \Big(\bigvee_{s=1}^{n} \theta_f(\widetilde{a}_{f,s}, \widetilde{b}_{f,s})\Big)|_{f^m(L)} \vee \Big(\bigvee_{t=1}^{n} \theta_g(\widetilde{a}_{g,t}, \widetilde{b}_{g,t})\Big)|_{f^m(L)}.$$

Finally, using [4, Lemma 2.4] and since $f^m(L) = g^m(L)$, we have

$$\theta(a,b)|_{f^m(L)} = \bigvee_{s=1}^{n} \theta_f(\widetilde{a}_{f,s}, \widetilde{b}_{f,s})|_{f^m(L)} \vee \bigvee_{t=1}^{n} \theta_g(\widetilde{a}_{g,t}, \widetilde{b}_{g,t})|_{f^m(L)}. \qquad \square$$

Given an algebra $\mathcal{L} \in \mathbf{O}$ (resp. $\mathcal{L} \in \mathbf{O}_2$), let $\operatorname{Con}' \mathcal{L}$ represent the lattice of complemented congruences on $\mathcal{L}$.

Lemma 2.5. *Let $\mathcal{L} = (L, f, g) \in \mathbf{DK}_{n,m}$ and $\theta \in \operatorname{Con} \mathcal{L}$. If $\theta \in \operatorname{Con}' \mathcal{L}$, then $\theta|_{f^m(L)} \in \operatorname{Con}' f^m(\mathcal{L})$. In fact, if θ' is the complement of θ in $\operatorname{Con} \mathcal{L}$, then $\theta'|_{f^m(L)}$ is the complement of $\theta|_{f^m(L)}$ in $\operatorname{Con} f^m(\mathcal{L})$.*

Proof. Let $\theta \in \mathrm{Con}' \mathcal{L}$ and θ' be the complement of θ in $\mathrm{Con}\,\mathcal{L}$. Then $\theta|_{f^m(L)}$ and $\theta'|_{f^m(L)}$ are elements of $\mathrm{Con}\, f^m(\mathcal{L})$. Since $\theta, \theta' \in \mathrm{Con}_f\,\mathcal{L}$, θ' is also the complement of θ in $\mathrm{Con}_f\,\mathcal{L}$. By [4, Lemma 2.5] we have that $\theta'|_{f^m(L)}$ is the complement of $\theta|_{f^m(L)}$ in $\mathrm{Con}_f\, f^m(\mathcal{L})$ and, consequently, in $\mathrm{Con}\, f^m(\mathcal{L})$. □

Lemma 2.6. *Let $\mathcal{L} = (L, f, g) \in \mathbf{DK}_{n,m}$ and $a, b \in L$ with $a \leq b$ and $k \in \mathbb{N}$ be such that $k \geq m$. Then*

i) $\theta\big(f^k(a), f^k(b)\big) = \theta_f\big(f^k(a), f^k(b)\big)$,
ii) $\theta\big(g^k(a), g^k(b)\big) = \theta_f\big(g^k(a), g^k(b)\big)$,
iii) $\theta\big(g^k(a), g^k(b)\big) = \theta_g\big(g^k(a), g^k(b)\big)$,
iv) $\theta\big(f^k(a), f^k(b)\big) = \theta_g\big(f^k(a), f^k(b)\big)$.

Proof. i) By Lemma 1.4 we have

$$\begin{aligned}\theta\big(f^k(a), f^k(b)\big) = \theta_{\mathrm{lat}}\big(f^k(a), f^k(b)\big) \ &\vee \bigvee_{i=1}^{2n+m-1} \theta_{\mathrm{lat}}\big(f^i(f^k(a)), f^i(f^k(b))\big)\\ &\vee \bigvee_{j=1}^{2n+m-1} \theta_{\mathrm{lat}}\big(g^j(f^k(a)), g^j(f^k(b))\big).\end{aligned}$$

Since $k = m + r$, for some $r \in \mathbb{N}_0$, it follows by Lemma 1.1 that, for all $x \in L$ and $j \in \{1, ..., 2n + m - 1\}$,

$$\begin{aligned}g^j\big(f^k(x)\big) &= g^j\big(f^m(f^r(x))\big) = f^{z_{j,m}}\big(f^r(x)\big)\\ &= f^{z_{j,m}-m}\big(f^m\big(f^r(x)\big)\big) = f^{z_{j,m}-m}\big(f^k(x)\big),\end{aligned}$$

with $z_{j,m} - m \in \{0, ..., 2n - 1\}$. Thus we have

$$\theta\big(f^k(a), f^k(b)\big) = \theta_{\mathrm{lat}}\big(f^k(a), f^k(b)\big) \ \vee \bigvee_{i=1}^{2n+m-1} \theta_{\mathrm{lat}}\big(f^i(f^k(a)), f^i(f^k(b))\big)$$

and by Lemma 1.3 we conclude that $\theta\big(f^k(a), f^k(b)\big) = \theta_f\big(f^k(a), f^k(b)\big)$; so i) follows. Since $f^m(L) = g^m(L)$ we have $g^k(a) = f^m(x)$ and $g^k(b) = f^m(y)$, for some $x, y \in L$. So case ii) is immediate from i). The proof of iii) is analogous to the one for case i). Case iv) follows from iii). □

Definition 2.7. By a *m-pair*, $m \in \mathbb{N}$, we shall mean the ordered pair (k, l) such that

$$(k, l) = \begin{cases} (m, m+1) & \text{if } m \text{ is even;}\\ (m+1, m) & \text{if } m \text{ is odd.}\end{cases}$$

It is useful to notice that, if (k, l) is a m-pair then k is always even, and l is always odd.

In what follows we consider $\mathcal{L} = (L, f, g) \in \mathbf{DK}_{n,m}$ and, for $z \in L$, $T = \{0, 1, ..., n - 1\}$, $s \in T$ and $i \in \mathbb{N}$, we represent the elements $f^i(\widetilde{z}_{f,s})$ and $g^i(\widetilde{z}_{g,s})$ by $f^i(\widetilde{z}_s)$ and $g^i(\widetilde{z}_s)$, respectively. Note that we never use the elements $f^i(\widetilde{z}_{g,s})$ and $g^i(\widetilde{z}_{f,s})$. Moreover, we denote by (k, l) an m-pair.

Let $a, b \in L$ be such that $a \leq b$ and suppose that $\theta(a,b)$ is complemented. As we will see, the description of the complement of $\theta(a,b)$ is, in fact, related to Theorem 1.9.

