Higher-order string effective actions and off-shell $d = 4$ supergravity

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Abstract

We start by a concise yet thorough revision of four-dimensional superspace supergravity. We present curved superspace geometry, for arbitrary $\mathcal{N}$, including torsion, curvature and Bianchi identities. We motivate the choice of torsion constraints.

We then consider the particular cases of $\mathcal{N} = 1, 2$. In both cases we show how Poincaré supergravity can be obtained from conformal supergravity. We see how to obtain the different versions of the Poincaré off-shell theory, with distinct compensating multiplets and sets of auxiliary fields. For those versions of $\mathcal{N} = 1, 2$ supergravities known as "old minimal", we present the solutions to the Bianchi identities, their field content and we show how to write superspace actions for these theories and their extensions using chiral densities and chiral projectors.

As concrete applications, we study the supersymmetrization of the two possible $\mathcal{R}^4$ terms in $d = 4$, which are both required as string corrections to supergravity.

We conclude by discussing possible applications of these results to open problems on black holes in string theory.

Contents

1 Introduction and plan

Remarkable results have been achieved recently on black hole physics in string theory, among which the microscopic interpretation of the entropy and the attractor mechanism. Supersymmetry has played a crucial role in these results.

Black holes can appear already at the supergravity level, when string theories are compactified and (nonperturbative) p-branes are wrapped around nontrivial cycles of the compactification manifold. But black holes can also be formed from elementary perturbative string excitations; however, in this case the area of their horizons vanishes at the supergravity level (these are called small black holes). In order to prevent a naked singularity and get a finite horizon area, one needs to consider the effect of higher-order string corrections to supergravity. These terms appear in string theory effective actions as α' corrections, both at string tree level and higher string loops. They also affect classical black holes, since they introduce corrections to the supergravity equations of motion.

These are some reasons that motivate us to study higher-derivative corrections to supergravity theories and their supersymmetrization. This is what we do in the following, concentrating on theories in $d = 4$.

We begin by reviewing four-dimensional superspace supergravity. We present curved superspace geometry, for arbitrary $\mathcal{N}$, including torsion, curvature and Bianchi identities. We motivate the choice of torsion constraints.

Next we move to the particular cases of $\mathcal{N} = 1, 2$. In both cases we show how Poincaré supergravity can be obtained from conformal supergravity by introducing a nonconformal constraint. We see how different choices of this nonconformal constraint lead to different versions of the Poincaré off-shell theory, with distinct compensating multiplets and sets of auxiliary fields. For those versions of $\mathcal{N} = 1, 2$ supergravities known as "old minimal", we present the solutions to the Bianchi identities, their field content and we show how to write superspace actions for these theories and their extensions using chiral densities and chiral projectors.

We then apply this formalism to the supersymmetrization of higher-derivative terms in $\mathcal{N} = 1, 2$ supergravities. As a concrete application, we study the supersymmetrization of $\mathcal{R}^4$ terms, which are required as string corrections to those theories. We write down the $\mathcal{R}^4$ terms which appear in the α'^3 type II and heterotic superstring effective actions. In $d = 4$ there are two of these terms. One of them is the square of the Bel-Robinson tensor. We work out its $\mathcal{N} = 1, 2$ supersymmetrizations, and we verify for both cases, with this term, that some auxiliary fields can be eliminated and some cannot. We identify these auxiliary fields and we interpret these results, which should be generalized to other supersymmetric higher-derivative terms, in terms of the breaking of conformal supergravity we discussed before.

The other $\mathcal{R}^4$ term cannot be directly supersymmetrized, as in $\mathcal{N}=1$ it violates chirality. We show how to circumvent this problem in $\mathcal{N}=1$ and we argue that it should not be possible in $\mathcal{N}=8$.

We conclude by discussing possible applications of these results to open problems on black holes in string theory.

2 Superspace geometry

2.1 Vielbein, connection, torsion and curvature

Curved superspace is a manifold parameterized by the usual commuting $x-$space coordinates x^μ, plus a set of anticommuting spinorial coordinates, their number depending on the number of space-time dimensions in question and the number of supersymmetries $\mathcal{N}$. In four dimensions, we have

$$z^\Pi = \left(x^\mu, \theta^a_A, \theta^a_{\dot{A}}\right) \tag{1}$$

with $\mu = 0, \cdots, 3$, $A, \dot{A} = 1, 2$, $a = 1, \cdots, \mathcal{N}$.

Symmetries that are manifest in curved superspace are general supercoordinate transformations, with parameters ξ^Λ, and tangent space (structure group) transformations, with parameters Λ^{MN}. Curved superspace coordinates transform under general reparameterizations as

$$z^\Pi \to z'^\Pi = z^\Pi + \xi^\Pi \tag{2}$$

with $\xi^\Pi = \left(\xi^\mu, \xi^a_A, \xi^a_{\dot{A}}\right)$ defined as arbitrary functions of z^Π. ξ^μ corresponds to the usual $x-$space diffeomorphisms (Einstein transformations); $\xi^a_A, \xi^a_{\dot{A}}$ are their supersymmetric extension: the local supersymmetry transformations.

The main geometric objects of curved superspace are the supervielbein E^M_Π and the superconnection $\Omega_{\Lambda N}{}^P$. These objects transform under general supercoordinate transformations as

$$\delta E_\Pi{}^N = \xi^\Lambda \partial_\Lambda E_\Pi{}^N + \left(\partial_\Pi \xi^\Lambda\right) E_\Lambda{}^N, \tag{3}$$

$$\delta \Omega_{\Lambda M}{}^N = \xi^\Pi \partial_\Pi \Omega_{\Lambda M}{}^N + \left(\partial_\Lambda \xi^\Pi\right) \Omega_{\Pi M}{}^N. \tag{4}$$

The supervielbein relates the curved indices to the tangent space group ones, which we take to be $\mathrm{SO}(1,3) \times \mathrm{U}(\mathcal{N})$, with parameters $\Lambda_{MN} = \left(\Lambda_{mn}, \Lambda_{BbAa}, \Lambda_{\dot{B}b\dot{A}a}\right)$. These parameters can still be decomposed in Lorentz and $\mathrm{U}(\mathcal{N})$ parts as

$$\Lambda_{BbAa} = \epsilon_{ba} \Lambda_{BA} + \epsilon_{BA} \widetilde{\Lambda}_{ba},\ \Lambda_{\dot{B}b\dot{A}a} = \epsilon_{ba} \Lambda_{\dot{B}\dot{A}} + \epsilon_{\dot{B}\dot{A}} \widetilde{\Lambda}_{ba}, \tag{5}$$

satisfying

$$\Lambda_{BA} = \Lambda_{AB},\, \Lambda_{\dot{B}\dot{A}} = \Lambda_{\dot{A}\dot{B}},\, \Lambda_{A\dot{A}B\dot{B}} = 2\varepsilon_{\dot{A}\dot{B}}\Lambda_{AB} + 2\varepsilon_{AB}\Lambda_{\dot{A}\dot{B}} = -\Lambda_{B\dot{B}A\dot{A}}.$$

The U($\mathcal{N}$) parameters can still be decomposed into SU($\mathcal{N}$) and U(1) parts:

$$\widetilde{\Lambda}_{ba} = \Lambda_{ba} - \frac{1}{2}\epsilon_{ba}\Lambda,\, \Lambda_a{}^a = 0. \tag{6}$$

About our choice of structure group, two remarks must be made. Although the superconformal algebra is SU($2,2|\,\mathcal{N}$), the superspace we have introduced is perfectly adequate for the description of conformal supergravity. This is because from the additional parameters of SU($2,2|\,\mathcal{N}$), special conformal boosts get absorbed into general coordinate transformations, while Weyl (dilatations) and special supersymmetry transformations will appear as extra symmetries.

In principle we could have chosen some other structure group: if we wanted a superspace formulation that mimicked the x-space formulation of general relativity, the natural choice of structure group would rather contain the orthosymplectic group OSp($1,3|\,4$) instead of the Lorentz group, but this would lead to problems. Indeed, any superspace formulation of supergravity requires the introduction of too many fields, through the supervielbeins and the superconnections. The gauge invariances of the theory allow one to eliminate some of the degrees of freedom, but that is still not enough. In order to have a plausible theory, in any superspace formulation one needs to put constraints on covariant objects, so that the excess of fields (some of them of spin exceeding two) can be eliminated. It can be shown (for instance, in [1]) that with such a choice of tangent group one would not be able to put an adequate set of constraints that could remove all the unwanted fields. The largest group that allows that set of constraints is precisely the one we took.

The supervielbein and superconnection transform under the structure group as

$$\begin{aligned}
\delta E_\Pi{}^N &= -E_\Pi{}^M \Lambda_M{}^N, & (7)\\
\delta\Omega_{\Lambda M}{}^N &= -\partial_\Lambda \Lambda_M{}^N + \Omega_{\Lambda M}{}^S \Lambda_S{}^N + \Omega_{\Lambda R}{}^N \Lambda_M{}^R \,(-)^{(M+R)(N+R)}\,. & (8)
\end{aligned}$$

The superconnection is a structure algebra-valued (i.e. in the Lie algebra of the structure group) object, which can of course also be decomposed in its Lorentz and U($\mathcal{N}$) parts. Specifically, the Lorentz part $\Omega^{Lor}_{\Lambda M}{}^N$ is written as

$$\Omega^{Lor}_{\Lambda M}{}^N = \begin{pmatrix} \Omega_{\Lambda m}{}^n & 0 & 0 \\ 0 & -\frac{1}{4}\Omega_\Lambda{}^{mn}\,(\sigma_{mn})_B{}^A & 0 \\ 0 & 0 & \frac{1}{4}\Omega_\Lambda{}^{mn}\,(\sigma_{mn})_{\dot{B}}{}^{\dot{A}} \end{pmatrix}. \tag{9}$$

Having the superconnection, we define a supercovariant derivative:

$$D_\Lambda = \partial_\Lambda + \frac{1}{2}\Omega_\Lambda{}^{MN} J_{MN},\ \nabla_M = E_M{}^\Lambda D_\Lambda. \tag{10}$$

J_{MN} are the generators of the structure group $((\sigma_{mn})_B{}^A, (\sigma_{mn})_{\dot{B}}{}^{\dot{A}}$ in the spinorial representation of the Lorentz group). We define the (super)torsions $T_{MN}{}^P$ and (super)curvatures $R_{MN}{}^{PQ}$ as

$$\begin{aligned} T_{MN}{}^R &= E_M{}^\Lambda \left(\partial_\Lambda E_N{}^\Pi\right) E_\Pi{}^R + \Omega_{MN}{}^R - (-)^{MN}(M \leftrightarrow N) \\ &= E_M{}^\Lambda \left(D_\Lambda E_N{}^\Pi\right) E_\Pi{}^R - (-)^{MN}(M \leftrightarrow N), \end{aligned} \tag{11}$$

$$\begin{aligned} R_{MN}{}^{RS} &= E_M{}^\Lambda E_N{}^\Pi \left\{\partial_\Lambda \Omega_\Pi{}^{RS} + \Omega_\Lambda{}^{RK}\Omega_{\Pi K}{}^S - (-)^{\Lambda\Pi}(\Lambda \leftrightarrow \Pi)\right\} \\ &= E_M{}^\Lambda E_N{}^\Pi \left\{D_\Lambda \Omega_\Pi{}^{RS} - (-)^{\Lambda\Pi}(\Lambda \leftrightarrow \Pi)\right\}. \end{aligned} \tag{12}$$

