Branch-and-Bound Reduction Type Method for Semi-Infinite Programming

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Abstract. Semi-infinite programming (SIP) problems can be efficiently solved by reduction type methods. Here, we present a new reduction method for SIP, where the multi-local optimization is carried out with a multi-local branch-and-bound method, the reduced (finite) problem is approximately solved by an interior point method, and the global convergence is promoted through a two-dimensional filter line search. Numerical experiments with a set of well-known problems are shown.

Keywords: Nonlinear Optimization. Semi-Infinite Programming. Global Optimization.

1 Introduction

A reduction type method for nonlinear semi-infinite programming (SIP) based on interior point and branch-and-bound methods is proposed. To allow convergence from poor starting points a backtracking line search filter strategy is implemented. The SIP problem is considered to be of the form

$$\min f(x) \text{ subject to } g(x,t) \leq 0, \text{ for every } t \in T \tag{1}$$

where $T \subseteq \mathbb{R}^m$ is a nonempty set defined by $T = \{t \in \mathbb{R}^m : a \leq t \leq b\}$. Here, we assume that the set T does not depend on x. The nonlinear functions $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \times T \to \mathbb{R}$ are twice continuously differentiable with respect to x and g is a continuously differentiable function with respect to t.

There are many problems in the engineering area that can be formulated as SIP problems. Approximation theory [14], optimal control [8], mechanical stress of materials and computer-aided design [37], air pollution control [31], robot trajectory planning [30], financial mathematics and computational biology and medicine [36] are some examples. For a review of other applications, the reader is referred to [5, 14, 23, 26, 32].

The numerical methods that are mostly used to solve SIP problems generate a sequence of finite problems. There are three main ways of generating the sequence: by discretization, exchange and reduction methods [8, 23, 30]. Methods that solve the SIP problem on the basis of the KKT system derived from the problem are emerging in the literature [11–13, 21, 22, 37].

This work aims to describe a reduction method for SIP. Conceptually, the method is based on the local reduction theory.

Our previous work on reduction type methods uses a stretched simulated annealing for the multi-local programming phase of the algorithm [19]. This is a stochastic method and convergence is guaranteed with probability one [10]. In this paper, we aim at analyzing the behavior of a reduction method that relies on a deterministic multi-local procedure, so that convergence to global solutions can be guaranteed in a finite number of steps. A practical comparison between both strategies is also carried out. Our proposal is focused on a multi-local procedure that makes use of a well-known deterministic global optimization method - the branch-and-bound method [6, 9]. In the reduction method context, the solution of the reduced finite optimization problem is achieved by an interior point method. To promote convergence from any initial approximation a two-dimensional filter methodology, as proposed in [4], is also incorporated into the reduction algorithm.

The paper is organized as follows. In Section 2, the basic ideas behind the local reduction of SIP to finite problems are presented. Section 3 is devoted to the multi-local procedure and Section 4 briefly describes an interior point method for solving the reduced optimization problem. Section 5 presents the filter methodology to promote global convergence, Section 6 lists the conditions for the termination of the algorithm, and Section 7 contains some numerical results and conclusions.

2 First-order optimality conditions and reduction method

In this section we present some definitions and the optimality conditions of problem (1). We denote the *feasible set* of problem (1) by X, where

$$X = \{x \in \mathbb{R}^n : g(x,t) \leq 0, \text{ for every } t \in T\}.$$

A feasible point $\bar{x} \in X$ is called a *strict local minimizer* of problem (1) if there exists a positive value ϵ such that

$$\forall x \in X : f(x) - f(\bar{x}) > 0 \;\wedge\; \|x - \bar{x}\| < \epsilon \wedge x \neq \bar{x}$$

where $\|.\|$ represents the euclidean norm. For $\bar{x} \in X$, the *active index set*, $T_0(\bar{x})$, is defined by

$$T_0(\bar{x}) = \{t \in T : g(\bar{x},t) = 0\}.$$

We first assume that:

Condition 1 *Let $\bar{x} \in X$. The linear independence constraint qualification (LICQ) holds at $\bar{x}$, i.e., $\{\nabla_x g(\bar{x},t),\, t \in T_0(\bar{x})\}$ is a linearly independent set.*