If we take $q = \lceil m/2n \rceil$, then $f^{q2n}(\widetilde{a}_s)$, $f^{q2n}(\widetilde{b}_s) \in f^m(L)$ and $g^{q2n}(\widetilde{a}_s)$, $g^{q2n}(\widetilde{b}_s) \in g^m(L)$. Consequently, taking into account that $f^m(L) = g^m(L)$, it follows by Lemma 2.4, [4, Lemma 2.1] and Lemma 1.2 that:

$$\begin{aligned}\theta(a,b)|_{f^m(L)} &= \bigvee_{s=1}^{n} \theta_f(\widetilde{a}_{f,s}, \widetilde{b}_{f,s})|_{f^m(L)} \vee \bigvee_{t=1}^{n} \theta_g(\widetilde{a}_{g,t}, \widetilde{b}_{g,t})|_{f^m(L)} \\ &= \bigvee_{s=1}^{n} \theta_f\big(f^{q2n}(\widetilde{a}_s), f^{q2n}(\widetilde{b}_s)\big)|_{f^m(L)} \vee \bigvee_{t=1}^{n} \theta_g\big(g^{q2n}(\widetilde{a}_t), g^{q2n}(\widetilde{b}_t)\big)|_{f^m(L)} \\ &= \bigvee_{s=1}^{n} \theta_{f,f^m(L)}\big(f^{q2n}(\widetilde{a}_s), f^{q2n}(\widetilde{b}_s)\big) \vee \bigvee_{t=1}^{n} \theta_{g,f^m(L)}\big(g^{q2n}(\widetilde{a}_t), g^{q2n}(\widetilde{b}_t)\big).\end{aligned}$$

Since $\widetilde{a}_{f,s}$, $\widetilde{b}_{f,s} \in L^f_{1,m}$, $\widetilde{a}_{g,t}$, $\widetilde{b}_{g,t} \in L^g_{1,m}$ and $q2n \geq m$, then $f^{q2n}(\widetilde{a}_s)$, $f^{q2n}(\widetilde{b}_s) \in L^f_{1,0}$, and $g^{q2n}(\widetilde{a}_t)$, $g^{q2n}(\widetilde{b}_t) \in L^g_{1,0}$. So, by Theorem 1.9, the congruences $\theta_{f,f^m(L)}\big(f^{q2n}(\widetilde{a}_s), f^{q2n}(\widetilde{b}_s)\big)$ and $\theta_{g,f^m(L)}\big(g^{q2n}(\widetilde{a}_t), g^{q2n}(\widetilde{b}_t)\big)$ are complemented, respectively, in $\mathrm{Con}_f\, f^m(\mathcal{L})$ and $\mathrm{Con}_g\, f^m(\mathcal{L})$.

Using Lemma 1.2 and [4, Lemma 2.4] it is proved in [4] that

$$\begin{aligned}\theta_{f,f^m(L)}(f^{q2n}(\widetilde{a}_s), f^{q2n}(\widetilde{b}_s))' &= \varphi_f(\widetilde{a}_{f,s}, \widetilde{b}_{f,s})|_{f^m(L)} \qquad \text{and} \\ \theta_{g,f^m(L)}(g^{q2n}(\widetilde{a}_t), g^{q2n}(\widetilde{b}_t))' &= \varphi_g(\widetilde{a}_{g,t}, \widetilde{b}_{g,t})|_{f^m(L)}\end{aligned}$$

where

$$\begin{aligned}\varphi_f(\widetilde{a}_{f,s}, \widetilde{b}_{f,s}) = \theta_f\big(f^k(\widetilde{b}_s) \vee f^l(\widetilde{a}_s), 1\big) &\vee \theta_f\big(f^k(\widetilde{b}_s), f^k(\widetilde{b}_s) \vee f^l(\widetilde{b}_s)\big) \\ &\vee \theta_f\big(f^l(\widetilde{a}_s), f^l(\widetilde{a}_s) \vee f^k(\widetilde{a}_s)\big) \qquad \text{and}\end{aligned}$$

$$\begin{aligned}\varphi_g(\widetilde{a}_{g,t}, \widetilde{b}_{g,t}) = \theta_g\big(g^k(\widetilde{b}_t) \vee g^l(\widetilde{a}_t), 1\big) &\vee \theta_g\big(g^k(\widetilde{b}_t), g^k(\widetilde{b}_t) \vee g^l(\widetilde{b}_t)\big) \\ &\vee \theta_g\big(g^l(\widetilde{a}_t), g^l(\widetilde{a}_t) \vee g^k(\widetilde{a}_t)\big)\end{aligned}$$

Since $k \geq m$ and $l \geq m$, it follows by Lemma 2.6 that each congruence $\theta_f\big(f^k(\widetilde{b}_s) \vee f^l(\widetilde{a}_s), 1\big)$, $\theta_f\big(f^k(\widetilde{b}_s), f^k(\widetilde{b}_s) \vee f^l(\widetilde{b}_s)\big)$ and $\theta_g\big(g^l(\widetilde{a}_t), g^l(\widetilde{a}_t) \vee g^k(\widetilde{a}_t)\big)$ is an element of $\mathrm{Con}\,\mathcal{L}$; so $\varphi_f(\widetilde{a}_{f,s}, \widetilde{b}_{f,s})$ is an element of $\mathrm{Con}\,\mathcal{L}$. Now, taking into account that $f^m(L)$ is a subuniverse of $\mathcal{L}$ we conclude that $\varphi_f(\widetilde{a}_{f,s}, \widetilde{b}_{f,s})|_{f^m(L)} \in \mathrm{Con}\, f^m(\mathcal{L})$. In a similar way we prove that $\varphi_g(\widetilde{a}_{g,t}, \widetilde{b}_{g,t})|_{f^m(L)} \in \mathrm{Con}\, f^m(\mathcal{L})$ and, also by Lemma 2.6, we have $\theta_{f,f^m(L)}\big(f^{q2n}(\widetilde{a}_s), f^{q2n}(\widetilde{b}_s)\big)$, $\theta_{g,f^m(L)}\big(g^{q2n}(\widetilde{a}_t), g^{q2n}(\widetilde{b}_t)\big) \in \mathrm{Con}\, f^m(\mathcal{L})$. So, both congruences $\theta_{f,f^m(L)}\big(f^{q2n}(\widetilde{a}_s), f^{q2n}(\widetilde{b}_s)\big)$ and $\theta_{g,f^m(L)}\big(g^{q2n}(\widetilde{a}_t), g^{q2n}(\widetilde{b}_t)\big)$ are complemented in $\mathrm{Con}\, f^m(\mathcal{L})$. Since