The curvatures are structure algebra-valued and, therefore, can also be decomposed in their Lorentz and U($\mathcal{N}$) parts. Because of (9), we have

$$\begin{aligned} R_{MNC\dot{C}D\dot{D}} &= 2\epsilon_{\dot{C}\dot{D}} R_{MNCD} + 2\epsilon_{CD} R_{MN\dot{C}\dot{D}}, \\ R_{MNmn} &= -\frac{1}{2}\sigma_{mn}^{CD} R_{MNCD} - \frac{1}{2}\sigma_{mn}^{\dot{C}\dot{D}} R_{MN\dot{C}\dot{D}}. \end{aligned} \tag{13}$$

From the definitions (10), (11) and (12) we have, for the supercommutator of covariant derivatives,

$$[\nabla_M, \nabla_N\} = T_{MN}{}^R \nabla_R + \frac{1}{2} R_{MN}{}^{RS} J_{RS}. \tag{14}$$

Torsions and curvatures satisfy Bianchi identities. One of the most important consequences of these identities is the fact that the curvatures can be expressed completely in terms of the torsions. This statement, known as Dragon's theorem [2], is also a consequence of the curvatures being Lie-algebra valued. This fact has no place in general relativity, where curvatures and torsions are independent, and one can constrain the torsion to vanish leaving a nonvanishing curvature. In superspace, the torsion is the main object determining the geometry. The curvature Bianchi identity is therefore redundant; all the information contained in it is also contained in the torsion Bianchi identity, which is written as

$$\begin{aligned} &- (-)^{(M+N)R} \nabla_R T_{MN}{}^F + (-)^{(N+R)M} T_{NR}{}^S T_{SM}{}^F + (-)^{(N+R)M} R_{NRM}{}^F \\ &+ (-)^{MN} \nabla_N T_{MR}{}^F - (-)^{NR} T_{MR}{}^S T_{SN}{}^F - (-)^{NR} R_{MRN}{}^F \\ &- \nabla_M T_{NR}{}^F + T_{MN}{}^S T_{SR}{}^F + R_{MNR}{}^F = 0. \end{aligned} \tag{15}$$

2.2 Variational equations

Arbitrary variations of supervielbein and superconnection are given by [3]

$$H_M{}^N = E_M{}^\Lambda \delta E_\Lambda{}^N,\ \Phi_{MN}{}^P = E_M{}^\Lambda \delta \Omega_{\Lambda N}{}^P. \tag{16}$$

From (11) and (16), we derive the arbitrary variation of the torsion:

$$\begin{aligned} \delta T_{MN}{}^R &= -H_M{}^S T_{SN}{}^R + (-)^{MN} H_N{}^S T_{SM}{}^R + T_{MN}{}^S H_S{}^R \\ &- \nabla_M H_N{}^R + (-)^{MN} \nabla_N H_M{}^R + \Phi_{MN}{}^R - (-)^{MN} \Phi_{NM}{}^R. \end{aligned} \tag{17}$$

By matching (16) to the variations under general coordinate and structure group transformations, one can solve for $H_M{}^N$ and $\Phi_{MN}{}^P$ in terms of the transformation parameters, torsions and curvatures as

$$H_M^N = \xi^P T_{PM}{}^N + \nabla_M \xi^N + \Lambda_M^N,\ \Phi_{MN}{}^P = \xi^Q R_{QMN}{}^P - \nabla_M \Lambda_N^P. \tag{18}$$

Until a gauge for the general coordinate and structure group transformations has not been fixed, any solution for $H_M{}^N$ and $\Phi_{MN}{}^P$ is valid up to the transformations

$$\delta H_M{}^N = \nabla_M \tilde{\xi}^N - \xi^P T_{PM}{}^N,\ \delta \Phi_{MN}{}^P = \tilde{\xi}^Q R_{QMN}{}^P, \tag{19}$$

$$\delta H_M^N = \widetilde{\Lambda}_M^N,\ \delta \Phi_{MN}{}^P = \nabla_M \widetilde{\Lambda}_N{}^P. \tag{20}$$

Even fixing those gauges does not fix all the degrees of freedom of $H_M{}^N$ [4, 5, 6]. Namely, $H = -\frac{1}{4} H_m^m$ remains an unconstrained superfield and parametrizes the super-Weyl transformations, which include the dilatations and the special supersymmetry transformations.

2.3 Choice of constraints

As we previously mentioned, the superspace formulation of supergravity requires the introduction of too many fields, some of those having spins higher than 2. The only natural way to eliminate the undesired fields and get only those belonging to an irreducible representation of supersymmetry is to place constraints in the theory. Since those constraints should be valid in any frame of reference, they should be put only in covariant objects; and since, as we saw, we can express the curvatures in terms of the torsions, we choose to put the constraints in the torsions. Therefore, using the gauge freedom from (17), we analyze, from lower to upper dimensions, which torsions we can constrain.

At dimension 0, we have the torsion parts T^{abm}_{AB}, $T^{abm}_{A\dot{B}}$ and their complex conjugates. Considering the flat superspace limit for $T^{abm}_{A\dot{B}}$, we write

$$T^{abm}_{A\dot{B}} = -2i\varepsilon^{ab}\sigma^m_{A\dot{B}} + \tilde{T}^{abm}_{A\dot{B}}. \tag{21}$$

From (17), one finds [7] that the only parts of the torsion which cannot be absorbed by $H_n{}^m$, H^{Ab}_{aB}, $H^{Ab}_{a\dot{B}}$ and their complex conjugates are

$$\tilde{T}^{ab}_{A\dot{B}C\dot{C}} = \tilde{T}^{ab}_{\underline{A\dot{B}C\dot{C}}},\ T^{ab}_{ABC\dot{C}} = T^{\underline{ab}}_{\underline{ABC}\dot{C}}, \tag{22}$$

$\tilde{T}^{ab}_{\underline{A\dot{B}C\dot{C}}}$ being traceless in a, b. Since these fields have spin greater than two and therefore it would be impossible to describe any dynamics in their presence, we set them to zero:

$$\tilde{T}^{ab}_{\underline{A\dot{B}C\dot{C}}} = 0,\ T^{\underline{ab}}_{\underline{ABC}\dot{C}} = 0. \tag{23}$$

One must emphasize that these are the *only* constraints which have to be postulated (i.e. no other choice could be made to these specific parts of the torsion). All the other constraints are *conventional*, which means they must exist, but other choices could have been made. Conventional constraints correspond to redefinitions of the supervielbein and superconnection.

We are then left with

$$T^{abm}_{A\dot{B}} = -2i\varepsilon^{ab}\sigma^m_{A\dot{B}},\ T^{abm}_{AB} = 0. \tag{24}$$

As we will see, in $\mathcal{N} = 1, 2$ theories the constraint $T^{abm}_{AB} = 0$ has a geometrical meaning, and will be called "representation preserving". The constraint in $T^{abm}_{A\dot{B}}$ is just conventional.

At dimension $\frac{1}{2}$, it can be shown that, by adequate choices of the suitable parts of H_N^M and $\Phi_{MN}{}^P$ [7], we may set

$$T_{Aa\dot{B}b\dot{C}c} = 0,\ T_{AaBbCc} = 0,\ T^{amn}_{A} = 0. \tag{25}$$

At dimension 1, an appropriate redefinition of the Lorentz connection through an adequate choice of $\Phi_{mn}{}^p$ gives the usual constraint in Riemannian geometry

$$T_{mn}{}^p = 0. \tag{26}$$

Also, an adequate choice of $\Phi_{ma}{}^b$ allow us to constrain $R^{c\dot{C}}_{Ccab}$, and to have

$$T_{C\dot{C}B}{}^{bCa} = \beta T_{C\dot{C}}{}^{Cba}{}_B. \tag{27}$$

This constraint establishes an identity between two a priori different superfields. The numerical parameter β depends on the choice that was made for $R^{c\dot{C}}_{Ccab}$, but it will have no impact on the theory, since this is a conventional constraint.

The Bianchi identities are valid, no matter which constraints we have. But once some of the torsions are constrained, the Bianchi identities become equations for the unconstrained torsions and curvatures. These equations are not independent, and need to be solved systematically. This has been achieved, in conformal supergravity, for arbitrary $\mathcal{N}$ [5]. One can conclude that off-shell conformal supergravity exists and is consistent for $\mathcal{N} \leq 4$. For $\mathcal{N} \geq 6$, an off-shell theory is not consistent [5, 6]. That does not rule out on-shell theories, but those have not been found. For $\mathcal{N} = 5$ nothing has been concluded. Thus for $\mathcal{N} > 4$ the situation is rather unclear. We will only review the $\mathcal{N} = 1, 2$ cases, because those are the ones we will need. For a more complete discussion the reader is referred to [6].

In $\mathcal{N} = 1, 2$ one can put chirality constraints in superfields. An antichiral superfield $\Phi_{\dots}$ satisfies

$$\nabla^a_A \Phi_{\dots} = 0 \tag{28}$$

(the hermitean conjugated equation defines a chiral superfield). This constraint on the superfield must be compatible with the solution to the Bianchi identities; an integrability condition must be verified (that is why general chiral superfields only exist for $\mathcal{N} = 1, 2$, as we will see; for other values of $\mathcal{N}$, a chirality condition may result only from the solution to the Bianchi identities, in the superfields introduced in this process).

$\mathcal{N} = 1, 2$ Poincaré supergravities can be obtained from the corresponding conformal theories by consistent couplings to compensating multiplets that break superconformal invariance and local U($\mathcal{N}$). There are different possible choices of compensating multiplets, leading to different formulations of the Poincaré theory. What is special about these theories is the existence of a completely off-shell formalism. This means that, for each of these theories, a complete set of auxiliary fields is known (actually, there exist three known choices for each theory). In superspace this means that, after imposing constraints on the torsions, we can completely solve the Bianchi identities without using the field equations [5, 8], and there is a perfect identification between the superspace and x-space descriptions. We will review how is this achieved for the "old minimal" $\mathcal{N} = 1, 2$ cases.

3 $\mathcal{N} = 1$ supergravity in superspace

3.1 $\mathcal{N} = 1$ superspace geometry and constraints

$\mathcal{N} = 1$ superspace geometry is a simpler particular case of the general $\mathcal{N}$ case we saw in the previous section. Namely, the internal group indices $a, b, \cdots$ do not exist. The structure group is at most $\mathrm{SO}(1,3) \times \mathrm{U}(1)$ (in the Poincaré theory we will consider, it is actually just the Lorentz group). To write any $\mathrm{U}(\mathcal{N})$ valued formula in the $\mathcal{N} = 1$ case, one simply has to decompose that formula under $\mathrm{U}(\mathcal{N})$ and take simply the group singlets.

Specific to $\mathcal{N} = 1, 2$ are the representation-preserving constraints, required by the above mentioned integrability condition for the existence of antichiral superfields, defined by (28). For $\mathcal{N} = 1$, these constraints are the following:

$$T_{AB}{}^{\dot{C}} = 0,\, T_{AB}{}^{m} = 0. \tag{29}$$

Conventional constraints allow us to express the superconnection in terms of the supervielbein. Namely, the constraint $T_{mnp} = 0$ allow us to solve for the bosonic connection $\Omega_{mn}{}^{p}$, exactly as in general relativity. Constraints $T_{AB}{}^{C} = 0$ allow us to solve for $\Omega_{AB}{}^{C}$, and $T_{A\underline{\dot{B}}}{}^{\underline{\dot{C}}} = 0$, for $\Omega_{A\dot{B}}{}^{\dot{C}}$. But in $\mathcal{N} = 1$ supergravity one can even go further, and solve for the supervielbein parts with bosonic tangent indices $E_{n}{}^{\Pi}$ in terms of the other parts of the supervielbein. The conventional constraints that allow for that are $T_{A\dot{B}}{}^{m} = -2i\sigma^{m}_{A\dot{B}}$, $T_{A\dot{B}}{}^{\dot{C}} - \frac{1}{4}T_{A}{}^{mn}\left(\sigma_{mn}\right)_{\dot{B}}^{\ \dot{C}} = 0$.