Since LICQ implies the Mangasarian-Fromovitz Constraint Qualification (MFCQ) [14], we can conclude that for $\bar{x} \in X$ there exists a vector $d \in \mathbb{R}^n$ such that for every $t \in T_0(\bar{x})$ the condition $\nabla_x g(\bar{x}, t)^T d < 0$ is satisfied. A direction d that satisfies this condition is called a *strictly feasible direction*. Further, the vector $d \in \mathbb{R}^n$ is a *strictly feasible descent direction* if the following conditions

$$\nabla f(\bar{x})^T d < 0, \ \nabla_x g(\bar{x}, t)^T d < 0, \text{ for every } t \in T_0(\bar{x}) \tag{2}$$

hold. If $\bar{x} \in X$ is a local minimizer of the problem (1) then there will not exist a strictly feasible descent direction $d \in \mathbb{R}^n \setminus \{0_n\}$, where 0_n represents the null vector of $\mathbb{R}^n$. A sufficient condition to identify a strict local minimizer of SIP can be described in the following theorem, that is based on Theorem 1 presented in [14].

Theorem 1. *Let $\bar{x} \in X$. Suppose that there is no direction $d \in \mathbb{R}^n \setminus \{0_n\}$ satisfying*

$$\nabla f(\bar{x})^T d \leq 0 \text{ and } \nabla_x g(\bar{x}, t)^T d \leq 0, \text{ for every } t \in T_0(\bar{x}).$$

Then $\bar{x}$ is a strict local minimizer of SIP.

Since Condition 1 is verified, the set $T_0(\bar{x})$ is finite. Suppose that $T_0(\bar{x}) = \{t_1, ..., t_p\}$, then $p \leq n$. If $\bar{x}$ is a local minimizer of problem (1) and if the MFCQ holds at $\bar{x}$, then there exist nonnegative values λ_i for $i = 1, ..., p$ such that

$$\nabla f(\bar{x}) + \sum_{i=1}^{p} \lambda_i \nabla_x g(\bar{x}, t_i) = 0_n. \tag{3}$$

This is the Karush-Kuhn-Tucker (KKT) condition of problem (1).

Many papers exist in the literature devoted to the reduction theory [2, 7, 8, 20, 23, 27]. The main idea is to describe, locally, the feasible set of the problem (1) by finitely many constraints. Assume that $\bar{x}$ is a feasible point and that each $t_l \in \bar{T} \equiv T(\bar{x})$ is a local maximizer of the so-called *lower level problem*

$$\max_{t \in T} g(\bar{x}, t), \tag{4}$$

satisfying the following condition

$$|g(\bar{x}, t_l) - g^*| \leq \delta^{ML}, \ l = 1, \ldots, \bar{L}, \tag{5}$$

where $\bar{L} \geq p$ and $\bar{L}$ represents the cardinality of $\bar{T}$, δ^{ML} is a positive constant and g^* is the global solution value of (4).

Condition 2 *For any fixed $\bar{x} \in X$, each $t_l \in \bar{T}$ is a strict local maximizer, i.e.,*

$$\exists \delta > 0, \forall t \in T : g(\bar{x}, t_l) > g(\bar{x}, t) \ \wedge \ \|t - t_l\| < \delta \ \ \wedge t \neq t_l.$$

Since the set T is compact, $\bar{x}$ is a feasible point and Condition 2 holds, then there exists a finite number of local maximizers of the problem (4) and the implicit function theorem can be applied, under some constraint qualifications [14]. So, it is possible to conclude that there exist open neighborhoods $\bar{U}$, of $\bar{x}$, and V_l, of t_l, and implicit functions $t_1(x), \ldots, t_{\bar{L}}(x)$ defined as:

i) $t_l : \bar{U} \to V_l \cap T$, for $l = 1, \ldots, \bar{L}$;
ii) $t_l(\bar{x}) = t_l$, for $l = 1, \ldots, \bar{L}$;
iii) $\forall x \in \bar{U}$, $t_l(x)$ is a non-degenerate and strict local maximizer of the lower level problem (4);

so that

$$\{x \in \bar{U} : g(x,t) \leq 0, \text{ for every } t \in T\} \Leftrightarrow \{x \in \bar{U} : g(x, t_l(x)) \leq 0, l = 1, ..., \bar{L}\}.$$