$$\theta(a,b)|_{f^m(L)} = \bigvee_{s=1}^{n} \theta_{f,f^m(L)}\big(f^{q2n}(\widetilde{a}_s), f^{q2n}(\widetilde{b}_s)\big) \vee \bigvee_{t=1}^{n} \theta_{g,f^m(L)}\big(g^{q2n}(\widetilde{a}_t), g^{q2n}(\widetilde{b}_t)\big)$$

it is obvious that $\theta(a,b)|_{f^m(L)}$ is also complemented in $\operatorname{Con} f^m(\mathcal{L})$ and we have:

$$\begin{aligned}(\theta(a,b)|_{f^m(L)})' &= \bigwedge_{s=1}^{n} \theta_{f,f^m(L)}\big(f^{q2n}(\widetilde{a}_s), f^{q2n}(\widetilde{b}_s)\big)' \wedge \bigwedge_{t=1}^{n} \theta_{g,f^m(L)}\big(g^{q2n}(\widetilde{a}_t), g^{q2n}(\widetilde{b}_t)\big)' \\ &= \Big(\bigwedge_{s=1}^{n} \varphi_f(\widetilde{a}_{f,s}, \widetilde{b}_{f,s})|_{f^m(L)}\Big) \wedge \Big(\bigwedge_{t=1}^{n} \varphi_g(\widetilde{a}_{g,t}, \widetilde{b}_{g,t})|_{f^m(L)}\Big) \\ &= \Big(\bigwedge_{s=1}^{n} \varphi_f(\widetilde{a}_{f,s}, \widetilde{b}_{f,s}) \wedge \bigwedge_{t=1}^{n} \varphi_g(\widetilde{a}_{g,t}, \widetilde{b}_{g,t})\Big)|_{f^m(L)}.\end{aligned}$$

Let $\varphi_{f,g}$ stand for $\bigwedge_{s=1}^{n} \varphi_f(\widetilde{a}_{f,s}, \widetilde{b}_{f,s}) \wedge \bigwedge_{t=1}^{n} \varphi_g(\widetilde{a}_{g,t}, \widetilde{b}_{g,t})$. From Lemma 2.6 we conclude that $\varphi_{f,g} \in \operatorname{Con}\mathcal{L}$.

Next lemma shows that $\varphi_{f,g}$ can be described as the join of a finite number of principal congruences and we use this result to determine the complement of $\theta(a,b)$. To obtain this description it is useful to remember facts R_1) an R_2) mentioned in [4] and to take into account the following:

Remark: Let $\mathcal{L} = (L,f,g) \in \mathbf{DK}_{n,m}$, (k,l) be an m-pair and $r \in \mathbb{N}_0$. Let $h \in \{f,g\}$. Then, for $x \in L^h_{1,m}$,

$$\begin{cases} f^r\big(h^k(x)\big) = h^l(x), \quad f^r\big(h^l(x)\big) = h^k(x) & \text{if } r \text{ is odd,} \\ f^r\big(h^k(x)\big) = h^k(x), \quad f^r\big(h^l(x)\big) = h^l(x) & \text{if } r \text{ is even,} \\ \\ g^r\big(h^k(x)\big) = h^l(x), \quad g^r\big(h^l(x)\big) = h^k(x) & \text{if } r \text{ is odd,} \\ g^r\big(h^k(x)\big) = h^k(x), \quad g^r\big(h^l(x)\big) = h^l(x) & \text{if } r \text{ is even.} \end{cases}$$

Lemma 2.8. *Let $\mathcal{L} = (L,f,g) \in \mathbf{DK}_{n,m}$ and $a,b \in L$ be such that $a \leq b$. Let $\widetilde{b}_{f,0} = \widetilde{b}_{g,0} = 0$ and $\widetilde{a}_{f,n+1} = \widetilde{a}_{g,n+1} = 1$ and (k,l) be an m-pair.*
Then,

$$\varphi_{f,g} = \bigvee_{i,p=1}^{n+1} \bigvee_{j=i-1}^{n} \bigvee_{q=p-1}^{n} \big[\theta_f(x_{i,j,p,q}, y_{i,j,p,q}) \vee \theta_g(w_{i,j,p,q}, z_{i,j,p,q})\big],$$

where

$$\begin{aligned} x_{i,j,p,q} &= f^l(\widetilde{a}_i) \vee f^k(\widetilde{b}_j) \vee g^l(\widetilde{a}_p) \vee g^k(\widetilde{b}_q), \\ y_{i,j,p,q} &= x_{i,j,p,q} \vee \big(f^k(\widetilde{a}_{j+1}) \wedge f^l(\widetilde{b}_{i-1}) \wedge g^k(\widetilde{a}_{q+1}) \wedge g^l(\widetilde{b}_{p-1})\big), \\ w_{i,j,p,q} &= f^l(\widetilde{a}_i) \vee f^k(\widetilde{b}_j) \vee \big(g^k(\widetilde{a}_p) \wedge g^l(\widetilde{b}_q) \wedge [g^l(\widetilde{a}_{q+1}) \vee g^k(\widetilde{b}_{p-1})]\big), \\ z_{i,j,p,q} &= w_{i,j,p,q} \vee \big(f^k(\widetilde{a}_{j+1}) \wedge f^l(\widetilde{b}_{i-1}) \wedge g^k(\widetilde{a}_p) \wedge g^l(\widetilde{b}_q)\big). \end{aligned}$$

Proof. We have $\varphi_{f,g} = \bigwedge_{s=1}^{n} \varphi_f(\widetilde{a}_{f,s}, \widetilde{b}_{f,s}) \wedge \bigwedge_{t=1}^{n} \varphi_g(\widetilde{a}_{g,t}, \widetilde{b}_{g,t})$ and from [4] we know that

$$\bigwedge_{s=1}^{n} \varphi_f(\widetilde{a}_{f,s}, \widetilde{b}_{f,s}) = \bigvee_{i=1}^{n+1} \bigvee_{j=i-1}^{n} \theta_f\big(f^l(\widetilde{a}_i) \vee f^k(\widetilde{b}_j), f^l(\widetilde{a}_i) \vee f^k(\widetilde{b}_j) \vee [f^k(\widetilde{a}_{j+1}) \wedge f^l(\widetilde{b}_{i-1})]\big)$$

and

$$\bigwedge_{t=1}^{n}\varphi_g(\widetilde{a}_{g,t},\widetilde{b}_{g,t})=\bigvee_{p=1}^{n+1}\bigvee_{q=p-1}^{n}\theta_g\big(g^l(\widetilde{a}_p)\vee g^k(\widetilde{b}_q),g^l(\widetilde{a}_p)\vee g^k(\widetilde{b}_q)\vee[g^k(\widetilde{a}_{q+1})\wedge g^l(\widetilde{b}_{p-1})]\big).$$