In section 2.3, we required a stronger constraint, which in $\mathcal{N} = 1$ language is written as $T_{A}^{\ mn} = 0$. We can still require that as a conventional constraint, if we take for structure group $\mathrm{SO}(1,3) \times \mathrm{U}(1)$. In the formulations in which $\mathrm{U}(1)$ is not gauged, only the constraints above are taken for the conformal theory, but an extra constraint will be necessary in order to obtain the Poincaré theory. We will analyze the possible cases next.

3.2 From conformal to Poincaré supergravity

To obtain $\mathcal{N} = 1$ Poincaré supergravity from conformal supergravity, we must adopt contraints which do not preserve the superconformal invariance. However, we must not break all superconformal invariance, since that would be equivalent to fixing all the superconformal gauges, and we would be left only with the fields which are inert under superconformal gauge choices, i.e. the fields of the Weyl multiplet e_{μ}^{m}, ψ_{μ}^{A} and A_{μ}. As we will see, this will be the case either with gauged or with ungauged $\mathrm{U}(1)$.

3.2.1 Ungauged U(1)

To determine the nonconformal constraints, we must first determine the transformation properties of the supervielbeins and superconnections.

In Lorentz superspace, the super Weyl parameter L is complex. We define

$$E'_A{}^{\Pi} = e^L E_A{}^{\Pi}, \; E'_{\dot{A}}{}^{\Pi} = e^{\overline{L}} E_{\dot{A}}{}^{\Pi}. \tag{30}$$

Since, with our choice of constraints, supervielbeins and superconnections can all be expressed in terms of the spinor vielbeins, we only need these transformation properties; conventional constraints are valid for any set of vielbeins and, therefore, they are automatically satisfied when one replaces $E_A{}^{\Pi}$, $E_{\dot{A}}{}^{\Pi}$ by their rescaled values. Then it can be proven [9] that the representation preserving constraints are invariant under (30). If these constraints were not invariant under Weyl transformations, then chiral multiplets could not exist in the background of conformal supergravity.

A complex scalar superfield can be decomposed in local superspace into chiral and linear parts. After breaking part of the super-Weyl group, the parameters $L, \overline{L}$ will be restricted such that a linear combination of them will be either chiral or linear. In the first case, one needs a dimension $\frac{1}{2}$ constraint; in ths second, one of dimension 1. The only left unconstrained objects of dimension $\frac{1}{2}$ and 1 are, respectively, the torsion component $T_{Am}{}^m$ and the superfield $R = R_{AB}{}^{AB}$. These superfields transform under the super-Weyl group as $\left(\nabla^2 = \nabla^A \nabla_A\right)$ [9, 10]

$$T'_{Am}{}^m = e^L \left(T_{Am}{}^m + 2\nabla_A \left(2L + \overline{L}\right)\right), \; R' = 3\left(\nabla^2 + \frac{1}{3}R\right) e^{2L}. \tag{31}$$

We can break the super Weyl invariance by imposing as a constraint

$$T_{Am}{}^m = 0. \tag{32}$$

For that to be consistent, we must impose that $2L + \overline{L}$ is antichiral:

$$\nabla_A \left(2L + \overline{L}\right) = 0. \tag{33}$$

What is left from the super-Weyl group is the so-called Howe-Tucker group [4]: the supervielbeins transforming as in (30), with $L, \overline{L}$ satisfying (33).

This constraint leads to the "old minimal" formulation of $\mathcal{N} = 1$ Poincaré supergravity [11, 12]. To the Weyl multiplet of conformal supergravity we are adding a compensating chiral multiplet with 8+8 components.

Another possibility to break the super Weyl invariance is to set the constraint $R = 0$; the remaining super Weyl invariance contains a parameter L that now is an antilinear superfield: $\nabla^2 L = 0$. This constraint leads to the nonminimal formulation of $\mathcal{N} = 1$ Poincaré supergravity [13]. To the Weyl multiplet of conformal supergravity we are adding a compensating linear multiplet having 12+12 components. This way, we have fermionic auxiliary fields.

Both constraints can be generalized. On dimensional grounds, the most general nonconformal constraint one may take is given by [9, 10]

$$C = -\frac{1}{3}R + \frac{n+1}{3n+1}\nabla^A T_{Am}{}^m - \left(\frac{n+1}{3n+1}\right)^2 T^{Am}{}_m T_{An}{}^n = 0. \tag{34}$$

n is a numerical parameter. This constraint transforms, for small L, as

$$\delta C = 2LC - 2\left(\nabla^2 - 2\frac{n+1}{3n+1}T^{Am}{}_m \nabla_A\right)\left(L - \frac{n+1}{3n+1}\left(2L + \overline{L}\right)\right). \tag{35}$$

For a generic choice of n, the constraint $R = 0$ is necessary and we have a nonminimal formulation. Taking $n = -\frac{1}{3}$ corresponds to the "old minimal" formulation we saw.

Another interesting case occurs by taking $n = 0$: only $L + \overline{L}$ appears in δC, such that the (axial) U(1) local gauge invariance, which we did not include in the structure group, is actually conserved, with parameter $L - \overline{L}$. This corresponds to the "new minimal" (also known as "axial") formulation of $\mathcal{N} = 1$ Poincaré supergravity [14], in which one introduces a compensating tensor multiplet having 8+8 components.

Whichever constraint we choose, the irreducible parameter invariance of the resulting geometry corresponds to the compensating multiplet. This invariance allows for redefinition of torsions and, after gauge-fixing, for the fields of the compensating multiplet to appear in the final theory, with the original symmetry completely broken. These are very generical features, which we will also meet in the formulation of the $\mathcal{N} = 2$ theory.

3.2.2 Gauged U(1)

Let's now start from a $\mathrm{SO}(1,3) \times \mathrm{U}(1)$ superspace. From (8), the fermionic part of the U(1) connection transforms under U(1) as (A is a "flat" index):

$$\delta\Omega_A = -\nabla_A \Lambda - \Lambda\Omega_A \tag{36}$$

while, from (16), under a general transformation we have

$$\delta\Omega_A = \Phi_A - H_A{}^M \Omega_M. \tag{37}$$

In U(1) superspace, after fixing the constraints it can be shown [5] that one has $H_{AB} = \frac{1}{2}\varepsilon_{AB}H$, $\Phi_A = \frac{3}{2}\nabla_A H$, $H = -\frac{1}{4}H_m^{\ m}$ being an unconstrained superfield defined in section 2.2 which parametrizes the super-Weyl transformations. Overall, Ω_A transforms as

$$\delta\Omega_A = \nabla_A\left(\frac{3}{2}H - \Lambda\right) + \left(\frac{1}{2}H - \Lambda\right)\Omega_A. \tag{38}$$

From this transformation law, by setting the constraint $\Omega_A = 0$, we see that we break the superconformal and local U(1) symmetries and restrict the combination $\frac{3}{2}H - \Lambda$ to a compensating chiral multiplet. This is the "old minimal" formulation of $\mathcal{N} = 1$ Poincaré supergravity [11, 12]. Other formulations have a treatment similar to the ungauged U(1) case. From now on, by $\mathcal{N} = 1$ Poincaré supergravity we always mean the "old minimal" formulation with $n = -1/3$.

3.3 The chiral compensator and the chiral measure

The superspace approach we have discussed has the inconvenience of involving a large number of fields and a large symmetry group. This way, one must put constraints and choose a particular gauge to establish the compatibility to the x-space theory (see section 3.5). There is an approach which uses from the beginning fewer fields and a smaller symmetry group (holomorphic general coordinate transformations) [1, 9, 10, 15, 16, 17]. In this approach we take two chiral superspaces with complex coordinates (y^μ, θ), $\left(\overline{y^\mu}, \overline{\theta}\right)$, which are related by complex conjugation. In four-component spinor notation, $\theta = \frac{1}{2}(1+\gamma_5)\Theta$, $\overline{\theta} = \frac{1}{2}(1-\gamma_5)\Theta$. One also has

$$\frac{1}{2}\left(y^\mu + \overline{y^\mu}\right) = x^\mu,\ y^\mu - \overline{y^\mu} = 2iH^\mu(x, \Theta). \tag{39}$$

This way, the imaginary part of the coordinates $y^\mu, \overline{y^\mu}$ is interpreted as an axial vector superfield, while the real part is identified with real spacetime. One has then in the combined $8+4$ dimensional space $\left(y^\mu, \overline{y^\mu}, \theta, \overline{\theta}\right)$ a $4+4$ dimensional hypersurface defined by $y^\mu - \overline{y^\mu} = 2iH^\mu\left(y^\mu + \overline{y^\mu}, \theta, \overline{\theta}\right)$. When one shifts points by a coordinate transformation, the hypersurface itself is deformed in such a way that the new points lie on the new hypersurface. These hypersurfaces, each characterized by the superfield $H^\mu\left(y^\mu + \overline{y^\mu}, \theta, \overline{\theta}\right)$, represent each a real superspace like the one we have been working with.

The holomorphic coordinate transformations form a supergroup. If one puts no further restriction on their parameters, one is led to conformal supergravity. However, Poincaré supergravity is described by the very natural subgroup of unimodular holomorphic transformations, which satisfy

$$\mathrm{sdet}\frac{\partial\left(y^{\mu\prime},\theta'\right)}{\partial\left(y^{\mu},\theta\right)}=1. \tag{40}$$

One can take Poincaré supergravity is a gauge theory with the gravitational superfield $H^{\mu}\left(x,\theta\right)$ as a dynamical object and the supergroup of holomorphic coordinate transformations being the gauge group [16]. But one can also remove the constraint (40) and handle arbitrary holomorphic transformations at the cost of the appearance of a compensating superfield. In the "old minimal" $n=-1/3$ theory, this superfield, which we define as $\varphi\left(y^{\mu},\theta\right)$, is holomorphic and is called the chiral compensator. It transforms as [9, 10]

$$\varphi\left(y^{\mu},\theta\right)=\left[\mathrm{sdet}\frac{\partial\left(y^{\mu\prime},\theta'\right)}{\partial\left(y^{\mu},\theta\right)}\right]^{\frac{1}{3}}\varphi\left(y^{\mu\prime},\theta'\right). \tag{41}$$

One can then find a coordinate system in which $\varphi\left(y^{\mu},\theta\right)=1$. Clearly, all the holomorphic coordinate transformations preserving this gauge are unimodular; this way, we recover the gauge group of Poincaré supergravity. Poincaré supergravity is then a theory of two dynamical objects [15] - the gravitational superfield $H^{\mu}\left(x^{\mu},\theta,\overline{\theta}\right)$ and the chiral compensator $\varphi\left(y^{\mu},\theta\right)$ -, transforming under the supergroup of holomorphic coordinate transformations, and defined in real superspaces, given by the hypersurfaces above.

The chiral compensator allows us to define an invariant chiral measure in superspace. Since

$$d^{4}yd^{2}\theta=\mathrm{sdet}\frac{\partial\left(y^{\mu},\theta\right)}{\partial\left(y^{\mu\prime},\theta'\right)}d^{4}y'd^{2}\theta', \tag{42}$$

we have

$$\varphi^{3}\left(y^{\mu},\theta\right)d^{4}yd^{2}\theta=\varphi'^{3}\left(y^{\mu\prime},\theta'\right)d^{4}y'd^{2}\theta'. \tag{43}$$

We define then the chiral density [3, 9, 10, 17] as

$$\epsilon=\varphi^{3}. \tag{44}$$

From the transformation law of φ, one can see that ϵ transforms under supercoordinate transformations with parameters ξ^{Λ} as

$$\delta\epsilon=-\partial_{\Lambda}\left(\epsilon\xi^{\Lambda}(-)^{\Lambda}\right). \tag{45}$$

Instead of choosing the gauge $\varphi\left(y^{\mu},\theta\right)=1$, it is more convenient to choose a Wess-Zumino gauge for H^{μ}, in which this superfield is expressed only in terms of the physical and auxiliary fields from the supergravity multiplet. After fixing the remaining gauge freedom, the same is valid for ϵ.