So it is possible to replace the infinite set of constraints by a finite set that is locally sufficient to define the feasible region. Thus the problem (1) is locally equivalent to the so-called *reduced (finite) optimization problem*

$$\min_{x \in \bar{U}} f(x) \text{ subject to } g_l(x) \equiv g(x, t_l(x)) \leq 0,\, l = 1, \ldots, \bar{L}. \tag{6}$$

A reduction method then emerges when any method for finite programming is applied to solve the locally reduced problem (6). This comes out as being an iterative process, herein indexed by k. Algorithm 1 below shows the main procedures of the proposed reduction method:

Algorithm 1 *Global reduction algorithm*

Given x^0 feasible, $\delta^{ML} > 0$, $k^{\max} > 0$, $\epsilon_g, \epsilon_f, \epsilon_x > 0$ and $i^{\max} > 0$; set $k = 0$.

1. *Based on x^k, compute the set T^k, solving problem*
$$\max_{t \in T} g(x^k, t), \tag{7}$$
with condition $\left|g(x^k, t_l) - g^*\right| \leq \delta^{ML}$, $t_l \in T^k$ (g^* is the global solution of (7)).
2. *Set $x^{k,0} = x^k$ and $i = 1$.*
   2.1. *Based on the set T^k, compute an approximation $x^{k,i}$, by solving the reduced problem*
   $$\min f(x) \text{ subject to } g_l(x) \equiv g(x, t_l) \leq 0,\, t_l \in T^k.$$
   2.2. *Stop if $i \geq i^{\max}$; otherwise set $i = i + 1$ and go to Step 2.1.*
3. *Based on $d^k = x^{k,i} - x^{k,0}$, compute a new approximation x^{k+1} that improves significantly over x^k using a globalization technique. If it is not possible, set $d^k = d^{k,1}$ ($d^{k,1}$ is the first computed direction in Step 2.1) and compute a new approximation x^{k+1} that improves significantly over x^k using a globalization technique.*
4. *Stop if termination criteria are met or $k \geq k^{\max}$; otherwise set $k = k + 1$ and go to Step 1.*

The remaining part of this paper presents our proposals for the Steps 1, 2, 3 and 4 of the Algorithm 1 for SIP.

An algorithm to compute the set T^k is known in the literature as a multi-local procedure. In this paper, a multi-local branch-and-bound (B&B) algorithm is implemented. The choice of a B&B type method is based on the fact that this is a deterministic method. Typically deterministic methods converge (with theoretical guarantee) to a global solution in a finite number of steps [6].

To solve the reduced problem (6) an interior point method is proposed. This type of methods have been implemented in robust software for finite optimization problems [29, 35]. They have shown to be efficient and robust in practice.

Finally, convergence of the overall reduction method to a SIP solution is encouraged by implementing a filter line search technique. The filter here aims to measure sufficient progress by using the constraint violation and the objective function value. This filter strategy has been shown to behave well for SIP problems when compared with merit function approaches ([17, 18]).

3 The multi-local procedure

The multi-local procedure is used to compute the set T^k, i.e., the local solutions of the problem (4) that satisfy (5). Some procedures to find the local maximizers of the constraint function consist of two phases: first, a discretization of the set T is made and all maximizers are evaluated on that finite set; second, a local method is applied in order to increase the accuracy of the approximations found in the first phase (e.g. [3]). Other proposal combines the function stretching technique, proposed in [16], with a simulated annealing (SA) type algorithm - the ASA variant of the SA in [10]. This is a stochastic point-to-point global search method that generates the elements of T^k sequentially [19].

In this work, to compute the solutions of (4) that satisfy (5), the branch-and-bound method is combined with strategies that keep the solutions that are successively identified during the process. The branch-and-bound method is a well-known deterministic technique for global optimization whose basic idea consists of a recursive decomposition of the original problem into smaller disjoint subproblems. The method avoids visiting those subproblems which are known not to contain a solution [6, 9].