By Lemma 1.3 and the remark we made before, it follows that:

$$\bigwedge_{s=1}^{n}\varphi_f(\widetilde{a}_{f,s},\widetilde{b}_{f,s})=\bigvee_{i=1}^{n+1}\bigvee_{j=i-1}^{n}\Big[\theta_{\mathrm{lat}}\big(f^l(\widetilde{a}_i)\vee f^k(\widetilde{b}_j),f^l(\widetilde{a}_i)\vee f^k(\widetilde{b}_j)\vee[f^k(\widetilde{a}_{j+1})\wedge f^l(\widetilde{b}_{i-1})]\big)$$
$$\vee\ \theta_{\mathrm{lat}}\big(f^k(\widetilde{a}_i)\wedge f^l(\widetilde{b}_j)\wedge[f^l(\widetilde{a}_{j+1})\vee f^k(\widetilde{b}_{i-1})],f^k(\widetilde{a}_i)\wedge f^l(\widetilde{b}_j)\big)\Big],$$

$$\bigwedge_{t=1}^{n}\varphi_g(\widetilde{a}_{g,t},\widetilde{b}_{g,t})=\bigvee_{p=1}^{n+1}\bigvee_{q=p-1}^{n}\Big[\theta_{\mathrm{lat}}\big(g^l(\widetilde{a}_p)\vee g^k(\widetilde{b}_q),g^l(\widetilde{a}_p)\vee g^k(\widetilde{b}_q)\vee[g^k(\widetilde{a}_{q+1})\wedge g^l(\widetilde{b}_{p-1})]\big)$$
$$\vee\ \theta_{\mathrm{lat}}\big(g^k(\widetilde{a}_p)\wedge g^l(\widetilde{b}_q)\wedge[g^l(\widetilde{a}_{q+1})\vee g^k(\widetilde{b}_{p-1})],g^k(\widetilde{a}_p)\wedge g^l(\widetilde{b}_q)\big)\Big].$$

Using [4, R_1) and R_2)] it is routine to prove the following identity (but we omit the proof since it is very long):

$$\varphi_{f,g}=\bigvee_{i,p=1}^{n+1}\ \bigvee_{j=i-1}^{n}\ \bigvee\nolimits_{q=p-1}^{n}\left(A_{i,j,p,q}\vee B_{i,j,p,q}\vee C_{i,j,p,q}\vee D_{i,j,p,q}\right),$$

with

$A_{i,j,p,q}=\theta_{\mathrm{lat}}(f^l(\widetilde{a}_i)\vee f^k(\widetilde{b}_j)\vee[g^k(\widetilde{a}_p)\wedge g^l(\widetilde{b}_q)\wedge(g^k(\widetilde{b}_{p-1})\vee g^l(\widetilde{a}_{q+1}))],$
$f^l(\widetilde{a}_i)\vee f^k(\widetilde{b}_j)\vee[g^k(\widetilde{a}_p)\wedge g^l(\widetilde{b}_q)\wedge(g^k(\widetilde{b}_{p-1})\vee g^l(\widetilde{a}_{q+1}))]\vee[f^l(\widetilde{b}_{i-1})\wedge f^k(\widetilde{a}_{j+1})\wedge g^k(\widetilde{a}_p)\wedge g^l(\widetilde{b}_q)]);$

$B_{i,j,p,q}=\theta_{\mathrm{lat}}(f^k(\widetilde{a}_i)\wedge f^l(\widetilde{b}_j)\wedge[g^l(\widetilde{a}_p)\vee g^k(\widetilde{b}_q)\vee(g^l(\widetilde{b}_{p-1})\wedge g^k(\widetilde{a}_{q+1}))],$
$f^k(\widetilde{a}_i)\wedge f^l(\widetilde{b}_j)\wedge[g^l(\widetilde{a}_p)\vee g^k(\widetilde{b}_q)\vee(g^l(\widetilde{b}_{p-1})\wedge g^k(\widetilde{a}_{q+1}))]\wedge[f^k(\widetilde{b}_{i-1})\vee f^l(\widetilde{a}_{j+1})\vee g^l(\widetilde{a}_p)\vee g^k(\widetilde{b}_q)]);$

$C_{i,j,p,q}=\theta_{\mathrm{lat}}(f^l(\widetilde{a}_i)\vee f^k(\widetilde{b}_j)\vee g^l(\widetilde{a}_p)\vee g^k(\widetilde{b}_q),$
$f^l(\widetilde{a}_i)\vee f^k(\widetilde{b}_j)\vee g^l(\widetilde{a}_p)\vee g^k(\widetilde{b}_q)\vee[f^l(\widetilde{b}_{i-1})\wedge f^k(\widetilde{a}_{j+1})\wedge g^l(\widetilde{b}_{p-1})\wedge g^k(\widetilde{a}_{q+1})]);$

$D_{i,j,p,q}=\theta_{\mathrm{lat}}(f^k(\widetilde{a}_i)\wedge f^l(\widetilde{b}_j)\wedge g^k(\widetilde{a}_p)\wedge g^l(\widetilde{b}_q)\wedge[f^k(\widetilde{b}_{i-1})\vee f^l(\widetilde{a}_{j+1})\vee g^k(\widetilde{b}_{p-1})\vee g^l(\widetilde{a}_{q+1})],$
$f^k(\widetilde{a}_i)\wedge f^l(\widetilde{b}_j)\wedge g^k(\widetilde{a}_p)\wedge g^l(\widetilde{b}_q)).$

Now, from Lemma 1.3 it follows that

$$\varphi_{f,g}=\bigvee_{i,p=1}^{n+1}\bigvee_{j=i-1}^{n}\bigvee_{q=p-1}^{n}\Big(\theta_f(x_{i,j,p,q},y_{i,j,p,q})\vee\theta_g(w_{i,j,p,q},z_{i,j,p,q})\Big),$$

where

$$x_{i,j,p,q}=f^l(\widetilde{a}_i)\vee f^k(\widetilde{b}_j)\vee g^l(\widetilde{a}_p)\vee g^k(\widetilde{b}_q),$$
$$y_{i,j,p,q}=x_{i,j,p,q}\vee\big(f^k(\widetilde{a}_{j+1})\wedge f^l(\widetilde{b}_{i-1})\wedge g^k(\widetilde{a}_{q+1})\wedge g^l(\widetilde{b}_{p-1})\big),$$
$$w_{i,j,p,q}=f^l(\widetilde{a}_i)\vee f^k(\widetilde{b}_j)\vee\big(g^k(\widetilde{a}_p)\wedge g^l(\widetilde{b}_q)\wedge[g^l(\widetilde{a}_{q+1})\vee g^k(\widetilde{b}_{p-1})]\big),$$
$$z_{i,j,p,q}=w_{i,j,p,q}\vee\big(f^k(\widetilde{a}_{j+1})\wedge f^l(\widetilde{b}_{i-1})\wedge g^k(\widetilde{a}_p)\wedge g^l(\widetilde{b}_q)\big).$$

□

Theorem 2.9. *Let* $\mathcal{L} = (L, f, g) \in \mathbf{DK}_{n,m}$ *and* $a, b \in L$ *be such that* $a \leq b$. *Let* (k, l) *be an* m*-pair.*
Then,

(a) $\theta(a, b) \vee \varphi_{f,g} = \mathbf{1}$,

(b) *if* $\theta(a, b)$ *is complemented, then necessarily* $\theta(a, b)' = \varphi_{f,g}$.