3.4 Solution to the Bianchi identities in "old minimal" $\mathcal{N} = 1$ Poincaré supergravity

The full off-shell solution to the Bianchi identities, given the representation-preserving and conventional constraints in section 3.1 and the nonconformal constraint $T_{Am}{}^{m} = 0$, is standard textbook material which we do not include here [8, 18]. The results, in our conventions, may be seen in [19]. It can be shown that, as a result of $T_{Am}{}^{m} = 0$ and the conventional constraint $T_{A\dot{B}}{}^{\dot{C}} - \frac{1}{4}T_{A}{}^{mn}\left(\sigma_{mn}\right)_{\dot{B}}{}^{\dot{C}} = 0$, one actually has simply $T_{Am}{}^{m} = 0$ and actually recovers the conventional constraint from the approach with gauged U(1).

The off-shell solutions are described in terms of the supergravity superfields $R = R_{AB}{}^{AB}$, G_m, W_{ABC}, their complex conjugates and their covariant derivatives. R and $W_{\dot{A}\dot{B}\dot{C}}$ are antichiral:

$$\nabla^{A} R = 0,\ \nabla_{A} W_{\dot{A}\dot{B}\dot{C}} = 0. \tag{46}$$

In $\mathcal{N} = 1$, chiral superfields may exist with any number of undotted indices (but no dotted indices). Chiral projectors exist; when acting with them on any superfield with only undotted indices, a chiral superfield always results. For scalar superfields the antichiral projector is given by $\left(\nabla^2 + \frac{1}{3}R\right)$.

The torsion constraints imply the following off-shell differential relations (not field equations) between the $\mathcal{N} = 1$ supergravity superfields:

$$\begin{aligned}
\nabla^{A} G_{A\dot{B}} &= \frac{1}{24}\nabla_{\dot{B}} R, &(47)\\
\nabla^{A} W_{ABC} &= i\left(\nabla_{B\dot{A}} G_{C}{}^{\dot{A}} + \nabla_{C\dot{A}} G_{B}{}^{\dot{A}}\right), &(48)
\end{aligned}$$

which, together with the torsion conventional constraints, imply the relation

$$\nabla^2 \overline{R} - \overline{\nabla}^2 R = 96 i \nabla^{n} G_{n}. \tag{49}$$

3.5 From superspace to x-space

Another special feature of pure $\mathcal{N} = 1$ four-dimensional supergravity is that its action in superspace is known. It is written as the integral, over the whole superspace, of the superdeterminant of the supervielbein [1, 3]:

$$\mathcal{L}_{SG} = \frac{1}{2\kappa^2}\int E\, d^4\theta\, , E = \mathrm{sdet} E_{\Lambda}{}^{M}. \tag{50}$$

On dimensional grounds, this is the only possible action. The $\frac{1}{2\kappa^2}$ factor is necessary to reproduce the x-space results; in principle, one could multiply this action by any dimension zero unconstrained scalar, but that object does not exist. In this action, and in actions written as $d^4\theta$ integrals, the indices of the θ-variables are curved, i.e. they vary under Einstein transformations.

In order to determine the component expansion of this action, the best is certainly not to determine directly all the components of E, but rather to determine the component expansion of the supergravity superfields. For that, we use the method of gauge completion [18, 20]. The basic idea behind it is to relate in superspace some superfields and superparameters at $\theta = 0$ (which we symbolically denote with a vertical bar on the right) with some x space quantities, and then to require compatibility between the x space and superspace transformation rules [11, 12].

We make the following identification for the supervielbeins at $\theta = 0$ $E_\Pi{}^N\big|$:

$$E_\Pi{}^N\big| = \begin{bmatrix} e_\mu{}^m & \frac{1}{2}\psi_\mu{}^A & \frac{1}{2}\psi_\mu{}^{\dot{A}} \\ 0 & \delta_B{}^A & 0 \\ 0 & 0 & \delta_{\dot{B}}{}^{\dot{A}} \end{bmatrix}. \tag{51}$$

In the same way, we gauge the fermionic superconnection at order $\theta = 0$ to zero and we can set its bosonic part equal to the usual spin connection:

$$\Omega_{\mu m}{}^n\big| = \omega_{\mu m}{}^n(e,\psi)\,,\, \Omega_{Am}{}^n\big|\,,\, \Omega_{\dot{A}m}{}^n\big| = 0. \tag{52}$$

The spin connection is given, in $\mathcal{N} = 1$ supergravity, by

$$\begin{aligned} \omega_{\mu m}{}^n(e,\psi) &= \omega_{\mu m}{}^n(e) - \frac{i}{4}\kappa^2\Big(\psi_{\mu A}\sigma_m^{A\dot{A}}\psi_{\dot{A}}^n - \psi_{\mu A}\sigma^{nA\dot{A}}\psi_{m\dot{A}} + \psi_{mA}\sigma_\mu^{A\dot{A}}\psi_{\dot{A}}^n \\ &+ \psi_{\mu\dot{A}}\sigma_m^{A\dot{A}}\psi_A^n - \psi_{\mu\dot{A}}\sigma^{nA\dot{A}}\psi_{mA} + \psi_{m\dot{A}}\sigma_\mu^{A\dot{A}}\psi_A^n\Big). \end{aligned} \tag{53}$$

$\omega_{\mu m}{}^n(e)$ is the connection from general relativity. We also identify, at the same order $\theta = 0$, the superspace vector covariant derivative (with an Einstein indice) with the curved space covariant derivative: $D_\mu| = \mathcal{D}_\mu$. These gauge choices are all preserved by supergravity transformations.

As a careful analysis using the solution to the Bianchi identities and the off-shell relations among the supergravity superfields $\overline{R}, G_n, W_{ABC}$ shows, the component field content of these superfields is all known once we know

$$\overline{R}\big|\,,\,\nabla_A\overline{R}\big|\,,\,\nabla^2\overline{R}\big|\,,\,G_{A\dot{A}}\big|\,,\,\nabla_{\underline{A}}G_{\underline{B}\dot{A}}\Big|\,,\,\nabla_{\underline{\dot{A}}}\nabla_{\underline{A}}G_{\underline{B\dot{B}}}\Big|\,,\,W_{ABC}\big|\,,\,\nabla_{\underline{D}}W_{\underline{ABC}}\big|\,.$$

All the other components and higher derivatives of $\overline{R}, G_{A\dot{A}}, W_{ABC}$ can be written as functions of these previous ones. In order to determine the "basic" components, first we solve for superspace torsions and curvatures in terms of supervielbeins and superconnections using (51) and (52); then we identify them with the off-shell solution to the Bianchi identities [9, 18, 20] [1]:

$$\overline{R}\big| = 4\,(M + iN)\,,\ G_{A\dot{A}}\big| = \frac{1}{3}A_{A\dot{A}},\ W_{ABC}\big| = -\frac{1}{4}\psi_{\underline{A}}{}^{\dot{C}}{}_{\underline{B}\dot{C}\underline{C}} - \frac{i}{4}A_{\underline{A}}{}^{\dot{C}}\psi_{\underline{B}\dot{C}\underline{C}}.$$

$R|$ and $G_m|$ are auxiliary fields. A_μ, a gauge field in conformal supergravity, is an auxiliary field in Poincaré supergravity. The (anti)chirality condition on $R, \overline{R}$ implies their $\theta = 0$ components (resp. the auxiliary fields $M - iN, M + iN$) lie in antichiral/chiral multiplets (the compensating multiplets); (47) shows the spin-1/2 parts of the gravitino lie on the same multiplets (because, as we will see in the next section, $\nabla_A G_{B\dot{B}}$, at $\theta = 0$, is the gravitino curl) and, according to (49), so does $\partial^\mu A_\mu$.

$\nabla_A \overline{R}\big|\,, \nabla_{\underline{A}} G_{\underline{B}\dot{A}}\Big|$ also come straightforwardly from comparison to the solution to the Bianchi identities [9, 19]. Finding $\nabla^2\overline{R}\big|\,, \nabla_{\dot{\underline{A}}}\nabla_{\underline{A}} G_{\underline{B\dot{B}}}\Big|\,, \nabla_{\underline{D}} W_{\underline{ABC}}\big|$ is a bit more involved: one must identify the (super)curvature $R_{\mu\nu}{}^{mn}$ with the x-space curvature $\mathcal{R}_{\mu\nu}{}^{mn}$, multiply by the inverse supervielbeins $E_M^{\ \mu} E_N^{\ \nu}$, identify with the solution to the Bianchi identities for R_{MN} and extract the field contents by convenient index manipulation. The field content of these components will include the Riemann tensor in one of its irreducible components, respectively the Ricci scalar, the Ricci tensor and the self-dual Weyl tensor ($\mathcal{W}_{ABCD} := -\frac{1}{8}\mathcal{W}^{+}_{\mu\nu\rho\sigma}\sigma^{\mu\nu}_{\underline{AB}}\sigma^{\rho\sigma}_{\underline{CD}}, \mathcal{W}^{\mp}_{\mu\nu\rho\sigma} := \frac{1}{2}\left(\mathcal{W}_{\mu\nu\rho\sigma} \pm \frac{i}{2}\varepsilon_{\mu\nu}{}^{\lambda\tau}\mathcal{W}_{\lambda\tau\rho\sigma}\right)$). The full results are derived in [19]; at the linearized level,

$$\nabla^2\overline{R}\big| = -8\mathcal{R} \ + \ \ldots,\ \nabla_{\underline{A}}\nabla_{\dot{\underline{A}}} G_{\underline{B\dot{B}}}\Big| = -\frac{1}{2}\sigma^{\mu}_{\underline{A\dot{A}}}\sigma^{\nu}_{\underline{B\dot{B}}}\left(\mathcal{R}_{\mu\nu} - \frac{1}{4}g_{\mu\nu}\mathcal{R}\right) + \ldots,$$

$$\nabla_{\underline{A}} W_{\underline{BCD}}\big| \ = \ -\frac{1}{8}\mathcal{W}^{+}_{\mu\nu\rho\sigma}\sigma^{\mu\nu}_{\underline{AB}}\sigma^{\rho\sigma}_{\underline{CD}} + \ldots,\ \nabla^2 W^2\big| = -2\mathcal{W}_{+}^2 + \ldots, \tag{54}$$

$$\nabla_{\dot{\underline{A}}} W_{\underline{\dot{B}\dot{C}\dot{D}}}\Big| \ = \ -\frac{1}{8}\mathcal{W}^{-}_{\mu\nu\rho\sigma}\sigma^{\mu\nu}_{\underline{\dot{A}\dot{B}}}\sigma^{\rho\sigma}_{\underline{\dot{C}\dot{D}}} + \ldots,\ \overline{\nabla}^2\overline{W}^2\Big| = -2\mathcal{W}_{-}^2 + \ldots \tag{55}$$

Knowing these components, we can compute, in x-space, any action which involves the supergravity multiplet. In order to do that, we need to know how to convert superspace actions to x-space actions.

[1] $\psi^B_{\mu\nu} = \mathcal{D}_\mu\psi^B_\nu - \mathcal{D}_\nu\psi^B_\mu$ is the gravitino curl.