So, given x^k, the main step of the multi-local B&B method is to solve a set of subproblems described as

$$\max g(x^k, t) \text{ for } t \in I^{i,j} \text{ for } i = 1, \ldots, n_j \tag{8}$$

where $I^{i,j} = [l_1^{i,j}, u_1^{i,j}] \times \ldots \times [l_m^{i,j}, u_m^{i,j}]$, and the sets $I^{i,j}$, for $i = 1, \ldots, n_j$, represent a list of sets, denoted by $\mathcal{L}^j$, that can have a local solution that satisfies condition (5).

The method starts with the list $\mathcal{L}^0$, with the set $I^{1,0} = T$, as the first element and stops at iteration j when the list $\mathcal{L}^{j+1}$ is empty. Furthermore, the algorithm will always converge due to the final check on the size of the set $I^{i,j}$. A fixed value, $\delta > 0$, is provided in order to guarantee a δ-optimal solution.

The generic scheme of the multi-local B&B algorithm can be formally described as in the Algorithm 2.

Algorithm 2 *Multi-local B&B algorithm*

Given x^k, $\epsilon > 0$, $\delta > 0$.

1. *Consider* g_0 *the solution of problem (8), for* $I^{1,0} = T$. *Set* $j = 0$ *and* $n_0 = 1$.
2. *Split each set* $I^{i,j}$ *into intervals, for* $i = 1, ..., n_j$;
   set $\mathcal{L}^{j+1} = \left\{ I^{1,j+1}, \ldots, I^{n_{j+1},j+1} \right\}$.
3. *Solve problem (8), for all sets in* $\mathcal{L}^{j+1}$. *Set* $g_1, \ldots, g_{n_{j+1}}$ *to the obtained maxima values.*
4. *Set* $g_0 = \max_i \{g_i\}$ *for* $i = 0, ..., n_{j+1}$. *Select the sets* $I^{i,j+1}$ *that satisfy the condition:*
$$|g_0 - g_i| < \epsilon.$$
5. *Reorganize the set* $\mathcal{L}^{j+1}$; *update* n_{j+1}.
6. *If* $\mathcal{L}^{j+1} = \emptyset$ *or* $\max_i \left\{ \left|\left| u^{i,j} - l^{i,j} \right|\right| \right\} < \delta$ *stop the process; otherwise set* $j = j + 1$ *and go to Step 2.*

4 Finite optimization procedure

The sequential quadratic programming method is the most used finite programming procedure in reduction type methods for solving SIP problems. L_1 and L_∞ merit functions and a trust region framework to ensure global convergence are usually proposed [3, 20, 27]. Penalty methods with exponential and hyperbolic penalty functions have already been tested with some success [18, 19]. However, to solve finite inequality constrained optimization problems, interior point methods [1, 24, 25, 28, 29] and interior point barrier methods [1, 33–35] have shown to be competitive and even more robust than sequential quadratic programming and penalty type methods. Thus, an interior point method is incorporated into the proposed reduction algorithm aiming to improve efficiency over previous reduction methods.

When using an interior point method, the reduced problem (6) is reformulated in a way that the unique inequality constraints are simple nonnegativity constraints. So, the first step is to introduce slack variables to replace all inequality constraints by equality constraints and simple nonnegativity constraints. Hence, adding nonnegative slack variables $w = (w_0, w_1, \ldots, w_{L^k+1})^T$ to the inequality constraints, the problem (6) is rewritten as follows

$$\min_{x \in U^k, w \in \mathbb{R}^{L^k+2}} f(x) \text{ subject to } g_l(x) + w_l = 0,\ w_l \geq 0,\ l = 0, \ldots, L^k + 1\,, \quad (9)$$

where $g_0(x) = g(x, a)$ and $g_{L^k+1}(x) = g(x, b)$ correspond to the values of the constraint function $g(x, t)$ at the lower and upper limits of set T. In an interior point barrier method, the solution of the problem (9) is obtained by computing approximate solutions of a sequence of (associated) barrier problems

$$\min_{x \in U^k, w \in \mathbb{R}^{L^k+2}} \Phi(x, w; \mu) \text{ subject to } g_l(x) + w_l = 0,\ l = 0, \ldots, L^k + 1\,, \quad (10)$$

for a decreasing sequence of positive barrier parameters $\mu \searrow 0$, while maintaining $w > 0$, where