Proof. (a) By Lemma 1.8, we have $\theta(a, b) = \bigvee_{s=1}^{n} \theta_f(\widetilde{a}_{f,s}, \widetilde{b}_{f,s}) \vee \bigvee_{t=1}^{n} \theta_g(\widetilde{a}_{g,t}, \widetilde{b}_{g,t})$ and from [4, Theorem 2.7] we know that, for all $s, t \in \{1, ..., n\}$, $\theta_f(\widetilde{a}_{f,s}, \widetilde{b}_{f,s}) \vee \varphi_f(\widetilde{a}_{f,s}, \widetilde{b}_{f,s}) = \mathbf{1}$ and $\theta_g(\widetilde{a}_{g,t}, \widetilde{b}_{g,t}) \vee \varphi_g(\widetilde{a}_{g,t}, \widetilde{b}_{g,t}) = \mathbf{1}$.
Consequently,

$$\begin{aligned}
\theta \vee \varphi_{f,g} &= \Big[\bigvee_{s=1}^{n} \theta_f(\widetilde{a}_{f,s}, \widetilde{b}_{f,s}) \ \vee \bigvee_{t=1}^{n} \theta_g(\widetilde{a}_{g,t}, \widetilde{b}_{g,t})\Big] \vee \Big[\bigwedge_{u=1}^{n} \varphi_f(\widetilde{a}_{f,u}, \widetilde{b}_{f,u}) \ \wedge \bigwedge_{v=1}^{n} \varphi_g(\widetilde{a}_{g,v}, \widetilde{b}_{g,v})\Big] \\
&= \bigwedge_{u=1}^{n} \Big(\varphi_f(\widetilde{a}_{f,u}, \widetilde{b}_{g,u}) \ \vee \theta_f(\widetilde{a}_{f,u}, \widetilde{b}_{f,u}) \vee \bigvee_{s=1, s\neq u}^{n} \theta_f(\widetilde{a}_{f,s}, \widetilde{b}_{f,s}) \ \vee \bigvee_{t=1}^{n} \theta_g(\widetilde{a}_{g,t}, \widetilde{b}_{g,t})\Big) \\
&\quad \wedge \bigwedge_{v=1}^{n} \Big(\varphi_g(\widetilde{a}_{g,v}, \widetilde{b}_{g,v}) \vee \theta_g(\widetilde{a}_{g,v}, \widetilde{b}_{g,v}) \ \vee \bigvee_{s=1}^{n} \theta_f(\widetilde{a}_{f,s}, \widetilde{b}_{f,s}) \ \vee \bigvee_{t=1, t\neq v}^{n} \theta_g(\widetilde{a}_{g,t}, \widetilde{b}_{g,t})\Big) \\
&= \mathbf{1}.
\end{aligned}$$

(b) Suppose now that $\theta(a, b)$ is complemented. From (a) it follows that $\theta(a, b)' \leq \varphi_{f,g}$. It remains to prove that $\varphi_{f,g} \leq \theta(a, b)'$.

As we have already seen $\big(\theta(a, b)|_{f^m(L)}\big)' = \varphi_{f,g}|_{f^m(L)}$.

Let $\widetilde{b}_{f,0} = \widetilde{b}_{g,0} = 0$ and $\widetilde{a}_{f,n+1} = \widetilde{a}_{g,n+1} = 1$. By Lemma 2.8 we have

$$\varphi_{f,g} = \bigvee_{i,p=1}^{n+1} \bigvee_{j=i-1}^{n} \bigvee_{q=p-1}^{n} \big[\theta_f(x_{i,j,p,q}, y_{i,j,p,q}) \vee \theta_g(w_{i,j,p,q}, z_{i,j,p,q})\big],$$

where

$$\begin{aligned}
x_{i,j,p,q} &= f^l(\widetilde{a}_i) \vee f^k(\widetilde{b}_j) \vee g^l(\widetilde{a}_p) \vee g^k(\widetilde{b}_q), \\
y_{i,j,p,q} &= x_{i,j,p,q} \vee \big(f^k(\widetilde{a}_{j+1}) \wedge f^l(\widetilde{b}_{i-1}) \wedge g^k(\widetilde{a}_{q+1}) \wedge g^l(\widetilde{b}_{p-1})\big), \\
w_{i,j,p,q} &= f^l(\widetilde{a}_i) \vee f^k(\widetilde{b}_j) \vee \big(g^k(\widetilde{a}_p) \wedge g^l(\widetilde{b}_q) \wedge [g^l(\widetilde{a}_{q+1}) \vee g^k(\widetilde{b}_{p-1})]\big), \\
z_{i,j,p,q} &= w_{i,j,p,q} \vee \big(f^k(\widetilde{a}_{j+1}) \wedge f^l(\widetilde{b}_{i-1}) \wedge g^k(\widetilde{a}_p) \wedge g^l(\widetilde{b}_q)\big).
\end{aligned}$$

From Lemma 2.6 we know that $\theta_f(x_{i,j,p,q}, y_{i,j,p,q})$ and $\theta_g(w_{i,j,p,q}, z_{i,j,p,q})$ are elements of $\operatorname{Con}\mathcal{L}$. So $\varphi_{f,g}$ is the least congruence of $\mathcal{L}$ that identifies each pair $(x_{i,j,p,q}, y_{i,j,p,q})$ and each pair $(w_{i,j,p,q}, z_{i,j,p,q})$.
Taking into account Lemma 2.5 we have $(\theta(a, b)|_{f^m(L)})' = \theta(a, b)'|_{f^m(L)}$. So $\varphi_{f,g}|_{f^m(L)} = \theta(a, b)'|_{f^m(L)}$ and, consequently, $\theta(a, b)'$ also identifies each of those pairs. Therefore $\varphi_{f,g} \leq \theta(a, b)'$ and we may conclude that $\theta(a, b)' = \varphi_{f,g}$. $\square$