Consider the coupling of a real scalar superfield to supergravity given by

$$\begin{aligned}\mathcal{L} &= \frac{1}{2\kappa^2}\int E\Phi d^4\theta = \frac{3}{4\kappa^2}\int \left[\frac{E}{\overline{R}}\left(\overline{\nabla}^2 + \frac{1}{3}\overline{R}\right) + \frac{E}{R}\left(\nabla^2 + \frac{1}{3}R\right)\right]\Phi d^4\theta \\ &= \frac{3}{4\kappa^2}\int \left(-\frac{1}{4}\frac{\overline{D}^2 E}{\overline{R}}\right)\left[\left(\overline{\nabla}^2 + \frac{1}{3}\overline{R}\right)\Phi\right] d^2\theta + \text{h.c.}.\end{aligned} \tag{56}$$

$D_A = (E^{-1})_A^{\ M}\nabla_M$ is the superspace covariant derivative with an Einstein index. In the previous equation, the operator $d^2\overline{\theta} = -\frac{1}{4}\overline{D}^2$ should apply to all the integrand, and not only to E. But, knowing that we can choose the gauge (51), we have $D_A| = \nabla_A|$ and therefore, to order $\theta = 0$, we have

$$D_{\dot{A}}\left[\frac{1}{\overline{R}}\left(\overline{\nabla}^2 + \frac{1}{3}\overline{R}\right)\right]\Bigg| = \nabla_{\dot{A}}\left[\frac{1}{\overline{R}}\left(\overline{\nabla}^2 + \frac{1}{3}\overline{R}\right)\right]\Bigg| = 0. \tag{57}$$

A "rigid" or "curved" superfield whose $\theta = 0$ component vanishes in any frame is identically zero (for a proof see [1]). Therefore, we conclude that we have $D_{\dot{A}}\left[\frac{1}{\overline{R}}\left(\overline{\nabla}^2 + \frac{1}{3}\overline{R}\right)\right] = 0$, and we may write (56).

In the particular gauge (51), we can write the chiral density (44) as

$$\epsilon = \frac{1}{4}\frac{\overline{D}^2 E}{\overline{R}}. \tag{58}$$

The proof of this fact requires the knowledge of the solution of the supergravity constraints in terms of unconstrained superpotentials [15]. Indeed, one of these prepotentials is identical to the chiral compensator. (58) is obtained from expressing the supertorsions in terms of the prepotentials [9, 10].

The expansion in components of the chiral density is derived, in the same gauge, by requiring that $2\epsilon| = e$ and using its transformation law (45) [17]. In its expression, the θ-variables carry Lorentz indices. In these new θ-variables, the coefficients of the θ-expansion of chiral superfields are precisely their covariant derivatives [6, 18]. A chiral superfield has no $\overline{\theta}$'s in its expansion. This makes superspace integration much easier. For $\mathcal{N} = 1, 2$, when we write full superspace integrals the θ-variables carry Einstein indices, but when the integrals are in half superspace ($d^2\theta$ in $\mathcal{N} = 1$, $d^4\theta$ in $\mathcal{N} = 2$), they carry Lorentz indices. Therefore, one finally has for (56)

$$\mathcal{L} = -\frac{3}{4\kappa^2}\int \epsilon\left[\left(\overline{\nabla}^2 + \frac{1}{3}\overline{R}\right)\Phi\right] d^2\theta + \text{h.c.}. \tag{59}$$

By writing (56) on this form, one can identify the lagrangian of supergravity minimally coupled to a chiral field [18, 21]. The lagrangian of pure supergravity is simply obtained by taking $\Phi = 1$.

4 $\mathcal{N} = 2$ supergravity in superspace

4.1 $\mathcal{N} = 2$ conformal supergravity

The $\mathcal{N} = 2$ Weyl multiplet has 24+24 degrees of freedom. Its field content is given by the graviton e_μ^m, the gravitinos ψ_μ^{Aa}, the U(2) connection $\widetilde{\Phi}_\mu^{ab}$, an antisymmetric tensor W_{mn} which we decompose as $W_{A\dot{A}B\dot{B}} = 2\varepsilon_{\dot{A}\dot{B}}W_{AB} + 2\varepsilon_{AB}W_{\dot{A}\dot{B}}$, a spinor Λ_A^a and, as auxiliary field, a dimension 2 scalar I. In superspace, a gauge choice can be made (in the supercoordinate transformation) such that the graviton and the gravitinos are related to $\theta = 0$ components of the supervielbein (symbolically $E_\Pi^{\ N}\big|$):

$$E_\Pi^{\ N}\Big| = \begin{bmatrix} e_\mu^{\ m} & \frac{1}{2}\psi_\mu^{\ Aa} & \frac{1}{2}\psi_\mu^{\ \dot{A}a} \\ 0 & -\delta_B^{\ A}\delta_b^{\ a} & 0 \\ 0 & 0 & -\delta_{\dot{B}}^{\ \dot{A}}\delta_b^{\ a} \end{bmatrix}. \tag{60}$$

In the same way, we gauge the fermionic part of the Lorentz superconnection at order $\theta = 0$ to zero and we can set its bosonic part equal to the usual spin connection:

$$\Omega_{\mu m}^{\ \ \ n}\Big| = \omega_{\mu m}^{\ \ \ n}(e, \psi^a), \Omega_{Aam}^{\ \ \ \ n}\big|, \Omega_{\dot{A}am}^{\ \ \ \ n}\Big| = 0. \tag{61}$$

The U(2) superconnection $\widetilde{\Phi}_\Pi^{ab}$ is such that $\widetilde{\Phi}_\mu^{ab}\Big| = \widetilde{\Phi}_\mu^{ab}$. The other fields are the $\theta = 0$ component of some superfield, which we write in the same way.

The chiral superfield W_{AB} is the basic object of $\mathcal{N} = 2$ conformal supergravity, in terms of which its action is written. Other theories with different $\mathcal{N}$ have an analogous superfield (e.g. W_{ABC} in $\mathcal{N} = 1$).

In U(2) $\mathcal{N} = 2$ superspace there is an off-shell solution to the Bianchi identities. The torsions and curvatures can be expressed in terms of superfields W_{AB}, Y_{AB}, $U_{A\dot{A}}^{ab}$, X_{ab}, their complex conjugates and their covariant derivatives. Of these four superfields, only W_{AB} transforms covariantly under super-Weyl transformations. The other three superfields transform non-covariantly; they describe all the non-Weyl covariant degrees of freedom in the transformation parameter H, and can be gauged away by a convenient (Wess-Zumino) gauge choice. Another nice feature of $\mathcal{N} = 2$ superspace is that there exists, analogously to the $\mathcal{N} = 1$ case, a chiral density ϵ which allows us to write chiral actions [22].

4.2 Degauging U(1)

The first step for obtaining the Poincaré theory is to couple to the conformal theory an abelian vector multiplet (with central charge), described by a vector A_μ, a complex scalar, a Lorentz-scalar SU(2) triplet and a spinorial SU(2) dublet. The vector A_μ is the gauge field of central charge transformations; it corresponds, in superspace, to a 1-form A_Π with a U(1) gauge invariance (the central charge transformation). This 1-form does not belong to the superspace geometry. Using the U(1) gauge invariance we can set the gauge $A_\Pi| = (A_\mu, 0)$. The field strength $F_{\Pi\Sigma}$ is a two-form defined as $F_{\Pi\Sigma} = 2D_{[\Pi}A_{\Sigma\}}$ or, with flat indices, $F_{MN} = 2\nabla_{[M}A_{N\}} + T_{MN}{}^P A_P$. It satisfies its own Bianchi identities $D_{[\Gamma}F_{\Pi\Sigma\}} = 0$ or, with flat indices,

$$\nabla_{[M}F_{NP\}} + T_{MN|}{}^Q F_{Q|P\}} = 0. \tag{62}$$

Here we split the U(2) superconnection $\widetilde{\Phi}^{ab}_\Pi$ into a SU(2) superconnection Φ^{ab}_Π and a U(1) superconnection φ_Π; only the later acts on A_Π: $\widetilde{\Phi}^{ab}_\Pi = \Phi^{ab}_\Pi - \frac{1}{2}\varepsilon^{ab}\varphi_\Pi$. One has to impose covariant constraints on its components (like in the torsions), in order to construct invariant actions:

$$F^{ab}_{AB} = 2\sqrt{2}\varepsilon_{AB}\varepsilon^{ab}F,\ \ F^{ab}_{A\dot{B}} = 0. \tag{63}$$

By solving the F_{MN} Bianchi identities with these constraints, we conclude that they define an off-shell $\mathcal{N} = 2$ vector multiplet, given by the $\theta = 0$ components of the superfields $A_\mu, F, F^a_A = \frac{i}{2}F^{\dot{A}a}{}_{A\dot{A}}, F^a_b = \frac{1}{2}\left(-\nabla^B_b F^a_B + F\overline{X}^a{}_b + \overline{F}X^a{}_b\right)$. $F^a_b|$ is an auxiliary field; $F^a_a = 0$ if the multiplet is abelian (as it has to be in this context). $\overline{F}$ is a Weyl covariant chiral superfield, with nonzero U(1) and Weyl weigths. A superconformal chiral lagrangian for the vector multiplet is

$$\mathcal{L} = \int \overline{\epsilon}F^2 d^4\overline{\theta} + \text{h.c.}. \tag{64}$$

In order to get a Poincaré theory, we must break the superconformal and local abelian (from the U(1) subgroup of U(2) - not the gauge invariance of A_μ) invariances. For that, we set the Poincaré gauge $F = \overline{F} = 1$. As a consequence, from the Bianchi and Ricci identities we get

$$\varphi^a_A = 0,\ \ F^A_a = 0. \tag{65}$$

Furthermore, $U^{ab}_{A\dot{A}}$ is an SU(2) singlet, to be identified with the bosonic U(1) connection (now an auxiliary field):

$$U^{ab}_{A\dot{A}} = \varepsilon^{ab}U_{A\dot{A}} = \varepsilon^{ab}\varphi_{A\dot{A}}. \tag{66}$$

Other consequences are

$$F_{A\dot{A}B\dot{B}} = \sqrt{2}i\left[\varepsilon_{AB}\left(W_{\dot{A}\dot{B}}+Y_{\dot{A}\dot{B}}\right)+\varepsilon_{\dot{A}\dot{B}}\left(W_{AB}+Y_{AB}\right)\right], \tag{67}$$

$$F_b^a = X_b^a, \tag{68}$$

$$\overline{X_{ab}} = X^{ab}. \tag{69}$$

(67) shows that W_{mn} is now related to the vector field strength F_{mn}. Y_{mn} emerges as an auxiliary field, like X_{ab} (from (68)). We have, therefore, the minimal field representation of $\mathcal{N}=2$ Poincaré supergravity, with a local SU(2) gauge symmetry and 32+32 off-shell degrees of freedom:

$$e_\mu^m, \psi_\mu^{Aa}, A_\mu, \Phi_\mu^{ab}, Y_{mn}, U_m, \Lambda_A^a, X_{ab}, I. \tag{70}$$

Although the algebra closes with this multiplet, it does not admit a consistent lagrangian because of the higher-dimensional scalar I [23].