$$\Phi(x, w; \mu) = f(x) - \mu \sum_{l=0}^{L^k+1} \log(w_l)$$

is the barrier function [1]. For a given fixed value of μ, the Lagrangian function for the problem is

$$\mathbf{L}(x, w, y) = \Phi(x, w; \mu) + y^T (g(x) + w)$$

where y is the Lagrange multiplier vector associated with the constraints $g(x) + w = 0$, and the KKT conditions for a minimum of (10) are

$$\begin{aligned} \nabla f(x) + \nabla g(x) y &= 0 \\ -\mu W^{-1} e + y &= 0 \\ g(x) + w &= 0 \end{aligned} \tag{11}$$

where $\nabla f(x)$ is the gradient vector of f, $\nabla g(x)$ is the matrix whose columns contain the gradients of the functions in vector g, $W = diag(w_0, \ldots, w_{L^k+1})$ is a diagonal matrix and $e \in \mathbb{R}^{L^k+2}$ is a vector of ones. Note that equations (11) are equivalent to

$$\begin{aligned} \nabla f(x) + \nabla g(x) y &= 0 \\ z - y &= 0 \\ -\mu e + W z &= 0 \\ g(x) + w &= 0, \end{aligned} \tag{12}$$

where z is the Lagrange multiplier vector associated with $w \geq 0$ in (9) and, for $\mu = 0$, together with $w, z \geq 0$ are the KKT conditions for the problem (9). They are the first-order optimality conditions for problem (9) if the LICQ is satisfied.

Applying Newton's method to solve the system (11), we obtain a linear system to compute the search directions $\Delta x, \Delta w, \Delta y$

$$\begin{bmatrix} H(x,y) & 0 & \nabla g(x) \\ 0 & \mu W^{-2} & I \\ \nabla g(x)^T & I & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta w \\ \Delta y \end{bmatrix} = - \begin{bmatrix} \nabla f(x) + \nabla g(x) y \\ y - \mu W^{-1} e \\ g(x) + w \end{bmatrix} \tag{13}$$

where $H(x, y) = \nabla^2 f(x) + \sum_{l=0}^{L^k+1} y_l \nabla^2 g_l(x)$.

Let the matrix $N \equiv N(x, w, y) = H(x, y) + \mu \nabla g(x) W^{-2} \nabla g(x)^T$ denote the dual normal matrix.

Theorem 2. *If N is nonsingular, then system (13) has a unique solution.*

Proof. From the second equation of (13), Δw can be eliminated giving

$$\Delta w = \mu^{-1} W^2 (-y - \Delta y) + W e$$

and the reduced system

$$\begin{bmatrix} H(x,y) & \nabla g(x) \\ \nabla g(x)^T & -\mu^{-1} W^2 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = - \begin{bmatrix} \nabla f(x) + \nabla g(x) y \\ g(x) + w + W e - \mu^{-1} W^2 y \end{bmatrix}. \tag{14}$$

Solving the second equation in (14) for Δy we obtain:

$$\Delta y = \mu W^{-2} \left((g(x) + w) + We - \mu^{-1} W^2 y + \nabla g(x)^T \Delta x \right) \tag{15}$$

and substituting in the first equation

$$H(x,y)\Delta x + \mu \nabla g(x) W^{-2} \left((g(x) + w) + We \; - \mu^{-1} W^2 y + \nabla g(x)^T \Delta x \right) \\ = -(\nabla f + \nabla g(x) y)$$

yields an equation involving only Δx that depends on N:

$$N(x,w,y)\Delta x = -\nabla f - \mu \nabla g(x) W^{-2} (g(x) + w) - \mu \nabla g(x) W^{-1} e.$$

From here, Δy and Δw could also be determined depending on N.