A double Ockham algebra $\mathcal{L} = (L, f, g)$ that satisfies id $\leq f^2$, $g^2 \leq$ id, $fg = g^2$ and $gf = f^2$ is called a double MS-algebra. Since every double MS-algebra is a double $\mathrm{K}_{1,1}$-algebra, we can establish Theorem 14.5 of [3] as a corollary of the previous theorem. Thus we have:

Corollary 2.10. *Let $\mathcal{L} = (L, f, g)$ be a double* MS*-algebra and let $a, b \in L$ be such that $a \leq b$.*
Let

$$\varphi_{f,g} = \big[\theta_f\big(f^2(b) \vee f(a), 1\big) \vee \ \theta_f\big(f^2(b), f^2(b) \vee f(b)\big) \vee \ \theta_f\big(f(a), f(a) \vee f^2(a)\big)\big]$$
$$\wedge \big[\theta_g\big(g^2(b) \vee g(a), 1\big) \vee \ \theta_g\big(g^2(b), g^2(b) \vee g(b)\big) \vee \ \theta_g\big(g(a), g(a) \vee g^2(a)\big)\big].$$

Then

(a) $\theta(a, b) \vee \varphi_{f,g} = \mathbf{1}$,
(b) *if $\theta(a, b)$ is complemented, then $\theta(a, b)' = \varphi_{f,g}$.*

We finish this paper establishing a necessary and sufficient condition for a principal congruence defined on a double $\mathrm{K}_{n,m}$-algebra to be complemented.

Theorem 2.11. *Let $\mathcal{L} = (L, f) \in \mathbf{DK}_{n,m}$ and $a, b \in L$ be such that $a \leq b$. Let (k, l) be an m-pair. Let $\widetilde{b}_{f,0} = \widetilde{b}_{g,0} = 0$ and $\widetilde{a}_{f,n+1} = \widetilde{a}_{g,n+1} = 1$. Then, $\theta(a, b)$ is complemented if and only if for all $s \in \{1, ..., n\}$, all $(x_s, y_s) \in \{(\widetilde{a}_{f,s}, \widetilde{b}_{f,s}), (\widetilde{a}_{g,s}, \widetilde{b}_{g,s})\}$ and all $i, p \in \{1, ..., n+1\}$ we have:*

$$y_s \wedge f^k(\widetilde{a}_{j+1}) \wedge f^l(\widetilde{b}_{i-1}) \wedge g^k(\widetilde{a}_{q+1}) \wedge g^l(\widetilde{b}_{p-1}) \leq x_s \vee f^l(\widetilde{a}_i) \vee f^k(\widetilde{b}_j) \vee g^l(\widetilde{a}_p) \vee g^k(\widetilde{b}_q),$$
for all $j \in \{i-1, ..., n\}$ and $q \in \{p-1, ..., n\}$,

$$y_s \wedge f^k(\widetilde{a}_{j+1}) \wedge f^l(\widetilde{b}_{i-1}) \wedge g^k(\widetilde{a}_p) \wedge g^l(\widetilde{b}_q) \leq x_s \vee f^l(\widetilde{a}_i) \vee f^k(\widetilde{b}_j) \vee g^l(\widetilde{a}_{q+1}) \vee g^k(\widetilde{b}_{p-1}),$$
for all $j \in \{i-1, ..., n\}$ and $q \in \{p, ..., n\}$,

$$y_s \wedge f^k(\widetilde{a}_i) \wedge f^l(\widetilde{b}_j) \wedge g^k(\widetilde{a}_{q+1}) \wedge g^l(\widetilde{b}_{p-1}) \leq x_s \vee f^l(\widetilde{a}_{j+1}) \vee f^k(\widetilde{b}_{i-1}) \vee g^l(\widetilde{a}_p) \vee g^k(\widetilde{b}_q),$$
for all $j \in \{i, ..., n\}$ and $q \in \{p-1, ..., n\}$,

$$y_s \wedge f^k(\widetilde{a}_i) \wedge f^l(\widetilde{b}_j) \wedge g^k(\widetilde{a}_p) \wedge g^l(\widetilde{b}_q) \leq x_s \vee f^l(\widetilde{a}_{j+1}) \vee f^k(\widetilde{b}_{i-1}) \vee g^l(\widetilde{a}_{q+1}) \vee g^k(\widetilde{b}_{p-1}),$$
for all $j \in \{i, ..., n\}$ and $q \in \{p, ..., n\}$.

Proof. By Lemma 1.8 we have $\theta(a, b) = \bigvee_{s=1}^{n} \theta_f(\widetilde{a}_{f,s}, \widetilde{b}_{f,s}) \vee \ \bigvee_{t=1}^{n} \theta_g(\widetilde{a}_{g,t}, \widetilde{b}_{g,t})$ and, from Theorem 2.9 it follows that $\theta(a, b)$ is complemented if and only if $\theta(a, b) \wedge \varphi_{f,g} = \mathbf{0}$. By Lemma 2.8 we know that

$$\varphi_{f,g} = \Big(\bigvee_{i=1}^{n+1} \bigvee_{j=i-1}^{n} \theta_f\big(f^l(\widetilde{a}_i) \vee f^k(\widetilde{b}_j), f^l(\widetilde{a}_i) \vee f^k(\widetilde{b}_j) \vee [f^k(\widetilde{a}_{j+1}) \wedge f^l(\widetilde{b}_{i-1})]\big)\Big)$$
$$\wedge \Big(\bigvee_{p=1}^{n+1} \bigvee_{q=p-1}^{n} \theta_g\big(g^l(\widetilde{a}_p) \vee g^k(\widetilde{b}_q), g^l(\widetilde{a}_p) \vee g^k(\widetilde{b}_q) \vee [g^k(\widetilde{a}_{q+1}) \wedge g^l(\widetilde{b}_{p-1})]\big)\Big)$$

with $\widetilde{b}_{f,0} = \widetilde{b}_{g,0} = 0$ and $\widetilde{a}_{f,n+1} = \widetilde{a}_{g,n+1} = 1$.