4.3 Degauging SU(2)

The second step is to break the remaining local SU(2) invariance. This symmetry can be partially broken (at most, to local SO(2)) through coupling to a compensating so-called "improved tensor multiplet" [24, 25], or broken completely. We take the later possibility. There are still two different versions of off-shell $\mathcal{N}=2$ supergravity without SO(2) symmetry, each with different physical degrees of freedom. In both cases we start by imposing a constraint on the SU(2) parameter L^{ab} which restricts it to a compensating nonlinear multiplet [26] (at the linearized level, $\nabla_A^{\underline{a}} L^{\underline{bc}}=0$). From the transformation law of the SU(2) connection $\delta\Phi_M^{ab}=-\nabla_M L^{ab}$ we can get the required condition for L^{ab} by imposing the following constraint on the fermionic connection:

$$\Phi_A^{abc} = 2\varepsilon^{a\underline{b}}\rho_A^{\underline{c}}. \tag{71}$$

This constraint requires introducing a new fermionic superfield ρ_A^a. We also introduce its fermionic derivatives P and H_m. The previous SU(2) connection Φ_μ^{ab} is now an unconstrained auxiliary field. The divergence of H_m is constrained, though, at the linearized level by the condition $\nabla^m H_m = \frac{1}{3}R - \frac{1}{12}I$, which is equivalent to saying that I is no longer an independent field. This constraint implies that only the transverse part of H_m belongs to the nonlinear multiplet; its divergence lies in the original Weyl multiplet. From the structure equation (12) and the definition (71), we can derive off-shell relations for the (still SU(2) covariant) derivatives of ρ_A^a. Altogether, these component fields form then the "old minimal" $\mathcal{N}=2$ 40+40 multiplet

[27]: $e_\mu^m, \psi_\mu^{Aa}, A_\mu, \Phi_\mu^{ab}, Y_{mn}, U_m, \Lambda_A^a, X_{ab}, H_m, P, \rho_A^a$. This is "old minimal" $\mathcal{N}=2$ supergravity, the formulation we are working with. The final lagrangian can be found in [26, 28]. The other possibility (also with SU(2) completely broken) is to further restrict the compensating non-linear multiplet to an on-shell scalar multiplet [29]. This reduction generates a minimal 32+32 multiplet (not to be confused with (70)) with new physical degrees of freedom. We will not further pursue this version of $\mathcal{N}=2$ supergravity.

4.4 From $\mathcal{N}=2$ SU(2) superspace to $x-$space

Our choices for torsion constraints in $\mathcal{N}=2$ are very similar to the ones for generic $\mathcal{N}$ presented in section 2.3, the only difference being that, like in $\mathcal{N}=1$, we have the representation-preserving constraints $T_{AB}^{abm}, T_{AaBb\dot{C}c}=0$. In conformal supergravity, all torsions and curvatures can be expressed in terms of the basic superfields W_{AB}, Y_{AB}, $U_{A\dot{A}}$, X_{ab}. After breaking of superconformal invariance and local U(2), the basic superfields in the Poincaré theory become the physical field W_{AB} and the auxiliary field ρ_A^a [30]. All torsions and curvatures can be expressed off-shell in terms of these superfields, their complex conjugates and derivatives [28]. $W_{AB}|$, at the linearized level, is related to the field strength of the physical vector field A_μ (the graviphoton): from (67),

$$W_{AB}| = -\frac{i}{2\sqrt{2}}\sigma_{AB}^{mn}F_{mn} - Y_{AB} - \frac{i}{4}\sigma_{AB}^{mn}\left(\psi_m^{Cc}\psi_{nCc} + \psi_m^{\dot{C}c}\psi_{n\dot{C}c}\right). \qquad (72)$$

$X^{ab} = \frac{1}{2}\left(\nabla^{\dot{A}\underline{a}} - 2\rho^{\dot{A}\underline{a}}\right)\rho_{\dot{A}}^{\underline{b}}$, $Y_{AB} = -\frac{i}{2}\left(\nabla_{\underline{A}}^a + 2\rho_{\underline{A}}^a\right)\rho_{\underline{B}a}$, $H_{A\dot{A}} = -i\nabla_A^a\rho_{\dot{A}a} + i\nabla_{\dot{A}}^a\rho_{Aa}$, $P = i\nabla^{\dot{A}a}\rho_{\dot{A}a}$, $\Phi_{A\dot{A}}^{ab} = \frac{i}{2}\left(\nabla_A^{\underline{a}}\rho_{\dot{A}}^{\underline{b}} - \nabla_{\dot{A}}^{\underline{a}}\rho_A^{\underline{b}} - 4\rho_A^{\underline{a}}\rho_{\dot{A}}^{\underline{b}}\right)$, $U_{A\dot{A}} = \frac{1}{4}\left(\nabla_A^a\rho_{\dot{A}a} + \nabla_{\dot{A}}^a\rho_{Aa} + 4\rho_A^a\rho_{\dot{A}a}\right)$, $\Lambda^{Aa} = -i\nabla_b^A X^{ab}$ are auxiliary fields at $\theta=0$; $I = i\nabla^{\dot{A}a}\Lambda_{\dot{A}a} - i\nabla^{Aa}\Lambda_{Aa}$ is a dependent field. In the linearized approximation,

$$\begin{aligned}
W_{BCAa}| &= \frac{i}{2}\nabla_{Ba}W_{CA}\,| - \frac{i}{6}\left(\varepsilon_{BC}\Lambda_{Aa} + \varepsilon_{BA}\Lambda_{Ca}\right)\Big| = -\frac{1}{4}\psi_{ABCc} + \ldots, \\
Y_{BC\dot{A}a}| &= -\frac{i}{2}\nabla_{\dot{A}a}Y_{BC}\Big| = -\frac{1}{8}\psi_{BC\dot{A}a} + \ldots, \\
W_{ABCD}| &= \left(\frac{i}{4}\nabla_{\underline{A}}^b\nabla_{\underline{B}b} - 2Y_{\underline{AB}}\right)W_{\underline{CD}}\Big| = -\frac{1}{8}\mathcal{W}_{\mu\nu\rho\sigma}^{+}\sigma_{\underline{AB}}^{\mu\nu}\sigma_{\underline{CD}}^{\rho\sigma} + \ldots, \qquad (73) \\
P_{AB\dot{A}\dot{B}}| &= \left(\frac{i}{8}\nabla_{\underline{A}}^b\nabla_{\underline{B}b}Y_{\dot{A}\dot{B}} + \text{h.c.}\right)\Big|\ldots = \frac{1}{2}\sigma_{\underline{A\dot{C}}}^{\mu}\sigma_{\underline{B\dot{D}}}^{\nu}\left(\mathcal{R}_{\mu\nu} - \frac{1}{4}g_{\mu\nu}\mathcal{R}\right)\ldots,
\end{aligned}$$

$$R| = \left(\frac{i}{4} \nabla^{\dot{A}a} \nabla_a^{\dot{B}} W_{\dot{A}\dot{B}} - \frac{1}{4} \nabla^{Aa} \nabla_A^b X_{ab} + \text{h.c.} \right) \Big| + \ldots = -\mathcal{R} + \ldots$$

4.5 The chiral density and the chiral projector

The action of $\mathcal{N} = 2, d = 4$ Poincaré supergravity is written in superspace as

$$\mathcal{L}_{SG} = -\frac{3}{4\kappa^2} \int \bar{\epsilon} d^4 \bar{\theta} + \text{h.c.}. \tag{74}$$

The expansion of the chiral density ϵ in components, which allows us to write chiral actions, can be seen in [28]. From the solution to the Bianchi identities one can check that the following object is an antichiral projector [6]:

$$\nabla^{Aa} \nabla_A^b \left(\nabla_a^B \nabla_{Bb} + 16 X_{ab} \right) - \nabla^{Aa} \nabla_a^B \left(\nabla_A^b \nabla_{Bb} - 16 i Y_{AB} \right). \tag{75}$$

When one acts with this projector on any scalar superfield, one gets an antichiral superfield (with the exception of W_{AB}, only scalar chiral superfields exist in curved $\mathcal{N} = 2$ superspace; other types of chiral superfields are incompatible with the solution to the Bianchi identities). Together with ϵ, this projector allows us to write more general actions in superspace.

5 Superstring α'^3 effective actions and $\mathcal{R}^4$ terms in $d = 4$

In $d = 4$, there are only two independent real scalar polynomials made from four powers of the Weyl tensor [31], given by

$$\mathcal{W}_+^2 \mathcal{W}_-^2 = \mathcal{W}^{ABCD} \mathcal{W}_{ABCD} \mathcal{W}^{\dot{A}\dot{B}\dot{C}\dot{D}} \mathcal{W}_{\dot{A}\dot{B}\dot{C}\dot{D}}, \tag{76}$$

$$\mathcal{W}_+^4 + \mathcal{W}_-^4 = \left(\mathcal{W}^{ABCD} \mathcal{W}_{ABCD} \right)^2 + \left(\mathcal{W}^{\dot{A}\dot{B}\dot{C}\dot{D}} \mathcal{W}_{\dot{A}\dot{B}\dot{C}\dot{D}} \right)^2. \tag{77}$$

We now write the effective actions for type IIB, type IIA and heterotic superstrings in $d = 4$, after compactification from $d = 10$ in an arbitrary manifold, in the Einstein frame (considering only terms which are simply powers of the Weyl tensor, without any other fields except their couplings to the dilaton, and introducing the $d = 4$

gravitational coupling constant κ):

$$\left.\frac{\kappa^2}{e}\mathcal{L}_{\mathrm{IIB}}\right|_{\mathcal{R}^4} = -\frac{\zeta(3)}{32}e^{-6\phi}\alpha'^3\mathcal{W}_+^2\mathcal{W}_-^2 - \frac{1}{2^{11}\pi^5}e^{-4\phi}\alpha'^3\mathcal{W}_+^2\mathcal{W}_-^2, \tag{78}$$

$$\begin{aligned}\left.\frac{\kappa^2}{e}\mathcal{L}_{\mathrm{IIA}}\right|_{\mathcal{R}^4} &= -\frac{\zeta(3)}{32}e^{-6\phi}\alpha'^3\mathcal{W}_+^2\mathcal{W}_-^2 \\ &- \frac{1}{2^{12}\pi^5}e^{-4\phi}\alpha'^3\left[\left(\mathcal{W}_+^4+\mathcal{W}_-^4\right)+224\mathcal{W}_+^2\mathcal{W}_-^2\right],\end{aligned} \tag{79}$$

$$\begin{aligned}\left.\frac{\kappa^2}{e}\mathcal{L}_{\mathrm{het}}\right|_{\mathcal{R}^2+\mathcal{R}^4} &= -\frac{1}{16}e^{-2\phi}\alpha'\left(\mathcal{W}_+^2+\mathcal{W}_-^2\right)+\frac{1}{64}\left(1-2\zeta(3)\right)e^{-6\phi}\alpha'^3\mathcal{W}_+^2\mathcal{W}_-^2 \\ &- \frac{1}{3\times 2^{12}\pi^5}e^{-4\phi}\alpha'^3\left[\left(\mathcal{W}_+^4+\mathcal{W}_-^4\right)+20\mathcal{W}_+^2\mathcal{W}_-^2\right].\end{aligned} \tag{80}$$

These are only the moduli-independent $\mathcal{R}^4$ terms from these actions. Strictly speaking not even these terms are moduli-independent, since they are all multiplied by the volume of the compactification manifold, a factor we omitted for simplicity. But they are always present, no matter which compactification is taken. The complete action, for every different manifold, includes many other moduli-dependent terms which we do not consider here: we are mostly interested in a $\mathbb{T}^6$ compactification.

At string tree level, for all these theories in $d=4$ only $\mathcal{W}_+^2\mathcal{W}_-^2$ shows up. Because of its well known $d=10$ $\mathrm{SL}(2,\mathbb{Z})$ invariance, in type IIB theory only the combination $\mathcal{W}_+^2\mathcal{W}_-^2$ is present in the $d=4$ effective action (78). In the other theories, $\mathcal{W}_+^4+\mathcal{W}_-^4$ shows up at string one loop level. For type IIA, the reason is the difference between the left and right movers in the relative GSO projection at one string loop, because of this theory being nonchiral. Heterotic string theories have $\mathcal{N}=1$ supersymmetry in ten dimensions, which allows corrections to the sigma model already at order α', including $\mathcal{R}^2$ corrections (forbidden in type II theories in $d=10$). Because of cancellation of gravitational anomalies, another $\mathcal{R}^4$ contribution is needed in heterotic theories, which when reduced to $d=4$ gives rise to (76) and (77).