■

It is known that if the initial approximation is close enough to the solution, methods based on the Newton's iteration converge quadratically under appropriate assumptions. For poor initial points, a backtracking line search can be implemented to promote convergence to the solution of problem (6) [15]. After the search directions have been computed, the idea is to choose $\alpha^i \in (0, \alpha^i_{\max}]$, at iteration i, so that $x^{k,i+1} = x^{k,i} + \alpha^i \Delta x^i$, $w^{k,i+1} = w^{k,i} + \alpha^i \Delta w^i$ and $y^{k,i+1} = y^{k,i} + \alpha^i \Delta y^i$ improve over an estimate solution $(x^{k,i}, w^{k,i}, y^{k,i})$ for problem (6). The index i represents the iteration counter of this inner cycle. The parameter $\alpha^i_{\max}$ represents the longest step size that can be taken along the directions before violating the nonnegativity conditions $w \geq 0$ and $y \geq 0$.

5 Globalization procedure

To achieve convergence to the solution within a local framework, line search methods use, in general, penalty or merit functions. A backtracking line search method based on a filter approach, as a tool to guarantee global convergence in algorithms for nonlinear constrained finite optimization [4, 33], avoids the use of a merit function. A filter method uses the concept of nondominance, from multi-objective optimization, to build a filter that is able to accept approximations if they improve either the objective function or the constraint violation, instead of a linear combination of those measures present in a merit function. So the filter replaces the use of merit functions, avoiding the update of penalty parameters.

This new technique has been combined with a variety of optimization methods to solve different types of optimization problems. Its use to promote global convergence to the solution of a SIP problem was originally presented in [17, 18]. We also extend its use to the herein proposed branch-and-bound reduction method. Its practical competitiveness with other methods in the literature suggests that this research is worth pursuing and the theoretical convergence analysis should be carried out in a near future.

To define the next approximation to the SIP problem, a two-dimensional filter line search method is implemented. Each entry in the filter has two components, one measures SIP-feasibility, $\Theta(x) = \| \max_{t\in T} (0, g(x,t)) \|_2$, and the other SIP-optimality, f (the objective function). First we assume that $d^k = x^{k,i} - x^k$, where i is the iteration index that satisfies the acceptance conditions that decide that an improvement over a previous estimate x^k is achieved. Based on d^k, the below described filter line search methodology computes the trial point $x^{k+1} = x^k + d^k$ and tests if it is acceptable by the filter. However, if this trial point is rejected, the algorithm recovers the direction of the first iteration, $d^k = x^{k,1} - x^k$, and tries to compute a trial step size α^k such that $x^{k+1} = x^k + \alpha^k d^k$ satisfies one of the below acceptance conditions and it is acceptable by the filter.

Here, a trial step size α^k is acceptable if a sufficient progress towards either the SIP-feasibility or the SIP-optimality is verified, i.e., if

$$\Theta^{k+1} \leq (1-\gamma)\Theta^k \;\text{ or }\; f^{k+1} \leq f^k - \gamma\Theta^k \tag{16}$$

holds, for a fixed $\gamma \in (0,1)$. Θ^{k+1} is the simplified notation of $\Theta(x^{k+1})$. On the other hand, if

$$\Theta^k \leq \Theta^{\min},\; (\nabla f^k)^T d^k < 0 \;\text{ and }\; \alpha^k \left[-(\nabla f^k)^T d^k\right]^{\iota} > \beta \left[\Theta^k\right]^r, \tag{17}$$

are satisfied, for fixed positive constants $\Theta^{\min}$, β and r and ι, then the trial approximation x^{k+1} is acceptable only if a sufficient decrease in f is verified

$$f^{k+1} \leq f^k + \eta\alpha^k (\nabla f^k)^T d^k \tag{18}$$

for $\eta \in (0, 0.5)$. The filter is initialized with pairs (Θ, f) that have $\Theta \geq \Theta^{\max} > 0$. If the acceptable approximation does not satisfy the condition (17), the filter is updated; otherwise (conditions (17) and (18) hold) the filter remains unchanged. The reader is referred to [17] for more details concerning the implementation of this filter strategy in the SIP context.

6 Termination criteria

As far as the termination criteria are concerned, in Step 5 of Algorithm 1, our reduction algorithm stops at a point x^{k+1} if the following conditions hold simultaneously:

$$\max\{g_l(x^{k+1}), l = 0, \ldots, L^{k+1}+1\} < \epsilon_g,\; \frac{|f^{k+1} - f^k|}{1 + |f^{k+1}|} < \epsilon_f$$

$$\text{and } \frac{\|x^{k+1} - x^k\|}{1 + \|x^{k+1}\|} < \epsilon_x$$

where $\epsilon_g, \epsilon_f, \epsilon_x > 0$ are given error tolerances.