Then $\theta(a,b)$ is complemented if and only if, for all $s,t \in \{1,...,n\}$, $i,p \in \{1,...,n+1\}$, $j \in \{i-1,...,n\}$ and $q \in \{p-1,...,n\}$,

$$\begin{aligned}\theta_f(\widetilde{a}_{f,s},\widetilde{b}_{f,s}) \wedge \theta_f\big(f^l(\widetilde{a}_i) \vee f^k(\widetilde{b}_j), f^l(\widetilde{a}_i) \vee f^k(\widetilde{b}_j) \vee [f^k(\widetilde{a}_{j+1}) \wedge f^l(\widetilde{b}_{i-1})]\big)\\ \wedge\, \theta_g\big(g^l(\widetilde{a}_p) \vee g^k(\widetilde{b}_q), g^l(\widetilde{a}_p) \vee g^k(\widetilde{b}_q) \vee [g^k(\widetilde{a}_{q+1}) \wedge g^l(\widetilde{b}_{p-1})]\big) = \mathbf{0}\end{aligned}$$

and

$$\begin{aligned}\theta_g(\widetilde{a}_{g,t},\widetilde{b}_{g,t}) \wedge \theta_f\big(f^l(\widetilde{a}_i) \vee f^k(\widetilde{b}_j), f^l(\widetilde{a}_i) \vee f^k(\widetilde{b}_j) \vee [f^k(\widetilde{a}_{j+1}) \wedge f^l(\widetilde{b}_{i-1})]\big)\\ \wedge\, \theta_g\big(g^l(\widetilde{a}_p) \vee g^k(\widetilde{b}_q), g^l(\widetilde{a}_p) \vee g^k(\widetilde{b}_q) \vee [g^k(\widetilde{a}_{q+1}) \wedge g^l(\widetilde{b}_{p-1})]\big) = \mathbf{0}.\end{aligned}$$

By Lemma 1.3 and since $\widetilde{a}_{f,s}$, $\widetilde{b}_{f,s} \in L^f_{1,m}$ and $\widetilde{a}_{g,t}$, $\widetilde{b}_{g,t} \in L^g_{1,m}$, it follows that, for all $s \in \{1,...,n\}$, i, $p \in \{1,...,n+1\}$, $j \in \{i-1,...,n\}$ and $q \in \{p-1,...,n\}$,

$$\begin{aligned}\theta_f(\widetilde{a}_{f,s},\widetilde{b}_{f,s}) \wedge \theta_f\big(f^l(\widetilde{a}_i) \vee f^k(\widetilde{b}_j), f^l(\widetilde{a}_i) \vee f^k(\widetilde{b}_j) \vee [f^k(\widetilde{a}_{j+1}) \wedge f^l(\widetilde{b}_{i-1})]\big)\\ \wedge\, \theta_g\big(g^l(\widetilde{a}_p) \vee g^k(\widetilde{b}_q), g^l(\widetilde{a}_p) \vee g^k(\widetilde{b}_q) \vee [g^k(\widetilde{a}_{q+1}) \wedge g^l(\widetilde{b}_{p-1})]\big) = \mathbf{0}\end{aligned}$$

if and only if

$$\begin{aligned}&\Big[\bigvee_{r=0}^{m+1}\theta_{\text{lat}}\big(f^r(\widetilde{a}_s), f^r(\widetilde{b}_s)\big)\Big]\\
&\wedge \Big[\theta_{\text{lat}}\big(f^l(\widetilde{a}_i) \vee f^k(\widetilde{b}_j), f^l(\widetilde{a}_i) \vee f^k(\widetilde{b}_j) \vee [f^k(\widetilde{a}_{j+1}) \wedge f^l(\widetilde{b}_{i-1})]\big)\\
&\qquad \vee\, \theta_{\text{lat}}\big(f^k(\widetilde{a}_i) \wedge f^l(\widetilde{b}_j) \wedge [f^l(\widetilde{a}_{j+1}) \vee f^k(\widetilde{b}_{i-1})], f^k(\widetilde{a}_i) \wedge f^l(\widetilde{b}_j)\big)\Big]\\
&\wedge \Big[\theta_{\text{lat}}\big(g^l(\widetilde{a}_p) \vee g^k(\widetilde{b}_q), g^l(\widetilde{a}_p) \vee g^k(\widetilde{b}_q) \vee [g^k(\widetilde{a}_{q+1}) \wedge g^l(\widetilde{b}_{p-1})]\big)\\
&\qquad \vee\, \theta_{\text{lat}}\big(g^k(\widetilde{a}_p) \wedge g^l(\widetilde{b}_q) \wedge [g^l(\widetilde{a}_{q+1}) \vee g^k(\widetilde{b}_{p-1})], g^k(\widetilde{a}_p) \wedge g^l(\widetilde{b}_q)\big)\Big] = \mathbf{0}.\end{aligned}$$

Now, using [4, R_1) and R_2)] it is easy we conclude that the previous identity follows if and only if, for all $r \in \{0,...,m+1\}$,

a) $$\begin{aligned}&f^r(\widetilde{b}_s) \wedge f^k(\widetilde{a}_{j+1}) \wedge f^l(\widetilde{b}_{i-1}) \wedge g^k(\widetilde{a}_{q+1}) \wedge g^l(\widetilde{b}_{p-1})\\ \le\ &f^r(\widetilde{a}_s) \vee f^l(\widetilde{a}_i) \vee f^k(\widetilde{b}_j) \vee g^l(\widetilde{a}_p) \vee g^k(\widetilde{b}_q),\end{aligned}$$

b) $$\begin{aligned}&f^r(\widetilde{b}_s) \wedge f^k(\widetilde{a}_{j+1}) \wedge f^l(\widetilde{b}_{i-1}) \wedge g^k(\widetilde{a}_p) \wedge g^l(\widetilde{b}_q)\\ \le\ &f^r(\widetilde{a}_s) \vee f^l(\widetilde{a}_i) \vee f^k(\widetilde{b}_j) \vee g^l(\widetilde{a}_{q+1}) \vee g^k(\widetilde{b}_{p-1}),\end{aligned}$$

c) $$\begin{aligned}&f^r(\widetilde{b}_s) \wedge f^k(\widetilde{a}_i) \wedge f^l(\widetilde{b}_j) \wedge g^k(\widetilde{a}_{q+1}) \wedge g^l(\widetilde{b}_{p-1})\\ \le\ &f^r(\widetilde{a}_s) \vee f^l(\widetilde{a}_{j+1}) \vee f^k(\widetilde{b}_{i-1}) \vee g^l(\widetilde{a}_p) \vee g^k(\widetilde{b}_q),\end{aligned}$$

and

d) $$\begin{aligned}&f^r(\widetilde{b}_s) \wedge f^k(\widetilde{a}_i) \wedge f^l(\widetilde{b}_j) \wedge g^k(\widetilde{a}_p) \wedge g^l(\widetilde{b}_q)\\ \le\ &f^r(\widetilde{a}_s) \vee f^l(\widetilde{a}_{j+1}) \vee f^k(\widetilde{b}_{i-1}) \vee g^l(\widetilde{a}_{q+1}) \vee g^k(\widetilde{b}_{p-1}).\end{aligned}$$