Next we consider the supersymmetrization of these $\mathcal{R}^4$ terms in $d=4$.

5.1 $\mathcal{N}=1,2$ supersymmetrization of $\mathcal{W}_+^2\mathcal{W}_-^2$

The supersymmetrization of the square of the Bel-Robinson tensor $\mathcal{W}_+^2\mathcal{W}_-^2$ has been known for a long time, in simple [19, 32] and extended [33, 34] four dimensional supergravity.

5.1.1 $\mathcal{N}=1$

In $\mathcal{N}=1$, the lagrangian to be considered is (α is a numerical constant)

$$\mathcal{L}_{SG}+\mathcal{L}_{\mathcal{R}^4}=\frac{1}{2\kappa^2}\int E\left(1+\alpha\kappa^6W^2\overline{W}^2\right)d^4\theta. \tag{81}$$

From (54) and (55), the α term represents the supersymmetrization of $\mathcal{W}_+^2\mathcal{W}_-^2$. To compute the variation of this action, we obviously need the constrained variation of W_{ABC}. The details of this calculation are presented in [19], and so is the final result for $\int\delta\left[E\left(1+\alpha\kappa^6W^2\overline{W}^2\right)\right]d^4\theta$, which we do not reproduce here again. From this result, the $R,\overline{R}$ field equations are given by

$$R=6\alpha\kappa^6\frac{\overline{W}^2\nabla^2W^2}{1-2\alpha\kappa^6W^2\overline{W}^2}=6\alpha\kappa^6\overline{W}^2\nabla^2W^2+12\alpha^2\kappa^{12}\overline{W}^4W^2\nabla^2W^2. \tag{82}$$

From (49), we can easily determine ∇^nG_n. This way, auxiliary fields belonging to the compensating chiral multiplet can be eliminated on-shell. This is not the case for the auxiliary fields which come from the Weyl multiplet (A_m), as we obtained, also in [19], a complicated differential field equation for G_m.

5.1.2 $\mathcal{N}=2$

Analogously to $\mathcal{N}=1$, we write the $\mathcal{N}=2$ supersymmetric $\mathcal{R}^4$ lagrangian in superspace, using the chiral projector and the chiral density, as a correction to the pure supergravity lagrangian [34] (α is again a numerical constant):

$$\begin{aligned}\mathcal{L}_{SG}+\mathcal{L}_{\mathcal{R}^4} &= \int\overline{\epsilon}\left[-\frac{3}{4\kappa^2}+\alpha\kappa^4\left(\nabla^{Aa}\nabla_A^b\left(\nabla_a^B\nabla_{Bb}+16X_{ab}\right)\right.\right.\\ &- \left.\left.\nabla^{Aa}\nabla_a^B\left(\nabla_A^b\nabla_{Bb}-16iY_{AB}\right)\right)W^2\overline{W}^2\right]d^4\overline{\theta}+\text{h.c.}.\end{aligned} \tag{83}$$

From the component expansion (73), the α term clearly contains $e\mathcal{W}_+^2\mathcal{W}_-^2$.

At this point we proceed with the calculation of the components of (83) and analysis of its field content. For that, we use the differential constraints from the solution to the Bianchi identities and the commutation relations. The process is straightforward but lengthy [34]. The results can be summarized as follows: with the correction (83), auxiliary fields X_{ab}, $\Lambda_{\dot{C}c}$, $Y_{\dot{A}\dot{B}}$, U_m and Φ_m^{ab} get derivatives, and the same should be true for their field equations; therefore, these superfields cannot be eliminated on-shell. We also fully checked that superfields P and H_m do not get derivatives (with the important exception of $\nabla^m H_m$) and, therefore, have algebraic field equations which should allow for their elimination on shell. The only auxiliary field remaining is ρ^a_A. We did not analyze its derivatives because that would require computing a big number of terms and, for each term, a huge number of different contributions. Its derivatives should cancel, though: otherwise, we would have a field (ρ^a_A) with a dynamical field equation while having two fields obtained from its spinorial derivatives (P and the transverse part of H_m) without such an equation. ρ^a_A, like P and transverse H_m, are intrinsic to the "old minimal" version of $\mathcal{N} = 2$ supergravity; they all belong to the same nonlinear multiplet. The physical theory does not depend on these auxiliary fields and, therefore, it seems natural that they can be eliminated from the classical theory and its higher-derivative corrections.

5.2 $\mathcal{N} = 1$ supersymmetrization of $\mathcal{W}^4_+ + \mathcal{W}^4_-$

For the term $\mathcal{W}^4_+ + \mathcal{W}^4_-$ there is a "no-go theorem", which goes as follows [35]: for a polynomial $I(\mathcal{W})$ of the Weyl tensor to be supersymmetrizable, each one of its terms must contain equal powers of $\mathcal{W}^+_{\mu\nu\rho\sigma}$ and $\mathcal{W}^-_{\mu\nu\rho\sigma}$. The whole polynomial must then vanish when either $\mathcal{W}^+_{\mu\nu\rho\sigma}$ or $\mathcal{W}^-_{\mu\nu\rho\sigma}$ do.

The derivation of this result is based on $\mathcal{N} = 1$ chirality arguments, which require equal powers of the different chiralities of the gravitino in each term of a superinvariant. The rest follows from the supersymmetric completion. That is why the only exception to this result is $\mathcal{W}^2 = \mathcal{W}^2_+ + \mathcal{W}^2_-$: in $d = 4$ this term is part of the Gauss-Bonnet topological invariant (it can be made equal to it with suitable field redefinitions). This term plays no role in the dynamics and it is automatically supersymmetric; its supersymmetric completion is 0 and therefore does not involve the gravitino.

The derivation of [35] has been obtained using $\mathcal{N} = 1$ supergravity, whose supersymmetry algebra is a subalgebra of $\mathcal{N} > 1$. Therefore, it should remain valid for extended supergravity too. But one must keep in mind the assumptions which were made, namely the preservation by the supersymmetry transformations of R-symmetry which, for $\mathcal{N} = 1$, corresponds to U(1) and is equivalent to chirality. In extended supergravity theories $R-$symmetry is a global internal U $(\mathcal{N})$ symmetry, which generalizes (and contains) U(1) from $\mathcal{N} = 1$.

Preservation of chirality is true for pure $\mathcal{N} = 1$ supergravity, but to this theory and to most of the extended supergravity theories one may add matter couplings and extra terms which violate U(1) R-symmetry and yet can be made supersymmetric, inducing corrections to the supersymmetry transformation laws which do not preserve U(1) R-symmetry.

Having this in mind [36], we consider a chiral multiplet, represented by a chiral superfield $\mathbf{\Phi}$ (we could take several chiral multiplets Φ_i, which show up after $d = 4$ compactifications of superstring and heterotic theories and truncation to $\mathcal{N} = 1$ supergravity, but we restrict ourselves to one for simplicity), and containing a scalar field $\Phi = \mathbf{\Phi}|$, a spin$-\frac{1}{2}$ field $\nabla_A \mathbf{\Phi}|$, and an auxiliary field $F = -\frac{1}{2} \nabla^2 \mathbf{\Phi}|$. This superfield and its hermitian conjugate couple to $\mathcal{N} = 1$ supergravity in its simplest version through a superpotential

$$P(\mathbf{\Phi}) = d + a\mathbf{\Phi} + \frac{1}{2} m \mathbf{\Phi}^2 + \frac{1}{3} g \mathbf{\Phi}^3 \tag{84}$$

and a Kähler potential $K\left(\mathbf{\Phi}, \overline{\mathbf{\Phi}}\right) = -\frac{3}{\kappa^2} \ln \left(-\frac{\Omega\left(\mathbf{\Phi}, \overline{\mathbf{\Phi}}\right)}{3} \right)$, with

$$\Omega\left(\mathbf{\Phi}, \overline{\mathbf{\Phi}}\right) = -3 + \mathbf{\Phi}\overline{\mathbf{\Phi}} + c\mathbf{\Phi} + \overline{c}\overline{\mathbf{\Phi}}. \tag{85}$$

In order to include the term (77), we take the following effective action:

$$\begin{aligned} \mathcal{L} &= -\frac{1}{6\kappa^2} \int E \left[\Omega\left(\mathbf{\Phi}, \overline{\mathbf{\Phi}}\right) + \alpha'^3 \left(b\mathbf{\Phi} \left(\nabla^2 W^2\right)^2 + \overline{b\mathbf{\Phi}} \left(\overline{\nabla}^2 \overline{W}^2 \right)^2 \right) \right] d^4\theta \\ &- \frac{2}{\kappa^2} \left(\int \epsilon P(\mathbf{\Phi})\, d^2\theta + \text{h.c.} \right). \end{aligned} \tag{86}$$

If one expands (86) in components, one does not directly get (77), but one should look at the auxiliary field sector. Because of the presence of the higher-derivative terms, the auxiliary field from the original conformal supermultiplet A_m also gets higher derivatives in its equation of motion, and therefore it cannot be simply eliminated [19, 34]. Because the auxiliary fields M, N belong to the chiral compensating multiplet, their field equation should be algebraic, despite the higher derivative corrections [19, 34]. That calculation should still require some effort; plus, those M, N auxiliary fields should not generate by themselves terms which violate U(1) R-symmetry: these terms should only occur through the elimination of the chiral multiplet auxiliary fields $F, \bar{F}$. This is why we will only be concerned with these auxiliary fields, which therefore can be easily eliminated through their field equations [21]. The final result, taking into account only terms up to order α'^3, is

$$\begin{aligned} \kappa^2 \mathcal{L}_{F,\overline{F}} &= -15e \frac{(3 + c\overline{c})}{(3 + 4c\overline{c})^2} \left(m\overline{a}\Phi + \overline{m}a\overline{\Phi} \right) \left(c\Phi + \overline{c}\overline{\Phi} \right) \\ &+ e \frac{2c^3\overline{c}^3 + 60c^2\overline{c}^2 + 117c\overline{c} - 135}{(3 + 4c\overline{c})^3} a\overline{a}\Phi\overline{\Phi} - 36\alpha'^3 e \left(b\overline{c} \left(\nabla^2 W^2\right)^2 \right| \end{aligned}$$

$$
\begin{aligned}
&+ \quad \bar{b}c\left(\overline{\nabla}^2\overline{W}^2\right)^2\Big|\Big)\frac{a\overline{a}+m\overline{a}\Phi+\overline{m}a\overline{\Phi}+g\overline{a}\Phi^2+\overline{g}a\overline{\Phi}^2+m\overline{m}\Phi\overline{\Phi}}{\left(3+4c\overline{c}\right)^2}\\
&- \quad 3\alpha'^3 a\overline{a}\frac{74c^2\overline{c}^2+192c\overline{c}-657}{\left(3+4c\overline{c}\right)^4}\Phi\overline{\Phi}\left(b\overline{c}\left(\nabla^2W^2\right)^2\Big|+\overline{b}c\left(\overline{\nabla}^2\overline{W}^2\right)^2\Big|\right)\\
&+ \quad 15\alpha'^3 e\frac{a\overline{a}+m\overline{a}\Phi+\overline{m}a\overline{\Phi}}{\left(3+4c\overline{c}\right)^3}\Big[\\
&\qquad \left(c^2\left(21+4c\overline{c}\right)\Phi+\left(-9+6c\overline{c}\right)\overline{\Phi}\right)\overline{b}\left(\overline{\nabla}^2\overline{W}\right)^2\Big|+\text{h.c.}\Big]+\ldots
\end{aligned}
\tag{87}
$$

This way we are able to supersymmetrize $\mathcal{W}_+^4+\mathcal{W}_-^4$, although we had to introduce a coupling to a chiral multiplet. Since from (54) and (55) the factor in front of $\mathcal{W}_+^4$ (resp. $\mathcal{W}_-^4$) in (87) is given by $\frac{-144b\overline{c}a\overline{a}}{(3+4c\overline{c})^2}$ (resp. $\frac{-144\overline{b}ca\overline{a}}{(3+4c\overline{c})^2}$), for this supersymmetrization to be effective, the factors a from $P(\Phi)$ in (84) and c from $\Omega\left(\Phi,\overline{\Phi}\right)$ in (85) (and of course b from (86)) must be nonzero.