7 Numerical results and conclusions

The proposed reduction method was implemented in the MatLab programming language on a Atom N280, 1.66Ghz with 2Gb of RAM. For the computational experiments we consider eight test problems from the literature [3, 13, 21, 22, 37]. Different initial points were tested with some problems so that a comparison with other results is possible [3, 37]. In the B&B type multi-local procedure we fix the following constants: $\epsilon = 5.0$, $\delta = 0.5$ and $\delta^{ML} = 1.0$.

In the globalization procedure context, the parameters in the filter line search technique are defined as follows [35]: $\gamma = 10^{-5}$, $\eta = 10^{-4}$, $\beta = 1$, $r = 1.1$, $\iota = 2.3$, $\Theta^{\max} = 10^4 \max\{1, \Theta^0\}$ and $\Theta^{\min} = 10^{-4} \max\{1, \Theta^0\}$.

In the termination criteria of the reduction method we fix the following constants: $\epsilon_g = \epsilon_f = \epsilon_x = 10^{-5}$. Other parameters present in Algorithm 1 are: $k^{\max} = 100$ and $i^{\max} = 5$.

The implementation of the interior point method in the MatLab Optimization Toolbox ™ was used.

Table 1. Computational results from B&B reduction method

$P\#$		n	f^*	$\mathbf{k}_{RM}$
1		2	$-2.50000E-01$	3
2		2	$2.43054E+00$	3
2	(1)	2	$1.94466E-01$	3
2	(2)	2	$1.94466E-01$	2
3		3	$8.64406E-01$	2
3	(2)	3	$8.64406E-01$	2
4		3	$6.49458E-01$	5
5		3	$4.30118E+00$	14
6		2	$9.71589E+01$	3
6	(2)	2	$9.71589E+01$	2
7		3	$1.00000E+00$	3
8	(2)	2	$-3.00000E+00$	2

Table 1 shows the results obtained by the proposed B&B reduction type method. In the table, $P\#$ refers to the problem number as reported in [3]. Problem 8 is from Liu's paper [13]. n represents the number of variables, f^* is the objective function value at the final iterate and $\mathbf{k}_{RM}$ gives the number of iterations needed by the reduction method. We used the initial approximations proposed in the above cited papers. Problems 2, 3, 6 and 8 were solved with the initial approximation proposed in [37] as well. They are identified in Table 1 with $^{(2)}$. When the initial $(0, 0)$ (see $^{(1)}$ in the table) is provided to problem 2 our algorithm reaches the solution obtained in [37].

We also include Table 2 to display the results from the literature, so that a comparison between the herein proposed reduction method and other reduction-type methods is possible. The compared results are taken from the cited papers [3, 17, 19, 20, 27]. In this table, "-" means that the problem is not in the test set of the paper.

Based on these preliminary tests, we may conclude that incorporating the B&B type method into a reduction method for nonlinear SIP, significantly re-

Table 2. Results from other reduction type methods

			in [19]	in [17]	in [20]	in [27]	in [3]
$P\#$	n	m	$\mathbf{k}_{RM}$	$\mathbf{k}_{RM}$	$\mathbf{k}_{RM}$	$\mathbf{k}_{RM}$	$\mathbf{k}_{RM}$
1	2	1	48	47	17	17	16
2	2	2	3	4	8	5	7
3	3	1	3	21	11	9	10
4	3	1	11	-	10	5	5
5	3	1	41	-	8	4	4
6	2	1	7	8	27	16	9
7	3	2	8	7	9	2	3

duces the total number of iterations required by the reduction method. The herein proposed method implements two new strategies in a reduction method context:

- a branch-and-bound method to identify the local solutions of a multi-local optimization problem;
- an interior point method to compute an approximation to the reduced (finite) optimization problem.

The comparison with other reduction type methods based on penalty techniques is clearly favorable to our proposal.

We remark that the assumptions that lie in the basis of the method (Conditions 1 and 2) are too strong and difficult to be satisfied in practice. In future work, they will be substituted by less strong assumptions.

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