These inequalities are trivial when r is odd. If r is even, we have already seen that, $f^r\big(f^k(x)\big) = f^k(x)$, $f^r\big(f^l(x)\big) = f^l(x)$, $f^r\big(g^k(y)\big) = g^k(y)$ and $f^r(g^l(y)) = g^l(y)$, for all $x \in L^f_{1,m}$ and $y \in L^g_{1,m}$. So, conditions a), b), c) and d) are equivalent, respectively, to 1), 2), 3) and 4) below:

$$1) \qquad \begin{aligned} &\widetilde{b}_{f,s} \wedge f^k(\widetilde{a}_{j+1}) \wedge f^l(\widetilde{b}_{i-1}) \wedge g^k(\widetilde{a}_{q+1}) \wedge g^l(\widetilde{b}_{p-1}) \\ &\leq \widetilde{a}_{f,s} \vee f^l(\widetilde{a}_i) \vee f^k(\widetilde{b}_j) \vee g^l(\widetilde{a}_p) \vee g^k(\widetilde{b}_q), \end{aligned}$$

$$2) \qquad \begin{aligned} &\widetilde{b}_{f,s} \wedge f^k(\widetilde{a}_{j+1}) \wedge f^l(\widetilde{b}_{i-1}) \wedge g^k(\widetilde{a}_p) \wedge g^l(\widetilde{b}_q) \\ &\leq \widetilde{a}_{f,s} \vee f^l(\widetilde{a}_i) \vee f^k(\widetilde{b}_j) \vee g^l(\widetilde{a}_{q+1}) \vee g^k(\widetilde{b}_{p-1}), \end{aligned}$$

$$3) \qquad \begin{aligned} &\widetilde{b}_{f,s} \wedge f^k(\widetilde{a}_i) \wedge f^l(\widetilde{b}_j) \wedge g^k(\widetilde{a}_{q+1}) \wedge g^l(\widetilde{b}_{p-1}) \\ &\leq \widetilde{a}_{f,s} \vee f^l(\widetilde{a}_{j+1}) \vee f^k(\widetilde{b}_{i-1}) \vee g^l(\widetilde{a}_p) \vee g^k(\widetilde{b}_q), \end{aligned}$$

$$4) \qquad \begin{aligned} &\widetilde{b}_{f,s} \wedge f^k(\widetilde{a}_i) \wedge f^l(\widetilde{b}_j) \wedge g^k(\widetilde{a}_p) \wedge g^l(\widetilde{b}_q) \\ &\leq \widetilde{a}_{f,s} \vee f^l(\widetilde{a}_{j+1}) \vee f^k(\widetilde{b}_{i-1}) \vee g^l(\widetilde{a}_{q+1}) \vee g^k(\widetilde{b}_{p-1}). \end{aligned}$$

Conditions 1) and 2) are equal when $q = p-1$ (the same happens with 3) and 4)). For $j = i-1$ we also have that 1) coincide with 3) and 2) coincide with 4)).

Given $t \in \{1, ..., n\}$, $i, p \in \{1, ..., n+1\}$, $j \in \{i-1, ..., n\}$ and $q \in \{p-1, ..., n\}$, we have

$$\begin{aligned} \theta_g(\widetilde{a}_{g,t}, \widetilde{b}_{g,t}) \wedge \theta_f\big(&f^l(\widetilde{a}_i) \vee f^k(\widetilde{b}_j), f^l(\widetilde{a}_i) \vee f^k(\widetilde{b}_j) \vee [f^k(\widetilde{a}_{j+1}) \wedge f^l(\widetilde{b}_{i-1})]\big) \\ &\wedge \theta_g\big(g^l(\widetilde{a}_p) \vee g^k(\widetilde{b}_q), g^l(\widetilde{a}_p) \vee g^k(\widetilde{b}_q) \vee [g^k(\widetilde{a}_{q+1}) \wedge g^l(\widetilde{b}_{p-1})]\big) = \mathbf{0}. \end{aligned}$$

if and only if are satisfied conditions analogous to 1), 2), 3) and 4).

Then $\theta(a, b)$ is complemented if and only if for all $s \in \{1, ..., n\}$, all $(x_s, y_s) \in \{(\widetilde{a}_{f,s}, \widetilde{b}_{f,s}), (\widetilde{a}_{g,s}, \widetilde{b}_{g,s})\}$ and all $i, p \in \{1, ..., n+1\}$ the following conditions hold:

$y_s \wedge f^k(\widetilde{a}_{j+1}) \wedge f^l(\widetilde{b}_{i-1}) \wedge g^k(\widetilde{a}_{q+1}) \wedge g^l(\widetilde{b}_{p-1}) \leq x_s \vee f^l(\widetilde{a}_i) \vee f^k(\widetilde{b}_j) \vee g^l(\widetilde{a}_p) \vee g^k(\widetilde{b}_q)$, for all $j \in \{i-1, ..., n\}$ and $q \in \{p-1, ..., n\}$,

$y_s \wedge f^k(\widetilde{a}_{j+1}) \wedge f^l(\widetilde{b}_{i-1}) \wedge g^k(\widetilde{a}_p) \wedge g^l(\widetilde{b}_q) \leq x_s \vee f^l(\widetilde{a}_i) \vee f^k(\widetilde{b}_j) \vee g^l(\widetilde{a}_{q+1}) \vee g^k(\widetilde{b}_{p-1})$, for all $j \in \{i-1, ..., n\}$ and $q \in \{p, ..., n\}$,

$y_s \wedge f^k(\widetilde{a}_i) \wedge f^l(\widetilde{b}_j) \wedge g^k(\widetilde{a}_{q+1}) \wedge g^l(\widetilde{b}_{p-1}) \leq x_s \vee f^l(\widetilde{a}_{j+1}) \vee f^k(\widetilde{b}_{i-1}) \vee g^l(\widetilde{a}_p) \vee g^k(\widetilde{b}_q)$, for all $j \in \{i, ..., n\}$ and $q \in \{p-1, ..., n\}$,

$y_s \wedge f^k(\widetilde{a}_i) \wedge f^l(\widetilde{b}_j) \wedge g^k(\widetilde{a}_p) \wedge g^l(\widetilde{b}_q) \leq x_s \vee f^l(\widetilde{a}_{j+1}) \vee f^k(\widetilde{b}_{i-1}) \vee g^l(\widetilde{a}_{q+1}) \vee g^k(\widetilde{b}_{p-1})$, for all $j \in \{i, ..., n\}$ and $q \in \{p, ..., n\}$. □

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