5.2.1 $\mathcal{W}_+^4+\mathcal{W}_-^4$ in extended supergravity

$\mathcal{W}_+^4+\mathcal{W}_-^4$ must also arise in extended $d=4$ supergravity theories, for the reasons we saw, but the "no-go" result of [35] should remain valid, since it was obtained for $\mathcal{N}=1$ supergravity, which can always be obtained by truncating any extended theory. For extended supergravities, the chirality argument should be replaced by preservation by supergravity transformations of U(1), which is a part of R-symmetry.

$\mathcal{N}=2$ supersymmetrization of $\mathcal{W}_+^4+\mathcal{W}_-^4$ should work in a way similar to what we saw for $\mathcal{N}=1$. $\mathcal{N}=2$ chiral superfields must be Lorentz and SU(2) scalars but they can have an arbitrary U(1) weight, which allows supersymmetric U(1) breaking couplings.

A similar result should be more difficult to implement for $\mathcal{N}\geq 3$, because there are no generic chiral superfields. Still, there are other multiplets than the Weyl, which one can consider in order to couple to $\mathcal{W}_+^4+\mathcal{W}_-^4$ and allow for its supersymmetrization. The only exception is $\mathcal{N}=8$ supergravity, a much more restrictive theory because of its higher amount of supersymmetry. In this case one can only take its unique multiplet, which means there are no extra matter couplings one can consider. We have shown that the $\mathcal{N}=8$ supersymmetrization of $\mathcal{W}_+^4+\mathcal{W}_-^4$, coupled to scalar fields from the Weyl multiplet, is not allowed even at the linearized level [37]. In $\mathcal{N}=8$ superspace one can only have SU(8) invariant terms, and we argued $\mathcal{W}_+^4+\mathcal{W}_-^4$ should be only SU(4) $\otimes$ SU(4) invariant. If that is the case, in order to supersymmetrize this term besides the supergravity multiplet one must introduce $U-$duality multiplets, with massive string states and nonperturbative states. The fact that one cannot supersymmetrize in $\mathcal{N}=8$ a term which string theory requires to be supersymmetric, together with the fact that one needs to consider nonperturbative states (from $U-$duality multiplets) in order to understand a perturbative contribution may be seen as indirect evidence that $\mathcal{N}=8$ supergravity is indeed in the swampland [38]. We believe that topic deserves further study.

6 Applications to black holes in string theory

String-corrected black holes have been a very active recent topic of research, for which one needs to know the string effective actions to a certain order in α'. Topics which have been studied include finding $\alpha'-$corrected black hole solutions by themselves, but also studying their properties like the entropy. One of the biggest successes of string theory was the calculation of the microscopic entropy of a class of supersymmetric black holes and the verification that this result corresponds precisely to the macroscopic result of Bekenstein and Hawking. Clearly it is very important to find out if and how this correspondence extends to the full string effective action, without α' corrections.

Because of different α' corrections each quantity gets, typically the entropy does not equal one quarter of the horizon area for black holes with higher derivative terms. In order to compute the entropy for these black holes, a formula has been developed by Wald [39]. When this formula is applied to extremal (not necessarily supersymmetric) black holes, one arrives at the entropy functional formalism developed by Sen (for a complete review see [40]). This formalism can be summarized as follows: one considers a black hole solution from a lagrangian $\mathcal{L}$ with gravity plus some gauge fields and massless scalars in d dimensions. The near horizon limit of such black hole corresponds to $AdS_2 \times S^{d-2}$ geometry, with two parameters v_1, v_2. Also close to the horizon, the gauge fields are parameterized by sets of electric (e_i) and magnetic (p_a) charges, and the scalar fields by constants u_s. The parameters $(\vec{u}, \vec{v}, \vec{e}, \vec{p})$ are up to now arbitrary and, therefore, the solution is off-shell. Next we define the function (to be evaluated in the near horizon limit)

$$f\left(\vec{u}, \vec{v}, \vec{e}, \vec{p}\right) = \int_{S^{d-2}} \sqrt{-g}\mathcal{L}\, d\Omega_{d-2}.$$

The on-shell values of $\vec{u}, \vec{v}, \vec{e}$ for a given theory are found through the relations

$$\frac{\partial f}{\partial u_s} = 0,\ \frac{\partial f}{\partial v_j} = 0,\ \frac{\partial f}{\partial e_i} = q_i,$$

which also reproduce the equations of motion. Then, using Wald's formulation, Sen derived the black hole entropy, given by

$$S = 2\pi \left(e_i \frac{\partial f}{\partial e_i} - f \right).$$

This process has been verified for extremal (supersymmetric or not) black holes in generic d dimensions. In particular, it has been tested with off shell formulations of supergravity [41] (these formulations are known for $\mathcal{N} = 1$ in $d = 4$ or $\mathcal{N} = 2$ in $d = 4, 5, 6$). When one considers black holes in these theories, auxiliary fields must also be considered in f, necessarily as independent fields (since for this functional we take an a-priori off-shell solution). As we have seen, when considering theories with higher-derivative corrections, some of these auxiliary fields can still be eliminated, but others become dynamical. Clearly a precise knowledge of the behavior of the different auxiliary fields, like we have studied, is essential if one wishes to determine the higher-derivative corrections to black hole properties such as the entropy.

A particularly well studied case [42] (which has been reviewed in this volume [43]) is that of BPS black holes in $d = 4, \mathcal{N} = 2$ supergravity coupled to n vector multiplets, to which are associated n scalar fields X^I and n vector fields A^I_μ. The holomorphic higher-derivative corrections associated to these black holes are given as higher genus contributions to the prepotential, in the form of a function

$$F(X^I, \hat{A}) = \sum_{g=0}^{\infty} F^{(g)}(X^I) \hat{A}^g, \tag{88}$$

$\hat{A}$ being a scalar field which, in our conventions, is given by $\hat{A} = W^{AB} W_{AB}\big|$. From (72), one sees that $\hat{A}$ is related to the square of the selfdual part of the graviphoton field strength $F_{\mu\nu}$, but also to the square of the auxiliary field Y_{AB} (which, as we saw, may become dynamical in the presence of higher-derivative terms). From (73), one immediately sees that a lagrangian containing $F(X^I, \hat{A})$ as an $F-$term includes $\mathcal{W}^2$ terms, each multiplied by terms depending on moduli and on powers of either $F_{\mu\nu}$ or Y_{mn}. These Y_{mn} factors may generate terms with higher powers of the Weyl tensor $\mathcal{W}_{\mu\nu\rho\sigma}$.

After some rescaling (in order to have manifest symplectic covariance), $\hat{A}$ becomes the variable Υ, which at the horizon takes a particular numerical value ($\Upsilon = -64$ in the conventions of [43]). This value is universal, independent of the model taken (i.e. for any function $F(X^I, \hat{A})$ of the form (88)), as long as the black hole solution under consideration is supersymmetric. There may exist other near-horizon configurations (corresponding to nonsupersymmetric black holes) which extremize the entropy function but correspond to different attractor equations and different values for Υ. These values are not universal: each solution has its own (constant) Υ.

The generalized prepotential (88) does not represent the full set of higher derivative corrections one must consider in a supersymmetric theory in $d = 4$, even for a black hole solution. There are also the nonholomorphic corrections, which are necessary for the entropy to be invariant under string dualities, as discussed in [43]. At the time, the way to incorporate these corrections into the attractor mechanism is still under study. On general grounds, if Υ is coupled to the nonholomorphic corrections, then it should in principle get a different value. This (still unknown) different value for Υ should also in principle depend on the model which we are taking. Because of this nonuniversality, we cannot simply take a general expression for the nonholomorphic corrections: we really need each term, to the order we are working, in the effective action. For that, in the cases when auxiliary fields (namely Υ) exist and are part of the higher derivative correction terms (as studied in [44]), we must know exactly their behavior in the presence of such corrections, in the way we presented on the first part of these notes.

7 Summary and discussion

We computed the $\mathcal{R}^4$ terms in the superstring effective actions in four dimensions. We showed that besides the usual square of the Bel-Robinson tensor $\mathcal{W}_+^2\mathcal{W}_-^2$, the other possible $\mathcal{R}^4$ term in $d = 4$, $\mathcal{W}_+^4 + \mathcal{W}_-^4$, was also part of two of those actions at one string loop. We then studied their supersymmetrization.

For $\mathcal{W}_+^2\mathcal{W}_-^2$ we wrote down its supersymmetrization directly in $\mathcal{N} = 1$ and $\mathcal{N} = 2$ superspace, taking advantage of the off-shell formulation of these theories. The terms we wrote down were off-shell; in both cases we tried to obtain the on-shell action by eliminating the auxiliary fields. We noticed that some auxiliary fields could be eliminated, while others couldn't.

A careful analysis shows that, in both cases we studied, the auxiliary fields that can be eliminated in the supersymmetrization of $\mathcal{W}_+^2\mathcal{W}_-^2$ come from multiplets which, on-shell, have no physical fields; while the auxiliary fields that get derivatives come from multiplets with physical fields on-shell (the graviton, the gravitino(s) and, in $\mathcal{N} = 2$, the vector). Our general conjecture for supergravity theories with higher derivative terms, which is fully confirmed in the "old minimal" $\mathcal{N} = 1, 2$ cases with $\mathcal{W}_+^2\mathcal{W}_-^2$, can now be stated: the auxiliary fields which come from multiplets with on-shell physical fields cannot be eliminated, but the ones that come from compensating multiplets that, on shell, have no physical fields, can. In order to get more evidence for it, the analysis we made should also be extended to the other different versions of these supergravity theories, and with other higher derivative terms.

We moved on to try to supersymmetrize $\mathcal{W}_+^4 + \mathcal{W}_-^4$, but we faced a previous result stating that supersymmetrization could not be achieved because in $\mathcal{N} = 1$ it would violate chirality, which is preserved in pure supergravity. The way we found to circumvent this problem was to couple $\mathcal{W}_+^4 + \mathcal{W}_-^4$ to a chiral multiplet and, after eliminating its auxiliary fields, obtain that same term on-shell. We worked this out in $\mathcal{N} = 1$ supergravity and the same should be possible in $\mathcal{N} = 2$. For $\mathcal{N} = 8$ that should not be possible any longer, because there are no other multiplets we could use to couple to $\mathcal{W}_+^4 + \mathcal{W}_-^4$ that could help us: the Weyl multiplet is the only one allowed in this theory. This is a sign that $\mathcal{N} = 8$ supergravity is indeed in the swampland.

We ended by discussing applications of these results to black holes in string theory, namely the attractor mechanism and the calculation of the black hole entropy in the presence of higher derivative terms. We considered extremal black holes in d dimensions, through Sen's entropy functional formalism, and in particular BPS black holes in $d = 4, \mathcal{N} = 2$ supergravity. In all cases we concluded that, having those applications in mind, when auxiliary fields exist, one needs to know exactly their behavior in the presence of such higher derivative corrections.

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